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ON THE NORTHCOTT PROPERTY FOR INFINITE EXTENSIONS

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ABSTRACT. We start with a brief survey on the Northcott property for subfields of the algebraic numbers $\overline{\mathbb{Q}}$. Then we introduce a new criterion for its validity (refining the author's previous criterion), addressing a problem of Bombieri. We show that Bombieri and Zannier's theorem, stating that the maximal abelian extension of a number field K contained in $K^{(d)}$ has the Northcott property, follows very easily from this refined criterion. Here $K^{(d)}$ denotes the composite field of all extensions of K of degree at most d.

1. INTRODUCTION

Heights are an important tool in Diophantine geometry to study the distribution of algebraic points on algebraic varieties, and in arithmetic dynamics to study preperiodic points under endomorphisms of algebraic varieties. There are various different heights but the most standard one is probably the Weil height on $\mathbb{P}^n$. However, there common fundamental property is that there are only finitely many points of bounded height over a given number field. To which fields of infinite degree does this finiteness property extend? This is the question we are concerned with in this article.

All algebraic field extensions of $\mathbb{Q}$ are considered subfields of some fixed algebraic closure $\overline{\mathbb{Q}}$. Let K be a number field, and for $P = (\alpha_0 : \cdots : \alpha_n) \in \mathbb{P}^n(K)$, with representative $(\alpha_0, \ldots, \alpha_n) \in K^{n+1}$, let

$$H(P) = \prod_{v \in M_K} \max\{|\alpha_0|_v, \ldots, |\alpha_n|_v\}^{\frac{d_v}{[K:\mathbb{Q}]}}$$

be the absolute multiplicative Weil height of P. Here M_K denotes the set of places of K. For each place v we choose the unique representative $|\cdot|_v$ that either extends the usual Archimedean absolute value on $\mathbb{Q}$ or a usual p-adic absolute value on $\mathbb{Q}$, and $d_v = [K_v : \mathbb{Q}_v]$ denotes the local degree at v. A standard reference for heights is [2]. We use $\mathbb{N} = \{1, 2, 3, \ldots\}$ for the set of positive natural numbers.

The unique prime factorisation of $\mathbb{Z}$ implies that $\prod_{M_\mathbb{Q}} |\alpha|_v^{d_v} = 1$ for every non-zero $\alpha \in \mathbb{Q}$. This identity is known as the product formula and extends to arbitrary number fields K [2, Proposition 1.4.4]. Consequently, the value of the height is independent of the representative $(\alpha_0, \ldots, \alpha_n)$ and thus defines a genuine function on $\mathbb{P}^n(K)$. Choosing a representative of P with a coordinate equal to 1 shows that $H(P) \geq 1$. The fundamental identity $\sum_{M_K} d_v = [K : \mathbb{Q}]$, valid for every number field (cf. [2, Corollary 1.3.2]), shows that the height $H(P)$ is also independent from the number field K containing the coordinates of P. Hence, $H(\cdot)$ is a well-defined function on $\mathbb{P}^n(\overline{\mathbb{Q}})$. D. G. Northcott [22, Theorem 1] proved the following simple but important result.

Theorem 1 (Northcott, 1950). *Given a number field K, $n \in \mathbb{N}$, and $X \geq 1$, there are only a finite number of points P in $\mathbb{P}^n(K)$ such that $H(P) \leq X$.*

For $P = (1 : \alpha_1 : \cdots : \alpha_n) \in \mathbb{P}^n(\overline{\mathbb{Q}})$ we obviously have $H(P) \geq \max_i H((1 : \alpha_i))$. Consequently, Theorem 1 holds true for a given field $K \subseteq \overline{\mathbb{Q}}$ if and only if it holds for $n = 1$. We define the height $H(\alpha)$ of an algebraic number α to be $H((1 : \alpha))$, and so we are led to the following notion, formally introduced in 2001 by Bombieri and Zannier [3].

Definition 1 (Northcott property). *A subset S of $\overline{\mathbb{Q}}$ has the* Northcott property *(or shorter:* Property (N)*) if*

$$\{\alpha \in S; H(\alpha) \leq X\}$$

is finite for every $X \geq 1$.

Theorem 1 was merely an intermediate step in Northcott's seminal work [22] from 1950 to show that for any morphism $f : \mathbb{P}^n \to \mathbb{P}^n$ of algebraic degree at least 2 and defined over a number field K there are only finitely many preperiodic points in $\mathbb{P}^n(K)$ under f. His proof also shows that one can replace number field by any field with Property (N).

Another somewhat surprising application of Property (N) builds on work of J. Robinson from 1962. It has been observed by Vidaux and Videla [29] that her work [24] implies the undecidability of each ring of totally real algebraic integers with Property (N). This connection was further exploited in [18] and in [28].

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These two applications extend interesting properties of number fields to fields with Property (N), suggesting that Property (N) fields behave similarly as number fields. However, this view was shattered by Fehm's discovery [11, Proposition 1.2] that some fields with Property (N) are pseudo algebraically closed (PAC).

Next we discuss two arithmetic properties with respect to which *all* fields of infinite degree with Property (N) behave radically different from number fields.

Gaudron and Rémond [15] introduced the notion of a Siegel field, which is a subfield of $\overline{\mathbb{Q}}$ over which Siegel's Lemma holds true (cf. [15, (∗) on p.189]). It is classical that number fields are Siegel fields, and work of Zhang [31], and independently of Roy and Thunder [26], shows that $\overline{\mathbb{Q}}$ is also a Siegel field. A priori it is not easy to find counterexamples but Gaudron and Rémond [15, Corollaire 1.2] proved that a field of infinite degree with Property (N) cannot be a Siegel field.

A very recent paper of Daans, Kala and Man [8] investigates the existence of universal quadratic forms over totally real fields of infinite degree. Whereas it is well-known that for totally real number fields such a form always exists, the existence of a universal quadratic form over a given totally real field of infinite degree is not clear at all. However, they prove [8, Theorem 1.2] that such a form cannot exists if the field has infinite degree and Property (N).

A point $P = (\alpha_0 : \cdots : \alpha_n) \in \mathbb{P}^n(\overline{\mathbb{Q}})$ defines a number field $\mathbb{Q}(\alpha_i/\alpha_j; \alpha_j \neq 0)$, and the degree of P is the degree of this number field. To prove Theorem 1 Northcott proved the stronger result [22, Lemma 2] that for any given $d \in \mathbb{N}$ and $X \geq 1$ there are only finitely many points $P \in \mathbb{P}^n(\overline{\mathbb{Q}})$ of degree d and height $H(P)$ at most X. The latter is a direct consequence of what nowadays is usually understood as "Northcott's Theorem" (cf. [2, Theorem 1.6.8]).

Theorem 2 (Northcott's Theorem)**.** *Let $d \in \mathbb{N}$, then the set $\{\alpha \in \overline{\mathbb{Q}}; [\mathbb{Q}(\alpha) : \mathbb{Q}] \leq d\}$ has Property (N).*

Northcott's Theorem already implies the existence of fields of infinite degree with Property (N). Indeed, let K be a number field and let $X \geq 1$ be given. Any two distinct quadratic extensions of K only intersect in K, and there are infinitely many such extensions. Hence, there must be one whose elements outside of K all have height bigger than X. Constructing an infinite tower $\mathbb{Q} = K_0 \subset K_1 \subset K_2 \subset \cdots$ where we choose a quadratic extension K_{i+1} of K_i whose elements outside of K_i all have height larger than i say, yields an infinite extension $L = \cup_i K_i$ with Property (N).

Dvornicich and Zannier [9] observed that Northcott's Theorem remains true when replacing the ground field $\mathbb{Q}$ by any field with Northcott property, i.e., if L is a field with Property (N) and $d \in \mathbb{N}$, then the set

$$\{\alpha \in \overline{\mathbb{Q}}; [L(\alpha) : L] \leq d\}$$

also has Property (N). In particular, Property (N) is preserved under finite field extensions. However, it is not always preserved under taking Galois closure over $\mathbb{Q}$, or taking compositum of two fields (cf. [30, Theorem 5]).

Bombieri and Zannier [3] were the first[1] authors that studied the Northcott property for infinite field extensions of $\mathbb{Q}$. In view of Northcott's Theorem it is very appealing to consider the field $\mathbb{Q}^{(d)}$ generated over $\mathbb{Q}$ by all algebraic numbers of degree at most d. Bombieri and Zannier [3] raised the following question.

Question 1 (Bombieri and Zannier, 2001)**.** *Let $d \in \mathbb{N}$. Does $\mathbb{Q}^{(d)}$ have Property (N)?*

There is a whole zoo of properties for subfields of $\overline{\mathbb{Q}}$ (including the properties (P), (SP), $(\overline{P})$, (R), $(\overline{R})$, (K), see [21, 17]; and (SB), (USB), see [12, 23]) in arithmetic dynamics, that are all implied by Property (N) (cf. [7, 23]). For some of these properties the analogue of Question 1 was posed, explicitly[2] or implicitly. We will not discuss any of these more exotic properties but let us mention that Pottmeyer [23, Theorem 4.3] showed that $\mathbb{Q}^{(d)}$ has the properties (USB) and (P) (solving a conjecture of Narkiewicz from 1963). However, (USB) and (P) are both strictly weaker than (N), as shown in [11, Proposition 1.3] and in [9, Theorem 3.3] respectively.

Question 1 is still open but a remarkable step was already made in [3]. For $d \in \mathbb{N}$ and K a number field we write $K^{(d)}$ for the composite field of all extensions of K of degree at most d. Then $K^{(d)}/K$ is a Galois extension, generated over K by all algebraic numbers of relative degree $[K(\alpha) : K]$ at most d. Let $K^{(d)}_{ab}$ be the composite field of all abelian extensions F/K with $F \subset K^{(d)}$. Then $K^{(d)}_{ab}$ is the maximal abelian subextension of $K^{(d)}/K$. If $d \geq 2$ then $\mathbb{Q}(\sqrt{n}; n \in \mathbb{Z}) \subset K^{(d)}_{ab} \subset K^{(d)}$, and so $K^{(d)}_{ab}$ and $K^{(d)}$ both have infinite degree over $\mathbb{Q}$, and thus also over K.

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[1] It is worthwhile mentioning that Julia Robinson [24] in 1962 proved that the ring of integers of $\mathbb{Q}(\sqrt{n}; n \in \mathbb{N})$ has the "Northcott property" with respect to the house (instead of Weil height), and deduced from this that $\mathbb{N}$ is first order definable in this ring.

[2] Narkiewicz [20, 19, Problem 10 (i)] conjectured that $K^{(d)}$ has (P) for all d. Further, for various pairs of these properties it was asked whether they are equivalent to each other, cf. [21, 7]

Theorem 3 (Bombieri, Zannier 2001)**.** *Let K be a number field and let $d \in \mathbb{N}$. The field $K_{ab}^{(d)}$ has the Northcott property. In particular, $K^{(2)}$ has the Northcott property.*

Taking $K = \mathbb{Q}(\zeta_d)$ for a primitive d-the root of unity, and applying Theorem 3 proves that the field

$$\mathbb{Q}(1^{1/d}, 2^{1/d}, 3^{1/d}, 4^{1/d}, \dots) \tag{1.1}$$

has the Northcott property.

Theorem 3 is a very interesting result for its own sake but it also has interesting applications. Specifically, to list some of the recent applications, Theorem 3 was used:

- in [29] to show that the maximal totally real subfield of $K_{ab}^{(d)}$ is undecidable, in [28] to show that $\mathbb{Q}_{ab}^{(d)}$ is undecidable, and in [18] as one of the ingredients that led the authors conjecture that $K^{(d)}$ is undecidable (proved for $\mathbb{Q}^{(2)}$ in the same paper).
- in [8] to deduce that if L is a totally real subfield of $K_{ab}^{(d)}$ of infinite degree, then no universal quadratic form exists over L. In particular, this holds if $L \subset \mathbb{Q}^{[d]}$ and d is a prime or a prime square, where $\mathbb{Q}^{[d]}$ denotes the compositum of all totally real Galois fields of degree exactly d over $\mathbb{Q}$.
- in [5, Corollary 1] to prove that if K is a number field, A is an abelian variety defined over K, and $K(A_{\mathrm{tors}})$ is the minimal field extension of K over which all torsion points of A are defined, then each subfield of $K(A_{\mathrm{tors}})$ which is Galois over K, and whose Galois group has finite exponent, has the Northcott property.

An abelian extension $L/\mathbb{Q}$ lies in $\mathbb{Q}^{(d)}$ for some d if and only if its Galois group has finite exponent (cf. [4, Theorem 1]). As pointed out in [5, Section 5] this remains true when replacing the ground field $\mathbb{Q}$ with an arbitrary number field K. Therefore Theorem 3 gives a purely Galois theoretic criterion for the Northcott property of a field, i.e., every abelian extension of a number field K with finite exponent has the Northcott property.

However, the restriction to abelian extensions (and finite exponent) in Theorem 3 is very rigid and rules out many interesting examples. In the survey article [1, p. 52] Bombieri states: *"It remains an open problem to determine whether the Northcott property holds for $K^{(d)}$ if $d \geq 3$ and, more generally, to determine workable conditions for its validity."* In this paper we are particularly concerned with the second part of Bombieri's statement.

Problem 1 (Bombieri, 2009)**.** *Determine workable conditions for the validity of the Northcott property for subfields of $\overline{\mathbb{Q}}$.*

In 2011 the author [30] gave a criterion which is robust and often easy to apply. For an extension M/K of number fields we write $D_{M/K}$ for the relative discriminant, and we write $N_{K/F}(\cdot)$ for the norm from K to F. If $F = \mathbb{Q}$ and $\mathfrak{A}$ is a non-zero ideal in the ring of integers $\mathcal{O}_K$ of K then we interpret $N_{K/F}(\mathfrak{A})$ as the unique positive rational integer that generates the principle ideal $N_{K/F}(\mathfrak{A})$.

Theorem 4 ([30, Theorem 3])**.** *Let K be a number field, let $K = K_0 \subsetneq K_1 \subsetneq K_2 \subsetneq \dots$. be a nested sequence of finite extensions and set $L = \bigcup_i K_i$. Suppose that*

$$\inf_{K_{i-1}\subsetneq M \subset K_i} \left(N_{K_{i-1}/\mathbb{Q}}(D_{M/K_{i-1}})\right)^{\frac{1}{[M:K_0][M:K_{i-1}]}} \longrightarrow \infty \tag{1.2}$$

as i tends to infinity where the infimum is taken over all intermediate fields M strictly larger than K_{i-1}. Then the field L has the Northcott property.

Theorem 4 implies the following refinement of (1.1). Let K be a number field, let $p_1 < p_2 < p_3 < \dots$ be a sequence of positive primes and let $d_1, d_2, d_3, \dots$ be a sequence of positive integers. Then the field

$$K(p_1^{1/d_1}, p_2^{1/d_2}, p_3^{1/d_3}, \dots)$$

has the Northcott property if and only if $\log p_i/d_i \longrightarrow \infty$ as i tends to infinity. The fact that every direct product of finite solvable groups can be realised over $\mathbb{Q}$ by a Galois extension with Property (N) can also easily be deduced from Theorem 4 (cf. [7, Theorem 4]). Fehm's aforementioned construction of PAC fields with Property (N) also used Theorem 4. And finally, Theorem 4 allows to construct fairly large non-abelian subfields of $\mathbb{Q}^{(d)}$ with Property (N) (cf. [30, Corollaries 3, 4, and 5]), providing another result on Question 1.

Theorem 4 is based on a fundamental height lower bound of Silverman [27, Theorem 2]. Here we give only a simplified version sufficient for our purposes. Let $\alpha \in \overline{\mathbb{Q}}$, let F be a number field, let $K = F(\alpha)$, $m = [F : \mathbb{Q}]$, and $d = [K : F]$. Then

$$H(\alpha) \geq \frac{1}{2} N_{F/\mathbb{Q}}(D_{K/F})^{\frac{1}{2md^2}}. \tag{1.3}$$

Using the optimal choice of F for given α to maximise the right hand-side in (1.3) plays an important role in our results. For the convenience of the reader we will give a proof of inequality (1.3) in Section 2.

Obviously Theorem 4 does not follow from Theorem 3. How does one prove Theorem 4? Let $\alpha \in L$ be of height at most X, and let K_{i_0} be the maximal field not containing α. Applying (1.3) with $F = K_{i_0}$, and using (1.2), shows that i_0 is bounded from above in terms of X and L, and thus, by Northcott's Theorem, the field L has the Northcott property.

However, the choice K_{i_0} for the ground field F can be far from optimal, and so we do not use the full force of (1.3). Therefore, Theorem 4 does not seem strong enough to deduce Theorem 3 either.

The aim of this short note is to provide a refined criterion, using the full force of (1.3), that easily implies Theorem 4 and Theorem 3. To this end we introduce the following invariant for an extension of number fields M/K:

$$\gamma(M/K) = \sup_{K \subset F} \left(N_{F/\mathbb{Q}}(D_{MF/F}) \right)^{\frac{1}{[MF:\mathbb{Q}][MF:F]}}, \tag{1.4}$$

where the supremum runs over all number fields F containing K, and MF denotes the composite field of M and F. We can now state a more powerful version of the criterion given in Theorem 4.

Theorem 5. *Let K be a number field, and let L be an infinite algebraic field extension of K. Suppose that*

$$\liminf_{K \subset M \subset L} \gamma(M/K) = \infty,$$

where M runs over all number fields in L containing K. Then L has the Northcott property.

Proof. Suppose that L does not have the Northcott property. Thus there exists $X \geq 1$ and a sequence $(\alpha_i)_i$ of pairwise distinct elements in L with $H(\alpha_i) \leq X$ for all i. By Northcott's Theorem the degrees of $M_i = K(\alpha_i)$ must tend to infinity. After passing to a subsequence we can assume all the M_i are distinct. Note that $M_i F = F(\alpha_i)$ for each F that contains K. We apply inequality (1.3) to get

$$\begin{aligned} 4X^2 &\geq \liminf_i (2H(\alpha_i))^2 \geq \liminf_i \left(\sup_{K \subset F} N_{F/\mathbb{Q}}(D_{M_iF/F})^{\frac{1}{[M_iF:\mathbb{Q}][M_iF:F]}} \right) \\ &\geq \liminf_{K \subset M \subset L} \left(\sup_{K \subset F} N_{F/\mathbb{Q}}(D_{MF/F})^{\frac{1}{[MF:\mathbb{Q}][MF:F]}} \right) = \liminf_{K \subset M \subset L} \gamma(M/K). \end{aligned}$$

□

Theorem 5 implies[3] Theorem 4, but why does it also imply Theorem 3, and how does this proof differ from the original one in [3]? We will discuss these questions in detail in Section 3.

Are there any known criteria for Property (N) for field extensions of infinite degree that we have not mentioned so far? The author is only aware of one such criterion. Let $L/\mathbb{Q}$ be a Galois extension and let $S(L)$ be the set of rational primes for which L can be embedded in a finite extension of $\mathbb{Q}_p$. For $p \in S(L)$ let e_p and f_p be the ramification index and the inertia degree above p. Bombieri and Zannier [3, Theorem 2] proved that

$$\liminf_{\alpha \in L} H(\alpha) \geq \exp\left(\frac{1}{2} \sum_{p \in S(L)} \frac{\log p}{e_p(p^{f_p}+1)} \right). \tag{1.5}$$

In particular, L has the Northcott property whenever the sum on the right hand-side of (1.5) diverges. The above criterion does not seem very workable. Bombieri and Zannier asked whether this sum can diverge for infinite extensions but considered this unlikely. However, it was shown by Checcoli and Fehm [6] in 2021 that there are Galois extensions $L/\mathbb{Q}$ of infinite degree for which the above sum diverges, and even such extensions for which neither Theorem 3 nor Theorem 4 applies, so it constitutes an independent criterion for the Northcott property, albeit one for which natural examples still need to be found.

2. Silverman's inequality

In this section we give a proof of Silverman's inequality (1.3). For the special case $F = \mathbb{Q}$ a very simple proof was given by Roy and Thunder [25, Lemma 1 and 2]. We extend the argument in [25] to arbitrary ground fields F, providing a slightly different proof from Silverman's original one in [27]. Yet another proof of Silverman's inequality was given by Ellenberg and Venkatesh [10, Lemma 2.2].

We first fix the notation and recall some basic facts. Let F be a number field of degree m, let K/F be a field extension of degree d, and let $\sigma_1, \ldots, \sigma_d : K \to K^{(G)}$ be the d distinct field homomorphisms of K to the Galois closure $K^{(G)}$ of K/F, fixing F. Let $(z_1, \ldots, z_d)$ be a d-tuple of elements in K. Then $D_{K/F}(z_1, \ldots, z_d) = \det[\sigma_i(z_j)]^2$, and for a non-zero ideal $\mathfrak{A}$ in $\mathcal{O}_K$ the discriminant $D_{K/F}(\mathfrak{A})$ is the ideal in $\mathcal{O}_F$ generated by the

[3] Let (M_j) be a sequence of distinct fields with $K \subset M_j \subset L$ and $\gamma(M_j/K) < X$, and let $i = i(j)$ be minimal with $M_j \subset K_i$. Set $M_j' = K_{i-1}M_j$ so that $K_{i-1} \subsetneq M_j' \subset K_i$. The choice $F = K_{i-1}$ on the right-hand side of (1.4) shows that (1.2) has a bounded subsequence.

numbers $D_{K/F}(z_1,\dots,z_d)$ as the tuples $(z_1,\dots,z_d)$ run over all F-bases of K and each basis element is contained in $\mathfrak{A}$. In particular, $D_{K/F}(\mathfrak{A})$ divides the principle ideal in $\mathcal{O}_F$ generated by $D_{K/F}(z_1,\dots,z_d)$ for each such tuple $(z_1,\dots,z_d)$ (see [16, III, §3]). Recall that we write $D_{K/F}$ for $D_{K/F}(\mathcal{O}_K)$. We will use the basic identity (cf. [13, III, §3, Proposition 13])

$$D_{K/F}(\mathfrak{A}) = D_{K/F}N_{K/F}(\mathfrak{A})^2. \tag{2.6}$$

Lemma 1 (Silverman, 1984)**.** *Let F be a number field of degree m. Let $\alpha \in \overline{\mathbb{Q}}\backslash F$, set $K = F(\alpha)$, and $d = [K : F]$. Then*

$$H(\alpha) \geq d^{-\frac{1}{2(d-1)}} N_{F/\mathbb{Q}}(D_{K/F})^{\frac{1}{2md(d-1)}}.$$

Proof. Choose $\omega_0, \omega_1 \in \mathcal{O}_K$ such that $\omega_0 \neq 0$ and $\alpha = \omega_1/\omega_0$. For $1 \leq j \leq d$ let $z_j = \omega_0^{d-j}\omega_1^{j-1}$, so that $P = (1 : \alpha : \cdots : \alpha^{d-1}) = (z_1 : \cdots : z_d) \in \mathbb{P}^{d-1}(K)$ and $H(\alpha)^{d-1} = H(P)$. We will bound

$$H(P)^{2md} = \prod_{v\nmid\infty} \max_j\{|z_j|_v\}^{2d_v} \prod_{v|\infty} \max_j\{|z_j|_v\}^{2d_v}$$

from below. Note that $z_1,\dots,z_d$ is an integral F-basis of K. Let $\mathfrak{A} = \sum_j z_j\mathcal{O}_K$ be the ideal in $\mathcal{O}_K$ generated by the z_j. For the non-Archimedean places of K we have

$$\prod_{v\nmid\infty} \max_j\{|z_j|_v\}^{2d_v} = N_{K/\mathbb{Q}}(\mathfrak{A})^{-2}.$$

For each embedding $\tau : F \to \mathbb{C}$ we choose an extension $\tilde{\tau} : K^{(G)} \to \mathbb{C}$ of τ to $K^{(G)}$. Then the d distinct maps $\tilde{\tau} \circ \sigma_i : K \to \mathbb{C}$ are precisely the d embeddings of K that extend τ. Ranging over all embeddings τ of F gives the full set of embeddings of K. Hence, for the Archimedean places of K we get

$$\prod_{v|\infty} \max_j\{|z_j|_v\}^{2d_v} = \prod_\tau \prod_{i=1}^d \max\{|\tilde{\tau}\circ\sigma_i(z_1)|,\dots,|\tilde{\tau}\circ\sigma_i(z_d)|\}^2.$$

Writing $z_{\tau,i}$ for the complex row vector $(\tilde{\tau}\circ\sigma_i(z_1),\dots,\tilde{\tau}\circ\sigma_i(z_d))$, and applying Hadamard's inequality yields

$$\prod_{i=1}^d \max\{|\tilde{\tau}\circ\sigma_i(z_1)|,\dots,|\tilde{\tau}\circ\sigma_i(z_d)|\}^2 \geq d^{-d}\prod_{i=1}^d |z_{\tau,i}|^2 \geq d^{-d}|\det[\tilde{\tau}\circ\sigma_i(z_j)]^2| = d^{-d}|\tilde{\tau}(\det[\sigma_i(z_j)]^2)| = d^{-d}|\tau(\det[\sigma_i(z_j)]^2)|,$$

where in the last step we used that $\det[\sigma_i(z_j)]^2 = D_{K/F}(z_1,\dots,z_d)$ lies in F. Taking the product over all τ, and using that $D_{K/F}(\mathfrak{A})$ divides the ideal generated by $\det[\sigma_i(z_j)]^2$ in $\mathcal{O}_F$, yields

$$\prod_{v|\infty} \max_j\{|z_j|_v\}^{2d_v} \geq d^{-md} N_{F/\mathbb{Q}}(D_{K/F}(\mathfrak{A})).$$

Now we use (2.6), and that $N_{F/\mathbb{Q}}(D_{K/F}N_{K/F}(\mathfrak{A}))^2) = N_{F/\mathbb{Q}}(D_{K/F})N_{K/\mathbb{Q}}(\mathfrak{A})^2$ to get

$$H(\alpha)^{2md(d-1)} = H(P)^{2md} \geq d^{-md}N_{F/\mathbb{Q}}(D_{K/F}),$$

which proves the claim. □

3. Theorem 5 implies Bombieri and Zannier's Theorem 3

In this section we show that Theorem 5 gives a short and straightforward proof of Theorem 3. We also compare this new proof with the original one from [3]. Both proofs have a common part which we extract and formulate below as a separate lemma.

Lemma 2 (Bombieri and Zannier [3])**.** *Let $d \in \mathbb{N}$, let K be a number field, and let M be a number field with $K \subset M \subset K_{ab}^{(d)}$. Then p_M, the largest prime that ramifies in M, tends to infinity as M runs over all such intermediate fields M. Further, if $p > d$ is prime and $\mathfrak{B}$ is a prime ideal in $\mathcal{O}_M$ above p and $\mathfrak{p} = \mathfrak{B} \cap K$, then the ramification index $e(\mathfrak{B}/\mathfrak{p})$ divides $d!$.*

Proof. We follow Bombieri and Zannier's argument from [3]. Let M be a number field with $K \subset M \subset K_{ab}^{(d)}$. Then M/K is an abelian extension of exponent[4] dividing $d!$, and thus $\mathrm{Gal}(M/K)$ is isomorphic to a direct product $A_1 \times \cdots \times A_r$ of cyclic groups of order dividing $d!$. Therefore M can be written as composite field of extensions E of K of degree at most $d!$. Indeed, let $\varphi : A_1 \times \cdots \times A_r \to \mathrm{Gal}(M/K)$ be an isomorphism and let $E_i = \mathrm{Fix}(\varphi(H_i))$ where H_i is the subgroup that picks the trivial group in the i-th component and the full A_j in all other components. By the Galois-correspondence we have $\mathrm{Gal}(M/E_1\cdots E_r) = \cap_i\varphi(H_i) = \varphi(\cap_i H_i) = \{id\}$. Hence, $M = E_1 \cdots E_r$. Now the largest power of a prime dividing the discriminant of E can be bounded solely

[4] If K_1, K_2 are two finite Galois extensions of K then $\sigma \to (\sigma_{|K_1}, \sigma_{|K_2})$ induces an injective group homomorphism from $\mathrm{Gal}(K_1K_2/K)$ to $\mathrm{Gal}(K_1/K) \times \mathrm{Gal}(K_2/K)$. This implies that for each Galois extension M/K with $M \subset K^{(d)}$ the Galois group $\mathrm{Gal}(M/K)$ has exponent dividing $d!$, and no prime $p > d$ divides the order of $\mathrm{Gal}(M/K)$.

in terms of K and d (cf. [2, Theorem B.2.12]). Thus, by Hermite's Theorem, p_M, the largest prime that ramifies in M, tends to infinity as M runs over all such intermediate fields M.

For the second claim note that the inertia group $I(\mathfrak{B}/\mathfrak{p})$ is a subgroup of $\mathrm{Gal}(M/K)$, and so its order is not divisible by p, whenever $p > d$ is prime. Since the ramification index $e(\mathfrak{B}/\mathfrak{p})$ is equal to the order of $I(\mathfrak{B}/\mathfrak{p})$ it follows that $\mathfrak{p}$ is tamely ramified in M. Hence (cf. [2, B.2.18 (e)]), $I(\mathfrak{B}/\mathfrak{p})$ is cyclic, and thus $e(\mathfrak{B}/\mathfrak{p})$ divides $d!$. □

Now let us show that Theorem 5 together with Lemma 2 implies Theorem 3.

Proof of Theorem 3. Let M be a number field with $K \subset M \subset K_{ab}^{(d)}$. Then M is abelian over K. By Lemma 2 $p_M > |D_{K/\mathbb{Q}}|+d$ for all but finitely many M, and thus we can assume p_M is unramified in K and $p_M > d$. Therefore, one of the prime ideal divisors of $p_M\mathcal{O}_K$, say $\mathfrak{p}$, must ramify in M. Let $\mathfrak{p}\mathcal{O}_M = (\mathfrak{B}_1 \cdots \mathfrak{B}_g)^e$ be the decomposition in $\mathcal{O}_M$ with $\mathfrak{B}_1, \ldots, \mathfrak{B}_g$ distinct prime ideals. Let T_i be the fixed field for the inertia group $I(\mathfrak{B}_i/\mathfrak{p})$, and let $\mathfrak{p}_i = \mathfrak{B}_i \cap T_i$. Then $e(\mathfrak{p}_i/\mathfrak{p}) = 1$, and $e = e(\mathfrak{B}_i/\mathfrak{p}_i) = [M : T]$. It follows that $\mathfrak{p}_i^{e-1}|D_{M/T_i}$, and that $f(\mathfrak{B}_i/\mathfrak{p}) = f(\mathfrak{p}_i/\mathfrak{p})$ for the residue degree. Now the $I(\mathfrak{B}_i/\mathfrak{p})$ are conjugated to each other and since M/K is abelian they are all equal, and thus all the fixed fields T_i are equal to T, say. Therefore $(\mathfrak{p}_1 \cdots \mathfrak{p}_g)^{e-1}|D_{M/T}$, which implies $p_M^{[M:K]/2} \leq N_{T/\mathbb{Q}}(D_{M/T})$. Choosing $F = T$ in (1.4) shows that $\gamma(M/K) \geq p_M^{1/(2e[K:\mathbb{Q}])}$ which, by Lemma 2, tends to infinity as M runs over all number fields with $K \subset M \subset K_{ab}^{(d)}$. Applying Theorem 5 completes the proof. □

Remark 1. *Alternatively, one can use the decomposition of M as compositum of extensions E of K of degree at most $d!$ as in the proof of Lemma 2. Hence, $\mathfrak{p}$ ramifies in at least one of the fields E, and thus $\mathfrak{p}|D_{E/K}$. Since $D_{M/K} = D_{E/K}^{[M:E]}N_{E/K}(D_{M/E})$ we conclude $\mathfrak{p}^{[M:E]}|D_{M/K}$. Since $\mathfrak{p}$ is unramified in T, and $D_{M/K} = D_{T/K}^{[M:T]}N_{T/K}(D_{M/T})$ we get $\mathfrak{p}^{[M:E]}|N_{T/K}(D_{M/T})$. Taking norms and using $[M : T] = e$ gives $\gamma(M/K) \geq p^{\frac{1}{[K:\mathbb{Q}]d!e}}$.*

To compare we now discuss Bombieri and Zannier's original proof of Theorem 3. We leave out some of the more technical details but the basic argument is as follows. We mostly use the notation of [3] (see also [2, Theorem 4.5.4] for a slightly more detailed approach).

Proof of Theorem 3 (after Bombieri and Zannier). By enlarging K we can assume K contains a primitive $d!$-th root of unity. Let $\alpha \in K_{ab}^{(d)}$ be of height at most X, and set $L = K(\alpha)$. Then L is abelian over K. Let $p > d$ be a prime unramified in K, let v be a place in K above p, and write e for the ramification index of v in L. Then e divides $d!$ by Lemma 2.

Now set $\theta = p^{1/e}$. Then $L(\theta)$ is again an abelian extension of K. Since $x^e - p \in K[x]$ is a v-Eisenstein polynomial it follows that $[K(\theta) : K] = e$ and v is totally ramified in $K(\theta)$. By Abhyankar's Lemma the ramification indices of the places in $L(\theta)$ above v are again e. As $\mathrm{Gal}(L(\theta)/K)$ is abelian the inertia groups of each place in $L(\theta)$ above v are equal, and of size e. Let U be their common fixed field, so that $[L(\theta) : U] = e$. Now v is unramified in U and totally ramified in $K(\theta)$ and thus $[U(\theta) : U] = e$. Hence, $L(\theta) = U(\theta)$, and thus

$$\alpha = \beta_0 + \beta_1\theta + \cdots + \beta_{e-1}\theta^{e-1}$$

for certain coefficients $\beta_i \in U$. Now the trace from $U(\theta)$ to U of $\alpha\theta^{-j}$ is the sum of the conjugates of $\alpha\theta^{-j}$ over U. It is not hard to see that this trace is also just $e\beta_j$. Combining both, and using standard height inequalities, gives an upper bound for the height of $\gamma_j := \beta_j p^{j/e}$ in terms of d and X.

Let us now assume that $1 \leq j \leq e-1$ and $b_j \neq 0$. Let u be a place in $U(\theta)$ above v, and let $\mathfrak{q} = \mathfrak{q}_u$ be the corresponding prime ideal in the ring of integers of $U(\theta)$. Then the exact order to which $\mathfrak{q}$ divides b_j is a (possibly negative) multiple of e, whereas the exact order to which it divides $p^{j/e}$ is j. This implies that the exact order to which $\mathfrak{q}$ divides γ_j is non-zero. Using this fact for all places in $U(\theta)$ above v yields a lower bound for the height of γ_j of the form $H(\gamma_j) \geq p^{1/(2e[K:\mathbb{Q}])}$, provided $1 \leq j \leq e-1$ and $b_j \neq 0$.

Combining the upper and lower bounds for the height of γ_j, and using that e is bounded in terms of d, gives an upper bound $B(K, d, X)$ for p in terms of d, $[K : \mathbb{Q}]$, and X, whenever one among $b_1, \ldots, b_{e-1}$ is non-zero.

This means that for each place v of K lying above a prime $p > B(K, d, X)$ we have $\alpha \in U$, and v is unramified in U. Therefore, $K(\alpha)$ is unramified at each prime p whenever $p > B(K, d, X)$ (assuming, as we can, $B(K, d, X) > |D_{K/\mathbb{Q}}|$). But, by Lemma 2, the largest prime p ramifying in $K(\alpha)$ tends to infinity when $K(\alpha)$ runs over an infinite set of subfields of $K_{ab}^{(d)}$. Hence, we conclude that α lies in a number field, depending only on K, d and X, and thus, by Northcott's Theorem, there are only finitely many possibilities for α. This completes the proof. □

The first proof of Theorem 3 (using Theorem 5) only requires e to be bounded in terms of d, whereas the second proof above requires the ramification index e to divide $d!$ to conclude that $L(\theta)/K$ is Galois (and abelian).

The fact that $K_{ab}^{(d)}/K$ is abelian is used in both proofs in three different ways, namely to ensure that:

(i) p_M in Lemma 2 tends to infinity,

(ii) M/K is Galois for every number field $K \subset M \subset K_{ab}^{(d)}$,

(iii) the inertia groups $I(\mathfrak{P}/\mathfrak{p})$ for the different prime ideals $\mathfrak{P} \subset \mathcal{O}_M$ above $\mathfrak{p} \subset \mathcal{O}_K$ are all equal.

The second claim of Lemma 2 remains true for $K^{(d)}/K$ (replace M by its Galois closure over K in the proof) but the proof of the first claim falls apart for $K^{(d)}$ when $d \geq 3$. This is because not all finite extensions of K in $K^{(d)}$ can be written as compositum of number fields of uniformly bounded degree over K as was shown by Checcoli [4, Theorem 1], at least if $d \geq 27$. Gal and Grizzard [14, Corollary 1.2] showed that $d \geq 3$ suffices.

However, they also showed [14, Theorem 1.3] that every number field in $K^{(3)}$ that is Galois over K can be written as a compositum of extensions of K of degree at most 3. This means that if we only consider α in the set $K_G^{(3)} = \{\alpha \in K^{(3)}; K(\alpha)/K \text{ is Galois}\}$ then (i) and (ii) are automatically satisfied for each $M = K(\alpha)$. This raises the question whether $K_G^{(3)}$ has the Northcott property. An affirmative answer would be a significant extension of the case $d = 3$ in Theorem 3.

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