

Lower Bounds for Maximum Weighted Cut

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Abstract

While there have been many results on lower bounds for Max Cut in unweighted graphs, the only lower bound for non-integer weights is that by Poljak and Turzík (1986). In this paper, we launch an extensive study of lower bounds for Max Cut in weighted graphs. We introduce a new approach for obtaining lower bounds for Weighted Max Cut. Using it, Probabilistic Method, Vizing's chromatic index theorem, and other tools, we obtain several lower bounds for arbitrary weighted graphs, weighted graphs of bounded girth and triangle-free weighted graphs. We pose conjectures and open questions.

1 Introduction

In this paper $G = (V(G), E(G), w)$ will denote a connected weighted graph with weight function $w : E(G) \rightarrow \mathbb{R}_+$, where \mathbb{R}_+ is the set of non-negative reals. Let A and B be a partition of $V(G)$. Then the *cut* (A, B) of G is the bipartite subgraph of G induced by the edges between A and B . The MAXIMUM WEIGHTED CUT problem (MWC) is a well-known NP-hard optimization problem on graphs [14], where given a weighted graph G , the aim is to find the maximum weight of a cut of G . This weight will be denoted by $\text{mac}(G)$.

Lower bounds for $\text{mac}(G)$ are of interest e.g. for designing heuristics and branch-and-bound algorithms for computing $\text{mac}(G)$. There are many publications where lower bounds on $\text{mac}(G)$ have been studied. However, almost all of them are either for the unweighted case i.e. the weight of every edge equals 1 (see e.g. [1, 3, 5, 7, 15, 16]) (In what follows, the weight of every edge

is an unweighted graph will be equal to 1) or for graphs with integral weights (see e.g. [2, 5]).

As far as we know, the only easily computable lower bound for the general weighted case is that of Poljak and Turzík [15]: $\text{mac}(G) \geq w(G)/2 + w(T_{\min})/4$, where T_{\min} is a minimum weight spanning tree of G . Note that here and in the rest of the paper, for a subgraph H of G , $w(H) = \sum_{e \in E(H)} w(e)$.

In Section 2, we introduce a generic lower bound for $\text{mac}(G)$ and show that the Poljak-Turzík bound can be easily obtained from the generic bound. We prove that unfortunately the lower bound is NP-hard to compute. However, the bound can be used to obtain other lower bounds which are computable in polynomial time, see Theorems 3.1, 4.1, 4.3, 5.4, Lemma 5.11 and Proposition 2.3.

In Section 3, we prove that the Poljak-Turzík bound can be improved by replacing a minimum weight spanning tree by a DFS tree (i.e. a tree that can be obtained by using a depth-first search algorithm): $\text{mac}(G) \geq w(G)/2 + w(D)/4$, where D is a DFS tree of G . Theorem 3.3 shows that we cannot replace D in the new bound (called below the *DFS bound*) by an arbitrary spanning tree. The DFS tree bound is stronger than the Poljak-Turzík bound because while the Poljak-Turzík bound requires the spanning tree to be of minimum weight, for the DFS bound we can use an arbitrary DFS tree. We also prove that unfortunately replacing an arbitrary DFS tree with a DFS tree of maximum weight would make the bound no longer computable in polynomial time unless $\text{P} = \text{NP}$. The last result holds even for triangle-tree graphs studied later in the paper. We complete Section 3 by observing another new bound: $\text{mac}(G) \geq (w(G) + w(M))/2$, where M is a maximum weight matching of G .

In Section 4, we study graphs of bounded girth. The girth of a graph is the length of its shortest cycle. We show that if the girth g of G is even then $\text{mac}(G) \geq \frac{w(G)}{2} + \frac{g-1}{2g} w(D_{\max})$, where D_{\max} is a maximum weight DFS tree of G . This bound can be extended to the case where the girth is odd by replacing g with $g-1$. We also prove that when G is triangle-free then the Poljak-Turzík bound can be improved as follows: $\text{mac}(G) \geq w(G)/2 + w(T_{\max})/4$, where T_{\max} is a maximum weight spanning tree. This is in sharp contrast with Theorem 3.3, which shows that the last bound does not hold for arbitrary graphs. Note that $w(T_{\max})$ can be computed in polynomial time while it is NP-hard to compute $w(D_{\max})$. We complete the section by a conjecture that $\text{mac}(G) \geq \frac{w(G)}{2} + \frac{3}{8} \cdot w(T)$ for a triangle-free graph G and a spanning tree T of G .

Section 5 is devoted to triangle-free graphs G with bounded maximum degree $\Delta(G)$. In Subsection 5.1 we study triangle-free graphs G with $\Delta(G) \leq 3$. Inspired by the result of Bondy and Locke [6] that a triangle-free graph G

with $\Delta(G) \leq 3$ has a bipartite subgraph with at least $\frac{4}{5}|E(G)|$ edges, we conjecture that $\text{mac}(G) \geq 4w(G)/5$ for a weighted triangle-free graph G with $\Delta(G) \leq 3$ (see Conjecture 5.3). Theorem 5.4 proved in Section 6 shows that $\text{mac}(G) \geq \frac{8}{11} \cdot w(G)$ for a weighted triangle-free graph G with $\Delta(G) \leq 3$. Theorem 5.4 allows us to prove Theorem 5.5 (also proved in Section 6) which states that $\text{mac}(G) \geq \frac{w(G)}{2} + 0.3193 \cdot w(T)$ for a triangle-free graph G with $\Delta(G) \leq 3$ and a spanning tree T of G . We show that Conjecture 5.3 implies the conjecture of Section 4 for triangle-free graphs of maximum degree at most 3 as well as Conjecture 5.8, which states that every triangle-free graph G with $\Delta(G) \leq 3$ has an edge set E' such that every 5-cycle of G contains exactly one edge from E' . Thus, if Conjecture 5.3 holds, it implies a somewhat unexpected structural result for unweighted graphs.

Subsection 5.2 is devoted to triangle-free graphs G with maximum degree bounded by arbitrary Δ . The main results of this section are Theorems 5.10 and 5.12, which give different bounds of the type $\text{mac}(G) \geq a_\Delta \cdot w(G)$, where a_Δ depends only on Δ . The proof of Theorem 5.10 easily follows from results of Shearer [16]. The bound of Theorem 5.12 is stronger than that of Theorem 5.10 if and only if $\Delta \leq 16$.

We conclude the paper in Section 7.

Our proofs rely in particular on the Probabilistic Method and Vizing's chromatic index theorem.

2 Generic Bound

The following theorem is a generic bound, which is used in the next section to obtain new lower bounds for $\text{mac}(G)$. These new lower bounds immediately imply a well-known lower bound of Poljak and Turzík [15].

Let $\mathcal{B}(G)$ denote the set of bipartite subgraphs R of G such that every connected component of R is an induced subgraph of G . Every graph in $\mathcal{B}(G)$ is called a \mathcal{B} -subgraph of G .

Theorem 2.1. *If $R \in \mathcal{B}(G)$, then $\text{mac}(G) \geq (w(G) + w(R))/2$.*

Proof. Let R_1, R_2, \dots, R_ℓ be connected components of R and let X_i, Y_i be partite sets of R_i , $i \in [\ell]$. For each $i \in [\ell]$, randomly and uniformly assign X_i color 1 or 2 and Y_i the opposite color. Note that this is a proper coloring of R . Let A be all vertices of color 1 and let B be all vertices of color 2. Now every edge in R deterministically lies in the cut induced by (A, B) and every edge not in R lies in the cut induced by (A, B) with probability $1/2$. Therefore the average weight of the cut (A, B) is $w(R) + w(E(G) - E(R))/2 = (w(G) + w(R))/2$. Thus, $\text{mac}(G) \geq (w(G) + w(R))/2$. \square

Using the well-known derandomization method of conditional probabilities [4, Section 15.1], given $R \in \mathcal{B}(G)$, in polynomial time we can find a cut of G of weight at least $(w(G) + w(R))/2$. Note that, in the definition of $\mathcal{B}(G)$, the requirement that every connected component of R is an induced subgraph of G , is necessary as otherwise the term $w(E(G) - E(R))/2$ in the average weight of the cut (A, B) is incorrect.

Let r_{\max} be the maximal weight of a \mathcal{B} -subgraph of G . By Theorem 2.1, $\text{mac}(G) \geq (w(G) + r_{\max})/2$. Unfortunately, it is NP-hard to compute r_{\max} , which follows from the next theorem.

Theorem 2.2. *Let H be an unweighted graph. It is NP-hard to compute the maximum number of edges in a \mathcal{B} -subgraph of H .*

Proof. The proof is by reduction from the INDEPENDENT SET problem. In this problem, given a graph F and a natural number k , we are to decide whether F contains an independent set of size at least k . It is well-known that INDEPENDENT SET is NP-complete.

Let F be an instance of the INDEPENDENT SET problem where we want to determine if F has an independent set of size k . Let n be the number of vertices in F and let Q be a set of n^2 vertices outside of F . Construct a new graph H by adding all edges between F and Q . We will show that H has a \mathcal{B} -subgraph with at least kn^2 edges if and only if F contains an independent set of size at least k . Indeed, if F contains an independent set I of size at least k , then H has a \mathcal{B} -subgraph with at least kn^2 edges, as the subgraph induced by $I \cup Q$ is such a subgraph.

Conversely, assume that H has a \mathcal{B} -subgraph, B , with at least kn^2 edges. As at most $\binom{n}{2} < n^2$ edges in $E(B)$ belong to $E(F)$ we must have at least $(k-1)n^2 + 1$ edges in $E(B)$ belonging to the cut $(V(F), Q)$. Now let C_1, C_2, \dots, C_l be the connected components of B that contain vertices from Q and let $X_i = V(F) \cap V(C_i)$ for all $i \in [l]$. Note that for all $i \in [l]$, X_i is an independent set in F , as if some edge uv belonged to $F[C_i]$ then u, v and any vertex from $Q \cap V(C_i)$ would form a 3-cycle in B . Let $x^{\max} = \max\{|X_i| \mid i \in [l]\}$ and note that no vertex in Q is incident with more than x^{\max} edges in $E(B)$. As there are at least $(k-1)n^2 + 1$ edges in $E(B)$ belonging to the cut $(V(F), Q)$, this implies that $x^{\max} > k-1$ which implies that F has an independent set of size at least k . This completes the proof. \square

The following is a simple corollary of Theorem 2.1 as a matching in G is a \mathcal{B} -subgraph of G .

Proposition 2.3. *Let M be a maximum weight matching of G . Then $\text{mac}(G) \geq (w(G) + w(M))/2$.*

The bound of Proposition 2.3 is tight. Indeed, let n be an even positive integer and let K_n be unweighted. Then clearly $\text{mac}(K_n) = n^2/4$. Also, $(w(K_n) + w(M))/2 = \frac{1}{2}\binom{n}{2} + n/4 = n^2/4$.

It is not hard to construct examples of weighted graphs for which the bound of Proposition 2.3 is larger than the Poljak-Turzík bound; e.g. consider a weighted graph with a spanning tree of weight zero and at least one edge of positive weight.

3 New Bounds for Arbitrary Connected Graphs

Throughout this section, G is connected. A *DFS tree* is a tree constructed by Depth First Search [8].

Theorem 3.1. *If D is a DFS tree of G then $\text{mac}(G) \geq w(G)/2 + w(D)/4$.*

Proof. Let D be rooted at vertex u . Let L_i be the set of vertices of D at distance i from u in D . Let H_i be the subgraph of D induced by the set of edges of D between vertices L_i and L_{i+1} . Note that each L_j is an independent set of G . Since D is a DFS tree, G has no cross edges with respect to D i.e. edges xy such that x is not a descendent of y and y is not a descendent of x in D [8]. Hence, each H_j is an induced bipartite subgraph of G . Let G_1 be the disjoint union of graphs H_i with odd i and G_2 the disjoint union of graphs H_i with even i . Since G_j ($j \in \{0, 1\}$) is a disjoint union of graphs from $\mathcal{B}(G)$, we have $G_0, G_1 \in \mathcal{B}(G)$. Hence, by Theorem 2.1, $\text{mac}(G) \geq (w(G) + w(G_j))/2$ for $j = 0, 1$. These bounds and $w(G_1) + w(G_2) = w(D)$ imply $\text{mac}(G) \geq w(G)/2 + w(D)/4$. \square

Note that the above bound immediately implies the following corollary, by Poljak and Turzík (as $w(D) \geq w(T_{\min})$ when D is a DFS tree and T_{\min} is a minimum weight spanning tree of G).

Corollary 3.2. *Let T_{\min} be a minimum weight spanning tree of G . Then $\text{mac}(G) \geq w(G)/2 + w(T_{\min})/4$.*

As some DFS trees may have much larger weight than a minimum weight spanning tree of G , the bound of Theorem 3.1 is, in general, stronger than that of Poljak and Turzík. The following theorem implies that D cannot be replaced by an arbitrary spanning tree T in Theorem 3.1.

Theorem 3.3. *Let $\varepsilon > 0$ be arbitrary. There exists (infinitely many) edge-weighted graphs G with a spanning tree T such that $\text{mac}(G) < w(G)/2 + \varepsilon w(T)$.*

Proof. Let W be any positive integer strictly larger than $\frac{1}{4\varepsilon}$. Let l be any integer such that:

$$l > \frac{W^2}{4W\varepsilon - 1}.$$

Let $G = K_{l+1}$ and let $x \in V(G)$ be arbitrary. Let $w : E(G) \rightarrow \mathbb{N}$ be a weight function such that $w(e) = W$ if e is an edge incident with x and let $w(e) = 1$ otherwise. Let T be the spanning star $K_{1,l}$ with x as the root. Note that $w(T) = Wl$. Let $\theta = w(G)/2 + \varepsilon w(T)$ and note that the following holds.

$$\theta = \frac{1}{2} \binom{l}{2} + Wl \left(\frac{1}{2} + \varepsilon \right) = \frac{l(l-1)}{4} + Wl \left(\frac{1}{2} + \varepsilon \right)$$

We will now bound $\text{mac}(G)$ from above. Let (A, B) be any partition of $V(G)$ and without loss of generality assume that $x \in A$. Let G' be the graph obtained from G by replacing x with an independent set, X , containing W vertices, such that $N(v) = V(G) \setminus \{x\}$ for all $v \in X$. Let all edge-weights of G' be one. Note that the weight of the cut $(X \cup A \setminus \{x\}, B)$ in G' is the same as the weight of the cut (A, B) in G and as (A, B) is an arbitrary cut in G we must have $\text{mac}(G') \geq \text{mac}(G)$. As the maximum number of edges in a cut in a graph on $W + l$ vertices is at most $((W + l)/2)^2$ the following must hold.

$$\text{mac}(G) \leq \text{mac}(G') \leq \left(\frac{W + l}{2} \right)^2 = \frac{(W + l)^2}{4}$$

Recall that $l > \frac{W^2}{4W\varepsilon - 1}$, which implies that $l(4W\varepsilon - 1) > W^2$. Adding $2Wl + l^2$ to both sides gives us the following:

$$l^2 + 4lW \left(\frac{1}{2} + \varepsilon \right) - l > 2Wl + W^2 + l^2$$

Dividing both sides by 4 and recalling our bounds for θ and $\text{mac}(G)$ implies the following:

$$\text{mac}(G) \leq \frac{2Wl + W^2 + l^2}{4} < \frac{l(l-1)}{4} + lW \left(\frac{1}{2} + \varepsilon \right) = \theta$$

□

Unfortunately, it is NP-hard to compute the maximum weight of a DFS tree in a weighted graph even in the class of *triangle-free* graphs, i.e. graphs that do not contain K_3 as a subgraph.

Theorem 3.4. *It is NP-hard to compute the maximum weight of a DFS tree in a weighted triangle-free graph.*

Proof. We will reduce from the Hamilton (x, y) -path problem proved to be NP-complete in [13] for triangle-free graphs (it is NP-complete already for grid graphs). In this problem, given a triangle-free graph G and two vertices x, y of G , we are to decide whether G has a Hamilton path with end-vertices x and

y . Given such a graph G and $x, y \in V(G)$ we create a new weighted graph G' as follows (see Figure 1).

Let $V(G) = \{v_1, v_2, \dots, v_n\}$ and let $V(G') = V(G) \cup Z \cup \{z\}$, where $Z = \{z_1, z_2, \dots, z_n\}$. Let $E(G') = E(G) \cup \{z_i v_i, z z_i \mid i = 1, 2, \dots, n\}$. Note that G' is triangle-free. Assume that $v_n = y$ and let the weight of all edges $z_i v_i$ be 1 for $i = 1, 2, \dots, n - 1$ and let the weight of all other edges in G' be 2. Note that the maximum weight of a spanning tree in G' is at most $2(|V(G')| - 1)$.

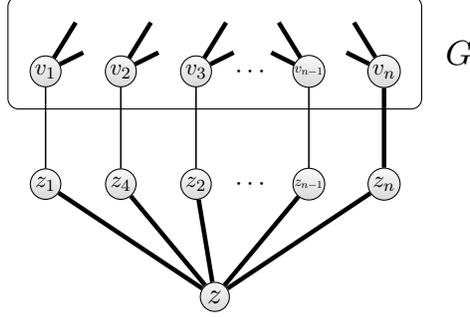


Figure 1: The graph G' in Theorem 3.4. The thick edges have weight 2 and the thin edges have weight 1.

We first show that there is a DFS tree rooted at x of weight $2(|V(G')| - 1)$ in G' if and only if G contains a Hamilton (x, y) -path. Assume that G contains a Hamilton (x, y) -path, P . Then $E(P)$ together with the edge $v_n z_n$ and all edges from z to Z form a DFS tree rooted at x where all edges have weight 2.

Conversely assume that there is a DFS tree, T' , in G' , rooted in x , of weight $2(|V(G')| - 1)$. This implies that all edges in T' have weight 2. As no edge of weight 1 is used in T' , the only edge between $T = T'[V(G)]$ and $T' - V(G)$ is yz_n . Thus, T is a tree. Note that T does not have a vertex v_j with $j < n$ as a leaf since otherwise T' would have edge $v_j z_j$, which is impossible. Thus, T is just a Hamilton path of G from x to y .

This shows that there is a DFS tree rooted at x of weight $2(|V(G')| - 1)$ if and only if G contains a Hamilton (x, y) -path.

Now create the graph G^* by taking two copies, G'_1 and G'_2 , of G' and adding an edge of weight 2 between the copy of vertex x in G'_1 and the copy of vertex x in G'_2 . Note that G^* is triangle-free. If there is a Hamilton (x, y) -path in G , then as shown above, there is a DFS tree in G' rooted at x , where all edges have weight 2, which implies that there is DFS tree in G^* where all edges have weight 2 (rooted in one of the copies of x).

Conversely assume that there is a DFS tree, T^* , in G^* where all edges have weight 2. Without loss of generality, assume the root of T^* lies in G'_1 . Then $T^*[V(G'_2)]$ is a DFS tree in G'_2 rooted at its copy of x , which by the above

implies that there is a Hamilton (x, y) -path in G .

Therefore, if we can decide whether there is a DFS-tree in G^* (which is triangle-free) only containing edges of weight 2, then we can decide whether G has a Hamilton (x, y) -path. This completes our proof. \square

4 Bounded Girth Families of Connected Graphs

Throughout this section, G is connected. The *girth* of G is the minimum number of edges in a cycle of G . The *depth* of a rooted tree is the maximum number of edges in a path from the root to a leaf.

Theorem 4.1. *Let k be a positive even integer. If the girth of G is at least k , then $\text{mac}(G) \geq \frac{w(G)}{2} + \frac{k-1}{2k}w(D)$, for every DFS tree D in G .*

Proof. Let G be any graph with girth at least k and let D be any DFS-tree in G . Let r denote the root of D and let L_i be the set of vertices of D reached from r by a path with i edges. Note that $L_0 = \{r\}$.

For $j = 0, 1, \dots, k-1$, let G_j be the subgraph induced by the edges of T minus those between L_i and L_{i+1} for all $i = j \pmod{k}$. Also add to G_j the edges of $G - V(D)$ linking vertices in the same connected component of G_j (see Figure 2 for an illustration of the G_j 's). Since D is a DFS tree, D has no cross edges i.e. edges xy such that x is not a descendent of y and y is not a descendent of x [8]. As the girth of G is at least k , every connected component of G_j consists of a tree of depth at most $k-1$ plus possibly some edges from the leaves to its root (if the leaves are at distance $k-1$ from the root). Since every connected component of G_j is an induced subgraph of G and k is even, we note that every connected component of G_j belongs to $\mathcal{B}(G)$ and and by Theorem 2.1, $\text{mac}(G) \geq (w(G) + w(G_j))/2$ for every $j = 0, 1, \dots, k-1$.

As every edge of T belongs to $k-1$ of the k subgraphs G_j 's, we note that summing the equations $\text{mac}(G) \geq (w(G) + w(G_j))/2$ for all $j = 0, 1, \dots, k-1$ gives us the following:

$$k \cdot \text{mac}(G) \geq k \frac{w(G)}{2} + \sum_{j=0}^{k-1} \frac{w(G_j)}{2} = k \frac{w(G)}{2} + (k-1) \frac{w(T)}{2}$$

Dividing the above inequality by k gives us the desired bound. \square

To see that the lower bound of Theorem 4.1 is tight, consider the unweighted cycle C_{k+1} , where k is even. Then $\text{mac}(C_{k+1}) = k$ and the lower bound of Theorem 4.1 equals

$$\frac{k+1}{2} + \frac{k-1}{2k}k = k.$$

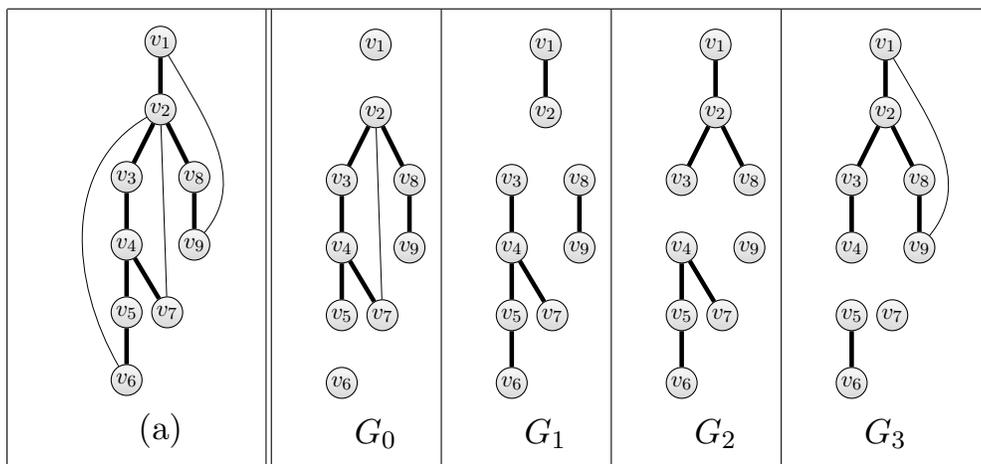


Figure 2: We illustrate the proof of Theorem 4.1 with the example shown in (a), where we see a DFS tree (thick edges) of a graph G . The remaining graphs depict G_0 , G_1 , G_2 and G_3 , respectively, where $k = 4$.

Note that in Theorem 4.1, k is assumed to be even. When the girth g is odd we can use Theorem 4.1 with $k = g - 1$. This implies the following:

Corollary 4.2. *Let g be a positive odd integer. If the girth of G is at least g , then $\text{mac}(G) \geq \frac{w(G)}{2} + \frac{g-2}{2(g-1)}w(D)$, for every DFS tree D in G .*

Corollary 4.2 is tight for the cycle C_g as the lower bounds for C_g and $\text{mac}(C_g)$ are both equal to $g - 1$ in this case.

Recall that a graph of girth at least 4 is called *triangle-free*.

By Theorem 4.1 for a triangle-free graph G , we have $\text{mac}(G) \geq \frac{w(G)}{2} + \frac{3}{8}w(D)$, where D is a DFS tree. However, by Theorem 3.4 finding a DFS tree of maximum weight is NP-hard.

The lower bound in the next theorem is of interest as it implies that for a maximum weight spanning tree T_{\max} of a triangle-free graph of G , $\text{mac}(G) \geq w(G)/2 + w(T_{\max})/4$, which is stronger than the Poljak-Turzík lower bound.

Theorem 4.3. *Let G be a triangle-free graph and let T be a spanning tree of G . Then $\text{mac}(G) \geq w(G)/2 + w(T)/4$.*

Proof. Let T be rooted at vertex u . Let L_i the set of vertices of T at distance i from u in D . Let H_i be the subgraph of T induced by the set of edges of T between vertices L_i and L_{i+1} . Since G is triangle-free, the children of any node in T form an independent set. Thus, every H_i is a disjoint union

of stars. Let G_0 be the disjoint union of graphs H_i with even i and G_1 the disjoint union of graphs H_i with odd i . Thus, $G_0, G_1 \in \mathcal{B}(G)$. Hence, by Theorem 2.1, $\text{mac}(G) \geq (w(G) + w(G_j))/2$ for $j = 0, 1$. These bounds and $w(G_0) + w(G_1) = w(T)$ imply $\text{mac}(G) \geq w(G)/2 + w(T)/4$. \square

Note that in Theorem 4.3, T can be any spanning tree. By Theorem 3.3 no similar bound holds if we drop the condition that G is triangle-free. In fact, for every $\varepsilon > 0$ there exist graphs G with spanning tree T such that $\text{mac}(G) < w(G)/2 + \varepsilon \cdot w(T)$.

Define θ to be the largest value such that $\text{mac}(G) \geq \frac{w(G)}{2} + \theta \cdot w(T)$ holds for all spanning trees T in a triangle-free graph G .

Proposition 4.4. $\frac{1}{4} \leq \theta \leq \frac{3}{8}$

Proof. Theorem 4.3 implies that $\theta \geq 1/4$. Now consider the cycle C_5 with weight 1 on all edges. Then $w(C_5) = 5$, $w(T) = 4$ for all spanning trees T and $\text{mac}(C_5) = 4$, which implies that $\theta \leq 3/8$. \square

We think that determining the optimal value of θ is an interesting open problem. In fact, we guess that $\theta = 3/8$.

Conjecture 4.5. *Let G be triangle-free and let T be a spanning tree of G . Then $\text{mac}(G) \geq \frac{w(G)}{2} + \frac{3w(T)}{8}$.*

5 Triangle-free Graphs with Bounded Maximum Degree

Another interesting problem is to determine what happens to θ if we restrict ourselves to fixed maximum degrees. That is, we let θ_Δ be defined as the largest number for which the following holds: If G is triangle-free graph with $\Delta(G) \leq \Delta$ then $\text{mac}(G) \geq \frac{w(G)}{2} + \theta_\Delta \cdot w(T)$.

It is not difficult to prove that $\theta_1 = 1/2$ (as if $\Delta(G) \leq 1$ then G is bipartite) and $\theta_2 = 3/8$ (due to C_5). This implies the following, by Proposition 4.4.

Proposition 5.1. $0.375 = \frac{3}{8} = \theta_2 \geq \theta_3 \geq \theta_4 \geq \dots \geq \theta \geq \frac{1}{4} = 0.25$

In the rest of this section, we will first study triangle-free graphs with maximum degree at most 3 (Subsection 5.1) and then those of arbitrary bounded maximum degree (Subsection 5.2).

5.1 Triangle-free subcubic graphs

A graph is *subcubic* if its maximum degree is at most 3. Triangle-free subcubic graphs have been widely studied, see e.g. [6, 12, 17]. We will discuss such graphs next, before moving to the more general case of triangle-free graphs of bounded maximum degree.

The following result is well-known.

Theorem 5.2. [6] *If G is triangle-free subcubic graph then there exists a bipartite subgraph of G containing at least $\frac{4}{5}|E(G)|$ edges.*

We conjecture that the above theorem can be extended to the weighted case as follows.

Conjecture 5.3. *Let G be a triangle-free subcubic graph, then $\text{mac}(G) \geq \frac{4}{5}w(G)$.*

Note that Theorem 5.2 implies that Conjecture 5.3 holds in the case when all weights equal 1. As a support of Conjecture 5.3, one can easily show that if G is a triangle-free subcubic graph, then $\text{mac}(G) \geq \frac{2}{3}w(G)$, as follows. As G is triangle-free with $\Delta(G) \leq 3$ it is known that G is 3-colorable, by Brook's Theorem. Let V_1, V_2, V_3 be the three color classes in a proper 3-coloring of G and assume that without loss of generality $w(V_1, V_2)$ is the maximum value in $\{w(V_1, V_2), w(V_1, V_3), w(V_2, V_3)\}$, where $w(V_i, V_j)$ is the total weight of all edges between V_i and V_j . For each vertex $v \in V_3$ add it to V_1 if $w(v, V_2) \geq w(v, V_1)$ and otherwise add it to V_2 . This results in a bipartition (V_1, V_2) with weight at least $\frac{2}{3}w(G)$.

In Section 6, we will provide a proof of the following theorem, which approaches Conjecture 5.3 even more.

Theorem 5.4. *Let G be an edge-weighted triangle-free subcubic graph. Then $\text{mac}(G) \geq \frac{8}{11} \cdot w(G)$.*

Theorem 5.4 will be used to prove the following theorem in Section 6.

Theorem 5.5. *Let G be an edge-weighted triangle-free subcubic graph and let T be an arbitrary spanning tree in G . Then $\text{mac}(G) \geq \frac{w(G)}{2} + 0.3193 \cdot w(T)$.*

The above theorem implies the following corollary.

Corollary 5.6. $0.3193 \leq \theta_3 \leq \frac{3}{8} = 0.375$.

Note that if Conjecture 4.5 holds for triangle-free subcubic graphs, then $\theta_3 = 3/8$.

Proposition 5.7. *Conjecture 5.3 implies Conjecture 4.5 for triangle-free subcubic graphs.*

Proof. Let G be a triangle-free subcubic graph and let T be any spanning tree in G . As any tree is bipartite, we note that $\text{mac}(G) \geq w(T)$. Conjecture 5.3 would imply that $\text{mac}(G) \geq \frac{4}{5}w(G)$. Thus, we have the following, which completes the proof.

$$\text{mac}(G) \geq \frac{3}{8}w(T) + \frac{5}{8} \left(\frac{4}{5}w(G) \right) = \frac{3}{8}w(T) + \frac{1}{2}w(G)$$

□

The well-known pentagon conjecture of Erdős [9] states that a triangle-free graph G on n vertices has at most $(n/5)^5$ 5-cycles. The bound is tight as every graph obtained from C_5 by replacing every vertex x by a set of $n/5$ independent vertices (with the same adjacencies as in C_5) has exactly $(n/5)^5$ 5-cycles. This conjecture was proved independently in [10] and [11].

Let us consider another conjecture, which is on 5-cycles in triangle-free subcubic graphs.

Conjecture 5.8. *Every triangle-free subcubic graph G contains a set E' of edges, such that every 5-cycle in G contains exactly one edge from E' .*

Note that Conjecture 5.8 holds for all triangle-free subcubic graphs, G , where every edge belongs to equally many 5-cycles, say k , by the following argument. By Theorem 5.2, there exists a bipartite subgraph of G with at least $4|E(G)|/5$ edges. Let E' be all the edges not in this subgraph. In G there are $k|E(G)|/5$ distinct 5-cycles and each of them is covered by (at least) one of the at most $|E(G)|/5$ edges in E' . As each edge in E' can cover at most k 5-cycles, we note that every 5-cycle is covered exactly once, which implies that Conjecture 5.8 holds for G . As a special case, Conjecture 5.8 holds for the Petersen Graph, where every edge lies in four 5-cycles. For example, in Figure 3, the edge set $E' = \{x_1y_1, y_3y_4, x_3x_4\}$ intersects every 5-cycle of the Petersen Graph exactly once.

The following result is a link between Conjectures 5.3 and 5.8, which provided Conjecture 5.3 holds, demonstrates that a lower bound on $\text{mac}(G)$ can be used to establish a structural result on unweighted graphs.

Proposition 5.9. *Conjecture 5.3 implies Conjecture 5.8.*

Proof. Let $G = (V, E)$ be a triangle-free graph with maximum degree 3. For each $e \in E$, define $w(e)$ to be the number of 5-cycles that contain e in G . If there are exactly c distinct 5-cycles in G , then $w(G) = 5c$. Let $B = (V, F)$ be a maximum weight bipartite subgraph in G and let $E' = E \setminus F$. Note that every 5-cycle in G must contain an edge from E' , which implies that $w(E') \geq c$. If Conjecture 5.3 holds, then $\text{mac}(G) \geq 4w(G)/5$, which implies that $w(E') \leq w(G)/5 = c$. Therefore, $w(E') = c$, which implies that every 5-cycle contains exactly one edge from E' , and thus Conjecture 5.8 holds. □

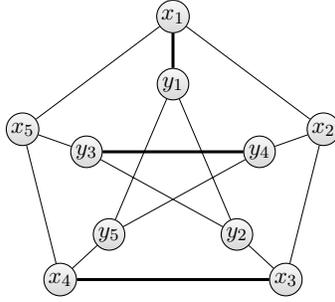


Figure 3: Petersen graph with E' in boldface

5.2 Triangle-free graphs with arbitrary bounded maximum degree

Using Shearer's randomized algorithm [16] we can obtain the following bound.

Theorem 5.10. *Let G be a weighted triangle-free graph with $\Delta(G) \leq \Delta$. Then $\text{mac}(G) \geq s_\Delta \cdot w(G)$, where $s_\Delta = \frac{1}{2} + \frac{1}{4\sqrt{2}\Delta}$.*

Proof. Shearer's randomized algorithm takes G as an input and constructs a random cut C of G as follows. In Step 1, we partition $V(G)$ randomly and uniformly into two sets A and B . Call a vertex x in G *good* if more than half of the vertices adjacent to it lie in the other set. If exactly half of the vertices adjacent to x lie in the other set call x *good* with probability $1/2$. Otherwise call x *bad*. In Step 2, leave the good vertices where they are and redistribute the bad vertices randomly and uniformly into A and B . The edges between A and B induce the cut C .

It is shown in the proof of Theorem 1 in [16] that an edge uv of G is in C with probability $\theta_{uv} = \frac{1}{2} + \frac{1}{4}(\rho_u - \frac{1}{2}) + \frac{1}{4}(\rho_v - \frac{1}{2})$, where ρ_x is the probability that $x \in \{u, v\}$ is good after Step 2 provided that u, v are put into different sets in Step 1. Let $f_x = \rho_x - \frac{1}{2}$, where $x \in \{u, v\}$. It follows from the proof of Theorem 1 and from Lemma 1 in [16] that for $x \in \{u, v\}$

$$f_x \geq \frac{1}{2\sqrt{2} \cdot \deg(x)} \geq \frac{1}{2\sqrt{2}\Delta}$$

Observe that the expected value of the weight of C is

$$\begin{aligned} \sum_{uv \in E} w(uv)\theta_{uv} &= \\ W(G)/2 + \sum_{uv \in E} w(uv)(f_u + f_v)/4 &\geq \\ W(G)/2 + (1/4\sqrt{2}\Delta)W(G) \end{aligned}$$

Thus, the expected weight of C is at least $s_\Delta \cdot w(G)$, where $s_\Delta = \frac{1}{2} + \frac{1}{4\sqrt{2\Delta}}$. Hence, G has a cut with at least such a weight. \square

Note that we can find a cut of G of weight at least $s_\Delta \cdot w(G)$ in polynomial time using the derandomization method of conditional probabilities [4]. Also note that for $\Delta(G) \leq 3$, the bound in Theorem 5.10 is as follows: $\text{mac}(G) \geq 0.602 \cdot w(G)$. However, Theorem 5.4 provides a significantly better bound: $\text{mac}(G) \geq 0.727 \cdot w(G)$. This indicates that Theorem 5.10 can be improved at least for small values of Δ and indeed Theorem 5.12 provides such an improvement Theorem 5.10 for small values of Δ . The following lemma will be used to prove Theorems 5.4 and 5.12.

Lemma 5.11. *Let G be a weighted triangle-free graph with $\Delta(G) \leq \Delta$ and let M be a matching in G . Then $\text{mac}(G) \geq \frac{\Delta}{2\Delta-1}(w(G) - w(M)) + w(M)$.*

Proof. Let G and M be defined as in the statement of the theorem and $\Delta(G) \leq \Delta$. Let G_1 be the graph obtained by contracting each edge $e_i \in M$ to a vertex x_i . If G_1 has parallel edges, for every pair u, v of vertices with parallel edges, delete all but one of the parallel edges and let the weight of the remaining edge between u and v be the sum of the original edges. We will denote the resulting graph by G_2 . Note that $w(G_2) = w(G) - w(M)$ and G_2 is a simple graph with $\Delta(G_2) \leq 2\Delta - 2$.

By Vizing's theorem, G_2 has a proper c -edge-coloring, where $c \leq 2\Delta - 1$. Let M_i'' denote the edges of color i in such a proper c -edge-coloring ($i \in [c]$). Note that M_i'' is a matching in G_2 . Observe that M_i'' corresponds to an induced matching in G (via G_1) denoted by M_i . Note that the parallel edges do not create a problem with the matchings since G is triangle-free. Every component C in $G[M \cup M_i]$ has at most four vertices (otherwise, contraction of edges of M in C would create a pair on non-parallel edges of M_i in G_1 , which is impossible). Hence, $G[M \cup M_i] \in \mathcal{B}(G)$ (as G is triangle-free). Thus, Theorem 2.1 implies that $\text{mac}(G) \geq \frac{w(G)+w(M)+w(M_i)}{2}$. Summing the above over $i = 1, 2, \dots, c$ and noting that every edge not in M belongs to one of the matching M_i , we have

$$c \cdot \text{mac}(G) \geq \frac{c}{2}(w(G) + w(M)) + \frac{w(G) - w(M)}{2}.$$

Hence,

$$\begin{aligned} \text{mac}(G) &\geq \frac{w(G)+w(M)}{2} + \frac{w(G)-w(M)}{2c} \\ &\geq \frac{w(G)+w(M)}{2} + \frac{w(G)-w(M)}{2(2\Delta-1)} \\ &= \frac{\Delta}{2\Delta-1}(w(G) - w(M)) + w(M) \end{aligned}$$

\square

Theorem 5.12. *Let G be a weighted triangle-free graph with $\Delta(G) \leq \Delta$. Then $\text{mac}(G) \geq t_\Delta \cdot w(G)$, where $t_\Delta = 1/2 + (3\Delta - 1)/(4\Delta^2 + 2\Delta - 2)$.*

Proof. By Vizing's theorem, G has a $(\Delta + 1)$ -coloring. Let M_i be the edges with color $i \in [\Delta + 1]$, and note that M_i is a matching. Using Lemma 5.11 for each M_i , we obtain $\text{mac}(G) \geq \frac{\Delta}{2\Delta-1}(w(G) - w(M_i)) + w(M_i)$ for every $i \in [\Delta + 1]$. By summing up these inequalities and simplifying the resulting inequality (by using the fact that $w(M_1) + w(M_2) + \dots + w(M_{\Delta+1}) = w(G)$), we obtain the following:

$$\begin{aligned} \text{mac}(G) &\geq \frac{1}{\Delta+1} \sum_{i=1}^{\Delta+1} \left(\frac{\Delta}{2\Delta-1}(w(G) - w(M_i)) + w(M_i) \right) \\ &= \frac{1}{\Delta+1} \sum_{i=1}^{\Delta+1} \left(\frac{\Delta}{2\Delta-1}w(G) + \frac{\Delta-1}{2\Delta-1}w(M_i) \right) \\ &= \frac{\Delta}{2\Delta-1}w(G) + \frac{1}{\Delta+1} \times \frac{\Delta-1}{2\Delta-1}w(G) \\ &= \left(\frac{1}{2} + \frac{3\Delta-1}{4\Delta^2+2\Delta-2} \right) w(G) \end{aligned}$$

□

Let us compare the bounds of Theorems 5.10 and 5.12. We have $t_\Delta > s_\Delta$ if and only if $\Delta \leq 16$. In fact, a selected number of values of s_Δ and t_Δ can be seen below.

Δ	s_Δ	t_Δ
1	0.6768	1.0000
2	0.6250	0.7778
3	0.6021	0.7000
4	0.5884	0.6571
...
16	0.5442	0.5446
17	0.5429	0.5421

Since $t_3 = 0.7$, we have $\text{mac}(G) \geq 0.7w(G)$ for a triangle-free G with $\Delta(G) \leq 3$. The gap between 0.7 and the coefficient 0.8 of Conjecture 4.5 is just 0.1, but it does not seem to be easy to bridge this gap as the proof of Theorem 5.4 in the next section shows.

Lemma 5.11 seems interesting in its own right and gives rise to the following open problem.

Open Problem 5.13. *For each $\Delta \geq 1$ determine the maximum value, c_Δ , such that for every edge-weighted triangle-free graph G with maximum degree at most Δ and matching M in G , the following holds.*

$$\text{mac}(G) \geq c_\Delta(w(G) - w(M)) + w(M)$$

The following proposition determines c_1 , c_2 and c_3 precisely.

Proposition 5.14. $c_1 = 1$, $c_2 = \frac{2}{3}$ and $c_3 = 0.6$.

Proof. By Lemma 5.11 we note that $c_1 \geq 1$, $c_2 \geq \frac{2}{3}$ and $c_3 \geq 0.6$. Clearly $c_1 \leq 1$ (consider unweighted K_2 and $M = \emptyset$), which implies that $c_1 = 1$.

If G is an unweighted 5-cycle and M is a matching of size two in G , then $\text{mac}(G) = 4 = \frac{2}{3}(|E(G)| - |E(M)|) + |E(M)|$, which implies that $c_2 \leq \frac{2}{3}$. Therefore $c_2 = \frac{2}{3}$.

Let G be the Petersen graph depicted in Figure 4. That is, the edge set consists of the edges of two 5-cycles, $C_x = x_1x_2x_3x_4x_5x_1$ and $C_y = y_1y_2y_3y_4y_5y_1$ and the matching $M = \{x_1y_1, x_2y_4, x_3y_2, x_4y_5, x_5y_3\}$. Let the weight of the edges in M be 10 and the weight of all other edges in G be 1. Then $w(M) = 50$ and $w(G) - w(M) = 10$. This implies that any maximum weight cut, (A, B) , must include all edges on M . As C_x and C_y are both 5-cycles at most 4 edges from each can belong to (A, B) . If four edges from C_x belongs to (A, B) then we note that at most 2 edges from C_y belong to (A, B) (if all edges in M belong to (A, B)) and analogously if four edges from C_y belongs to (A, B) then we note that at most 2 edges from C_x belong to (A, B) . This implies that $\text{mac}(G) \leq w(M) + 6 = w(M) + 0.6(w(G) - w(M))$. Therefore $c_3 \leq 0.6$, which implies that $c_3 = 0.6$. \square

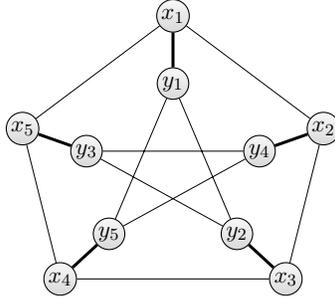


Figure 4: Petersen graph G and matching M in boldface

Proposition 2.3 implies that $c_\Delta \geq 0.5$ for all $\Delta \geq 1$, which by Theorem 5.14 implies that the following holds.

$$0.6 = c_3 \geq c_4 \geq c_5 \geq \dots \geq 0.5$$

We will finish this section with the following asymptotic result for c_Δ .

Proposition 5.15. *We have $\lim_{\Delta \rightarrow \infty} c_\Delta = 0.5$.*

Proof. Since the sequence is monotonically decreasing and bounded by 0.5, the limit exists and it is at least 0.5. Suppose the limit L is larger than 0.5.

Alon [1] proved that there exists some absolute positive constant c' so that for every m there exists a triangle-free graph G with m edges for which no bipartite subgraph has $m/2 + c' \cdot m^{4/5}$ edges. Let all edges of G be of weight 1 and let M be a matching of G . If $|E(G)|$ is large enough for $c'm^{4/5} < (L-0.5)m$ to hold, we have

$$\begin{aligned} \text{mac}(G) &\leq m/2 + c'm^{4/5} \\ &< Lm \\ &\leq Lm + (1-L)|M| \\ &= L(m - |M|) + |M| \\ &\leq c_{\Delta(G)}(m - |M|) + |M| \end{aligned}$$

This contradiction completes the proof. \square

6 Proofs of Theorems 5.4 and 5.5

We first need the following lemmas.

Lemma 6.1. *Let k be any positive integer and let T be any spanning tree in a graph G and let $e^* \in E(T)$ be arbitrary. If for all $e \in E(G) \setminus E(T)$ we have that $T + e$ contains no odd cycle of length $2k - 1$ or less, then $\text{mac}(G) \geq \frac{w(G)}{2} + \frac{k-1}{2k}w(T) + \frac{1}{2k}w(e^*)$.*

Proof. Let G , T , e^* and k be defined as in the lemma. Let $e^* = r_1r_2$ and let L_i contain the vertex u if and only if $i = \min\{d_T(r_1, u), d_T(r_2, u)\}$, where $d_T(r_i, u)$ denotes the distance from r_i to u in T . Note that $L_0 = \{r_1, r_2\}$.

For $j = 0, 1, \dots, k-1$, let G_j be the subgraph induced by the edges of T minus those between L_i and L_{i+1} for all $i = j \pmod{k}$.

Let T^* be an arbitrary component in G_j (for any $j \in \{0, 1, \dots, k-1\}$). As T^* is a subgraph of T we note that T^* is a tree. Let $e \in E(G) \setminus E(T)$ have both end-points in $V(T^*)$. We will now show that $T^* + e$ contains no odd-cycle. Let C be the unique cycle in $T^* + e$. By the construction of G_j we note that the path $C - e$ has length at most $2k - 1$ (in fact, if C does not contain e^* , then the length is at most $2k - 2$) and that the cycle C therefore has length at most $2k$. As $T + e$ contains no odd cycle of length $2k - 1$ or less (and $2k$ is even) we note that $T^* + e$ contains no odd-cycle.

Therefore $G[V(T^*)]$ is bipartite, which implies that $G[V(T^*)]$ belongs to $\mathcal{B}(G)$ and and by Theorem 2.1, $\text{mac}(G) \geq (w(G) + w(G_j))/2$ for every $j = 0, 1, \dots, k-1$.

As e^* belongs to all the k subgraphs G_j and every edge of $T - e^*$ belongs to $k - 1$ of the k subgraphs G_j , we note that summing the equations $\text{mac}(G) \geq$

$(w(G) + w(G_j))/2$ for all $j = 0, 1, \dots, k-1$ gives us the following:

$$k \cdot \text{mac}(G) \geq k \frac{w(G)}{2} + \sum_{j=0}^{k-1} \frac{w(G_j)}{2} = k \frac{w(G)}{2} + (k-1) \frac{w(T)}{2} + \frac{w(e^*)}{2}$$

Dividing the above inequality by k gives us the desired bound. \square

Lemma 6.2. *Let G be a subcubic edge-weighted graph and let T be a spanning tree in G . Let r be the length of a shortest odd cycle in $T + e$ for any edge $e \in E(G) \setminus E(T)$. Let p be arbitrary such that $0 \leq p \leq 1$. Then the following bound holds.*

$$\text{mac}(G) \geq \frac{p+1}{2} w(T) + \frac{1-p^{r-1}}{2} (w(G) - w(T)).$$

Proof. Let G, T, r and p be defined as in the lemma. Pick each edge of T with probability p and denote the resulting set of edges by E^* . Let $T_1^*, T_2^*, \dots, T_l^*$ be the connected components of the graph $G^* = (V(G), E^*)$. Note that all T_i^* are subtrees of T (some of which may contain only one vertex).

Let (X_i, Y_i) be a bipartition of T_i^* for all $i = 1, 2, \dots, l$. For each $i \in [l]$, randomly and uniformly assign X_i color 1 or 2 and Y_i the opposite color. Let A be all vertices of color 1 and let B be all vertices of color 2. Now every edge in E^* lies in the cut induced by (A, B) . Consider an arbitrary edge $e \in E(G) \setminus E^*$ and let $e = uv$. Assume that $u \in V(T_i^*)$ and $v \in V(T_j^*)$. We will show the following claims.

Claim 1: *If $e \in E(T) \setminus E^*$, then e lies in the cut induced by (A, B) with probability $\frac{1}{2}$ (given E^*).*

Proof of Claim 1: Note that $i \neq j$, as adding the edge e to G^* does not create a cycle (all edges belong to the tree T). Therefore u and v will be assigned colors 1 and 2 randomly and independently. This completes the proof of Claim 1.

Claim 2: *If $e \in E(G) \setminus E(T)$, then e lies in the cut induced by (A, B) with probability at least $\frac{1-p^{r-1}}{2}$.*

Proof of Claim 2: Let C_e denote the unique cycle in $T + e$.

First consider the case when $i = j$ and $|E(C_e)|$ is even. In this case e lies in the cut induced by (A, B) with probability 1 (given E^*).

Now consider the case when $i = j$ and $|E(C_e)|$ is odd. By the definition of r in the statement of the lemma we must have $|E(C_e)| \geq r$. Furthermore all edges in $C_e - e$ belong to E^* . So this case only happens with probability at most p^{r-1} (as each edge in $C_e - e$ has probability p of belonging to E^*).

Finally, if $i \neq j$ then e lies in the cut induced by (A, B) with probability $\frac{1}{2}$ (given E^*).

So with probability at least $1 - p^{r-1}$, we are in a case where e will belong to the cut induced by (A, B) with probability at least $1/2$. Therefore e will belong to the cut induced by (A, B) with probability at least $\frac{1}{2} \times (1 - p^{r-1})$. This completes the proof of Claim 2.

We now return to the proof of Lemma 6.2. If $e \in E(T)$ then by Claim 1 and the law of total probability, the probability that e belongs to the cut induced by (A, B) is $p + (1 - p)\frac{1}{2} = \frac{1+p}{2}$. By Claim 2, if $e \in E(G) \setminus E(T)$ then e lies in the cut induced by (A, B) with probability at least $\frac{1-p^{r-1}}{2}$. This completes the proof. \square

Recall the statement of Theorem 5.4.

Theorem 5.4. *Let G be an edge-weighted triangle-free graph with $\Delta(G) \leq 3$. Then $\text{mac}(G) \geq \frac{8}{11}w(G)$.*

Proof. We will first show that we may restrict our attention to 3-regular triangle-free graphs. Let G' be equal to $K_{3,3}$ where one edge uv has been subdivided. That is, uv has been replaced by a path uvw . For every vertex, $s \in G$ we add $3 - d_G(s)$ copies of G' to G and add an edge from each w -vertex in the G' s to s . Note that the resulting graph is 3-regular and triangle-free. Furthermore giving all the new edges a weight of zero, shows that if the theorem holds for this new graph then it also holds for G . We may therefore without loss of generality assume that G is 3-regular.

By Brook's Theorem we note that $\chi(G) \leq 3$. Let $c : V(G) \rightarrow \{1, 2, 3\}$ be a proper 3-coloring of G and let V_1, V_2, V_3 be the color classes of c . For a given vertex $v \in V(G)$, if a color appears exactly once in $N(v)$ (i.e. one vertex of $N(v)$ has a color i and the other two vertices of $N(v)$ have a color $j \neq i$), we let $s(v)$ be the neighbor of v with that color. Otherwise, $s(v)$ is not defined.

Define the digraph D^* such that $V(D^*) = V(G)$ and the arc set of D^* is as follows:

$$A(D^*) = \{vs(v) \mid v \in V(G) \text{ \& } s(v) \text{ is defined}\}$$

Let $\Delta^+(D^*)$ denote the maximum out-degree of D^* . Note that $\Delta^+(D^*) \leq 1$ by the construction of D^* and that D^* may contain 2-cycles. If uvu is a 2-cycle in D^* then $s(u) = v$ and $s(v) = u$. Let $G^* = UG(D^*)$ i.e. G^* is the underlying graph of D^* and contains all edges of the form $vs(v)$, where $v \in V(G)$.

For every edge $e \in E(G)$, let $V^*(e) = \{v \mid vs(v) = e\}$ and let $A_i = \{e \mid |V^*(e)| = i\}$ for $i = 0, 1, 2$. That is, A_0 contains all edges e that are not of the form $vs(v)$ for any v , A_2 contains all edges uv where $s(u) = v$ and $s(v) = u$ and $A_1 = E(G) \setminus (A_0 \cup A_2)$. Also note that $E(G^*) = A_1 \cup A_2$. We will now prove the following claims.

Claim A: $\text{mac}(G) \geq w(A_0) + \frac{2}{3}w(A_1) + \frac{1}{3}w(A_2)$.

Proof of Claim A: For all $i \in \{1, 2, 3\}$, we define

$$C_i = E(G) \setminus \{vs(v) \mid v \in V_i \text{ \& } s(v) \text{ is defined}\}.$$

Note that C_i induces a bipartite graph as every vertex in V_i only has edges to one of the two other sets. So $\text{mac}(G) \geq w(C_i)$ for $i = 1, 2, 3$. Note that every edge in A_i appears in $3 - i$ of the sets C_1, C_2, C_3 , which implies the following:

$$\text{mac}(G) \geq \frac{1}{3}(w(C_1) + w(C_2) + w(C_3)) = w(A_0) + \frac{2}{3}w(A_1) + \frac{1}{3}w(A_2).$$

This completes the proof of Claim A.

Claim B: *If $p_1p_2 \in E(G)$ and $p_2p_3 \in A(D^*)$, where $p_1 \neq p_3$, then $\{c(p_1), c(p_2), c(p_3)\} = \{1, 2, 3\}$. This implies that $c(p_1) = 6 - c(p_2) - c(p_3)$.*

Furthermore, if $R = r_1, r_2, r_3, \dots, r_s$ is a directed path in D , then $c(r_1) = c(r_4) = c(r_7) = \dots$ and $c(r_2) = c(r_5) = c(r_8) = \dots$ and $c(r_3) = c(r_6) = c(r_9) = \dots$

Proof of Claim B: As all edges in G , and therefore also arcs in D^* , go between different V_i -sets, we note that $c(p_1) \neq c(p_2)$ and $c(p_2) \neq c(p_3)$. As $p_2p_3 \in A(D^*)$ we have $s(p_2) = p_3$, which implies that there is only one edge from p_2 to $V_{c(p_3)}$. Furthermore this edge is p_2p_3 . Therefore $p_1 \notin V_{c(p_3)}$ as otherwise p_2 would have two edges to $V_{c(p_3)}$. So, $c(p_1) \neq c(p_3)$. This implies that $c(p_1), c(p_2)$ and $c(p_3)$ take on three distinct values, which completes the proof of the first part of Claim B. The second part of Claim B follows immediately from the first part.

Claim C: *Let $e \in A_0$ and assume that C is a cycle in $G^* + e$ containing e . Then $|E(C)| = 0 \pmod{3}$.*

Proof of Claim C: Let C be a cycle in $G^* + e$ containing e , where $e \in A_0$. Let P be the path $C - e$ and assume $P = p_1p_2p_3 \dots p_l$. Note that $e = p_1p_l$. Assume without loss of generality that $c(p_1) = 1$ and $c(p_l) = 2$. We now consider the following two cases.

Case C.1. $p_1p_2p_3 \dots p_l$ or $p_l p_{l-1} p_{l-2} \dots p_1$ is a directed path in D^* .

Assume without loss of generality that $p_1p_2p_3 \dots p_l$ is a directed path in D^* . As $c(p_1) = 1$ and $c(p_l) = 2$, we note that $c(p_2) = 3$ and $c(p_3) = 2$, by Claim B. Therefore, as $c(p_l) = 2 = c(p_3)$ we must have that l is divisible by 3, by Claim B, which completes the proof of Case C.1.

Case C.2. $p_1p_2p_3 \dots p_l$ and $p_l p_{l-1} p_{l-2} \dots p_1$ are not directed paths in D^* .

As $\Delta^+(D^*) \leq 1$ this implies that there exists a $p_i \in \{p_2, p_3, \dots, p_{l-1}\}$ such that p_i has two in-neighbors in P . Let $P' = p_1p_2 \dots p_i$ and let $P'' = p_l p_{l-1} p_{l-2} \dots p_i$ and note that P' and P'' are both directed paths in D^* . As $c(p_1) = 1$ and $c(p_l) = 2$ we note that, by Claim B, $c(p_{l-1}) = c(p_2) = 3$.

If $c(p_i) = 1$ then $|E(P')| = 0 \pmod{3}$ and $|E(P'')| = 2 \pmod{3}$, implying that $|E(C)| = 0 \pmod{3}$ as desired. If $c(p_i) = 2$ then $|E(P')| = 2 \pmod{3}$ and $|E(P'')| = 0 \pmod{3}$, implying that $|E(C)| = 0 \pmod{3}$ as desired. And finally if $c(p_i) = 3$ then $|E(P')| = 1 \pmod{3}$ and $|E(P'')| = 1 \pmod{3}$ again implying that $|E(C)| = 0 \pmod{3}$ as desired. This completes the proof for Case C.2.

Claim D: *If C is a cycle in G^* then $|E(C)| = 0 \pmod{3}$ and C contains no chord in G .*

Proof of Claim D: Let $C = p_1p_2p_3 \dots p_l p_1$ be a cycle of length l in G^* . As $\Delta^+(D^*) \leq 1$ we note that C is a directed cycle in D^* . Without loss of generality assume that $c(p_1) = 1$ and $c(p_2) = 2$ (otherwise rename the V_i 's). By Claim B we note that $c(p_l) = 3$ (due to the path $p_l p_1 p_2$) and $c(p_3) = 3$ (due to the path $p_1 p_2 p_3$). Continuing using Claim B we note that the following holds.

$$\begin{aligned} c(p_l) &= 3, c(p_1) = 1, c(p_2) = 2, c(p_3) = 3, \\ c(p_4) &= 1, c(p_5) = 2, c(p_6) = 3, c(p_7) = 1, \dots \end{aligned}$$

We see that $c(p_j) = 3$ if and only if j is divisible by three, which implies that l is divisible by three (as $c(p_l) = 3$). Therefore $|E(C)| = 0 \pmod{3}$.

For the sake of contradiction assume that C has a chord, $p_i p_j$ in G , where $i < j$. Consider the two cycles

$$C_1 = p_i p_{i+1} \dots p_j p_i \text{ and } C_2 = p_j p_{j+1} \dots p_l p_1 p_2 \dots p_i p_j$$

By Claim C we note that $|E(C_1)|$ and $|E(C_2)|$ are both divisible by three. However $|E(C_1)| + |E(C_2)| = |E(C)| + 2$ (as the edge $p_i p_j$ is counted twice). This is a contradiction as $|E(C_1)| + |E(C_2)|$ is divisible by three, but $|E(C)| + 2$ is not (as $|E(C)|$ is divisible by three, by our above arguments), which completes the proof of Claim D.

Claim E: *If P is a (u, x) -path in D^* , Q is a (v, x) -path in D^* and there exists an arc $xy \in D^*$, such that $|A(P)|, |A(Q)| \geq 1$ and yx is not an arc on either P or Q , then $uv \notin E(G)$.*

Proof of Claim E: Let $P = p_1 p_2 \dots p_a$ ($a \geq 2$) and $Q = q_1 q_2 \dots q_b$ ($b \geq 2$), where $p_1 = u$, $q_1 = v$ and $p_a = q_b = x$ and let $xy \in A(D^*)$ be defined as in the statement of Claim E.

Without loss of generality assume that $c(x) = 2$ and $c(y) = 3$. Using Claim B on the paths P_y and Q_y we note that all arcs in P and in Q go from V_1 to V_2 or from V_2 to V_3 or from V_3 to V_1 in D^* .

Assume for the sake of contradiction that $uv \in E(G)$ (i.e. $p_1 q_1 \in E(G)$). As $p_1 p_2, q_1 q_2 \in A(D^*)$ and $p_1 q_1 \in E(G)$, Claim B implies the following:

$$c(p_2) = 6 - c(p_1) - c(q_1) = c(q_2)$$

As all arcs in P and in Q go from V_1 to V_2 or from V_2 to V_3 or from V_3 to V_1 in D^* this implies that $c(p_1) = c(q_1)$ (as $c(p_2) = c(q_2)$), a contradiction to $p_1q_1 \in E(G)$. This completes the proof of Claim E.

Claim F: $\text{mac}(G) \geq \frac{1}{2}w(A_0) + \frac{7}{8}w(A_1) + w(A_2)$.

Proof of Claim F: For a set U of vertices of G , let $G[U]$ denote the subgraph of G induced by U . Let $G_1^*, G_2^*, \dots, G_r^*$ denote the connected components in G^* and let $D_1^*, D_2^*, \dots, D_r^*$ be the maximal subgraphs in D^* such that $UG(D_i^*) = G_i^*$. Let F_i denote all edges in $E(G) \setminus E(G^*)$ with both endpoints in $V(G_i^*)$. Note that the edge set of $G[V(G_i^*)]$ is exactly $E(G_i^*) \cup E(F_i)$. Furthermore note that each D_i^* is either an in-tree (an *in-tree* is an oriented tree, where exactly one vertex, the root, has out-degree 0 and all other vertices have out-degree 1) or D_i^* contains one directed cycle (possibly a 2-cycle) and all vertices have out-degree one in D_i^* . We now consider the following cases (for $i = 1, 2, \dots, r$):

Case F.1. D_i^* is an in-tree. If $f \in F_i$ then by Claim C we note that any cycle in $G_i^* + f$ has length divisible by three. As G is triangle-free this implies that any odd cycle in $G_i^* + f$ has length at least nine. Lemma 6.1 (with $k = 4$) now implies the following.

$$\begin{aligned} \text{mac}(G[V(G_i^*)]) &\geq \frac{1}{2}(w(F_i) + w(G_i^*)) + \frac{k-1}{2k}w(G_i^*) \\ &\geq \frac{1}{2}w(F_i) + \frac{7}{8}w(G_i^*) \end{aligned} \quad (1)$$

Case F.2. D_i^* contains a directed 2-cycle. Let $C = c_1c_2c_1$ be the cycle in D_i^* . As in the proof of Case F.1 we note that if $f \in F_i$ then, by Claim C, any cycle in $G_i^* + f$ has length divisible by three. As G is triangle-free this again implies that any odd cycle in $G_i^* + f$ has length at least nine. Lemma 6.1 (with $k = 4$) now implies the following, where we let e^* in Lemma 6.1 be the edge c_1c_2 .

$$\begin{aligned} \text{mac}(G[V(G_i^*)]) &\geq \frac{1}{2}(w(F_i) + w(G_i^*)) + \frac{k-1}{2k}w(G_i^*) + \frac{1}{2k}w(c_1c_2) \\ &\geq \frac{1}{2}w(F_i) + \frac{7}{8}w(G_i^*) + \frac{1}{8}w(c_1c_2) \end{aligned} \quad (2)$$

Case F.3. D_i^* contains a directed cycle of length greater than two. Let $C = c_1c_2 \dots c_l c_1$ be the cycle in D_i^* . Note that by Claim D, l is divisible by 3 and C contains no chord in G . Note that every $u \in V(D_i^*) \setminus V(C)$ has a unique directed path from u to $V(C)$ (as G_i^* is connected and $\Delta^+(D_i^*) \leq 1$). Let $\text{pe}(u) = c_j$ if and only if the unique path from u to $V(C)$ ends in c_j . Define C_j as follows (see Figure 5 for an illustration).

$$C_j = \{v \mid v \in V(D_i^*) \setminus V(C) \ \& \ \text{pe}(v) = c_j\} \cup \{c_j\}$$

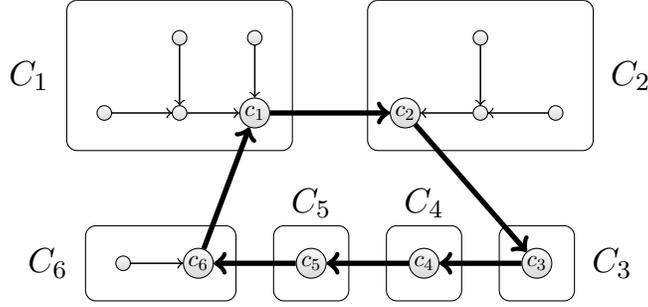


Figure 5: An illustration of the sets C_i 's in Claim F.3.

We will now show that if $u \in C_j$ and $v \in C_k$, where $j \neq k$, then $uv \notin E(G) \setminus E(C)$. For the sake of contradiction assume that $uv \in E(G) \setminus E(C)$. As $uv \notin E(C)$ and C has no chords in G we note that $u \neq c_j$ or $v \neq c_k$. Without loss of generality assume that $u \neq c_j$. Let P be the unique path from u to c_j in D_i^* and let Q be the unique path from v to c_k followed by the path from c_k to c_j using the arcs of C . As $c_j c_{j+1} \in D_i^*$ (and $c_{j+1} c_j$ is not an arc in D_i^* , as $l \geq 3$, and therefore also not an arc on $A(P) \cup A(Q)$), Claim E implies that $uv \notin E(G)$, a contradiction. Therefore $uv \notin E(G) \setminus E(C)$, as desired.

Also, $G_i^*[C_j]$ is a tree (possibly containing only one vertex) for all j . Analogously to Case F.1 we note that, for every $f \in F_i$, any odd cycle in $G_i^*[C_j] + f$ has length at least nine. Lemma 6.1 (with $k = 4$) now implies the following:

$$\begin{aligned} \text{mac}(G[C_j]) &\geq \frac{1}{2}(w(G_i^*[C_j]) + w(F_i \cap E(G[C_j]))) + \frac{k-1}{2k}w(G^*[C_j]) \\ &\geq \frac{1}{2}w(F_i \cap E(G[C_j])) + \frac{7}{8}w(G^*[C_j]) \end{aligned}$$

Recall that either l is even or $l \geq 9$. In this case picking an optimal weighted cut in each C_j and adding all edges of C if $|E(C)|$ is even or all edges of C except the cheapest one if $|E(C)|$ is odd, we obtain the following (as if $|E(C)|$ is odd then $|E(C)| \geq 9$).

$$\text{mac}(G[V(G_i^*)]) \geq \frac{8}{9}w(C) + \frac{1}{2}w(F_i) + \frac{7}{8}w(E(G_i^*) \setminus E(C))$$

As $8/9 > 7/8$ this implies the following:

$$\text{mac}(G[V(G_i^*)]) \geq \frac{1}{2}w(F_i) + \frac{7}{8}w(G_i^*) \quad (3)$$

This completes Case F.3.

Note that every edge in $A_2 \cap E(G_i^*)$ is considered in Case F.2 above and in this case Inequality (2) holds. Therefore any edge in $A_2 \cap E(G_i^*)$ is counted

$\frac{7}{8} + \frac{1}{8}$ times. Now combining Inequality (1), Inequality (2) and Inequality (3) we obtain the following.

$$\text{mac}(G[V(G_i^*)]) \geq w(A_2 \cap E(G_i^*)) + \frac{1}{2}w(F_i) + \frac{7}{8}w(E(G_i^*) \setminus A_2)$$

Let (X_i, Y_i) be a maximum weight cut of G_i^* and for each $i \in [r]$, randomly and uniformly assign X_i color 1 or 2 and Y_i the opposite color. Let A be all vertices of color 1 and let B be all vertices of color 2. Now every edge in (X_i, Y_i) lies in the cut induced by (A, B) and every edge between different G_i^* 's lies in the cut induced by (A, B) with probability $1/2$. Let W be the weight of all edges between different G_i^* 's and note that the average weight of the cut (A, B) is as follows.

$$\frac{W}{2} + \sum_{i=1}^r \text{mac}(G[V(G_i^*)])$$

If $e \in E(A_0)$ then e either belongs to some F_i or is an edge between different G_i^* 's (and in this case is counted in W), while if $e \in E(A_1) \cup E(A_2)$ then e belongs to some G_i^* . Therefore the following holds (as $\text{mac}(G)$ is greater than or equal to the average weight of the cut (A, B)).

$$\text{mac}(G) \geq \frac{1}{2}w(A_0) + \frac{7}{8}w(A_1) + w(A_2)$$

This completes the proof of Claim F.

Claim G: $\text{mac}(G) \geq \frac{3}{5}w(A_0) + \frac{3}{5}w(A_1) + w(A_2)$.

Proof of Claim G: As $\Delta^+(D^*) \leq 1$ we note that all edges in A_2 form a matching in G (i.e. they have no endpoints in common). By Lemma 5.11 (with $\Delta = 3$ and $M = A_2$) we note that $\text{mac}(G) \geq \frac{3}{5}(w(G) - w(A_2)) + w(A_2)$. As $w(G) - w(A_2) = w(A_0) + w(A_1)$ this implies Claim G.

We now return to the proof of Theorem 5.4. By Claims A, F and G, the following three inequalities hold.

(1) $\text{mac}(G) \geq w(A_0) + \frac{2}{3}w(A_1) + \frac{1}{3}w(A_2)$.

(2) $\text{mac}(G) \geq \frac{1}{2}w(A_0) + \frac{7}{8}w(A_1) + w(A_2)$.

(3) $\text{mac}(G) \geq \frac{3}{5}w(A_0) + \frac{3}{5}w(A_1) + w(A_2)$.

Taking $\frac{9}{22}$ times inequality (1) plus $\frac{8}{22}$ times inequality (2) plus $\frac{5}{22}$ times

inequality (3), implies the following:

$$\begin{aligned} \text{mac}(G) &\geq \left(\frac{9}{22} + \frac{8 \cdot 1}{22 \cdot 2} + \frac{5 \cdot 3}{22 \cdot 5}\right) w(A_0) + \left(\frac{9 \cdot 2}{22 \cdot 3} + \frac{8 \cdot 7}{22 \cdot 8} + \frac{5 \cdot 3}{22 \cdot 5}\right) w(A_1) \\ &\quad + \left(\frac{9 \cdot 1}{22 \cdot 3} + \frac{8}{22} + \frac{5}{22}\right) w(A_2) \\ &= \frac{16}{22}(w(A_0) + w(A_1) + w(A_2)) = \frac{8}{11} \cdot w(G) \end{aligned}$$

□

Recall the statement of Theorem 5.5.

Theorem 5.5. *Let G be an edge-weighted triangle-free graph with $\Delta(G) \leq 3$ and let T be an arbitrary spanning tree in G . Then $\text{mac}(G) \geq \frac{w(G)}{2} + 0.3193 \cdot w(T)$.*

Proof. By Lemma 6.2, with $p = 0.85$ and $r = 5$ and Theorem 5.4, we obtain the following inequalities.

$$\begin{aligned} \text{(a)} \quad \text{mac}(G) &\geq \frac{p+1}{2}w(T) + \frac{1-p^{r-1}}{2}(w(G) - w(T)) \\ &\geq 0.925 \cdot w(T) + 0.23899687(w(G) - w(T)) \end{aligned}$$

$$\text{(b)} \quad \text{mac}(G) \geq \frac{8}{11} \cdot w(T) + \frac{8}{11}(w(G) - w(T))$$

Taking 0.46545 times inequality (a) plus 0.53455 times inequality (b) gives us the following inequality, which completes the proof.

$$\text{mac}(G) \geq 0.8193 \cdot w(T) + \frac{w(G) - w(T)}{2}.$$

□

7 Conclusion

In this paper, we study lower bounds of the maximum weight $\text{mac}(G)$ of a cut in a weighted graph G . We obtain lower bound for arbitrary graphs, graphs of bounded girth, and triangle-free graphs of bounded maximum degree. We posed a number of conjectures and an open problem. We conjecture that if G is a weighted triangle-free graph and T is a spanning tree of G , then $\text{mac}(G) \geq \frac{w(G)}{2} + \frac{3w(T)}{8}$. We also conjecture that $\text{mac}(G) \geq 4w(G)/5$ for a weighted triangle-free subcubic graph G . Bondy and Locke [6] proved that the last conjecture holds for unweighted triangle-free subcubic graphs, in other words for weighted triangle-free subcubic graphs where each edge has the same weight. The following conjecture is related to the main topic of the paper: every triangle-free subcubic graph G contains a set E' of edges, such that every 5-cycle in G contains exactly one edge from E' .

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