Some Aspects of Directed Graphs: Paths and Cycles in Digraphs

Problem Session

Yacong Zhou[1]
(PhD Student of Prof. Gregory Gutin)
Yacong.Zhou@rhul.ac.uk

[1]Department of Computer Science, Royal Holloway University of London, London, UK

2nd School on Graph Theory, 06/10/2022

Outline

1. Hamilton Cycles in Tournaments
2. Hamilton Cycles in Degree-Constrained Digraphs
   - The Multi-Insertion Technique
   - the proof of the theorem
3. 4-kings in Semicomplete Multipartite Digraphs

Outline

1. Hamilton Cycles in Tournaments

2. Hamilton Cycles in Degree-Constrained Digraphs
   - The Multi-Insertion Technique
   - the proof of the theorem

3. 4-kings in Semicomplete Multipartite Digraphs

Definitions

Definition 1

In a digraph D, a vertex y is **reachable** from a vertex x if D has an (x, y)-path. In particular, a vertex is reachable from itself.

Definition 2

D is **strong** if every vertex of D is reachable from every other vertex of D.

Definition 3

A **tournament** is a digraph where there is exactly one arc between every pair of the vertices.

Definition 4

A **Hamilton cycle** is a cycle that contains all vertices in D. A digraph is said to be **Hamiltonian** if it has a Hamilton cycle.

Clearly, Hamiltonian digraphs are neccessarily strong.

Vertex-Pancyclicity of tournaments

Camion proved that, for tournaments, the sufficiency is also true.

Theorem 5 ([P. Camion., 1959])

Every strong tournament is hamiltonian.

Camion proved that, for tournaments, the sufficiency is also true.

Theorem 5 ([P. Camion., 1959])

Every strong tournament is hamiltonian.

Moon showed the following stronger result is also true.

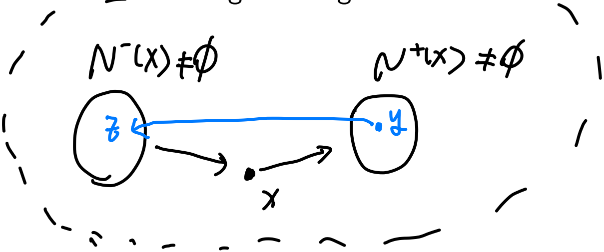
Definition 6

A digraph D is **vertex-pancyclic** if for every $x \in V(D)$ and every integer $k \in \{3, 4, \ldots, n\}$, there exists a k-cycle through x in D.

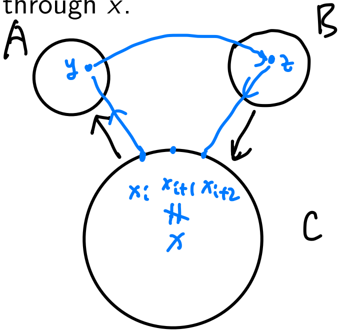
Theorem 7 ([J.W. Moon, 1966])

Every strong tournament is vertex-pancyclic.

- By induction on the length of the cycle. For any vertex $x \in V(D)$, we first show that there is a triangle through x in D.

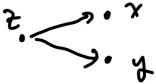


- Suppose there is a $k-1$-cycle through x ($k \geq 4$). We show that there is a k cycle through x.



Outline

Hamilton Cycles in Degree-Constrained Digraphs



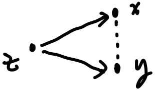
The pair $\{x, y\}$ is dominated by a vertex z if $z \rightarrow x$ and $z \rightarrow y$; in this case we say that the pair $\{x, y\}$ is a **dominated pair**.

Hamilton Cycles in Degree-Constrained Digraphs

The pair $\{x, y\}$ is dominated by a vertex z if $z \to x$ and $z \to y$; in this case we say that the pair $\{x, y\}$ is a **dominated pair**.

Theorem 8 ([J. Bang-Jensen, G. Gutin and H. Li, 1996])

Let D be a strong digraph of order $n \geq 2$. Suppose that, for every dominated pair of non-adjacent vertices $\{x, y\}$, either $d(x) \geq n$ and $d(y) \geq n-1$ or $d(x) \geq n-1$ and $d(y) \geq n$. Then D is hamiltonian.



Outline

The Multi-Insertion Technique

Definition 9

Let $P = u_1u_2\dots u_s$ be a path in a digraph D and let $Q = v_1v_2\dots v_t$ be a path in $D - V(P)$. The path P **can be inserted into** Q if there is a subscript $i \in [t-1]$ such that $v_i \to u_1$ and $u_s \to v_{i+1}$.

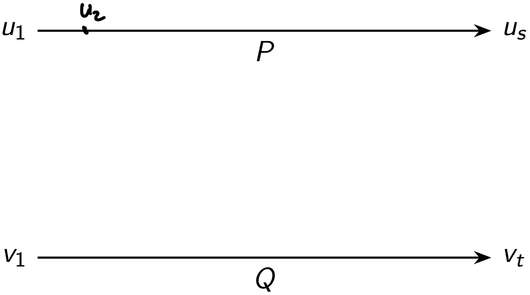


Figure: Inserting P into Q

The Multi-Insertion Technique

Definition 9

Let $P = u_1u_2\ldots u_s$ be a path in a digraph D and let $Q = v_1v_2\ldots v_t$ be a path in $D - V(P)$. The path P **can be inserted into** Q if there is a subscript $i \in [t-1]$ such that $v_i \to u_1$ and $u_s \to v_{i+1}$.

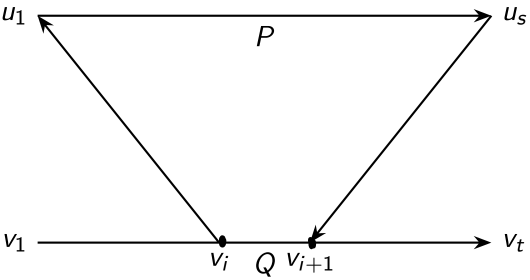
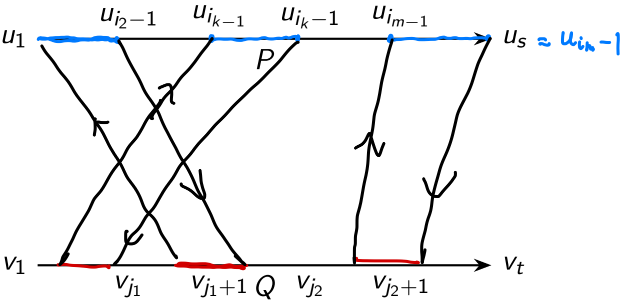


Figure: Inserting P into Q

The Multi-Insertion Technique

Definition 10

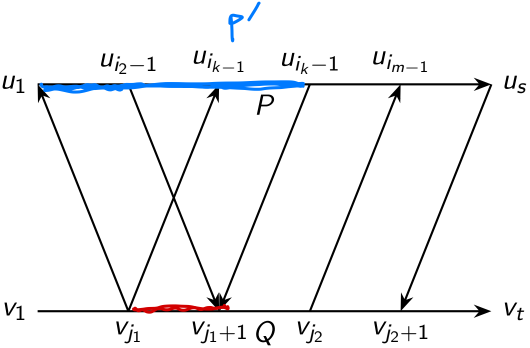
The path P **can be multi-inserted into** Q if there are integers $i_1 = 1 < i_2 < \cdots < i_m = s+1$ such that, for every $k = 2, 3, \ldots, m$, the subpath $P[u_{i_{k-1}}, u_{i_k - 1}]$ can be inserted into Q. The sequence of subpaths $P[u_{i_{k-1}}, u_{i_k - 1}]$, $k = 2, \ldots, m$, is a **multi-insertion partition** of P.



The Multi-Insertion Technique

Definition 10

The path P **can be multi-inserted into** Q if there are integers $i_1 = 1 < i_2 < \cdots < i_m = s+1$ such that, for every $k = 2, 3, \ldots, m$, the subpath $P[u_{i_{k-1}}, u_{i_k-1}]$ can be inserted into Q. The sequence of subpaths $P[u_{i_{k-1}}, u_{i_k-1}]$, $k = 2, \ldots, m$, is a **multi-insertion partition** of P.



The following lemma is a simple extension of a lemma by Bang-Jensen, Gutin and Li [J. Bang-Jensen, G. Gutin and H. Li, 1996].

Lemma 11

Let P be a path in D and let $Q = v_1 v_2 \dots v_t$ be a path in $D - V(P)$. If P can be multi-inserted into Q, then there is a (v_1, v_t)-path R in D so that $V(R) = V(P) \cup V(Q)$. Given a multi-insertion partition of P, the path R can be found in time $O(|V(P)||V(Q)|)$.

A proof for the Lemma

Proof.

Let $P = u_1 u_2 \dots u_s$. Suppose that integers $i_1 = 1 < i_2 < \cdots < i_m = s+1$ are such that the subpaths $P[u_{i_{k-1}}, u_{i_k - 1}]$, $k = 2, 3, \dots, m$, form a multi-insertion partition with **the minimum** m of P. Then, every subpath can be inserted into a different arc in Q. This completes the proof of the first part.

A proof for the Lemma

Proof.

Let $P = u_1 u_2 \dots u_s$. Suppose that integers $i_1 = 1 < i_2 < \cdots < i_m = s+1$ are such that the subpaths $P[u_{i_{k-1}}, u_{i_k - 1}]$, $k = 2, 3, \dots, m$, form a multi-insertion partition with **the minimum** m of P. Then, every subpath can be inserted into a different arc in Q. This completes the proof of the first part.

In the beginning, find for each subpath all arcs it can be inserted into taking at most $2(|V(Q)| - 1)$ steps. There are at most $|V(P)|$ subpaths. So this process takes $2|V(P)|(|V(Q)| - 1)$ steps.

A proof for the Lemma

Proof.

Let $P = u_1 u_2 \dots u_s$. Suppose that integers $i_1 = 1 < i_2 < \cdots < i_m = s+1$ are such that the subpaths $P[u_{i_{k-1}}, u_{i_k - 1}]$, $k = 2, 3, \dots, m$, form a multi-insertion partition with **the minimum** m of P. Then, every subpath can be inserted into a different arc in Q. This completes the proof of the first part.

In the beginning, find for each subpath all arcs it can be inserted into taking at most $2(|V(Q)| - 1)$ steps. There are at most $|V(P)|$ subpaths. So this process takes $2|V(P)|(|V(Q)| - 1)$ steps. Changing the initial partition into a new partition with each subpath inserted into different arcs in Q takes at most $2|V(P)|$ steps. So, in total, it will take $O(|V(P)||V(Q)|)$ time to construct this (v_1, v_t) path. □

Lemma 12

$t = d_Q(w) \le t-1$

Let $Q = v_1 v_2 \dots v_t$ be a path in D, and let $w \in V(D) - V(Q)$. If w cannot be inserted into Q, then $d_Q(w) = |N^+(w) \cap Q| + |N^-(w) \cap Q| \le t+1$. If, in addition, v_t does not dominate w, then $d_Q(w) \le t$.

Proof.

a $\quad I(a)$ an indicator $\quad I(a) = \begin{cases} 1, & a \in V(D) \\ 0, & \text{otherwise.} \end{cases}$

$$\underset{\underset{t\ge}{t+1}}{t-1+2} \ge \sum_{i=1}^{t-1} \Big(\underbrace{I(v_i w) + I(w v_{i+1})}_{\le 1}\Big) + \overset{0.}{I(v_t w)} + \underset{0.}{I(w v_1)}$$

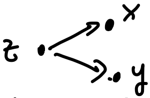
$$= d_Q^-(w) + d_Q^+(w) = d_Q(w).$$

□

From the proof, we can see that, if we further require $w \not\to v_1$, we have $d_Q(w) \le t-1$. One can use the modified lemma to prove the fact that every tournament has a Hamilton path [L. Rédei, 1934].

Outline

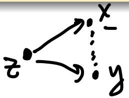
Hamilton Cycles in Degree-Constrained Digraphs



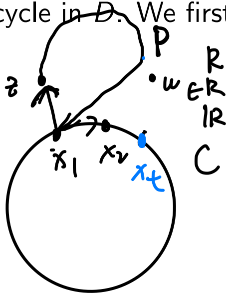
The pair $\{x, y\}$ is dominated by a vertex z if $z \to x$ and $z \to y$; in this case we say that the pair $\{x, y\}$ is a **dominated pair**.

Theorem 13
([J. Bang-Jensen, G. Gutin and H. Li, 1996])

Let D be a strong digraph of order $n \geq 2$. Suppose that, for every dominated pair of non-adjacent vertices $\{x, y\}$, either $d(x) \geq n$ and $d(y) \geq n-1$ or $d(x) \geq n-1$ and $d(y) \geq n$. Then D is hamiltonian.



- Assume that D is non-hamiltonian and $C = x_1x_2\dots x_mx_1$ is a longest cycle in D. We first show that D contains a C-bypass.



$R = D - V(P \cup C)$

$w \in R$

$|R| = n - |P| - |C| + 1$

$d(z) + d(x_2)$

$= d_{P\cup C}(z) + d_{P\cup C}(x_2) + d_R(z) + d_R(x_2)$

z cannot have neighbours in $C - x_1$

$\leq 2(\cancel{|P|} - 1) + 2(\cancel{|C|} - \cancel{1}) + 2 \cdot |R|$

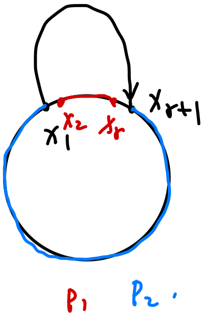
$|R| = 2(n - \cancel{|P|} \cancel{-|C|} \cancel{+1})$

$= 2(n-1)$

$x_1 \to \{z, x_2\}$, $z \not\to x_2$

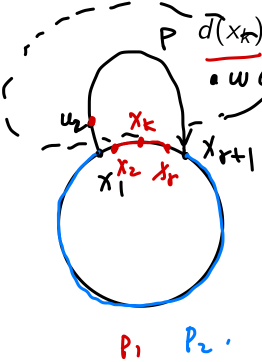
$d(z) + d(x_2) \geq 2n-1$

- Let $P = u_1 u_2 \dots u_s$ be a C-bypass ($s \geq 3$). Without loss of generality, let $u_1 = x_1$, $u_s = x_{\gamma+1}$, $0 < \gamma < m$. Suppose also that the gap γ of P is minimum among the gaps of all C-bypasses. Since C is a longest cycle of D, $\gamma \geq 2$. Let $P_1 = C[x_2, x_\gamma]$ and $P_2 = C[x_{\gamma+1}, x_1]$.



the Proof of theorem 13

Let $R = D - V(C)$. Let x_k be an arbitrary vertex in P_1. We first prove that



$$d(x_k) + d(u_2) \le d_{P_2}(x_k) + 2n - |V(P_2)| - 3. \tag{1}$$

$w \in R - u_2$.

by lemma 12

$$d(x_k) + d(u_2)$$

$$= \underline{d_{P_1}(x_k)} + \underline{d_{P_1}(u_2)} + \underbrace{d_R(x_k) + d_R(u_2)}_{=0}$$

$$+ \underline{d_{P_2}(x_k)} + \underline{d_{P_2}(u_2)} \quad \le |P_2| + 1$$

$$\le 2(|P_1| - 1) + 0 + 2 \cdot \cancel{(|R| - 1)}^{\,= n - |P_1| - |P_2|} + d_{P_2}(x_k)$$

u_2 cannot be inserted into P_2.

$$+ |P_2| + 1$$

$$= 2n - |P_2| - 3 + d_{P_2}(x_k).$$

Let $R = D - V(C)$. Let x_k be an arbitrary vertex in P_1. We first prove that

u_2

$x_1 \to x_j$

$$d(x_k) + d(u_2) \leq d_{P_2}(x_k) + 2n - |V(P_2)| - 3. \quad (1)$$

In particular, **for any** $x_j \in P_1$ **such that** $x_1 \to x_j$. By the assumption we have

$$2n - 1 \leq d(x_j) + d(u_2) \leq d_{P_2}(x_j) + 2n - |V(P_2)| - 3$$

and therefore,

$$d_{P_2}(x_j) \geq |V(P_2)| + 2. \quad (2)$$

$\Rightarrow$ by Lemma 12

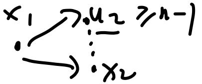
x_j can be inserted into P_2

- By (2) and Lemma 12, x_2 can be inserted into P_2.

the Proof of theorem 13

- By (2) and Lemma 12, x_2 can be inserted into P_2. Since C is a longest cycle, it follows from Lemma 11 that there exists $\beta \in \{3, \ldots, \gamma\}$ so that the subpath $C[x_2, x_{\beta-1}]$ can be multi-inserted into P_2, but $C[x_2, x_\beta]$ cannot. In particular, x_β cannot be inserted into P_2. Now, We show that

$$d(x_\beta) \leq n-2. \tag{3}$$



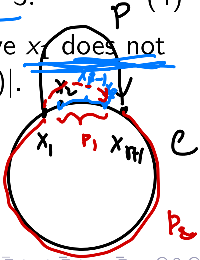
By (1), we have

$$d(x_\beta) + d(u_2) \leq d_{P_2}(x_\beta) + 2n - |V(P_2)| - 3. \tag{4}$$

Because x_β cannot be inserted into P_2 and (2), we have x_1 does not dominate x_β. By Lemma 12 we have $d_{P_2}(x_\beta) \leq |V(P_2)|$.

$$d(x_\beta) + d(u_2) \leq 2n-3.$$

$$d(x_\beta) \leq n-2.$$

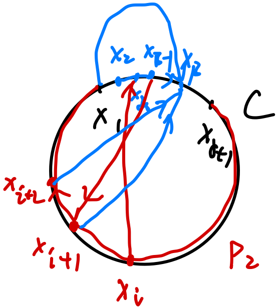


- Let $C[x_\alpha, x_{\beta-1}]$ ($\alpha \in \{3, \dots, \gamma\}$)be the last subpath in P_1 that can be inserted into P_2. $x_i \to x_\alpha$ and $x_{\beta-1} \to x_{i+1}$. Observe that the pair $\{x_\beta, x_{i+1}\}$ is dominated by $x_{\beta-1}$. Thus, by (3) and the assumption of the theorem, either $x_\beta \to x_{i+1}$ or $x_{i+1} \to x_\beta$.
- $x_{i+1} \to x_\beta$.

$x_{\beta-1} \to \{\underbrace{x_\beta, x_{i+1}}\}$

$d(x_\beta) \le n-2 \quad (3)$

$x_{i+1} \to \{x_{i+2}, x_\beta\}$



- $x_1 \to x_\beta$.

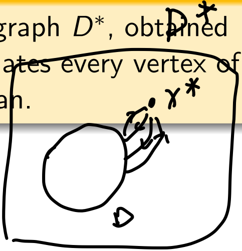
Suffficent conditions for containing a Hamilton path

Definition 14

A **Hamilton path** of D is a path in D that visits all vertices.

proposition

A digraph D has a Hamilton path if and only if the digraph D^*, obtained from D by adding a new vertex x^* such that x^* dominates every vertex of D and is dominated by every vertex of D, is hamiltonian.



Sufficent conditions for containing a Hamilton path

Definition 14

A **Hamilton path** of D is a path in D that visits all vertices.

proposition

A digraph D has a Hamilton path if and only if the digraph D^*, obtained from D by adding a new vertex x^* such that x^* dominates every vertex of D and is dominated by every vertex of D, is hamiltonian.

Using this proposition and Theorem 13, one can prove the following sufficient condition for a digraph to have a Hamilton path.

Theorem 15

Let D be a digraph of order n. Suppose that, for every dominated pair of non-adjacent vertices $\{x, y\}$, either $d(x) \geq n-1$ and $d(y) \geq n-2$ or $d(x) \geq n-2$ and $d(y) \geq n-1$. Then D has a Hamilton path.

Outline

1. Hamilton Cycles in Tournaments

2. Hamilton Cycles in Degree-Constrained Digraphs
   - The Multi-Insertion Technique
   - the proof of the theorem

3. 4-kings in Semicomplete Multipartite Digraphs

Definition 16

For a pair of vertices $u, v \in V(D)$, **the distance from u to v**, denoted by $dist(u, v)$ is the length of a shortest path from u to v

Definition 17

A r-**king** is a vertex $u \in V(D)$ such that for every vertex v in D, $dist(u, v) \leq r$.

A **source** is a vertex of in-degree zero.

Remark

For any integer r, If a digraph D has at least two sources, then it has no r-king. Thus, **we always consider digraphs with at most one source**.

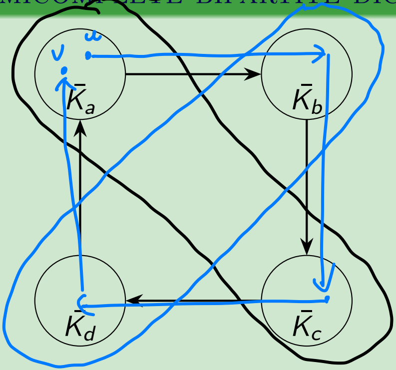
Semicomplete Multipartite Digraphs

Semicomplete Multipartite Digraphs

Definition 18

A digraph is **semicomplete multipartite** if it is obtained from a complete multipartite graph by replacing every edge by an arc or a pair of opposite arcs.

Example 19 (A semicomplete bipartite digraph)



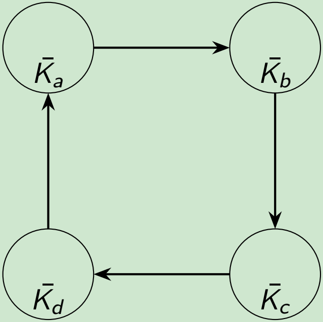
$a, b, c, d \geq 2.$

Semicomplete Multipartite Digraphs

Definition 18

A digraph is **semicomplete multipartite** if it is obtained from a complete multipartite graph by replacing every edge by an arc or a pair of opposite arcs.

Example 19 (A semicomplete bipartite digraph)



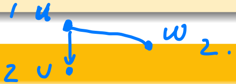
This source-free semicomplete bipartite digraph do not have 3-kings!

Theorem 20 ([G. Gutin and A. Yeo, 2000])

Every semicomplete multipartite digraph with at most one source has a 4-king.

THEOREM 20 ([G. GUTIN AND A. YEO, 2000])

Every semicomplete multipartite digraph with at most one source has a 4-king.



OBSERVATION

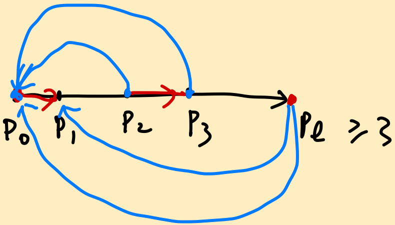
Let D be a semicomplete multipartite digraph. Then, for every adjacent pair $\{u, v\}$ and a vertex w different from them, there is at least one arc in $(\{u, v\}, w) \cup (w, \{u, v\})$.

Some useful lemmas

Lemma 21

If $P = p_0 p_1 \ldots p_l$ is a shortest path from p_0 to p_l in a semicomplete multipartite digraph D, and $l \geq 3$, then there is a (p_l, p_0)-path of length at most 4 in $D[V(P)]$.

Proof.



□

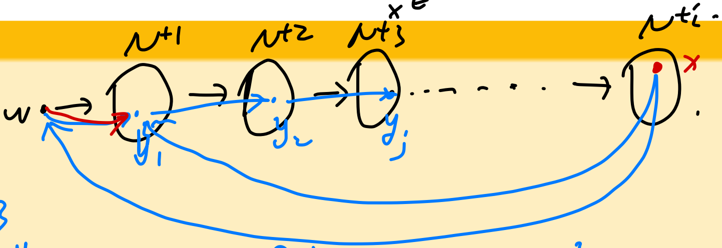
Let $N^{+i}(x) = \{y \in V(D) : dist(x, y) = i\}$ and $N^{+i}[x] = \cup_{j=0}^{i} N^{+j}(x)$

Lemma 22

Let D be a strong semicomplete multipartite digraph and let w be a vertex in D. For $i \geq 3$, if $N^{+i}(w) \neq \emptyset$, then $dist(N^{+i}(w), \boxed{N^{+i}[w]}) \leq 4$.

$x \in$

Proof.



$y_j \in N^{+j}$

① $1 \leq j \leq 3$

$w y_1 y_2 \cdots y_j$

② $j \geq 4$, $\{y_{j-3}, y_{j-2}\}$, x :

□

Proof.

- Let D be a semicomplete multipartite digraph with at most one source. If D has a vertex x of in-degree zero, then clearly x is a 2-king in D. Thus, we assume that D has no source.

Proof.

- Let D be a semicomplete multipartite digraph with at most one source. If D has a vertex x of in-degree zero, then clearly x is a 2-king in D. Thus, we assume that D has no source.
- Then, every initial strong component Q of D has at least two vertices. Therefore, D only has one initial strong component and we denote it by Q.

Proof.

- Let D be a semicomplete multipartite digraph with at most one source. If D has a vertex x of in-degree zero, then clearly x is a 2-king in D. Thus, we assume that D has no source.
- Then, every initial strong component Q of D has at least two vertices. Therefore, D only has one initial strong component and we denote it by Q. Observe that every 4-king of Q is a 4-king of D.

Proof.

- Let D be a semicomplete multipartite digraph with at most one source. If D has a vertex x of in-degree zero, then clearly x is a 2-king in D. Thus, we assume that D has no source.
- Then, every initial strong component Q of D has at least two vertices. Therefore, D only has one initial strong component and we denote it by Q. Observe that every 4-king of Q is a 4-king of D.
- It remains to show Q has a 4-king. lemma 22.

□

Thank you for your attention!

Bang-Jensen, Gutin, and Li
Sufficient Condition for a digraph to be Hamiltonian.
J. Graph Theory, 22(2): 181-187, 1996.

L. Rédei
Ein kombinatorischer Satz.
Acta. Litt. Sci. Szeged 7, 39–43.

Camion
Chemins et circuits hamiltoniens des graphes complets.
C. R. Acad. Sci. Paris, 249:2151–2152, 1959.

Gutin and Yeo
Kings in semicomplete multipartite digraphs.
J. Graph Theory, 33:177–183, 2000.

Moon
Solution to problem 463.
Math. Mag., 35:189, 1962.

Moon

On subtournaments of a tournament.
Can. Math. Bull., 9:297–301, 1966.