# Lower Bounds for Maximum Weight Bisections

## of Graphs with Bounded Degrees

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# Outline

1. Graphs and Weighted Graphs
2. Cuts, Bisections and colorings
3. Bounds for Maximum weight bisection
   - of graphs with even maximum degrees
   - of graphs with maximum degree three
4. triangle-free subcubic graphs
   - 2-connected
   - all cases except a claw

### DEFINITION (GRAPHS)

A graph $G$ is usually denoted by an ordered pair $G = (V(G), E(G))$, where $V(G)$ is the vertex sets and $E(G)$ is the edge set.

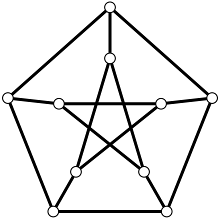

FIGURE: The Petersen graph

# Definitions

## Definition (Weighted Graphs)

A weighted graph $G$ is usually denoted by an ordered triple $G = (V(G), E(G), w)$ where $w : E \to \mathbb{R}_{\geq 0}$ is the weight function. In addition, we usually use $w(G) = \sum_{e \in E(G)} w(e)$ to denote the total weight of $G$.

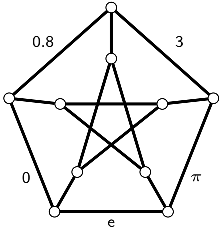


Figure: A weighted Petersen graph

### Definition (neighbours and degrees)

Let $G$ be a graph and $v$ a vertex in $V(G)$. The neighbours of $v$ are those vertices which are adjacent to $v$, and the degree of $v$, denoted by $d_G(v)$, is the number of neighbours of $v$.

### Definition (the maximum degree)

Let $G$ be a graph. The maximum degree of $G$ denoted by $\Delta(G)$ is the maximum degree over all vertices, i.e., $\Delta(G) = \max\{d_G(v) : v \in V(G)\}$.

### Definition (cubic and subcubic)

A graph is subcubic iff $\Delta(G) \leq 3$, and cubic iff the degree of any vertex equal three.

# Cuts, Bisections and their values (or weights)

### Definition

A cut of a weighted graph $G = (V, E, w)$ is a partition of the vertex set $V$ into two disjoint subsets $X$ and $Y$. The weight of the cut, denoted by $w(X, Y)$, refers to the sum of the weights over all edges between (i.e. edges with one endpoint in each partite set).

### Definition

A bisection of $G$ is a cut $(X, Y)$ where the cardinality of $X$ and $Y$ differs by at most one.

The following is a well known result in graph theory by Erdős.

**Theorem ([P. Erdős, 1965])**

*Every weighted graph $G = (V, E, w)$ has a cut with weight at least $\frac{w(G)}{2}$.*

**Proof.**

State on the white board. □

Every graph can be seen as a weighted complete graph.

**Lemma**

*Let $G = (V, E, w)$ be a weighted complete graph with $n$ vertices. Then, $G$ has a bisection with weight at least $\frac{n}{2(n-1)} w(G)$ when $n$ is even and $\frac{n+1}{2n} w(G)$ when $n$ is odd.*

**Proof.**

State on the white board. □

# Vertex Coloring

## Definition (Vertex coloring)

A $k$-vertex-coloring of a graph $G$ is a partition of the vertex set into $k$ disjoint subsets such that there is no edge between any two vertices with the same color. A graph is said to be $k$-vertex-colorable if $G$ has a $k$-vertex-coloring. The chromatic number $\chi(G)$ of a graph $G$ refers to smallest positive integer $k$ such that $G$ is $k$-vertex-colorable.

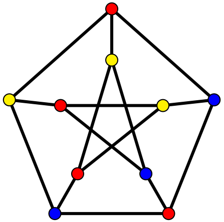

Figure: The Petersen graph

**Theorem ([L. Anderson, D. Grant and N. Linial, 1983, J. Lehel and Zs. Tuza, 1982, S. Locke, 1982])**

*Let $G = (V(G), E(G), w)$ be a weighted graph and $\chi = \chi(G)$. Then, $G$ admits a cut of weight at least $\frac{\chi+1}{2\chi}w(G)$ when $\chi$ is odd and $\frac{\chi}{2(\chi-1)}w(G)$ when $\chi$ is even.*

**Proof.**

State on the white board. $\square$

[1] $\chi(G) \leq \Delta(G) + 1$

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### Proof.

State on the white board. □

By Brook's theorem[1], we have the following holds.

### Corollary

*Let $G = (V(G), E(G), w)$ be a weighted graph. If $\Delta(G) \leq k$, then there exists a cut of weight at least $\frac{k+1}{2k} w(G)$ if $k$ is odd and $\frac{k+2}{2(k+1)} w(G)$ if $k$ is even.*

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# Our conjecture and past results

**Conjecture ([S. Gerke, G. Gutin, A. Yeo and Y. Zhou, 2024+])**

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**Theorem ([B. Bollobás and A. Scott, 2004])**

*Let $G = (V(G), E(G))$ be a graph. If $G$ is $k$-regular, then there exists a bisection of size at least $\frac{k+1}{2k}|E(G)|$.*

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**Theorem ([C. Lee, P. Loh and B. Sudakov, 2013])**

*Let $G = (V(G), E(G))$ be a graph. If $\Delta(G) \leq k$, then there exists a bisection of size at least $\frac{k+1}{2k}|E(G)| - \frac{k(k+1)}{4}$ if $k$ is odd and $\frac{k+2}{2(k+1)}|E(G)| - \frac{k(k+2)}{4}$ if $k$ is even.*

### DEFINITION (EDGE COLORING)

A $k$-edge-coloring of a graph $G$ is a partition of the edge set into $k$ disjoint matchings. A graph is said to be $k$-edge-colorable if $G$ has a $k$-edge-coloring. The chromatic index $\chi'(G)$ of a graph $G$ refers to smallest positive integer $k$ such that $G$ is $k$-edge-colorable.

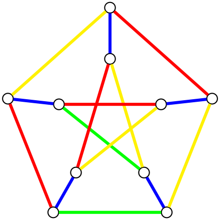


FIGURE: The Petersen graph

# A Useful lemma

We say that $B \in \mathcal{B}_b(G)$, if $B$ is the union of vertex-disjoint bipartite subgraphs $B_i$'s (not necessarily connected) of $G$ with partite sets $(X_i, Y_i)$ where $G[X_i]$ and $G[Y_i]$ have no edges and $|X_i| = |Y_i|$.

### Lemma

*Let $G = (V, E, w)$ be a weighted graph and $B \in \mathcal{B}_b(G)$. Then, $G$ has a bisection with weight at least $\frac{w(G)+w(B)}{2}$.*

### Proof.

State on the white board. $\square$

Since any matching $M$ of $G$ is clearly in $\mathcal{B}_b(G)$, we have the following corollary.

**Corollary**

*Let $G = (V, E, w)$ be a weighted graph and $M$ its maximum weight matching. Then, $G$ has a bisection with weight at least $\frac{w(G)+w(M)}{2}$.*

**Theorem**

*Every weighted graph $G = (V, E, w)$ has a bisection with weight at least $\frac{\chi'(G)+1}{2\chi'(G)} w(G)$.*

**Proof.**

State on the white board. □

The following bound for chromatic index is known as Vizing's Theorem.

**THEOREM ([V. G. VIZING, 1964])**

*For any simple graph $G$, $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$.*

**COROLLARY**

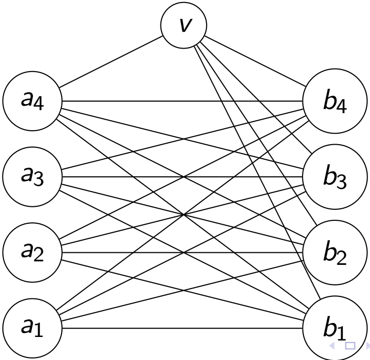
*Let $k$ be a positive integer. Any weighted graph $G$ with $\Delta(G) \leq k$ has a bisection with weight at least*

$$\frac{k+2}{2(k+1)} w(G).$$

*In particular, the conjecture holds when $k$ is even.*

### Conjecture ([S. Gerke, G. Gutin, A. Yeo and Y. Zhou, 2024+])

*Let $G = (V(G), E(G), w)$ be a weighted graph. If $\Delta(G) \le k$, then there exists a bisection of weight at least $\frac{k+1}{2k} w(G)$ if $k$ is odd and $\frac{k+2}{2(k+1)} w(G)$ if $k$ is even.*



# CONNECTED COMPONENTS, TREES AND FORESTS

1. A graph is connected if between any two vertices there is a path. Connected components of a graph are maximal connected subgraphs of it.
2. A tree is a connected graph with no cycle.
3. A forest is a graph of which all connected components are trees.

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*Trees are bipartite.*

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3. A forest is a graph of which all connected components are trees.

**Theorem**

*Trees are bipartite.*

**Theorem**

*Let $F$ be a forest and $c(F)$ the number of connected components in $F$. Then,*

$$|E(G)| = |V(G)| - c(F).$$

### LEMMA

*If $F$ is a forest with at most $|V(F)|/2$ edges, then there is a biparite subgraph $B \in \mathcal{B}_b(F)$ such that $E(B) = E(F)$.*

### PROOF.

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## PROOF.

Let $c(F)$ be the number of connected components in $F$, $m_i$ the number of components with $i$ edges and $t$ the maximum size of a component in $F$. Since $F$ is a forest $|E(F)| = |V(F)| - c(F) \leq |V(F)|/2$, which implies that $|V(F)|/2 \leq c(F) = \sum_{i=0}^{t} m_i$. Thus,

$$\sum_{i=1}^{t} i \cdot m_i = |E(F)| \leq |V(F)|/2 \leq \sum_{i=0}^{t} m_i,$$

which implies $\sum_{i=2}^{t} m_i(i-1) \leq m_0$, and therefore we have enough isolated vertices to use. □

### Definition

For any integer $k > 0$, a *k-bisection* of a graph $G$ is a bisection $(X, Y)$ where every component in $G[X] \cup G[Y]$ is a tree with at most $k$ vertices. Mattiolo and Mazzuoccolo showed the following result.

## Lemma ([D. Mattiolo and G. Mazzuoccolo, 2021])

*Every cubic multigraph has a 3-bisection.*

## Lemma

*Every weighted cubic multigraph $G$ has a bisection $(X, Y)$ such that the following holds.*

(i) $G[X] \cup G[Y]$ *is a forest;*

(ii) $|(X, Y)|$ *attains the maximum value among all bisection that satisfy (I);*

(iii) $\Delta(G[X]) \leq 1$ *and* $|E(G[Y])| \leq |Y|/2$.

## Proof.

State on the white board. □

**Theorem**

**([S. Gerke, G. Gutin, A. Yeo and Y. Zhou, 2024+])**

*Every weighted subcubic graph $G$ has a bisection with weight at least $\frac{2}{3}w(G)$.*

# Sketch of the Proof

We may assume that $G$ has at most one vertex with degree not equal to 3 as we may add edges of weight 0 between any two vertices with degree less than 3. We consider three cases.

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1. $G$ is cubic.
2. $G$ has one vertex with degree one, say $d_G(z) = 1$.
3. If $G$ has a vertex with degree two, say $d_G(z) = 2$.

# TRIANGLE-FREE SUBCUBIC GRAPHS (2-CONNECTED)

**Theorem ([S. Gerke, G. Gutin, A. Yeo and Y. Zhou, 2024+])**

*Every bridgeless triangle-free subcubic graph $G$ has a bisection with weight at least $\theta \cdot w(G)$, where $\theta = \frac{613}{855} \approx 0.716959$.*

# TRIANGLE-FREE SUBCUBIC GRAPHS (ALL CASES EXCEPT A CLAW)

**Theorem**
**([S. Gerke, G. Gutin, A. Yeo and Y. Zhou, 2024+])**

*Every triangle-free subcubic graph $G$, different than the claw, has a bisection with weight at least $\theta \cdot w(G)$.*

**CONJECTURE**
**([S. GERKE, G. GUTIN, A. YEO AND Y. ZHOU, 2024+])**

*Every triangle-free subcubic graph $G$, different than the claw, has a bisection with weight at least $\frac{11}{15} \cdot w(G)$.*

Thank you for your attention!

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