On Bankruptcy in General Equilibrium with Uncertainty*

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Abstract

In a general equilibrium model with time and uncertainty the possibility of bankruptcy cannot be excluded in general, when short sale constraints are too loose. Tight short trading constraints, on the other hand, are inefficient. Bankruptcies turn security payoffs endogenous and destroy convexity of the induced preferences over portfolios. The latter raises existence issues for competitive equilibrium, as illustrated in this paper by an example.

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1 Introduction

In a combustion engine a revolution limiter constrains the rotation speed of the drive shaft to avoid overheating of the engine. In the short run this may come with a loss of efficiency, but it does ensure durability of the vehicle. This example illustrates a trade-off between efficiency and resilience. In engineering this trade-off is well understood, giving rise to built-in tolerances. This paper illustrates that in economics there is a similar trade-off: Bounds on actions can lead to inefficiencies but unbounded actions can cause negative externalities. More precisely, we show that in a general equilibrium model with uncertainty binding short-trading constraints yield inefficiencies, while lifting such constraints may entail a failure of existence of competitive equilibrium.

A main substantive result of the dominant theory of general competitive equilibrium concentrates on efficiency by asserting that market outcomes are

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always efficient—the first welfare theorem. While uncertainty had been incorporated in general equilibrium theory (Arrow and Debreu, 1954) early on (Arrow, 1953), a conflict between efficiency and resilience has not been explicitly addressed. This also applies to the extensions of the basic static model to a sequence of (complete or incomplete) markets (Radner, 1966, 1968, 1972).

The conflict between efficiency and resilience illustrated in this paper comes in the form of a trade-off between inefficient short-selling constraints and the possibility of bankruptcy and default. Yet, among the original assumptions that define the model with uncertainty there is one that precludes bankruptcy: It requires " ... that the trader not plan to deliver at any date-event pair more than he would have available from his resources after subtracting his consumption." (Radner, 1972, p. 292)—effectively an "obedience assumption." Of course, for the conceptual exercise of formalizing a sequence of competitive markets—which was Radner's goal—this assumption is perfectly acceptable, as it rules out bankruptcy and the ensuing chain reaction when contractual obligations are abrogated. Nevertheless, its descriptive content is questionable.

If you sell your homegrown potatoes at the farmer's market on Saturday morning, you will not be able to sell more than what you brought, because each customer can easily verify how much you have left. But when you sell the senior tranche of a pool of asset-backed securities to an anonymous market, matters are different. You may not have an accurate assessment of default risks for the underlying, and even if you do, it is difficult for the buyer to verify your estimates. In particular, if you are protected by limited liability—or a nonnegativity constraint on future consumption—there is an incentive to overstate how much you will be able to deliver. In short, by promising more today than what you will be able to deliver tomorrow you make yourself richer today, while being protected by limited liability tomorrow.

As a matter of realism one may introduce an institution that assesses a borrower's creditworthiness. For instance, agents who default could be excluded from contingent claims markets for the entire future (as in Kehoe and Levine, 1993; see also Alvarez and Jermann, 2000). Such personalized scrutiny is however at odds with the anonymity assumption of general equilibrium theory, which is institution-free. Indeed, the latter suggests that any limit on borrowing can depend only on the security but not on the identity of the issuer. Hence, one is naturally led to impose short selling constraints as a substitute for the obedience assumption. Yet, short trading constraints need not be consistent with the borrower's ability to deliver.

The fact that short selling constraints depend only on the security but not on the borrower's ability to repay has two consequences. First, if the constraints are very tight, the economy works smoothly but the resulting allocation may be inefficient in the sense that relaxing these constraints may result in a Pareto improvement. On the other hand, if the constraints are too loose, some borrowers may have an incentive to promise more than they can deliver, even if they wanted to, and consequently may be bankrupt in some future states. Therefore, the trade-off between efficiency and resilience re-emerges in general equilibrium. Constraining agents so tightly so that nobody ever goes bankrupt is inefficient, whereas relaxing the constraints entails the risk of future bankruptcies.

This paper considers a general equilibrium model with unsecured contingent claims but with short selling constraints, similar to Araujo and Páscoa (2002) and Sabarwal (2003). While Sabarwal (2003) employs personalized short trading constraints, in this paper short selling constraints depend only on the security but not on the identity of the seller, as in Araujo and Páscoa (2002). In particular, we use the same equilibrium notion as these papers do. It is shown that short trading constraints are in fact necessary for the agents' portfolio choice problems to result in bounded solutions. On the other hand, we also argue that too tight short selling constraints are inefficient. However, loose short selling constraints introduce the possibility of bankruptcies in future states. Nonetheless, bankruptcies can be accounted for in a general equilibrium setting and will not affect the operation of future spot markets. Even a cascade of bankruptcies caused by a single default can be integrated into the accounting identities of the theory. The only consequence for future spot markets (which is well understood in the literature; see Dubey, Geanakoplos, and Shubik, 1989, 2005; Zame, 1993) is that security payoffs become endogenous, in a similar fashion as they are in multi-period models with long-term securities (see Magill and Quinzii, 1996, chp. 4, pp. 211).

The problems caused by future bankruptcies emerge at the security markets *prior* to the resolution of uncertainty. The fact that today there is room for bankruptcies occurring tomorrow may destroy the convexity of the induced preferences over portfolios. And the latter has dire consequences for the existence of equilibrium: Some excess demand functions may be discontinuous so that no market-clearing price exists—a problem to which the literature has not paid much attention.

This raises the question of how to interpret non-existence of competitive equilibrium. One possibility is to say that some market may shut down and, as a consequence, markets may become endogenously incomplete. Another is that in such cases general equilibrium theory makes no predictions. We remain agnostic on this issue but simply point out the problem.

1.1 Relations to the Literature

The failure of equilibrium to exist, as identified in the present paper, is radically different from what has been studied in the literature on existence of general equilibrium with incomplete markets (GEI). The discussion about existence of equilibrium in GEI models has revolved around discontinuities of excess demand caused by a sudden drop of rank in the security payoff matrix. The latter may be due to a coincidental discontinuity of the budget correspondence, as in Hart's (1975) original example; or, as a more robust phenomenon, to the presence of production (Momi, 2001) or derivatives (Polemarchakis and Ku, 1990). In contrast, in this paper markets may well be complete, and yet existence may fail even when the rank of the security payoff matrix does not change at all. In the present paper it is not a change in the asset span that causes a discontinuous excess demand function, but a change in the behavior of suppliers of securities:

They may suddenly decide to go bankrupt.

Since Shubik's (1973) seminal contribution bankruptcy and default in a general equilibrium setting have been studied, by e.g., Shubik and Wilson, 1977; Dubey, Geanakoplos, and Shubik, 1989, 2005; Zame, 1993; Kehoe and Levine, 1993; Araujo and Páscoa, 2002; Sabarwal, 2003; Eichberger, Rheinberger, and Summer, 2014; Ben-Ami and Geanakoplos, 2021; Martins-da-Rocha and Rosa, 2022. This literature distinguishes between default and bankruptcy (see Araujo and Páscoa, 2002). An agent *defaults* if she chooses not to honor her contracts, even if she could. By contrast, an agent is *bankrupt* if she cannot honor her commitments.

Hence, no-default corresponds to a weak obedience assumption, that the agent always delivers what she promised if she can. This assumption is maintained throughout in this paper. No-bankruptcy, on the other hand, corresponds to a strong obedience assumption, that the agent never promises more than she can deliver. This stronger version is dropped in the present paper. The distinction between bankruptcy and default also concerns which securities are affected. Default can happen promise by promise, while filing for insolvency affects all promises by a given agent. (In practise, of course, the distinction between default and bankruptcy is less clear, because if an agent does not deliver, her creditors may initiate bankruptcy procedures.)

In this terminology the main argument of the present paper is that while default can be, and has been, integrated into a general equilibrium model, by stipulating appropriate penalties for a refusal to deliver, bankruptcy raises problems for existence of equilibrium. This is neither because a cascading effect of bankruptcies renders the accounting identities inconsistent, nor because of different rates of return on issued and purchased securities. Instead, the possibility to go bankrupt in some future states introduces a non-convexity in the investors' preferences over portfolios. And it is precisely this non-convexity that may result in a discontinuous aggregate demand and a subsequent failure of existence of equilibrium. Since the existence of market-clearing prices justifies the assumption of price-taking behavior, the conclusion emerges that general equilibrium theory with uncertainty runs into conceptual difficulties when the strong obedience assumption is removed.

Among the papers on default in GEI only few deal with bankruptcy in the sense explained above. Eichberger, Rheinberger, and Summer (2014) allow negative consumption, which acts as a default penalty when evaluated by an increasing utility function. Martins-da-Rocha and Rosa (2022) show that when default penalties are pecuniary, rather than in "utils," equilibrium, if it exists, always entails bankruptcies.

The closest precursors to our argument are Araujo and Páscoa (2002) and Sabarwal (2003), the first dealing with nominal assets and the second with real assets. Both also identify an existence problem, which they refer to as a "non-convexity of the budget set." What they have in mind is that the graph of the correspondence that maps portfolios into future budget-feasible consumption does not form a convex subset of the ambient product space, as it kinks at the point where liabilities exhaust the future endowment.[1] Such a kink in this graph corresponds to a non-convexity of the induced preferences over portfolios.

Araujo and Páscoa (2002) prove an existence theorem that does not rely on continuity of excess demand but uses a continuum of agents (see Remark 1 in Section 5 below). Sabarwal (2003) also uses a continuum of agents but in addition posits personalized credit constraints as part of the solution concept. That is, agents "... lose their anonymity to a credit-setting financial intermediary like a bank, a brokerage house or some other lending institution." (Sabarwal, 2003, p. 14) While the personalized credit constraints are determined as part of the equilibrium, they may still leave room for bankruptcies, giving rise to what Sabarwal calls "Bankruptcy equilibrium." He finds that in the case of incomplete markets such a Bankruptcy equilibrium may Pareto-dominate the standard GEI equilibrium.

In dynamic stochastic general equilibrium (DSGE) models of the macroeconomy bankruptcy is typically excluded by imposing an Inada condition—that marginal utility diverges as consumption goes to zero—under which agents always *choose* not to go bankrupt. In the few models that allow default it is introduced mechanically, by assuming that a certain fraction of entrepreneurs defaults (Bernanke, Gertler, and Gilchrist, 1999; Carlstrom and Fuerst, 1997; Suarez and Sussman, 2007; Christiano, Motto, and Rostagno, 2010). Similarly, in the model by Cúrdia and Woodford (2010) impatient households default on their loans exogenously; in the one by Meh and Moran (2010) a predetermined fraction of entrepreneurs and bankers exit the economy each period and are replaced by new ones without any assets. By contrast, the present paper abstracts from default, but allows agents to make promises that may lead to bankruptcy.

The next section reviews the model and explains the necessity of short sale constraints in detail. Section 3 considers tight constraints that preclude bankruptcy and argues why this is inefficient. Section 4 shows that with loose short selling constraints, which allow for bankruptcies, the accounting at future spot markets still works. Section 5 identifies the problems emerging at the security markets in the initial period, and finally Section 6 concludes.

2 The Economy

The argument will be cast in terms of the simplest possible finance economy with two dates, $t = 0, 1$, finitely many states $s = 1, \ldots, S$ (with $S \geq 1$) in the second period, no production, and one consumption good per state. Nothing in the argument would change, except for notation, if there were multiple consumption goods per state. State $s = 0$ refers to the initial period $t = 0$, and consumption in each state serves as the numeraire.

The (common) consumption set of all agents $i = 1, ..., I$ (with $I \geq 1$) is the nonnegative orthant $\mathbb{R}_+^{S+1}$. Their endowments are given by the (column) vectors $\boldsymbol{\omega}_i = (\omega_{i0}, \omega_{i1}, ..., \omega_{iS})' \in \mathbb{R}_+^{S+1} \setminus \{\mathbf{0}\}$, one for each $i = 1, ..., I$, such that

[1] We are grateful to Mário Páscoa for clarifying this point in private communication.

$\sum_{i=1}^{I} \omega_{is} > 0$ and $\omega_{is} \geq \kappa_s \geq 0$ for all $s = 1, ..., S$ and all $i = 1, ..., I$. (All vectors are typeset in boldface.) The vector $\boldsymbol{\kappa} \in \mathbb{R}_+^S$ denotes, possibly state-dependent, subsistence consumption, applicable when agents go bankrupt. Agent i's preferences over consumption bundles $\boldsymbol{c} \in \mathbb{R}_+^{S+1}$ are represented by utility functions $u_i : \mathbb{R}_+^{S+1} \to \mathbb{R}$, which are assumed to be continuously differentiable, strictly increasing in all arguments, and quasi-concave. There are $J \geq 0$ securities $j = 1, \ldots, J$ available for trade in the first period $s = 0$. In the standard model, securities would be specified by the $S \times J$ matrix $Z \in \mathbb{R}^{S \times J}$ of security payoffs.[2] Agent i would then solve the problem

$$\max_{\boldsymbol{c} \in \mathbb{R}_+^{S+1},\, \boldsymbol{x} \in \mathbb{R}^J} u_i(\boldsymbol{c}) \text{ s.t. } c_0 + \boldsymbol{p} \cdot \boldsymbol{x} \leq \omega_{i0} \text{ and } c_s \leq \omega_{is} + \boldsymbol{z}_s \cdot \boldsymbol{x} \; \forall s = 1, ..., S \quad (1)$$

where $\boldsymbol{z}_s \in \mathbb{R}^J$ denotes the s-th row of the matrix Z and $\boldsymbol{p} \in \mathbb{R}_+^J$ the security price (row) vector. With multiple commodities per state the optimization problems on the spot markets in the second period $t = 1$ would have to be considered separately. With one good per state, i's consumption in state $s = 1, ..., S$ is simply $c_{si} = m_s + \max\{0, \omega_{is} + \boldsymbol{z}_s \cdot \boldsymbol{x} - m_s\} \equiv m_s + (\omega_{is} + \boldsymbol{z}_s \cdot \boldsymbol{x} - m_s)^+$. Therefore, writing $\boldsymbol{\omega_{i1}} = (\omega_{i1}, ..., \omega_{iS}) \in \mathbb{R}_+^S$ for the (column) vector of future endowments and $\boldsymbol{y}^+ \in \mathbb{R}_+^S$ for a vector defined by $y_s^+ = \max\{0, y_s\}$ for all $s = 1, ..., S$, agent i's problem could be reduced to the portfolio choice problem

$$\max_{\boldsymbol{x} \in \mathbb{R}^J} u_i \left(\omega_{i0} - \boldsymbol{p} \cdot \boldsymbol{x}, (\boldsymbol{\omega_{i1}} + Z \cdot \boldsymbol{x} - \boldsymbol{\kappa})^+ + \boldsymbol{\kappa} \right) \quad (2)$$

This problem contains neither short selling constraints nor the no-bankruptcy, or "strong obedience," condition that $\boldsymbol{\omega_{i1}} + Z \cdot \boldsymbol{x} \geq \boldsymbol{0}$.[3]

Proposition 1 *If at least one security $j = 1, ..., J$ has a positive price $p_j > 0$, then for any agent $i = 1, ..., I$, for whom $u_i(\boldsymbol{c})$ is bounded from below for all $\boldsymbol{c} \in \mathbb{R}_+^{S+1}$ and $u_i(c_0, \boldsymbol{\kappa})$ is unbounded from above for all $c_0 \in \mathbb{R}_+$, there exists no solution to problem (2).*

Proof. If $u_i(\boldsymbol{c})$ is bounded from below for all $\boldsymbol{c} \in \mathbb{R}_+^{S+1}$, then $u_i(c_0, \boldsymbol{m})$ is finite for all $c_0 \in \mathbb{R}_+$. Suppose problem (2) has a solution that yields utility u_i^*. Since u_i is strictly increasing in all arguments and $u_i(c_0, \boldsymbol{\kappa})$ is unbounded from above, there is some $c_0^* \in \mathbb{R}_+$ such that $u_i(c_0^*, \boldsymbol{\kappa}) > u_i^*$. Setting $x_k = 0$ for all $k \neq j$ and $x_j = \omega_{i0}/p_j - c_0^*/p_j$ yields $u_i(\omega_{i0} - p_j x_j, \boldsymbol{\kappa}) = u_i(c_0^*, \boldsymbol{\kappa}) > u_i^*$, which contradicts the assumption that (2) has a solution. ∎

Models, in which bankruptcy is possible but does not occur in equilibrium, may suggest that short selling constraints can be dispensed with. The proposition says that this is not the case as long as the agents' portfolio choice problems are meant to have finite solutions. A consequence of this observation is that

[2] As usual, markets are said to be *complete* if the rank of Z is S. Otherwise, they are *incomplete*.

[3] Vector inequalities are defined as follows. For $\boldsymbol{x}, \hat{\boldsymbol{x}} \in \mathbb{R}^m$ say that $\boldsymbol{x} \leq \hat{\boldsymbol{x}}$ if $x_j \leq \hat{x}_j$ for all $j = 1, \ldots, m$; further, $\boldsymbol{x} < \hat{\boldsymbol{x}}$ if $\boldsymbol{x} \leq \hat{\boldsymbol{x}}$ and $\boldsymbol{x} \neq \hat{\boldsymbol{x}}$; finally, $\boldsymbol{x} \ll \hat{\boldsymbol{x}}$ if $x_j < \hat{x}_j$ for all $j = 1, \ldots, m$.

some constraint on short sales is indeed *necessary* for the model to be well defined. There are several possibilities. For instance, collateral requirements could be imposed, as e.g. by Geanakoplos (2003) or Geanakoplos and Zame (2014), as these will induce constraints on short selling. In this paper we explore the simpler alternative of imposing quantity constraints on short sales, as e.g. in Dubey, Geanakoplos, and Shubik (1989, 2005) or Araujo and Páscoa (2002). In line with the anonymity assumption of general equilibrium theory, these short sales constraints will depend only on the security, but not on the identity of the trader.

The latter, however, raises an additional difficulty. The maximum quantity of security j, which an agent i can short, times the payoff of that security in state $s \geq 1$ may or may not be below the endowment of agent i in state s. If it is, the agent is solvent and will pay obediently. If not, the agent is bankrupt and the model needs to specify what will happen in this case. This is because whenever an agent cannot pay what she is supposed to, the payoff of the security, to which she owes, is affected. As a consequence, what other agents earn on their holdings will also be affected and more agents may go bankrupt. In short, with bankruptcies security payoffs become endogenous and their ultimate payoffs, while foreseen, may differ from their face values, that is, from what they promised to pay.

This calls for a slightly more general specification of securities than usual. More precisely, in the presence of short sales constraints the following formalization of securities is needed (see also Dubey, Geanakoplos, and Shubik, 1989, 2005). Each security $j = 1, \ldots, J$ is given by a pair $(\boldsymbol{v}_j, \boldsymbol{z}_j)$, where $\boldsymbol{v}_j \in \mathbb{R}^S_+$ is an exogenous (column) vector of face values $v_{sj} \geq 0$ comprising principal plus interest that an issuer promises to repay in the second period in state $s = 1, \ldots, S$, and $\boldsymbol{z}_j \in \mathbb{R}^S_+$ denotes the endogenous vector of realized and correctly foreseen (ex-post) payoffs z_{sj} that security j actually pays per unit in state $s = 1, \ldots, S$.[4] Note that, while $\boldsymbol{v}_j$ is part of the exogenous data of the economy, the realized payoffs $\boldsymbol{z}_j$ are determined in equilibrium. The $S \times J$ matrices $V = [v_{sj}]$ and $Z = [z_{sj}]$ summarize the payoff information for securities. This specification nests the traditional one, which amounts to $V = Z$. All securities are in zero net supply.

In multi-period general equilibrium models with long-term securities the security payoff matrix is naturally endogenous, as it depends on the security price vectors $\boldsymbol{p} \in \mathbb{R}^J_+$ through capital gains or losses on long-term securities (see Magill and Quinzii, 1996, chp. 4, pp. 211). In two-period models endogenous security payoffs have always been linked with the possibility of default. This is the route followed here, since the inevitable possibility of bankruptcy compels endogenous security payoffs.

For any vector $\boldsymbol{x} \in \mathbb{R}^J$ denote by $\boldsymbol{x}^+ \in \mathbb{R}^J_+$ its positive part, the vector given by $x_j^+ = \max\{0, x_j\}$, and by $\boldsymbol{x}^- \in \mathbb{R}^J_-$ its negative part, the vector given

[4] Dubey, Geanakoplos, and Shubik (2005) refer to $\boldsymbol{v}_j \in \mathbb{R}^S_+$ as the "promise" of security j. In their model the distinction between $\boldsymbol{v}_j$ and $\boldsymbol{z}_j$ is specified as z_{sj} being a "delivery" fraction $K_{sj} \in [0, 1]$ of v_{sj} in state $s = 1, ..., S$.

by $x_j^- = \min\{0, x_j\}$ for all $j = 1, ..., J$. Further, for $\boldsymbol{c} = (c_0, \boldsymbol{c_1}) \in \mathbb{R}_+^{S+1}$ let $\boldsymbol{c_1} = (c_1, ..., c_S) \in \mathbb{R}_+^S$ denote the part referring to the future. With this notation each agent $i = 1, ..., I$ now chooses a consumption vector $\boldsymbol{c} \in \mathbb{R}_+^{S+1}$ and a portfolio $\boldsymbol{x} \in \mathbb{R}^J$ so as to solve the problem

$$\max_{\boldsymbol{c}\in\mathbb{R}_+^{S+1},\, \boldsymbol{x}\in\mathbb{R}^J} u_i(\boldsymbol{c}) \qquad \text{s.t. } c_0 + \boldsymbol{p}\cdot\boldsymbol{x} \leq \omega_{i0}, \tag{3}$$

$$\boldsymbol{c_1} \leq \boldsymbol{\kappa} + \left(\boldsymbol{\omega_{i1}} + Z\cdot\boldsymbol{x}^+ + V\cdot\boldsymbol{x}^- - \boldsymbol{\kappa}\right)^+, \text{ and } \boldsymbol{0} \leq \boldsymbol{\xi} + \boldsymbol{x}$$

taking the security price vector $\boldsymbol{p} \in \mathbb{R}_+^J$, the matrix V of face values, and the security payoff matrix Z as given. In problem (3) the (column) vector $\boldsymbol{\xi} \in \mathbb{R}_+^J$ specifies the short sale constraints, independently of the agent's identity.

Taking into account that $\boldsymbol{x}^+ + \boldsymbol{x}^- = \boldsymbol{x}$ for any vector $\boldsymbol{x} \in \mathbb{R}^J$, the second term on the right-hand side of the budget constraints for future states $s = 1, ..., S$ can be rewritten as

$$\left(\boldsymbol{\omega_{i1}} + Z\cdot\boldsymbol{x}^+ + V\cdot\boldsymbol{x}^-\right)^+ = \left(\boldsymbol{\omega_{i1}} + V\cdot\boldsymbol{x} - (V - Z)\cdot\boldsymbol{x}^+\right)^+$$

This expression makes it explicit that the value of a portfolio $\boldsymbol{x}$ consists of its face value $V\cdot\boldsymbol{x}$ minus capital losses due to bankruptcies, $(V - Z)\cdot\boldsymbol{x}^+$. The realized security payoffs Z due to bankruptcies will be specified below.

At this point we only complete the description of the equilibrium concept. A *feasible allocation* is a matrix $C = (\boldsymbol{c}_i)_{i=1,...,I} \in \mathbb{R}_+^{(S+1)\times I}$ of consumption vectors such that $\sum_{i=1}^I \boldsymbol{c}_i \leq \sum_{i=1}^I \boldsymbol{\omega}_i$. A *competitive equilibrium* is a triplet $(\boldsymbol{p}, Z, C)$ consisting of a security price vector $\boldsymbol{p} \in \mathbb{R}_+^J$, a security payoff matrix $Z \in \mathbb{R}_+^{S\times J}$ with $Z \leq V$, and a feasible allocation $C \in \mathbb{R}_+^{(S+1)\times I}$ satisfying that for every agent $i = 1, ..., I$ there is a portfolio $\boldsymbol{x}_i \in \mathbb{R}^J$ such that $(\boldsymbol{c}_i, \boldsymbol{x}_i) \in \mathbb{R}_+^{S+1}\times\mathbb{R}^J$ solves problem (3) and $\sum_{i=1}^I \boldsymbol{x}_i = \boldsymbol{0}$, given the vector $\boldsymbol{\xi} \in \mathbb{R}_+^J$ of short sale constraints.

There are now two cases to be considered. The first is the case where no bankruptcies can occur, because the short trading constraints are so tight that no agent can ever promise to deliver more than her endowment. The second case, described later, concerns the case when short sale constraints are loose enough that agents may promise more than they can deliver and bankruptcies can occur in some future states.

3 Tight Short Sale Constraints

To study the first case, assume that the short sale constraints $\boldsymbol{\xi} \in \mathbb{R}_+^J$ satisfy

$$\sum_{j=1}^J \xi_j v_{sj} \leq \min_{i=1,...,I} \omega_{is} \text{ for all } s = 1, ..., S \tag{4}$$

In this case every agent can pay the face value of any security that she issues from her future endowment, even if she shorts all securities. Therefore, no agent ever goes bankrupt.

To show existence of an equilibrium is then a standard exercise. The constraint sets are compact and convex, and the utility functions are continuous and quasi-concave. By the maximum theorem the security excess demand correspondences are hence upper hemi-continuous, nonempty- and convex-valued. Consequently, by Kakutani's fixed point theorem a competitive equilibrium always exists.

It is well known that in a two-period finance economy, with a single consumption good per state, any competitive equilibrium is constrained Pareto optimal, in the sense that there is no redistribution of portfolios in the initial period that leaves all agents at least as well off and makes some better off—*fixing* the short sale constraints $\boldsymbol{\xi} \in \mathbb{R}_+^J$. Relaxing the short trading constraints may make agents better off, though. This can be seen by adopting the following terminology.

Definition 1 *(a) A* ***solvency equilibrium*** *is a competitive equilibrium* $(\boldsymbol{p}, Z, C)$ *that involves no bankruptcies (hence* $Z = V$*) such that for some security* $j = 1, ..., J$ *and some small* $\varepsilon > 0$ *the constraint* ξ_j *can be relaxed to* $\xi_j' = \xi_j + \varepsilon$ *without inducing any bankruptcies, i.e., such that at the new equilibrium still* $\boldsymbol{\omega_{i1}} + V \cdot \boldsymbol{x}_i' \geq \mathbf{0}$ *holds for all* $i = 1, ..., I$.
(b) A solvency equilibrium is ***weakly constrained efficient relative to*** $\boldsymbol{\xi} \in \mathbb{R}_+^J$ *if there is no* $j = 1, ..., J$ *such that slightly relaxing the constraint* ξ_j *makes at least one agent strictly better off and nobody worse off.*

Unlike the stronger criterion of constrained Pareto optimality, where the planner may redistribute portfolios, the test for weak constrained efficiency relative to $\boldsymbol{\xi}$ allows the planer only to relax short sales constraints. Portfolio adjustments are left to the market.

Proposition 2 *If at a solvency equilibrium there is at least one agent who would be willing to supply more of security* $j = 1, ..., J$ *than* ξ_j*, then the solvency equilibrium is not weakly constrained efficient relative to* $\boldsymbol{\xi}$.

Proof. At a solvency equilibrium $Z = V$ holds, because there are no bankruptcies. Therefore, agent i's problem is

$$\max_{\boldsymbol{x} \in \mathbb{R}^J} u_i \left(\omega_{i0} - \boldsymbol{p} \cdot \boldsymbol{x}, \boldsymbol{\omega_{i1}} + V \cdot \boldsymbol{x} \right) \quad \text{s.t.} \quad \boldsymbol{x} + \boldsymbol{\xi} \geq 0$$

Denote by $\nabla u_i (\boldsymbol{c}_i)$ the (normalized row) vector of marginal rates of substitution (or "stochastic discount factors") in equilibrium,

$$\nabla u_i (\boldsymbol{c}_i) = \left(\frac{\partial u_i (\boldsymbol{c}_i) / \partial c_{is}}{\partial u_i (\boldsymbol{c}_i) / \partial c_{i0}} \right)_{s=1,\ldots,S}$$

where $\boldsymbol{c}_i = (\omega_{i0} - \boldsymbol{p} \cdot \boldsymbol{x}, \boldsymbol{\omega_{i1}} + V \cdot \boldsymbol{x}) \in \mathbb{R}_+^{S+1}$ for all $i = 1, ..., I$. If the solvency equilibrium $(\boldsymbol{p}, V, C)$ is weakly constrained efficient relative to $\boldsymbol{\xi}$, then the familiar first-order condition

$$\nabla u_i (\boldsymbol{c}_i) \cdot V = \boldsymbol{p} \tag{5}$$

must hold in equilibrium, that is, all agents' marginal rates of substitution are equalized.

Yet, suppose that at least one agent i is willing to supply more than ξ_j of some security j at the equilibrium. Let

$$\mathcal{L}_i(\boldsymbol{x}, \boldsymbol{\lambda}) = u_i(\omega_{i0} - \boldsymbol{p} \cdot \boldsymbol{x}, \boldsymbol{\omega}_{i\mathbf{1}} + V \cdot \boldsymbol{x}) + \boldsymbol{\lambda} \cdot (\boldsymbol{x} + \boldsymbol{\xi})$$

be the Lagrangian for agent i's problem, where $\boldsymbol{\lambda} \in \mathbb{R}_+^J$ is the (row) vector of Lagrange multipliers. Then for security j the first-order condition

$$\frac{\partial \mathcal{L}_i(\boldsymbol{x}, \boldsymbol{\lambda})}{\partial x_j} = -\frac{\partial u_i(\boldsymbol{c}_i)}{\partial c_0} p_j + \sum_{s=1}^{S} \frac{\partial u_i(\boldsymbol{c}_i)}{\partial c_{is}} v_{sj} + \lambda_j = 0$$

holds with $\lambda_j > 0$, since by the envelope theorem λ_j is the shadow price of the j-th constraint. Therefore,

$$\nabla u_i(\boldsymbol{c}_i) \cdot V < \boldsymbol{p}$$

violates the necessary condition (5) for weak constrained efficiency relative to $\boldsymbol{\xi}$. ■

This observation says that weak constrained efficiency relative to $\boldsymbol{\xi}$ can only hold if the constraints do not bind for any agent and any security (or bind coincidentally at an unconstrained optimum). Otherwise, it is socially desirable to relax binding constraints. Hence, too tight short sale constraints tend to be inefficient. While the proposition refers to a local change of the constraints, the following example looks at a global change and illustrates how dramatic this inefficiency may become. This example will also prove useful later on.

Example 1 *There are two states in the second period, $S = 2$, and $\boldsymbol{\kappa} = \mathbf{0}$. The economy is populated by $n \geq 1$ agents of the first type with utility functions*

$$u(\boldsymbol{c}) = c_0 + \frac{1}{2}\ln(1 + c_1) + \frac{1}{2}\ln(c_2)$$

and with endowments $\boldsymbol{\omega} = (\omega_0, \omega_1, \omega_2) = (1, 1, 0)$ and by $m > n$ agents of the second type with utility functions

$$\hat{u}(\boldsymbol{c}) = c_0 + \frac{1}{2}\ln(c_1) + \frac{1}{2}\ln(1 + c_2)$$

and with endowments $\hat{\boldsymbol{\omega}} = (\hat{\omega}_0, \hat{\omega}_1, \hat{\omega}_2) = (1, 0, 1)$. Securities are given by the two Arrow-Debreu securities, $J = 2$, that is,

$$V = Z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and both securities are in zero net supply. Each type of agent has no endowment in the future state in which she desires it most. Hence, "hatted" agents wish to sell the second Arrow-Debreu security and "unhatted" agents want to sell the first. Therefore, any trade is beneficial. However, under condition (4) no trade *is possible, i.e., the only equilibrium is autarky. This is because the minimum endowment in each future state across agents is zero, that is, $\xi_1 = \xi_2 = 0$ is the only solution to (4), so no sales are possible.*

If the short selling constraints were relaxed to $\xi_1 = \xi_2 = 1$, there would still be no bankruptcies. Instead, the excess demand functions would be

$$\begin{aligned} x_1(\boldsymbol{p}) &= \max\left\{-1, \frac{1}{2p_1} - 2\right\} \text{ and } x_2(\boldsymbol{p}) = \frac{1}{2p_2} \\ \hat{x}_1(\boldsymbol{p}) &= \frac{1}{2p_1} \text{ and } \hat{x}_2(\boldsymbol{p}) = \max\left\{-1, \frac{1}{2p_2} - 2\right\} \end{aligned}$$

giving rise to a unique equilibrium with security prices

$$p_1 = \frac{m}{2n} > \frac{1}{2} \text{ and } p_2 = \frac{n+m}{4m} < \frac{1}{2}$$

Since markets are complete, the resulting consumption allocation,

$$\boldsymbol{c} = \left(\frac{n+m}{2n}, 0, \frac{2m}{m+n}\right) \text{ and } \hat{\boldsymbol{c}} = \left(\frac{n+m}{2m}, \frac{n}{m}, \frac{m-n}{m+n}\right)$$

would in fact be Pareto optimal. That is, with the relaxed constraints all agents are better off than in autarky.

It follows that, while condition (4) rules out bankruptcies, it also generates inefficiencies. If markets are meant to be efficient, it is therefore necessary to relax condition (4) and allow for bankruptcy. That this is possible, even in the presence of cascades, is shown in the next section.

4 Loose Short Sale Constraints

When agents can promise more than what they are able to deliver, their earnings may be garnished and redistributed to creditors, when the need arises. This is what a bankruptcy code achieves. Since such a code depends on the legal framework, it is desirable to formulate it in most general terms. For that reason we impose only four abstract conditions, which most bankruptcy codes in practise will satisfy.

4.1 Bankruptcy Code

Consider some state $s = 1, ..., S$ in the second period. An agent has bought a portfolio $\boldsymbol{x} \in \mathbb{R}^J$ in the previous period, at $s = 0$ (dropping the subscript for the agent's identity). This portfolio now induces a vector of liabilities $\boldsymbol{\ell}_s = -\operatorname{diag}(\boldsymbol{v}_s) \cdot \boldsymbol{x}^-$ towards the J securities, where $\operatorname{diag}(\boldsymbol{v})$ denotes writing a vector $\boldsymbol{v} \in \mathbb{R}^J$ as a diagonal matrix and $\boldsymbol{v}_s$ denotes the s-th rows of the matrix V of face values.

As for notation, let $\mathbf{1}_J \in \mathbb{R}^J_{++}$ denote the (summation row) vector $\mathbf{1}_J = (1, \ldots, 1)$ and denote transposition by a prime. In what follows the (column) vector $\boldsymbol{\ell} \in \mathbb{R}_+^J$ is to be thought of as the liabilities that an agent has towards the J securities in a particular state $s \geq 1$ and $y \geq 0$ denotes the agent's disposable income in that state, net of subsistence consumption $m = m_s \geq 0$. The vector f specifies what the agent ultimately delivers to the securities at $t = 1$.[5]

Definition 2 *A* ***bankruptcy code*** *is a continuous function* $f : \mathbb{R}_+^{J+1} \to \mathbb{R}_+^J$ *such that, for all* $\boldsymbol{\ell} \in \mathbb{R}_+^J$ *and all* $y, \hat{y} \in \mathbb{R}_+$*,*
(BC1) $f(\boldsymbol{\ell}, y) \leq \boldsymbol{\ell}$*;*
(BC2) *if* $y < \hat{y}$*, then* $f(\boldsymbol{\ell}, y) \leq f(\boldsymbol{\ell}, \hat{y})$*;*
(BC3) *if* $f(\boldsymbol{\ell}, y) < \boldsymbol{\ell}$*, then* $\mathbf{1}_J \cdot \boldsymbol{\ell} > y$*;*
(BC4) *if* $\mathbf{1}_J \cdot \boldsymbol{\ell} > y$*, then* $\mathbf{1}_J \cdot f(\boldsymbol{\ell}, y) = y$*.*

The function $f(\boldsymbol{\ell}, y) = \min\{1, y/(\mathbf{1}_J \cdot \boldsymbol{\ell})\}\boldsymbol{\ell}$ is an example of a bankruptcy code that captures a pro-rata rule (cf. Eisenberg and Noe, 2001; and also Part 1 of Araujo and Páscoa, 2002; and Ben-Ami and Geanakoplos, 2021). This example is popular in the literature and obviously not excluded by (BC1-4), but in practice many other codes can be found,[6] some of which are documented in the Talmud (see Aumann and Maschler, 1985). The following auxiliary result makes explicit the most important properties of a bankruptcy code.

Lemma 1 *If* f *is a bankruptcy code, then, for all* $(\boldsymbol{\ell}, y) \in \mathbb{R}_+^{J+1}$*,*
(a) $f(\boldsymbol{\ell}, 0) = \mathbf{0} \in \mathbb{R}_+^J$ *and* $\ell_j = 0 \Rightarrow f_j(\boldsymbol{\ell}, y) = 0$ *for all* $j = 1, ..., J$*;*
(b) if $\mathbf{1}_J \cdot \boldsymbol{\ell} \leq y$*, then* $f(\boldsymbol{\ell}, y) = \boldsymbol{\ell}$*;*
(c) $y - \mathbf{1}_J \cdot f(\boldsymbol{\ell}, y) = \max\{0, y - \mathbf{1}_J \cdot \boldsymbol{\ell}\}$*.*

Proof. (a) The first part follows, because either $\boldsymbol{\ell} = \mathbf{0}$ and then (BC1) and nonnegativity imply $f(\mathbf{0}, 0) = \mathbf{0}$, or $\boldsymbol{\ell} > \mathbf{0}$ and then (BC4) and nonnegativity imply $f(\boldsymbol{\ell}, 0) = \mathbf{0}$. The second part follows directly from (BC1) and nonnegativity.

(b) Under (BC1) this statement is equivalent to (BC3).

(c) If $y > \mathbf{1}_J \cdot f(\boldsymbol{\ell}, y)$, then $y \geq \mathbf{1}_m \cdot \boldsymbol{\ell}$ by (BC4), in which case (BC3) and (BC1) imply that $f(\boldsymbol{\ell}, y) = \boldsymbol{\ell}$, hence, $y - \mathbf{1}_m \cdot f(\boldsymbol{\ell}, y) = y - \mathbf{1}_m \cdot \boldsymbol{\ell} > 0$. Otherwise, $y \leq \mathbf{1}_J \cdot f(\boldsymbol{\ell}, y) \leq \mathbf{1}_J \cdot \boldsymbol{\ell}$ by (BC1) implies $y - \mathbf{1}_J \cdot f(\boldsymbol{\ell}, y) = \max\{0, y - \mathbf{1}_J \cdot \boldsymbol{\ell}\} = 0$ by (BC4), which completes the proof. ■

In particular, property (b) states that a solvent agent will honor her commitments. This is the weaker obedience assumption that is maintained in the present paper in order to focus on bankruptcy. By property (c), on the other hand, the income of an insolvent agent is a constant (subsistence consumption m).

A bankruptcy code applies to an individual case. The failure to deliver on a promise however also affects others, who may themselves go bankrupt, possibly causing a chain reaction. The next subsection focuses on the systemic effects of bankruptcy.

[5] If there were multiple goods in which securities pay off, the vector $\boldsymbol{\ell}$ would have to be replaced by a matrix and the scalar y by a vector. All results below would still go through.

[6] For example, Part 2 of Araujo and Páscoa (2002) considers non-linear rules that favor small claims.

4.2 Accounting for Bankruptcy

Recall that at any state $s = 1, \ldots, S$ in the second period each agent i arrives with an endowment $\omega_{is} \geq 0$, a (column) vector $\boldsymbol{\ell}_{is} \in \mathbb{R}_+^J$ of liabilities towards the J securities, and a (column) vector of claims $\boldsymbol{x}_i^+ \in \mathbb{R}_+^J$ towards securities. (Note that liabilities are nonnegative and depend on the state.) Liabilities are derived from the agents' portfolios $\boldsymbol{x}_i \in \mathbb{R}^J$ and the face values of securities according to

$$\boldsymbol{\ell}_{is} = -\operatorname{diag}(\boldsymbol{v}_s) \cdot \boldsymbol{x}_i^- \tag{6}$$

for all $i = 1, ..., I$, where again $\boldsymbol{v}_s$ denotes the s-th row of V. So, apart from face values, the common parameter of liabilities for all agents is the $J \times I$ matrix $X = (\boldsymbol{x}_1, ..., \boldsymbol{x}_I) \in \mathbb{R}^{J \times I}$ of portfolios purchased at $s = 0$.

Fix a state $s = 1, \ldots, S$ in the second period, a security price vector $\boldsymbol{p} \in \mathbb{R}_+^J$, and a (state dependent) bankruptcy code $f_s : \mathbb{R}_+^{J+1} \to \mathbb{R}_+^J$. Despite the potential dependence of the bankruptcy code on the state the subscript for the state s will be dropped in the remainder of this subsection to avoid cluttering the notation. Define for each $i = 1, ..., I$ the function $f_i : \mathbb{R}^{J \times I} \times \mathbb{R}^J \to \mathbb{R}_+^J$ by

$$f_i(X, \boldsymbol{z}) = f_s\left(\boldsymbol{\ell}_i, \omega_i + \boldsymbol{z} \cdot \boldsymbol{x}_i^+\right)$$

Then, fixing X, for each security j, which has been traded at $s = 0$ in the sense that $\sum_{i=1}^I x_{ij}^+ > 0$, define the function $\Phi_j : \mathbb{R}^J \to \mathbb{R}$ (which depends on the state s, because f, $\boldsymbol{\ell}_i$, and ω_i do) by

$$\Phi_j(\boldsymbol{z}) = \frac{1}{\sum_{i=1}^I x_{ij}^+} \left(\sum_{i=1}^I f_{ij}(X, \boldsymbol{z}) \right) \tag{7}$$

For an untraded security j with $\sum_{i=1}^I x_{ij}^+ = 0$ let $\Phi_j(\boldsymbol{z}) = v_j$ be the constant function. The proportionality of reimbursement to the size of the claim in (7) is a direct consequence of the notion of a security. It is unknown who lent to whom; only the individual portfolios count.

Finally, define the product function $\Phi = \times_{j=1}^J \Phi_j : \mathbb{R}^J \to \mathbb{R}^J$. Then, the vector $\boldsymbol{z} \in \mathbb{R}^J$ of security payoffs in state $s \geq 1$ is a fixed point of the function Φ, that is, $\boldsymbol{z} = \Phi(\boldsymbol{z})$. This is because, for each traded security j, what is paid to asset holders must equal the payments by issuers, i.e.

$$z_j \sum_{i=1}^I x_{ij}^+ = \sum_{i=1}^I f_{ij}(X, \boldsymbol{z}) \tag{8}$$

As pointed out before, at this point the anonymity assumption of general equilibrium theory bites. While a bankruptcy code may treat distinct creditors differently, a security pays the same amount per unit outstanding, which is what (7) and (8) state.

The agents' incomes net of subsistence $\kappa = \kappa_s$ in state $s \geq 1$ are their asset payoffs minus repayments for issued securities, i.e., incomes are determined by the functions

$$\begin{aligned} y_i(\boldsymbol{z}) &= \omega_i - \kappa + \boldsymbol{z} \cdot \boldsymbol{x}_i^+ - \mathbf{1}_J \cdot f_i(X, \boldsymbol{z}) \\ &= \omega_i - \kappa + \boldsymbol{z} \cdot \boldsymbol{x}_i^+ - \mathbf{1}_J \cdot f_s\left(\boldsymbol{\ell}_i, \omega_i + \boldsymbol{z} \cdot \boldsymbol{x}_i^+\right) \end{aligned} \tag{9}$$

for all $i = 1, ..., I$. In particular, if agent $i \in I$ is *insolvent*, that is, if $\mathbf{1}_J \cdot \boldsymbol{\ell}_i > \omega_i + \boldsymbol{z} \cdot \boldsymbol{x}_i^+$, then $y_i(\boldsymbol{z}) = 0$ by (BC4), hence i's consumption is $\kappa = \kappa_s$.

Lemma 2 *Suppose that $\Phi(\boldsymbol{z}) = \boldsymbol{z} \in \mathbb{R}^J$ for some state $s = 1, ..., S$. Then,*
(a) the aggregate income of the economy in state s equals aggregate endowments, i.e.

$$\sum_{i=1}^{I} (y_i(\boldsymbol{z}) + \kappa) = \sum_{i=1}^{I} \omega_i \tag{10}$$

(b) if there are no bankruptcies in state $s \geq 1$, then traded securities pay their face values, i.e. $z_j = v_j$ for all securities j with $\sum_{i=1}^{I} x_{ij}^+ > 0$.[7]

Proof. (a) A direct consequence of the determination of incomes (9) together with (8) is that

$$\begin{aligned} \sum_{i=1}^{I} (y_i(\boldsymbol{z}) + \kappa) &= \sum_{i=1}^{I} \omega_i + \sum_{j=1}^{m} z_j \sum_{i+1}^{I} x_{ij}^+ - \sum_{i=1}^{I} \mathbf{1}_J \cdot f_i(X, \boldsymbol{z}) \\ &= \sum_{i=1}^{I} \omega_i - \sum_{i=1}^{I} \mathbf{1}_J \cdot f_i(X, \boldsymbol{z}) + \sum_{j=1}^{m} \left(\sum_{i=1}^{I} f_{ij}(X, \boldsymbol{z}) \right) \\ &= \sum_{i=1}^{I} \omega_i \end{aligned}$$

which verifies the first statement.

(b) If there are no bankruptcies in state $s \geq 1$, then it follows from Lemma 1(b) that $f_i(X, \boldsymbol{z}_s) = \boldsymbol{\ell}_i$. Consequently, for all traded securities $j = 1, ..., J$,

$$\boldsymbol{z}_j = \Phi_j(\boldsymbol{z}) = \frac{1}{\sum_{i=1}^{I} x_{ij}^+} \left(\sum_{i=1}^{I} \ell_{ij} \right)$$

Since $\boldsymbol{\ell}_i = -\operatorname{diag}(\boldsymbol{v}) \cdot \boldsymbol{x}_i^+$, this amounts to

$$z_j \sum_{i=1}^{I} x_{ij}^+ = -v_j \sum_{i=1}^{I} x_{ij}^-$$

Market clearing at $s = 0$ implies that $\sum_{i=1}^{I} \left(x_{ij}^+ + x_{ij}^- \right) = 0$ for all traded securities j. Therefore,

$$z_j = \frac{-v_j \sum_{i=1}^{I} x_{ij}^-}{\sum_{i=1}^{I} x_{ij}^+} = \frac{-v_j \sum_{i=1}^{I} x_{ij}^-}{-\sum_{i=1}^{I} x_{ij}^-} = v_j$$

[7] Untraded securities "pay" their face values by definition.

for all traded securities j, as claimed. ■

The second part of this result reconstructs the traditional case of $Z = V$ within the present framework. That is, without bankruptcies in state $s \geq 1$ the vector or realized payoffs $\boldsymbol{z} = \boldsymbol{z}_s \in \mathbb{R}^J$ for traded securities equals precisely the vector of face values $\boldsymbol{v} = \boldsymbol{v}_s \in \mathbb{R}^J_+$ in state s. The next result shows that the accounting works consistently.

Proposition 3 *For every $s = 1, \ldots, S$ the equation system $\boldsymbol{z} = \Phi(\boldsymbol{z})$ has a solution and, in particular, a greatest solution. Furthermore, for all agents their net incomes $y_i(\boldsymbol{z})$ from (9) are unique, that is, the same at all solutions to $\boldsymbol{z} = \Phi(\boldsymbol{z})$.*

Proof. Fix $s = 1, ..., S$ and denote by $(\mathcal{Z}, \leq)$ the J-dimensional rectangle

$$\mathcal{Z} = \times_{j=1}^{J} [0, v_{sj}] \subseteq \mathbb{R}^J_+$$

partially ordered by $\leq$. If C is a chain of vectors in $(\mathcal{Z}, \leq)$, then the vector $\bar{\boldsymbol{y}} \in \mathcal{Z}$ defined by $\bar{y}_j = \sup_{\boldsymbol{y} \in C} y_j$ for all $j = 1, \ldots, J$ forms a supremum for the chain. Since $\mathbf{0} \leq \Phi(\mathbf{0})$ by (BC1) and (BC4) and Φ is monotone by (BC2), the Knaster-Tarski fixed point theorem (Aliprantis and Border, 2006, p. 16) implies that the set of fixed points is nonempty and has a maximal element.

Denote by $\boldsymbol{z}^* \in \mathcal{Z}$ the greatest fixed point and by $\boldsymbol{z} \in \mathcal{Z}$ some other fixed point of Φ (still suppressing the subscript for the state), hence $\boldsymbol{z} \leq \boldsymbol{z}^*$. Then, for every agent $i = 1, ..., I$, by Lemma 1(c)

$$\begin{aligned} y_i(\boldsymbol{z}^*) &= \max\left\{0, \omega_i + \boldsymbol{z}^* \cdot \boldsymbol{x}_i^+ - \mathbf{1}_J \cdot \boldsymbol{\ell}_i\right\} \\ &\geq y_i(\boldsymbol{z}) = \max\left\{0, \omega_i + \boldsymbol{z} \cdot \boldsymbol{x}_i^+ - \mathbf{1}_J \cdot \boldsymbol{\ell}_i\right\} \end{aligned}$$

If this inequality were strict, i.e., $y_i(\boldsymbol{z}^*) > y_i(\boldsymbol{z})$, for some $i = 1, ..., I$, then summation over all agents would yield $\sum_{i=1}^I y_i(\boldsymbol{z}^*) > \sum_{i=1}^I y_i(\boldsymbol{z})$ in contradiction with $\sum_{i=1}^I y_i(\boldsymbol{z}^*) = \sum_{i=1}^I \omega_i = \sum_{i=1}^I y_i(\boldsymbol{z})$ by (10) from Lemma 2(a). Therefore, $y_i(\boldsymbol{z}^*) = y_i(\boldsymbol{z})$ for all agents i, as desired. ■

Even though the solution to $\boldsymbol{z} = \Phi(\boldsymbol{z})$ may not be unique, there are no economic consequences of multiplicity, because the net incomes of all agents are the same at all solutions. In particular, the multiplicity of solutions to $\boldsymbol{z} = \Phi(\boldsymbol{z})$ only affects the gross incomes of insolvent agents, but the net incomes of agents are unaffected.

The following example, adapted from Eisenberg and Noe (2001), illustrates that multiplicity can arise in states where no agent has any endowment.

Example 2 *Still dropping the subscript for the state, let $I = J = 2$, $x_1 = (1, -1)'$, $x_2 = (-1, 1)'$, $\boldsymbol{\kappa} = \mathbf{0}$, and $v_1 = v_2 = v > 0$, hence $\boldsymbol{\ell}_1 = (0, v)'$ and $\boldsymbol{\ell}_2 = (v, 0)'$. Then, by Lemma 1(c) and the definition of Φ the fixed point problem defining the vector $\boldsymbol{z} \in \mathbb{R}^2_+$ boils down to*

$$z_1 = \min\{v, \omega_2 + z_2\} \text{ and } z_2 = \min\{v, \omega_1 + z_1\}$$

If $\boldsymbol{\omega} = \mathbf{0}$, then every vector $z = (\theta, \theta)$ with $\theta \in [0, v]$ solves $\boldsymbol{z} = \Phi(\boldsymbol{z})$ with both agents bankrupt (if $\theta < v$). For, $\min\{v, z_{3-j}\} = z_j$ *yields $z_j = \theta$ for $j = 1, 2$.*

The multiplicity arises because $\boldsymbol{\omega} = \mathbf{0}$. If $\boldsymbol{\omega} > \mathbf{0}$, the picture changes. Suppose that at a solution $\omega_1 + z_1 < v$, i.e. agent 1 is bankrupt, hence $z_2 = \omega_1 + z_1$. Then, by $\omega_1 + \omega_2 > 0$,

$$z_1 = \min\{v, \omega_2 + z_2\} = \min\{v, \omega_1 + \omega_2 + z_1\} = v$$

must hold, implying that agent 2 is solvent, i.e. $\omega_2 + z_2 \geq v$. Therefore, $z_2 = \min\{v, \omega_1 + v\} = v$, since $\omega_1 \geq 0$, implies that agent 1 is also solvent in contradiction to the hypothesis. Exploiting the symmetry of the example, an analogous argument shows that agent 2 must also be solvent at any solution. Thus, at any solution both agents must be solvent, implying that $z_j = v$ for $j = 1, 2$ is the only solution.

Proposition 3 establishes that bankruptcies do not pose any problems at the spot markets in the future, i.e., at $s = 1, ..., S$. Even when a cascade of bankruptcies occurs, the (net) incomes of all agents are uniquely determined. The only aspect that changes, as compared to the standard model, is that security payoffs are endogenous. Yet, this assumes that security prices are determined at the security markets in the first period, at $s = 0$. We turn to this issue in the next section.

5 Security Markets

The security markets in the first period determine with which portfolios agents enter the second period. Hence, they determine assets and liabilities at all future states. While future spot markets can be cleared even in the presence of bankruptcies by Proposition 3, loose short trading constraints (that violate (4)) cause problems in the initial trading period. The reason is that with loose short sale constraints the induced preferences over portfolios may not be convex anymore.[8] A failure of convexity of preferences may lead to discontinuous excess demand functions, so that market clearing prices at the security markets may not exist. The following extended example illuminates this claim.

5.1 The Example

The economy is as in Example 1, with $S = J = 2$ and $\boldsymbol{\kappa} = \mathbf{0}$. There are $n \geq 1$ agents of the first type with induced preferences over portfolios represented by

$$\begin{aligned} v(\boldsymbol{x}) &= \omega_0 - \boldsymbol{p} \cdot \boldsymbol{x} + \frac{1}{2} \ln\left(1 + \left(\omega_1 + \boldsymbol{z}_1 \cdot \boldsymbol{x}^+ + \boldsymbol{v}_1 \cdot \boldsymbol{x}^-\right)^+\right) \\ &\quad + \frac{1}{2} \ln\left(\left(\omega_2 + \boldsymbol{z}_2 \cdot \boldsymbol{x}^+ + \boldsymbol{v}_2 \cdot \boldsymbol{x}^-\right)^+\right) \end{aligned}$$

[8] Dubey, Geanakoplos, and Shubik (2005) claim in the proof of their Theorem 1 (p. 31) that their payoff functions are concave, but they do not give an argument for why this is the case.

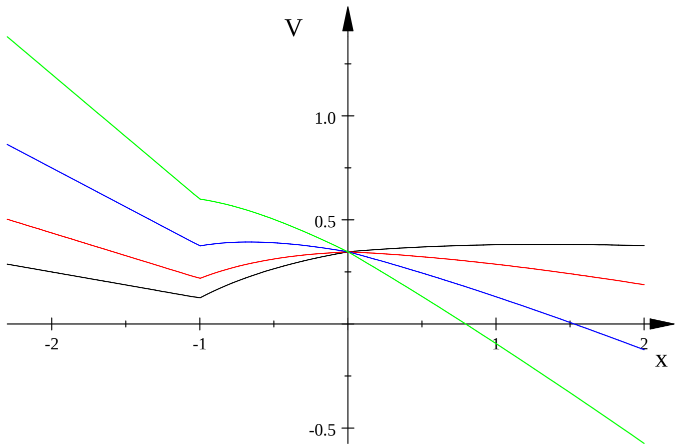

Figure 1: $V(x, p, .75)$ for $p = .125$ (black), .21875 (red), .375 (blue), and .60 (green).

and with endowments $\boldsymbol{\omega} = (1, 1, 0)$; and there are $m > n$ agents of the second type with induced preferences over portfolios represented by

$$\begin{aligned} \hat{v}(\boldsymbol{x}) &= \hat{\omega}_0 - \boldsymbol{p} \cdot \boldsymbol{x} + \frac{1}{2} \ln \left(\left(\hat{\omega}_1 + \boldsymbol{z}_1 \cdot \boldsymbol{x}^+ + \boldsymbol{v}_1 \cdot \boldsymbol{x}^- \right)^+ \right) \\ &\quad + \frac{1}{2} \ln \left(1 + \left(\hat{\omega}_2 + \boldsymbol{z}_2 \cdot \boldsymbol{x}^+ + \boldsymbol{v}_2 \cdot \boldsymbol{x}^- \right)^+ \right) \end{aligned}$$

and with endowments $\hat{\boldsymbol{\omega}} = (1, 0, 1)$. The "unhatted" resp. the "hatted" agents maximize $v(\boldsymbol{x})$ resp. $\hat{v}(\boldsymbol{x})$ subject to $\boldsymbol{\xi} + \boldsymbol{x} \geq 0$ where $\boldsymbol{\xi} \in \mathbb{R}_+^2$ is arbitrary, in particular need not satisfy (4). The two securities are the Arrow-Debreu securities given by

$$V = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } Z = \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix}$$

where z_1 and z_2 are endogenous variables that depend on $\boldsymbol{p}$ and $\boldsymbol{\xi}$, to be determined below.

Maximizing $v(\boldsymbol{x})$ resp. $\hat{v}(\boldsymbol{x})$ with respect to x_2 resp. x_1 at $\boldsymbol{p} \gg 0$ is straightforward, because both have interior maxima with respect to these variables, where the short selling constraints are slack. This gives the demand functions

$$x_2(\boldsymbol{p}) = \frac{1}{2p_2} > 0 \text{ and } \hat{x}_1(\boldsymbol{p}) = \frac{1}{2p_1} > 0$$

Finding maxima for $v(\boldsymbol{x})$ resp. $\hat{v}(\boldsymbol{x})$ with respect to x_1 resp. x_2 is more difficult. It amounts to maximizing the function $V : \mathbb{R} \times \mathbb{R}_{++} \times (0, 1] \to \mathbb{R}$ defined by

$$V(x, p, z) = \frac{1}{2} \ln \left(1 + \left(1 + zx^+ + x^- \right)^+ \right) - px$$

subject to the constraint $\xi + x \geq 0$, with $\xi \geq 0$. Figure 1 depicts $V(\cdot)$ as a function of x for $z = 3/4$ and four different values of $p > 0$. At $x \in \{-1, 0\}$ the function $V(\cdot)$ is continuous, but not differentiable, as it kinks at these points. Directional derivatives exist, though, and are given by

$$\lim_{x \nearrow -1} \frac{\partial V}{\partial x} = -p, \ \lim_{x \searrow -1} \frac{\partial V}{\partial x} = \frac{1}{2} - p, \ \lim_{x \nearrow 0} \frac{\partial V}{\partial x} = \frac{1}{4} - p, \text{ and } \lim_{x \searrow 0} \frac{\partial V}{\partial x} = \frac{z}{4} - p$$

To derive the excess demand correspondence, define the function $\varphi : \mathbb{R}_{++} \times (0, 1] \to \mathbb{R}$ by

$$\varphi(p, z) = \frac{2}{z} - \frac{1}{2p} (1 + \ln(2p) - \ln(z))$$

Working out all the possible cases finally yields the correspondence of maximizers of $V(\cdot)$ under the constraint $x + \xi \geq 0$,

$$X(p) = \begin{cases} 1/(2p) - 2/z > 0 & \text{if} \quad p < z/4 \text{ and } \xi < \varphi(p, z) \\ \{-\xi, 1/(2p) - 2/z\} & \text{if} \quad p < z/4 \text{ and } 1 < \xi = \varphi(p, z) \\ 0 & \text{if} \quad z/4 \leq p \leq 1/4 \text{ and } \xi < \varphi(p, 4p) \\ \{-\xi, 0\} & \text{if} \quad z/4 \leq p \leq 1/4 \text{ and } \xi = \varphi(p, 4p) \\ 1/(2p) - 2 < 0 & \text{if} \quad \frac{1}{4} < p < \frac{1}{2} \text{ and } 2 - \frac{1}{2p} \leq \xi < \varphi(p, 1) \\ \{-\xi, 1/(2p) - 2\} & \text{if} \quad 1/4 < p < 1/2 \text{ and } \xi = \varphi(p, 1) \\ -\xi & \quad \text{otherwise} \end{cases}$$

(where $\varphi(p, 4p) = 0.34657/p$), which potentially has three points of discontinuity. Luckily, because "hatted" agents always demand the first security and "unhatted" agents always demand the second security, only a part of this correspondence is relevant for the determination of an equilibrium. In particular, the first and the third line are irrelevant.

Since "unhatted" agents must issue the first security in equilibrium, their aggregate supply (strictly speaking, excess demand) must be either $-n\xi_1$ or $n/(2p_1) - 2n$, where the latter can only hold if $1/4 < p_1 < 1/2$ and $2 - 1/(2p_1) \leq \xi_1 \leq \varphi(p_1, 1)$. If the aggregate supply of the first security were given by $n/(2p_1) - 2n$, then the equilibrium price would be $p_1 = (n + m)/(4n) > 1/2$, which would contradict that $1/4 < p_1 < 1/2$. Therefore, the aggregate supply of the first security must be $-n\xi_1$, hence the equilibrium price must be $p_1 = m/(2\xi_1 n)$. As long as $\xi_1 \leq 1$ this does not involve any bankruptcies and implies that $p_1 = m/(2\xi_1 n) \geq m/(2n) > 1/2$ and $z_1 = 1$. When $\xi_1 > 1$, all "unhatted" agents are bankrupt in state $s = 1$ and $z_1 = 2np_1/m = 1/\xi_1 < 1$.

Similarly, since "hatted" agents must issue the second security, their aggregate supply (excess demand) must be either $-m\xi_2$ or $m/(2p_2) - 2m$, where the latter takes $1/4 < p_2 < 1/2$ and $2 - 1/(2p_2) \leq \xi_2 \leq \varphi(p_2, 1)$. If the aggregate supply of the second security is $m/(2p_2) - 2m$, then the equilibrium price is $p_2 = (n + m)/(4m) \in (1/4, 1/2)$. If the aggregate supply of the second security is $-m\xi_2$, then the equilibrium price is $p_1 = n/(2\xi_2 m)$. Unlike in the previous case, here neither possibility can be excluded. Yet, bankruptcies of "hatted" agents occur only when $p_2 = m/(2\xi_2 n)$ and $\xi_2 > 1$, in which case $z_2 = 2mp_2/n = 1/\xi_2 < 1$ (otherwise $z_2 = 1$).

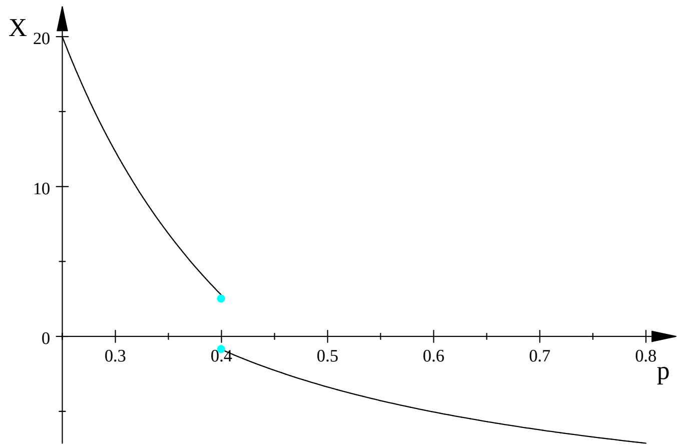

Figure 2: Aggregate excess demand is discontinuous at $p_2 = 0.4$.

These arguments apply under the assumption that an equilibrium exists. This is not guaranteed, though. To illustrate this claim, we provide the following numerical example.

Example 3 *Let $n = 10$ and $m = 13$, and set $\xi_1 = 1.1$ and $\xi_2 = 1.0289$. Then $p_1 = m/(2\xi_1 n) = 0.59091 > 1/2$ is the only possible equilibrium price for the first security, implying that in equilibrium $z_1 = 2np_1/m = 0.90909$, because all "unhatted" agents are bankrupt in state $s = 1$.*

For the second security matters are a bit more subtle. There are five possibilities for "hatted" agents to issue the second security. First, if $p_2 < z_2/4$ and $\xi_2 = \varphi(p_2, z_2)$, then $\hat{x}_2(\boldsymbol{p}) = -\xi_2$ can hold. But, if $p_2 < z_2/4$, then $\varphi(\cdot)$ is strictly decreasing in p_2, hence $\varphi(p_2, z_2) > \varphi(z_2/4, z_2) = \ln(4)/z_2 \geq \ln(4) = 1.3863 > \xi_2$, a contradiction. Second, if $z_2/4 \leq p_2 \leq 1/4$ and $\xi_2 = \varphi(p_2, 4p_2) = \ln(2)/2p_2 = 0.34657/p_2$, then $\hat{x}_2(\boldsymbol{p}) = -\xi_2$ can hold. But, if $p_2 \leq 1/4$, then $\varphi(p_2, 4p_2) \geq 1.3863 > \xi_2$, a contradiction. This leaves the following three cases in which "hatted" agents may issue the second security:

$$\hat{x}_2(p) = \begin{cases} 1/(2p_2) - 2 & \text{if } \frac{1}{4} < p_2 < \frac{1}{2} \text{ and } 2 - \frac{1}{2p_2} \leq \xi_2 < \varphi(p_2, 1) \\ \{-\xi_2, 1/(2p_2) - 2\} & \text{if } \frac{1}{4} < p_2 < \frac{1}{2} \text{ and } \xi_2 = \varphi(p_2, 1) \\ -\xi_2 & \text{otherwise} \end{cases}$$

In particular, at equilibrium $p_2 > 1/4$. Notice that $\xi_2 = 1.0289 = \varphi(p_2, 1) = 2 - (1 + \ln(2p_2))/(2p_2)$ if and only if $p_2 = 2/5 = 0.4$. Since for $p_2 < 1/2$ the function $\varphi(p_2, 1)$ is strictly decreasing in p_2, the inequality $\xi_2 < \varphi(p_2, 1)$ holds if and only if $p_2 < 0.4$. (That $2 - 1/(2p_2) \leq \xi_2 = 1.0289$ holds if and only if $p_2 \leq 0.51488$.) Therefore, the "supply correspondence" is discontinuous *at* $p_2 = 0.4$. *The aggregate excess demand correspondence over the domain* $p_2 > 1/4$,

$$n \cdot x_2(\boldsymbol{p}) + m \cdot \hat{x}_2(\boldsymbol{p}) = \frac{10}{2p_2} + 13 \cdot \hat{x}_2(\boldsymbol{p})$$

$$= \begin{cases} 23/(2p_2) - 26 & \textit{if} \quad 1/4 < p_2 < 0.4 \\ \{-0.8757, 2.75\} & \textit{if} \quad p_2 = 0.4 \\ 10/(2p_2) - 13.376 & \textit{if} \quad 0.4 < p_2 \end{cases}$$

is strictly decreasing in p_2 *wherever it is continuous, but has no intersection with zero (see Figure 2). In brief, no equilibrium exists at the market for the second security.*

Remark 1 *In the model with a continuum of agents by Araujo and Páscoa (2002) the problem could be repaired (along the lines of their Lemma 2) as follows. Think of the two sets of agents as two continua, with masses n and m, respectively, compute a mixed-strategy Nash equilibrium of a game between the agents and an auctioneer (who maximizes the value of excess demand), and then purify the mixed-strategy equilibrium for the "hatted" agents. That is, an "umpire" instructs exactly the right fraction (namely* 0.75847*) of "hatted" agents to choose* $\hat{x}_2 = -1.0289$ *and the remaining "hatted" agents to choose* $\hat{x}_2 = -0.75$ *at the security price* $p_2 = 0.4$. *This construction moves the value of excess demand at the price* $p_2 = 0.4$ *exactly to zero (without removing the discontinuity, though). How this coordination is achieved by an anonymous market remains open, though.*

Hence, a continuum of agents can be a way around the non-existence problem by introducing additional coordination among agents beyond the price system. Another possible reaction is to observe that endogenous security payoffs tend to make market (in)completeness endogenous, just like in multi-period GEI models. In this view the example would exhibit incomplete markets with a single security.

The reason for why in this example no equilibrium exists is the non-convexity of preferences over portfolios caused by loose short trading constraints. At the point, where the constraint ξ_2 just exceeds the future endowment $\hat{\omega}_2 = 1$ of issuers, the utility function kinks, because the nonnegativity constraint on future consumption kicks in. Even though the issuers anticipate that they will be bankrupt in state $s = 2$, they make themselves richer at $s = 0$ by promising more than they can deliver, while being protected by limited liability in the future.

Indeed, bankruptcy and non-convex preferences over portfolios are two sides of the same coin. If short sale constraints were so tight that no issuer can ever promise more than she can deliver, bankruptcy cannot occur and preferences over portfolios would be convex. In this case only strategic default can occur, that is, an issuer choosing not to deliver even if she could. Hence, whenever preferences over portfolios are convex, bankruptcy has been ruled out by assumption.

Yet, Example 3 only numerically illustrates that equilibrium may not exist. For other parameter values equilibrium may in fact exist and then the question arises whether or not an equilibrium involving bankruptcy can be more efficient than tight short selling constraints, which rule out bankruptcy. To that end returning to Example 1 is instructive. In this example tight short trading constraints imply no trade at all. It follows that relaxing the constraints, even if this involves bankruptcies, can only be welfare improving.[9] Under the hypothesis that an equilibrium exists, therefore, an equilibrium with bankruptcy can be more efficient than tight short selling constraints that make bankruptcies impossible.

6 Conclusions

General equilibrium theory with time and uncertainty rests on a strong obedience assumption, that no agent will ever promise more than she can deliver. If this assumption is lifted—while maintaining the weaker assumption that agents will deliver if they can—the induced preferences over portfolios need not be convex anymore and bankruptcy can occur in some future states. Clearing bankruptcies when they occur does not pose a problem for the theory. But the non-convexity of preferences does. It may lead to discontinuous excess demand correspondences and, as a consequence, with finitely many agents competitive equilibrium may not exist.

Short sale constraints may avoid this problem, if they are tight enough. But there is a trade-off, since too tight constraints are inefficient. Relaxing short trading constraints promotes efficiency but runs the risk that some markets may be inactive. While this is an observation about the theory, it concurs with the practical experience from the 2007-8 global financial crisis. When credit constraints become too loose, in the sense of allowing bankruptcies and foreclosures, some markets for "promises" may disappear.

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[9] We thank Tilman Börgers for alerting us to this point.

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