TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 360, Number 11, November 2008, Pages 5999–6015
S 0002-9947(08)04701-6
Article electronically published on June 26, 2008

THE STEINBERG SYMBOL AND SPECIAL VALUES OF L-FUNCTIONS

CECILIA BUSUIOC

ABSTRACT. The main results of this article concern the definition of a compactly supported cohomology class for the congruence group $\Gamma_0(p^n)$ with values in the second Milnor K-group (modulo 2-torsion) of the ring of p-integers of the cyclotomic extension $\mathbb{Q}(\mu_{p^n})$. We endow this cohomology group with a natural action of the standard Hecke operators and discuss the existence of special Hecke eigenclasses in its parabolic cohomology. Moreover, for $n = 1$, assuming the non-degeneracy of a certain pairing on p-units induced by the Steinberg symbol when (p, k) is an irregular pair, i.e. $p|\frac{B_k}{k}$, we show that the values of the above pairing are congruent mod p to the L-values of a weight k, level 1 cusp form which satisfies Eisenstein-type congruences mod p, a result that was predicted by a conjecture of R. Sharifi.

1. Introduction

Let $p^n > 1$ be a power of a positive prime p, $R_n := \mathbb{Z}[\mu_{p^n}, \frac{1}{p}]$, and $G_n := \mathrm{Gal}\,(\mathbb{Q}(\mu_{p^n})/\mathbb{Q})$. Then G_n acts naturally on Milnor's K-group, $K_2^M(R_n)$. We let the congruence group $\Gamma_0 := \Gamma_0(p^n)$ act on $K_2^M(R_n)$ via the homomorphism $\Gamma_0 \to G_n$ sending $\gamma \in \Gamma_0$ to σ_a where a is the upper left hand entry of γ and $\sigma_a \in G_n$ is given by $\sigma_a(\zeta) = \zeta^a$ for every $\zeta \in \mu_{p^n}$. In this paper, we will define a modular symbol

$$\phi_n \in H_c^1\left(\Gamma_0, \widetilde{K}_2(R_n)\right),$$

where $\widetilde{K}_2(R_n) = K_2^M(R_n)$ (mod 2-torsion). We note that the action of G_n on R_n induces a natural action of G_n on the above cohomology group. Moreover, we will endow this cohomology group with a natural action of the Hecke operators T_ℓ, $\ell \neq p$, and will prove the following theorem.

Theorem 1.1. *Let $\varphi_n \in H^1_{par}(\Gamma_0, \widetilde{K}_2(R_n))$ be the image of ϕ_n under the canonical map. Then*

(1) $\varphi_n|T_2 = (\sigma_2 + 2)\varphi_n$ *if* $p \neq 2$; *and*
(2) $\varphi_n|T_3 = (\sigma_3 + 3)\varphi_n$ *if* $p \neq 3$.

Now set $n = 1$, let $k \geq 2$ be an even integer and suppose $p > 3$. Let $R = R_1$, $G = G_1$. Let $E = R^\times/R^{\times p}$ be the group of units modulo p^{th} powers of units in R. Then we may decompose E as a direct sum

$$E = \bigoplus_{i=0}^{p-2} E^{(1-i)},$$

Received by the editors October 27, 2006.
2000 *Mathematics Subject Classification.* Primary 11F67.

where $E^{(1-i)}$ denotes the $\mathbb{F}_p$-submodule of E on which G acts via ω^{1-i} and $\omega : G \longrightarrow \mathbb{F}_p^\times$ is the canonical isomorphism. It is well known that $E^{(1)} = \mu_p$, and $E^{(1-i)} = 0$ for even i satisfying $0 < i \leq p-3$. For i odd, let $\eta_i \in E^{(1-i)}$ be the image of the cyclotomic p-unit $1-\zeta_p$ under the canonical projection $R^\times \longrightarrow E^{(1-i)}$. For $i = 1, 3, \ldots, p-2$, let $\xi_i = \{\eta_{k-i}, \eta_i\} \in (K_2^M(R)/pK_2^M(R))^{(2-k)}$. We remark that Vandiver's conjecture implies that the cyclotomic p-units generate E, therefore also that the symbols ξ_i $(i = 1, \ldots, p-2)$ generate $(K_2^M(R)/pK_2^M(R))^{(2-k)}$. In what follows, we will often assume the following hypothesis:

Hypothesis (H_k): There exists a non-zero G-equivariant map

$$\rho : K_2^M(R)/pK_2^M(R) \to \mathbb{F}_p(\omega^{2-k})$$

and an odd integer $i \neq p$ with $1 < i < k-1$ such that the pairing induced by the composition of ρ with the restriction of the Steinberg Symbol on the p-units

$$E^{(1+i-k)} \times E^{(1-i)} \to \mathbb{F}_p(\omega^{2-k})$$

is non-degenerate.

We have the following theorem.

Theorem 1.2. *Let $p > 3$ be a prime and k an even integer, $2 \leq k < 2p$, for which hypothesis H_k holds. Then there exists a non-zero parabolic cohomology class $\psi \in H^1_{par}(SL_2(\mathbb{Z}), Symm^{k-2}(\mathbb{F}_p^2))$ such that*

(1) *$\psi|T_q = (1+q^{k-1})\psi$ for $q = 2, 3$.*
(2) *For $3 \leq i \leq k-3$, $i \neq p$, we have $L(\psi, i) = \rho(\xi_i)$.*

The above results were motivated by the joint work of W. McCallum and R. Sharifi [MS03], [MS] and by a well-publicized conjecture of Sharifi (see [Sh1-04]). Indeed, McCallum and Sharifi [MS03] predict that, assuming Vandiver's Conjecture, Hypothesis H_k holds whenever p is irregular with $p|\frac{B_k}{k}$. In this case, Sharifi's conjecture [Sh1-04] predicts the truth of Theorem 1.2 with the Hecke condition (1) strengthened to include all primes $q \neq p$ (not just $q = 2, 3$). In future work of the author with G. Stevens, we will generalize Theorem 1.2 by constructing a universal Eisenstein cohomology class Ψ that specializes to the class ψ of Theorem 1.2. The general statement (1) about Hecke eigenvalues will follow from the properties of Ψ. Details will appear later.

Now, for a set S of positive primes $q \neq p$, let

$$H^+_{k,eis,S} \subseteq H^1_{par}(SL_2(\mathbb{Z}), Symm^{k-2}(\mathbb{F}_p))$$

denote the subspace of all vectors that are fixed by complex conjugation and on which the operators T_q for $q \in S$ act with eigenvalue $1 + q^{k-1}$. It follows from Theorem 1.2 that $H^+_{k,eis,\{2,3\}}$ is positive dimensional whenever hypothesis H_k holds. The following theorem is a consequence of Theorem 1.2.

Theorem 1.3. *Let (p,k) be an irregular pair with $k < p$ such that hypothesis H_k holds and assume $H^+_{k,eis,\{2,3\}}$ is one-dimensional. Then Sharifi's conjecture is true for the pair (p,k).*

Finally, we remark that Romyar Sharifi has recently and independently also proved Theorem 1.1.

I would like to mention that the present work would not have been possible without the insightful advice and wealth of ideas from my thesis advisor, Glenn

Stevens. I am also extremely grateful for his immense help with the expository part of the paper and Lemma 7.5. I would also like to thank Romyar Sharifi for his generosity in sharing copies of transparencies from talks he has given on his conjectures and also for a number of helpful conversations.

2. Modular symbols

Let $\Delta = Div(\mathbb{P}^1(\mathbb{Q}))$ and $\Delta_0 \subseteq \Delta$ be the subgroup of divisors of degree 0. The group $GL_2(\mathbb{Q})$ acts by fractional linear transformations on Δ and Δ_0. Let $\Sigma_0(p^n) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_2^+(\mathbb{Z}) | (a,p) = 1, p^n | c\}$ and $\Sigma_1(p^n) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_2^+(\mathbb{Z}) | a \equiv 1 (\operatorname{mod} p^n), p^n | c\}$.

For any right $\Sigma_0(p^n)$-module M we define a right action of $\Sigma_0(p^n)$ on $\operatorname{Hom}_{\mathbb{Z}}(\Delta_0, M)$ by

$$\big(\phi|\sigma\big)(D) = \phi(\sigma D)|\sigma$$

for all $\sigma \in \Sigma_0(p^n)$, $D \in \Delta_0$. The group of M-valued modular symbols over $\Gamma_0(p^n)$ is defined to be the group

$$\operatorname{Symb}_{\Gamma_0(p^n)}(M) := \operatorname{Hom}_{\mathbb{Z}}(\Delta_0, M)^{\Gamma_0(p^n)}.$$

For each positive integer m we define the Hecke operator

$$\begin{aligned} T_m : \operatorname{Symb}_{\Gamma_0(p^n)}(M) &\longrightarrow \operatorname{Symb}_{\Gamma_0(p^n)}(M) \\ \phi &\longmapsto \phi|T_m := \textstyle\sum_i \phi|\delta_i, \end{aligned}$$

where the sum is over a complete set of representatives $\{\delta_i\}_i$ for the left $\Gamma_0(p^n)$-cosets in the double coset $\Gamma_0(p^n)\left(\begin{smallmatrix} 1 & 0 \\ 0 & m \end{smallmatrix}\right)\Gamma_0(p^n)$.

For a cusp $r \in \mathbb{P}^1(\mathbb{Q})$, let Γ_r be the stabilizer subgroup in $\Gamma_0(p^n)$ of r. Then for any $m \in M^{\Gamma_r}$ we define $\widetilde{\phi}_{r,m} : \mathbb{P}^1(\mathbb{Q}) \to M$ by defining

$$\widetilde{\phi}_{r,m}(s) = \begin{cases} m|\gamma^{-1} & \text{if } s = \gamma r, \text{ with } \gamma \in \Gamma_0(p^n), \\ 0 & \text{otherwise.} \end{cases}$$

We extend $\widetilde{\phi}_{r,m}$ by linearity to an additive function

$$\widetilde{\phi}_{r,m} : \Delta \longrightarrow M$$

and note that the restriction of $\widetilde{\phi}_{r,m}$ to Δ_0 is an M-valued modular symbol over $\Gamma_0(p^n)$, which we denote $\phi_{r,m}$.

Definition 2.1. A modular symbol of the form $\phi_{r,m}$ will be called a boundary symbol supported on the r-cusps, i.e. the cusps that are $\Gamma_0(p^n)$-equivalent to r. We define the group of M-valued boundary symbols over $\Gamma_0(p^n)$ to be the subgroup

$$\operatorname{Bound}_{\Gamma_0(p^n)}(M) \subseteq \operatorname{Symb}_{\Gamma_0(p^n)}(M)$$

generated by the set of all $\phi_{r,m}$, where r, m run over all pairs with $r \in \mathbb{P}^1(\mathbb{Q})$ and $m \in M^{\Gamma_r}$.

There is also a canonical map $\operatorname{Symb}_{\Gamma_0(p^n)}(M) \longrightarrow H^1(\Gamma_0(p^n), M)$. If $\phi \in \operatorname{Symb}_{\Gamma_0(p^n)}(M)$ and $r \in \mathbb{P}^1(\mathbb{Q})$, then the map $\Gamma_0(p^n) \longrightarrow M$ defined by $\gamma \longmapsto \phi\big((\gamma r) - (r)\big)$ is a 1-cocycle, whose cohomology class is independent of the choice of r. We let π_ϕ be that cohomology class. From the definition it is clear that for any $r \in \mathbb{P}^1(\mathbb{Q})$, the restriction of π_ϕ to Γ_r is trivial. Thus, we have

$$\pi_\phi \in H^1(\Gamma_0(p^n), M).$$

We have the following theorem of Ash and Stevens [AS86].

Theorem 2.2. *If multiplication by* 6 *is invertible on* M*, then there is a canonical isomorphism* $H^1_c(\Gamma_0(p^n), M) \cong \mathrm{Symb}_{\Gamma_0(p^n)}(M)$*. Moreover, there is a canonical commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Bound}_{\Gamma_0(p^n)}(M) & \longrightarrow & \mathrm{Symb}_{\Gamma_0(p^n)}(M) & \xrightarrow{\pi_\phi} & H^1_{par}(\Gamma_0(p^n), M) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & H^1_\partial(\Gamma_0(p^n), M) & \longrightarrow & H^1_c(\Gamma_0(p^n), M) & \longrightarrow & H^1_{par}(\Gamma_0(p^n), M) & \longrightarrow & 0 \end{array}$$

in which the rows are exact, the vertical arrows are isomorphisms, and all maps commute with the natural action of the Hecke operators T_m $(m \in \mathbb{N})$*. Here,* H^1_∂ *is the "boundary cohomology", which is defined by the exactness of the second row.*

3. Manin symbols

In the special case where the subgroup $\Gamma_1(p^n) \subseteq \Gamma_0(p^n)$ acts trivially on M we can give a simple description of $\mathrm{Symb}_{\Gamma_0(p^n)}(M)$ in terms of "Manin symbols". We recall that description in this section.

Let A be a commutative ring. A group homomorphism

$$\chi : \Gamma_0(p^n) \longrightarrow A^\times$$

will be called a nebentype character if χ is trivial on $\Gamma_1(p^n)$. Any nebentype character χ extends uniquely to a multiplicative map

$$\chi : \Sigma_0(p^n) \longrightarrow A^\times$$

that is trivial on $\Sigma_1(p^n)$.

Definition 3.1. Let A be a ring and M be an A-module endowed with a right action of $\Sigma_0(p^n)$. We say that $\Sigma_0(p^n)$ acts via the nebentype character χ if for all $\gamma \in \Sigma_0(p^n)$ and all $m \in M$ we have $m|\gamma = \chi(\gamma) \cdot m$.

The group $SL_2(\mathbb{Z})$ acts by right matrix multiplication on the additive group of row vectors $(\mathbb{Z}/p^n\mathbb{Z})^2$. The orbit of $(0,1)$ is the set

$$X_n := \left((\mathbb{Z}/p^n\mathbb{Z})^2\right)' = \left\{(x,y) \in (\mathbb{Z}/p^n\mathbb{Z})^2 \;\middle|\; (x,y) = (1)\right\}.$$

The stabilizer of $(0,1)$ is the subgroup $\Gamma_1(p^n)$, which is a normal subgroup of $\Gamma_0(p^n)$. Thus $\Gamma_0(p^n)$ also acts on X_n on the left. In fact, if $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, then the left action of γ on X_n is given by scalar multiplication by d: $\gamma\mathbf{x} \mapsto d \cdot \mathbf{x}$. For the rest of this section M will be an A-module on which $\Sigma_0(p^n)$ acts via the nebentype character $\chi : \Sigma_0(p^n) \longrightarrow A^\times$.

Definition 3.2. A function

$$e : X_n \longrightarrow M$$

is called an M-valued Manin symbol over $\Gamma_0(p^n)$ if e satisfies the following "Manin relations" for all $\mathbf{x} = (x,y) \in X_n$ and $\lambda \in (\mathbb{Z}/p^n\mathbb{Z})^\times$:

(1) $e(\lambda\mathbf{x}) = \chi(\lambda) \cdot e(\mathbf{x})$;
(2) $e(x,y) + e(y,-x) = 0$; and
(3) $e(x,y) + e(y,-x-y) + e(-x-y,x) = 0$.

We denote by

$$\mathrm{Manin}_{\Gamma_0(p^n)}(M)$$

the group of all M-valued Manin symbols over $\Gamma_0(p^n)$.

Remark 3.3. Two applications of the second Manin condition show that if e is a Manin symbol, then $e(\mathbf{x}) = e(-\mathbf{x})$ for every $\mathbf{x} \in X_n$. So if $e \neq 0$, then the first condition implies χ must be *even*, i.e. $\chi(-1) = 1$.

Now fix a section $X_n \longrightarrow SL_2(\mathbb{Z})$, $\mathbf{x} \mapsto \gamma_{\mathbf{x}}$, so that

$$(0,1)\gamma_{\mathbf{x}} = \mathbf{x}$$

for every $\mathbf{x} \in X_n$. Also let $D_{\mathbf{x}} \in \Delta_0$ be given by

$$D_{\mathbf{x}} := \gamma_{\mathbf{x}} \cdot \big((\infty) - (0)\big).$$

If M is a right $\Gamma_0(p^n)$-module and $\phi \in \mathrm{Symb}_{\Gamma_0(p^n)}(M)$, then we define $\mathbf{e}_\phi : X_n \longrightarrow M$ by

$$\mathbf{e}_\phi(\mathbf{x}) := \phi(D_{\mathbf{x}})$$

and note that this is well defined independent of our choices of the $\gamma_{\mathbf{x}}$. We have the following reformulation of a theorem of Manin [Ma72].

Theorem 3.4. *The map $\phi \longmapsto \mathbf{e}_\phi$ induces an isomorphism*

$$\mathbf{e} : \mathrm{Symb}_{\Gamma_0(p^n)}(M) \longrightarrow \mathrm{Manin}_{\Gamma_0(p^n)}(M)$$

for every right $\Gamma_0(p^n)$-module M.

We use this isomorphism to transfer the action of the Hecke operators T_m on $\mathrm{Symb}_{\Gamma_0(p^n)}(M)$ to an action on $\mathrm{Manin}_{\Gamma_0(p^n)}(M)$. The following theorem of Merel [Mer94] gives an "explicit" description of this action.

Theorem 3.5. *Let m be a positive integer and let*

$$H_m := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; \begin{array}{l} a,b,c,d \in \mathbb{Z} \\ a > b \geq 0,\ d > c \geq 0 \\ ad - bc = m \end{array} \right\}.$$

Then for every $e \in \mathrm{Manin}_{\Gamma_0(p^n)}(M)$ we have

$$(e|T_m)(\mathbf{x}) = \sum_{\delta \in H_m} e(\mathbf{x}\delta).$$

For future reference we record the following corollary.

Corollary 3.6. *For arbitrary $e \in \mathrm{Manin}_{\Gamma_0(p^n)}(M)$ we have*

$$\begin{aligned} (e|T_2)(\mathbf{x}) &= e(x,2y) + e(2x,y) + e(x+y,2y) + e(2x,x+y), \\ (e|T_3)(\mathbf{x}) &= e(x,3y) + e(3x,y) + e(x+y,3y) + e(3x,x+y) \\ &\quad + e(x-y,3y) + e(3x,x-y). \end{aligned}$$

Proof. We easily verify that

$$\begin{aligned} H_2 &= \left\{ \left(\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right), \left(\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 1 & 2 \end{smallmatrix}\right), \left(\begin{smallmatrix} 2 & 1 \\ 0 & 1 \end{smallmatrix}\right) \right\} \quad \text{and} \\ H_3 &= \left\{ \left(\begin{smallmatrix} 1 & 0 \\ 0 & 3 \end{smallmatrix}\right), \left(\begin{smallmatrix} 3 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 1 & 3 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 2 & 3 \end{smallmatrix}\right), \left(\begin{smallmatrix} 3 & 1 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 3 & 2 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right) \right\}. \end{aligned}$$

The description of T_2 is then an immediate consequence of Merel's theorem. On the other hand, Merel's theorem implies

$$\begin{aligned} (e|T_3)(\mathbf{x}) &= e(x,3y) + e(3x,y) + e(x+y,3y) + e(3x,x+y) \\ &\quad + e(x+2y,3y) + e(3x,2x+y) + e(2x+y,2y+x). \end{aligned}$$

Now consider the following matrix:

$$\begin{pmatrix} e(x+2y,3y) & e(x-y,x+2y) & 0 \\ e(3x,2x+y) & 0 & e(2x+y,x-y) \\ e(2x+y,x+2y) & e(x+2y,x-y) & e(x-y,2x+y) \end{pmatrix}.$$

From the Manin relations, we see that the sum of the three rows are $e(x-y,3y)$, $e(3x,x-y)$, and 0, respectively. Thus the sum of all the entries of the matrix is $e(x-y,3y)+e(3x,x-y)$. On the other hand, the second and third columns sum to 0. Hence the sum of all the entries of the matrix is the sum of the first column. We therefore have

$$e(x+2y,3y)+e(3x,2x+y)+e(2x+y,x+2y)=e(x-y,3y)+e(3x,x-y).$$

Substituting this into the above expression for $e|T_3$ gives us

$$\begin{aligned}(e|T_3)(\mathbf{x}) \quad = \quad & e(x,3y)+e(3x,y)+e(x+y,3y)+e(3x,x+y) \\ & +e(x-y,3y)+e(3x,x-y),\end{aligned}$$

proving our claim for $e|T_3$. □

We conclude this section with a discussion of boundary symbols supported on the ∞-cusps (see Definition 2.1).

Definition 3.7. We say that a Manin symbol $e \in \mathrm{Manin}_{\Gamma_0(p^n)}(M)$ is supported at ∞ if e satisfies the following condition:

$$e(\mathbf{x})=0 \quad \text{for all } \mathbf{x}=(x,y)\in X_n \text{ with } xy\neq 0.$$

We denote by

$$\mathrm{Manin}^{\infty}_{\Gamma_0(p^n)}(M)\subseteq \mathrm{Manin}_{\Gamma_0(p^n)}(M)$$

the subgroup of all Manin symbols supported at ∞.

It is easy to describe the action of the Hecke operators on $\mathrm{Manin}^{\infty}_{\Gamma_0(p^n)}(M)$. The result is the following.

Proposition 3.8. *The subgroup* $\mathrm{Manin}^{\infty}_{\Gamma_0(p^n)}(M)$ *is an eigen-submodule for the action of the Hecke operators* T_m *on* $\mathrm{Manin}_{\Gamma_0(p^n)}(M)$*. Moreover, we have:*

- *if* $\ell\neq p$ *is prime, then the eigenvalue of* T_ℓ *is* $\ell+\chi(\ell)$*;*
- *the eigenvalue of* T_p *is* p*.*

The proof is an easy computation from the definitions. We note that the formal Dirichlet series of Hecke acting on $\mathrm{Manin}^{\infty}_{\Gamma_0(p^n)}(M)$ is given by the formal identity

$$\sum_{m=1}^{\infty} T_m m^{-s} = \left(1-p^{1-s}\right)^{-1}\prod_{\ell\neq p}\left(1-\ell^{1-s}\right)^{-1}\left(1-\chi(\ell)\ell^{-s}\right)^{-1},$$

which we may write suggestively in the form

$$\sum_{m=1}^{\infty} T_m m^{-s} = \zeta(s-1)\cdot L(s,\chi).$$

The right hand side is the L-function of an Eisenstein series E_χ defined over the ring A.

4. K_2 of cyclotomic integer rings

First, we recall the definition of Milnor's K_2-group of a commutative ring R.

Definition 4.1. The second Milnor K-group of R is defined as

$$K_2^M(R) := (R^\times \otimes_{\mathbb{Z}} R^\times)/I_2,$$

where I_2 is the subgroup of $R^\times \otimes_{\mathbb{Z}} R^\times$ generated by the set

$$\left\{ a_1 \otimes a_2 \in R^\times \otimes_{\mathbb{Z}} R^\times \;\middle|\; a_1 + a_2 \in \{0,1\} \right\}.$$

The Steinberg Symbol is defined as the canonical map

$$\{\ ,\ \} : R^\times \times R^\times \longrightarrow K_2^M(R).$$

We write the multiplication in $K_2^M(R)$ additively and note that for $a, b \in R^\times$ we have

$$\{a,b\} + \{b,a\} = 0$$

as an easy consequence of the relation $\{ab, -ab\} = 0$. This skew-symmetry of the Steinberg symbol will be used throughout the paper.

Denote by μ_n the group of p^n-th roots of unity in $\mathbb{C}^\times$ and fix $\zeta_n \in \mu_n$ a primitive p^n-th root of unity. Let $K_n := \mathbb{Q}(\mu_n)$, $R_n = \mathbb{Z}[\mu_n, \frac{1}{p}]$, and $G_n := \mathrm{Gal}(K_n/\mathbb{Q})$. We associate to each $a \in (\mathbb{Z}/p^n\mathbb{Z})^\times$ the element $\sigma_a \in G_n$ for which $\zeta_n^{\sigma_a} = \zeta^a$. As in the introduction, we let $\widetilde{K}_2(R_n) = K_2^M(R_n)/(2\text{-torsion})$. Note that $\widetilde{K}_2(R_n)$ has a natural action of the Galois group G_n.

In the next section, we will need the following lemma.

Lemma 4.2. *For any $x, y \in \mathbb{Z}/p^n\mathbb{Z}$ with $x \neq 0$ we have*

$$\{1 - \zeta_n^x, \zeta_n^y\} = 0.$$

Proof. If $x \in (\mathbb{Z}/p^n\mathbb{Z})^\times$, then we may choose $a \in \mathbb{Z}$ such that $ax = 1$ in $(\mathbb{Z}/p^n\mathbb{Z})^\times$. But then $\{1-\zeta_n^x, \zeta_n^y\} = \{1-\zeta_n^x, \zeta_n^{axy}\} = ay \cdot \{1-\zeta_n^x, \zeta_n^x\} = 0$, proving the lemma in this special case. In the general case we may write $x \equiv p^k u \pmod{p^n}$ for $u, k \in \mathbb{Z}$ with $p \nmid u$ and $0 \leq k < n$. Then,

$$1 - \zeta_n^x = \prod_{\substack{\alpha \in (\mathbb{Z}/p^n\mathbb{Z})^\times \\ \alpha \equiv 1 \ (p^{n-k})}} 1 - \zeta_n^{u\alpha},$$

and the relation will now follow from the special case. □

5. A $\widetilde{K}_2(R_n)$-valued Manin symbol

Define the *Artin nebentype character* to be the character

$$\begin{aligned} \chi : \Sigma_0(p^n) &\longrightarrow \mathbb{Z}[G_n]^\times \\ \gamma &\longmapsto \sigma_a, \end{aligned}$$

where, as always, a is the upper left corner of γ. We let $\Sigma_0(p^n)$ act on $\widetilde{K}_2(R_n)$ via the Artin nebentype χ.

Theorem 5.1. *The function $e_n : X_n \longrightarrow \widetilde{K}_2(R_n)$ defined by*

$$e_n(x,y) = \begin{cases} \{1-\zeta_n^x, 1-\zeta_n^y\} & \textit{if } x, y \neq 0, \\ 0 & \textit{otherwise,} \end{cases}$$

is a Manin symbol: $e_n \in \mathrm{Manin}_{\Gamma_0(p^n)}(\widetilde{K}_2(R_n))$. *Moreover, for* $q = 2, 3$,

$$e_n \,\Big|\, \Big(T_q - \big(q + \chi(q)\big)\Big) \in \mathrm{Manin}^{\infty}_{\Gamma_0(p^n)}(\widetilde{K}_2(R_n)).$$

Proof. We begin by verifying that e_n satisfies the three Manin conditions (see Definition 3.2). Let $\mathbf{x} = (x, y) \in X_n$ be fixed and note that the first two Manin conditions are trivially satisfied if $xy = 0$.

For $\lambda \in (\mathbb{Z}/p^n\mathbb{Z})^\times$ we have

$$\begin{aligned} e_n(\lambda\mathbf{x}) &= \{1 - \zeta_n^{\lambda x}, 1 - \zeta_n^{\lambda y}\} = \{1 - \zeta_n^{x}, 1 - \zeta_n^{y}\}^{\sigma_\lambda} \\ &= \chi(\lambda) \cdot e_n(\mathbf{x}). \end{aligned}$$

So the first Manin condition is satisfied.

We let $\sim$ denote the congruence modulo 2-torsion in $K_2^M(R_n)$ (recall that $\widetilde{K}_2(R_n) = K_2^M(R_n)/(\text{2-torsion})$). From the properties of the Steinberg symbol we have, for $xy \neq 0$,

$$\begin{aligned} e_n(x, y) &\sim \{\zeta_n^x - 1, 1 - \zeta_n^y\} = -\{1 - \zeta_n^y, \zeta_n^x - 1\} \\ &= -\{1 - \zeta_n^y, \zeta_n^x\} \cdot \{1 - \zeta_n^y,\ 1 - \zeta_n^{-x}\} \\ &= -\{1 - \zeta_n^y, 1 - \zeta_n^{-x}\} \\ &= -e_n(y, -x). \end{aligned}$$

We used Lemma 4.2 to derive the second to last equality. This proves the second Manin condition.

To verify the third Manin condition,

$$e_n(x, y) + e_n(y, -x - y) + e_n(-x - y, x) = 0,$$

we consider cases. If $x = 0$, then $y \neq 0$, and in that case $e_n(y, -y) = -e_n(y, y)$ by the second Manin condition. But by the skew-symmetry of the Steinberg symbol we have $e_n(y, y) \sim 0$, so $e_n(y, -y) \sim 0$ and the third Manin condition is satisfied in this case. Similarly, it is satisfied if either $y = 0$ or $x + y = 0$. So we may assume x, y, and $x + y$ are all non-zero. In this case, we have the identity

$$\frac{\zeta_n^y(1 - \zeta_n^x)}{1 - \zeta_n^{x+y}} + \frac{1 - \zeta_n^y}{1 - \zeta_n^{x+y}} = 1.$$

From the Steinberg relations we then have

$$\left\{ \frac{\zeta_n^y(1 - \zeta_n^x)}{1 - \zeta_n^{x+y}}, \frac{1 - \zeta_n^y}{1 - \zeta_n^{x+y}} \right\} = 0.$$

Bimultiplicativity of the Steinberg symbol, Lemma 4.2, and the skew symmetry of e_n imply

$$e_n(x, y) - e_n(x + y, y) - e_n(x, x + y) = 0.$$

Now apply the second Manin condition to the last two terms to obtain

$$e_n(x, y) + e_n(y, -x - y) + e_n(-x - y, x) = 0,$$

and the third Manin condition is proved. This proves e_n is a Manin symbol.

To compute the Hecke operators, we use Corollary 3.6 to Merel's Theorem 3.5.

Lemma 5.2. *Let $q = 2$ or $q = 3$. Then for all $\mathbf{x} = (x, y)$ with $xy \neq 0$ we have*

$$(e_n|T_q)(\mathbf{x}) = (q + \chi(q)) \cdot e_n(\mathbf{x}).$$

Proof. For $q = 2$ we have, from Corollary 3.6,

$$(e_n|T_2)(x,y) = e_n(x,2y) + e_n(2x,y) + e_n(x+y,2y) + e_n(2x,x+y).$$

If $x+y=0$ this says $(e_n|T_2)(x,-x) = e_n(x,-2x)+e_n(2x,-x)$, which vanishes by skew symmetry and the second Manin condition. But also $e_n(x,-x)=0$, so we have

$$(e_n|T_2)(x,-x) = (2+\chi(2))\cdot e_n(x,-x)$$

since both sides of this equation vanish.

If $x+y \neq 0$, then we use the following identity, which was discovered by W. McCallum and R. Sharifi [MS03]:

$$\frac{(1-\zeta_n^{x+y})(1-\zeta_n^x)}{1-\zeta_n^{2x}} + \zeta_n^x \frac{(1-\zeta_n^{2y})(1-\zeta_n^x)}{(1-\zeta_n^{2x})(1-\zeta_n^y)} = 1.$$

This implies

$$\begin{aligned}&e_n(x,2y)+e_n(2x,y)+e_n(x+y,2y)-e_n(x+y,2x)\\&\quad = e_n(x,y)+e_n(2x,2y)+e_n(x+y,y)-e_n(x+y,x)+e_n(x,2x)+e_n(2x,x).\end{aligned}$$

According to the above, the left hand side of this equality is $(e_n|T_2)(x,y)$. So we have

$$\begin{aligned}(e_n|T_2)(x,y) \quad = \quad & 2e_n(x,y)+e_n(2x,2y)\\ & -\big(e_n(x,y)+e_n(y,x+y)+e_n(x+y,x)\big)\\ & +\big(e_n(x,2x)+e_n(2x,x)\big).\end{aligned}$$

But the last two lines of the right side of this equation vanish by the third and second Manin conditions, so we have

$$(e_n|T_2)(\mathbf{x}) = (2+\chi(2))\cdot e_n(\mathbf{x})$$

and the assertion for T_2 is proved.

For $q=3$ we again use Corollary 3.6 to obtain

$$\begin{aligned}(e_n|T_3)(x,y) \quad = \quad & e_n(x,3y)+e_n(3x,y)+e_n(x+y,3y)\\ & +e_n(x-y,3y)+e_n(3x,x+y)+e_n(3x,x-y).\end{aligned}$$

If $x+y=0$, then the right hand side simplifies to $e_n(x,-3x)+e_n(3x,-x)+e_n(-x,-3x)+e_n(3x,x)$, which vanishes by the skew symmetry of e_n. A similar calculation shows that the right hand side vanishes when $x-y=0$. Thus in either case, we have

$$(e_n|T_3)(x,\pm x) = (3+\chi(3))\cdot e_n(x,\pm x)$$

since both sides vanish.

So we may assume $x+y, x-y \neq 0$. In that case, we have the following identity (see [Sh2-04]):

$$\zeta_n^{y-x}(\zeta_n^{2x}+\zeta_n^x+1) + (1-\zeta_n^{y-x})(1-\zeta_n^{x+y}) = \frac{1-\zeta_n^{3y}}{1-\zeta_n^y},$$

which may be rewritten as

$$\frac{\zeta_n^{y-x}(1-\zeta_n^{3x})(1-\zeta_n^y)}{(1-\zeta_n^x)(1-\zeta_n^{3y})} + \frac{(1-\zeta_n^{y-x})(1-\zeta_n^y)(1-\zeta_n^{x+y})}{1-\zeta_n^{3y}} = 1.$$

This implies

$$\begin{gathered} e_n(x,3y)+e_n(3x,y)-e_n(3y,x+y)-e_n(3y,y-x)+e_n(3x,x+y) \\ +e_n(3x,y-x)=e_n(3x,3y)-e_n(y,y-x)-e_n(y,x+y)+e_n(x,y-x) \\ +e_n(x,y)+e_n(x,x+y). \end{gathered}$$

The left hand side of this equality is $(e_n|T_3)(x,y)$. So we have

$$\begin{aligned} (e_n|T_3)(x,y) &= 3e_n(x,y)+e_n(3x,3y) \\ &\quad + e_n(y,x)+e_n(x,y-x)+e_n(y-x,y) \\ &\quad + e_n(y,x)+e_n(x,x+y)+e_n(x+y,y). \end{aligned}$$

Using the third Manin relation, we see that the bottom two rows of the right hand side vanish. Hence

$$(e_n|T_3)(\mathbf{x})=(3+\chi(3))\cdot e_n(\mathbf{x})$$

and the lemma is proved. □

We now return to the proof of the theorem. Let $q=2$ or $q=3$. It follows from the lemma that the Manin symbol $e := e_n|(T_q-(q+\chi(q)))$ vanishes on all $\mathbf{x}=(x,y)\in X_n$ with $xy\neq 0$. Thus from Proposition 3.8 we see that $e\in \mathrm{Manin}^{\infty}_{\Gamma_0(p^n)}(\widetilde{K}_2(R_n))$ and the theorem is proved. □

6. The parabolic cohomology class φ_n

For each positive integer n we let

$$B_n := \mathrm{Bound}_{\Gamma_0(p^n)}(\widetilde{K}_2(R_n)) \quad\text{and}\quad S_n := \mathrm{Symb}_{\Gamma_0(p^n)}(\widetilde{K}_2(R_n))$$

and consider the exact sequence

$$0 \longrightarrow B_n \longrightarrow S_n \xrightarrow{\ \pi\ } H^1_{par}(\Gamma_0(p^n),\widetilde{K}_2(R_n)) \longrightarrow 0.$$

Let $\phi_n\in S_n$ be the modular symbol associated to the Manin symbol e_n defined in the previous section. We define

$$\varphi_n := \pi_{\phi_n}\in H^1_{par}(\Gamma_0(p^n),\widetilde{K}_2(R_n))$$

to be the image of ϕ_n. Theorem 1.1 of the Introduction follows now immediately from Theorem 5.1.

We raise a number of key questions.

Questions:

(1) Is φ_n an eigenclass for all the Hecke operators?
(2) Under what conditions can we say $\varphi_n\neq 0$?
(3) What relations exist among the φ_n as n varies?
(4) Assuming $\varphi_n\neq 0$, is there a Hecke eigensymbol $\psi_n\in S_n$ lifting φ_n?

In fact, in future work, we will draw a connection between the modular symbols ϕ_n and the Eisenstein distribution [St89] and use this connection to show that φ_n is an eigenclass satisfying

$$\varphi_n|T_q=(q+\chi(q))\cdot\varphi_n$$

for all primes q, where we understand that $\chi(q)=0$ when $q=p$. Thus (1) has an affirmative answer.

The answers to questions (2) to (4) are closely tied to some very beautiful recent results of Romyar Sharifi [Sh3-05], which in turn are motivated by work of Ohta (see [Oh03]). Sharifi's ideas suggest that (2) is closely connected to the structure of the class group of $K_n := \mathbb{Q}(\zeta_n)$. The answer to (3) should be given (for $m \leq n$) in terms of the transfer map $\widetilde{K}_2(R_n) \longrightarrow \widetilde{K}_2(R_m)$ on K-theory composed with corestriction of the cohomology of $\Gamma_0(p^n)$ to $\Gamma_0(p^m)$. Finally, we expect the answer to (4) to be negative, which corresponds to an expectation that there should be lots of fusion between the boundary cohomology and the parabolic cohomology.

7. Special values of L-functions

In this section, following [AS86], we define the universal L-value of a modular symbol and describe a few of its properties. In particular we will see that the Manin symbols are universal L-values.

Let R be a ring and M be an R-module endowed with a right action of $SL_2(\mathbb{Z})$.

Definition 7.1 (Universal L-values)**.** Let $\phi \in H^1_c(SL_2(\mathbb{Z}), M)$ be a modular symbol. We define $\Lambda(\phi) \in M$ by

$$\Lambda(\phi) := \phi\big((\infty) - (0)\big)$$

and call $\Lambda(\phi)$ the universal L-value of ϕ.

We define M^* to be the R-dual of M: $M^* := \mathrm{Hom}_R(M, R)$ with $SL_2(\mathbb{Z})$ acting on the right as $(\lambda|\sigma)(m) = \lambda(m|\sigma')$ where $\sigma \mapsto \sigma'$ is the adjugate involution $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$. For example, let

$$W_r(R) = \left\{ F \in R[X, Y] \;\middle|\; F \text{ is homogeneous of degree } r \right\}$$

with $M_2^+(\mathbb{Z})$ acting by the formula $(F|\sigma)(X, Y) = F((X, Y)\sigma')$ and let

$$V_r(R) := W_r(R)^*.$$

As in [AS86] we make the following definition.

Definition 7.2. Let $\phi \in H^1_c(SL_2(\mathbb{Z}), V_r(R))$. We define the special L-values $L(\phi, i+1) \in R$ for $i = 0, 1, \ldots, r$ by

$$L(\phi, i+1) := \big\langle \Lambda(\phi), (-1)^i X^{r-i} Y^i \big\rangle$$

where $\langle\ ,\ \rangle : V_r(R) \times W_r(R) \longrightarrow R$ is the canonical pairing.

If $r!$ is invertible in R, we have from Lemma 3.2 of [AS86] a unique $M_2^+(\mathbb{Z})$-equivariant perfect pairing

$$\langle\ ,\ \rangle : W_r(R) \times W_r(R) \longrightarrow R$$

with respect to which

$$\big\langle \left(\begin{smallmatrix} r \\ i \end{smallmatrix}\right) X^i Y^{r-i}, (-1)^j X^{r-j} Y^j \big\rangle = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, when $r!$ is invertible in R we have $V_r(R) \cong W_r(R)$ and we may regard $\Lambda(\phi)$ as an element of $W_r(R)$. With the above identifications we then have

$$\Lambda(\phi) = \sum_{i=0}^{r} \left(\begin{smallmatrix} r \\ i \end{smallmatrix}\right) L(\phi, i+1) X^i Y^{r-i},$$

consistent with the conventions of [AS86]. Note however that Definition 7.2 is meaningful for any commutative ring R – we do not need to assume $r!$ is invertible in R.

Now consider the general case. Let Γ be any congruence subgroup of $SL_2(\mathbb{Z})$, let M be a Γ-module, and let $\varphi \in \mathrm{Symb}_\Gamma(M)$ be an M-valued modular symbol over Γ. To define the universal L-value of φ we first induce to $SL_2(\mathbb{Z})$ using Shapiro's Lemma and then take the universal L-value of the induced modular symbol.

More precisely, we define the induced module of M to be the module

$$I(M) := \left\{ f : SL_2(\mathbb{Z}) \to M \;\middle|\; f(\gamma x) = f(x)|\gamma^{-1},\ \forall \gamma \in \Gamma,\ x \in SL_2(\mathbb{Z}) \right\}$$

with $SL_2(\mathbb{Z})$ acting by $(f|g)(x) = f(xg^{-1})$. The Shapiro isomorphism gives us a canonical isomorphism

$$H^1_c(\Gamma, M) \xrightarrow{I} H^1_c(SL_2(\mathbb{Z}), I(M)),$$

which is given explicitly on modular symbols by $I : \varphi \longmapsto I(\varphi)$, where $I(\varphi) : \Delta_0 \longrightarrow I(M)$ is given by

$$I(\varphi)(D)(x) = \varphi(xD)$$

for $D \in \Delta_0$ and $x \in SL_2(\mathbb{Z})$.

Definition 7.3. Let $\varphi \in H^1_c(\Gamma, M)$. Then the universal L-value of φ is defined to be

$$\Lambda(\varphi) := I(\varphi)\big((\infty) - (0)\big).$$

In other words we set $\Lambda(\varphi) = \Lambda(I(\varphi))$.

In the special case where M is a $\Gamma_0(p^n)$-module on which $\Gamma_0(p^n)$ acts via a nebentype character χ we may identify $I(M)$ with the module of functions $f : X_n \longrightarrow M$ satisfying $f(d\mathbf{x}) = \chi(d) \cdot f(\mathbf{x})$, which we will denote by $I_\chi(M)$. Thus we have a natural inclusion

$$\mathrm{Manin}_{\Gamma_0(p^n)}(M) \hookrightarrow I(M).$$

In fact, we have the following simple proposition, whose proof we leave to the reader.

Proposition 7.4. *Let M be an R-module on which $\Sigma_0(p^n)$ acts via a nebentype character χ and let $\phi \in H^1_c(\Gamma_0(p^n), M)$ be a modular symbol. Then with the above identifications, we have*

$$\Lambda(\phi) = \mathbf{e}_\phi.$$

Finally, we turn to the problem of defining special L-values of parabolic cohomology classes. For this we need to understand the module $V_r(R)^{\Gamma_\infty}$. For the rest of this section R will be a ring of characteristic p and $r \geq 0$ will an even integer. Let $\{\lambda_i\}_{i=0}^r$ in $V_r(R)$ be the dual basis to $\{(-1)^i X^{r-i} Y^i\}_{i=0}^r$ in $W_r(R)$. We leave the simple proof of the following lemma to the reader.

Lemma 7.5. *Let r be an even integer. If $r < p$, then λ_r spans $V_r(R)^{\Gamma_\infty}$. If $p \leq r < 2p$, then λ_r and λ_{p-1} span $V_r(R)^{\Gamma_\infty}$.*

Proposition 7.6. *Let $\Gamma = SL_2(\mathbb{Z})$, let i, r be positive integers with r even and $0 \leq i \leq r$. Let $\varphi \in \mathrm{Bound}_\Gamma(V_r(R))$ be a boundary symbol. Then*

(a) *If $r < 2p$, then $L(\varphi, i+1) = 0$* for $i \not\equiv 0, r \pmod{p-1}$.
(b) *For arbitrary r, if $\varphi|T_p = \varphi$, then $L(\varphi, i+1) = 0$* for $i \neq 0, r$.

Proof. Fix i, r as in the statement of the proposition and let $\varphi \in \mathrm{Bound}_\Gamma(V_r(R))$. Since $SL_2(\mathbb{Z})$ has only one cusp, we have $\varphi = \varphi_{\infty,\lambda}$ for some $\lambda \in V_r(R)^{\Gamma_\infty}$. By Lemma 7.5 there are constants $a, b \in R$ such that $\lambda = a\lambda_r$, if $r < p$, and $\lambda = a\lambda_r + b\lambda_{p-1}$ if $p < r < 2p$. But then

$$\Lambda(\varphi) = \lambda - \lambda | \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) = \begin{cases} a(\lambda_r - \lambda_0) & \text{if } r < p, \\ a(\lambda_r - \lambda_0) + b(\lambda_{p-1} - \lambda_{r-p+1}) & \text{if } p < r < 2p. \end{cases}$$

For $0 \leq i \leq r$ we have that $L(\varphi, i+1)$ is the coefficient of λ_i, and by inspection we have, for $r < 2p$,

$$L(\varphi, i+1) = 0$$

for $i \not\equiv 0, r \pmod{p-1}$. This proves (a).

To prove (b) we write $\lambda = \sum_{i=0}^r a_i \lambda_i$ with coefficients in R and let m be the smallest index for which $a_m \neq 0$.

We remark that

$$\lambda_i | T_p = \lambda_i(p^{i+1} + p^{r-i}) + \sum_{j=i+1}^{r} (-1)^{j-i} \lambda_j \binom{r-i}{r-j} \sum_{k=0}^{p-1} k^{j-i}.$$

Since $\varphi | T_p = \varphi$ we have

$$\lambda = (\varphi | T_p)(\infty) = \left(\varphi | \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)\right)(\infty) + \sum_{k=0}^{p-1} \left(\varphi | \left(\begin{smallmatrix} 1 & k \\ 0 & p \end{smallmatrix}\right)\right)(\infty) = \lambda \left| \left(\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) + \sum_{k=0}^{p-1} \left(\begin{smallmatrix} 1 & k \\ 0 & p \end{smallmatrix}\right) \right) \right. .$$

Comparing the coefficients of λ_m on both sides of this equation we obtain $p^{m+1} + p^{r-m} = 1$ in the ring R. But since $p = 0$ in R this can only happen if $r = m$. □

Definition 7.7. For $\psi \in H^1_{par}(\Gamma, V_r(R))$, with $r < 2p$, we choose $\widetilde{\psi} \in \mathrm{Symb}_\Gamma(V_r(R))$ to be an arbitrary lift of ψ and define

$$L(\psi, i+1) = L(\widetilde{\psi}, i+1) \ \text{ for } 0 \leq i \leq r \text{ with } i \not\equiv 0, r \pmod{p-1}.$$

We call these the special L-values of ψ.

By (a) of Proposition 7.6, these special L-values are well defined, independent of the choice of $\widetilde{\psi}$. In fact, by (b) of the proposition, if $\widetilde{\psi} | T_p = \widetilde{\psi}$, then *all* of the L-values $L(\psi, i+1)$ are well defined in the range $0 < i < r$.

8. Proof of Theorem 1.2

In this section we prove Theorem 1.2 of the introduction. We take $n = 1$ and suppress the subscript n from the notation. Thus $\zeta = \zeta_1$ is a primitive p^{th} root of unity, $R = R_1 = \mathbb{Z}\left[\zeta, \frac{1}{p}\right]$, $G = G_1$ is the Galois group of $\mathbb{Q}(\zeta)/\mathbb{Q}$, $\phi = \phi_1$, $\varphi = \varphi_1$, $e = e_1$, and $X = X_1 = (\mathbb{F}_p^2)'$, and for arbitrary d we let $I_d = I_{\omega^d}(\mathbb{F}_p)$, $V_d = V_d(\mathbb{F}_p)$, and $W_d = W_d(\mathbb{F}_p)$.

We also let $k \geq 2$ be an even integer and set $g = 2 - k$. The semigroup $M_2^+(\mathbb{Z})$ acts on W_{k-2} and I_d (for any d) by

$$\begin{aligned} (F|\sigma)(X, Y) &:= F((X, Y)\sigma'), \\ (f|\sigma)(\mathbf{x}) &:= f(\mathbf{x}\sigma') \end{aligned}$$

for $F \in W_{k-2}$, $f \in I_d$, and $\sigma \in M_2^+(\mathbb{Z})$, where in the latter case we take $f(\mathbf{x}\sigma') = 0$ in case $\mathbf{x}\sigma' = 0$.

We will use the notation $M(g) := M \otimes \det^g$ for any $M_2^+(\mathbb{Z})$-module M.

The pairing

$$\begin{aligned} I_{k-2} \times I_g &\longrightarrow \mathbb{F}_p \\ (f_1, f_2) &\longmapsto \sum_{\mathbf{x}\in X} f_1(\mathbf{x}) f_2(\mathbf{x}) \end{aligned}$$

can be seen to induce the isomorphism

$$I_g \cong I_{k-2}^*(g).$$

There is a natural map of $M_2^+(\mathbb{Z})$-modules: $W_{k-2} \longrightarrow I_{k-2}$ defined by sending a polynomial to the function it represents. In [AS86] it is shown that this map is injective if $k-2 < p$ and is surjective otherwise. By duality we obtain a natural map

$$I_g \xrightarrow{\beta} V_{k-2}(g),$$

which is surjective if $k-2 < p$ and injective otherwise.

One can easily check that the map β is given explicitly by

$$\beta(f) = \sum_{i=0}^{k-2} \lambda_i \sum_{(x,y)\in X} (-1)^i x^{k-2-i} y^i f(x,y),$$

where the λ_i's were defined in the previous section.

For more details, see Lemma 3.2 of [AS86].

We now turn to the proof of Theorem 1.2. So, suppose $2 \le k \le 2p$ and that Hypothesis H_k from the introduction is satisfied. Thus we have a G-equivariant map

$$\rho : K_2(R) \to \mathbb{F}_p(\omega^g)$$

and an odd integer $i \neq p$ with $1 < i < k-1$ such that $\rho(\xi_i) \neq 0$. Here $\xi_i := \{\eta_{k-i}, \eta_i\}$ as in the introduction.

We let $\Gamma := SL_2(\mathbb{Z})$ and define the map

$$\lambda : H_c^1(\Gamma_0, \widetilde{K}_2(R)) \longrightarrow H_c^1(\Gamma, V_{k-2})$$

to be the composition

$$\lambda : H_c^1(\Gamma_0, \widetilde{K}_2(R)) \xrightarrow{Sh_\rho} H_c^1(\Gamma, I_g) \xrightarrow{\nu} H_c^1(\Gamma, V_{k-2}),$$

where Sh_ρ is the map induced by ρ and the Shapiro isomorphism, and ν is the composition of β and the "twist map"

$$H_c^1(\Gamma, V_{k-2}(g)) \xrightarrow{\tau} H_c^1(\Gamma, V_{k-2}),$$

which is defined as the identity map on the underlying cohomology groups. Note, however, that τ does not commute with the action of the Hecke operators T_m. Indeed, we have $\tau(\varphi|T_m) = m^g \cdot \tau(\varphi)|T_m$ for every $\varphi \in H_c^1(\Gamma, V_{k-2}(g))$.

Now let ϕ be the modular symbol defined in Section 5 and set

$$\widetilde{\psi} := \lambda(\phi) \in H_c^1(\Gamma, V_{k-2}) \quad \text{and} \quad \psi := \text{the image of } \widetilde{\psi} \text{ in } H_{par}^1(\Gamma, V_{k-2}).$$

It is proved in [AS86] that Sh_ρ is Hecke equivariant, so from the last paragraph we have

$$m^g \cdot \widetilde{\psi}|T_m = \lambda(\phi|T_m)$$

for every $m \in \mathbb{N}$. In particular, for $q = 2, 3$ we have

$$\begin{aligned} \psi|T_q &= q^{k-2}\lambda\left((q+\chi(q))\cdot\phi\right) = q^{k-2}(q+q^g)\cdot\psi \\ &= (1+q^{k-1})\psi, \end{aligned}$$

proving (1) of Theorem 1.2.

To compute the special L-values $L(\psi, i+1)$ for $2 \le i < k-2$ we use Proposition 7.4 to conclude that $\Lambda(Sh_\rho(\phi)) = \rho \circ e$, where e was defined in Theorem 5.1. Thus $\Lambda(Sh_\rho(\phi))$ is the function

$$\begin{array}{rccl} \Lambda(Sh_\rho(\phi)) & : & X & \longrightarrow & \mathbb{F}_p \\ & & (x,y) & \longmapsto & \rho\big(\{1-\zeta^x, 1-\zeta^y\}\big) \text{ if } xy \neq 0. \end{array}$$

Then $\Lambda(\widetilde{\psi}) = \beta\left(\Lambda(Sh_\rho(\phi))\right)$. Thus, from the explicit description of the map β, we have

$$\Lambda(\widetilde{\psi}) = \sum_{i=0}^{k-2}\lambda_i \sum_{(x,y)\in X}(-1)^i x^{k-2-i}y^i\rho(e(x,y)),$$

which by the definition of L-values (Definition 7.2) means

$$\begin{aligned} (-1)^i L(\widetilde{\psi}, i+1) &= \sum_{(x,y)\in(\mathbb{F}_p^\times)^2} y^i x^{k-2-i}\rho\big(\{1-\zeta^x, 1-\zeta^y\}\big) \\ &= \rho\left(\left\{\prod_{x\in\mathbb{F}_p^\times}(1-\zeta^x)^{x^{k-2-i}}, \prod_{y\in\mathbb{F}_p^\times}(1-\zeta^y)^{y^i}\right\}\right) \\ &= \rho\left(\left\{\prod_{\sigma\in G}(1-\zeta)^{\omega^{k-2-i}(\sigma)\sigma}, \prod_{\sigma\in G}(1-\zeta)^{\omega^i(\sigma)\sigma}\right\}\right). \end{aligned}$$

Finally, we recall that $\xi_j \in K_2(R)$ was defined in the Introduction as $\xi_j := \{\eta_{k-j}, \eta_j\}$ where $\eta_j \in E_j$ is the projection of $(1-\zeta)$ to $E_j := E^{(1-j)}$. Recalling that the idempotent projecting to $E^{(1-j)}$ is $\frac{1}{p-1}\sum_{\sigma\in G}\omega^{j-1}(\sigma)\sigma$, we conclude that

$$\eta_j^{-1} = \prod_{\sigma\in G}(1-\zeta)^{\omega^{j-1}(\sigma)\sigma}.$$

Thus

$$(-1)^i L(\widetilde{\psi}, i+1) = \rho\left(\left\{\eta_{k-1-i}^{-1}, \eta_{i+1}^{-1}\right\}\right) = \rho(\xi_{i+1})$$

for $i = 0, \ldots, k-2$. The L-values of ψ are the same as those for $\widetilde{\psi}$, but with the values at $i+1$ with $i \equiv 0, k-2 \pmod{p-1}$ excluded. Moreover, $\xi_j = 0$ for even j, so for $2 \le i < k-2$, $i \neq p$, we have proved

$$L(\psi, i) = \begin{cases} \rho(\xi_i) & \text{if } i \text{ is odd;} \\ 0 & \text{otherwise.} \end{cases}$$

This completes the proof of Theorem 1.2.

9. Sharifi's Conjecture

Let f be a weight $k \ge 2$ cusp form whose Fourier expansion is given by $f(z) = \sum_{n\ge1} a_n e^{2\pi i n z}$. Let $L(f, s)$ be the complex L-function of f. Then, f gives rise to a class $\phi_f \in H_c^1(SL_2(\mathbb{Z}), V_{k-2}(\mathbb{C}))$ given by

$$\phi_f((x)-(y)) = \int_y^x f(z)(zX+Y)^{k-2}dz.$$

It is well known that $L(\phi_f, \alpha) = \frac{(-1)^{\alpha-1}(\alpha-1)!}{(2\pi i)^\alpha} L(f, \alpha)$ for all integers α with $1 \leq \alpha \leq k-1$.

We will denote by ψ_f the parabolic cohomology class associated to ϕ_f in $H^1_{par}(SL_2(\mathbb{Z}), V_{k-2}(\mathbb{C}))$.

Now let p be an irregular prime and choose k, $2 \leq k < p$, such that $p \mid \frac{B_k}{k}$. Let f be a normalized weight k newform of level 1 and let $\mathcal{O}_f$ be the ring of integers of the number field K_f generated by the Fourier coefficients of f. Let $\wp$ be a place above p and assume $f \equiv G_k \pmod{\wp}$, where G_k is the Eisenstein series of level 1 and weight k: $G_k = -\frac{B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$ with $\sigma_{k-1}(n) = \sum_{d|n, d>0} d^{k-1}$.

The following is a Conjecture of Romyar Sharifi, which was mentioned in this form in [Sh1-04].

Conjecture (Sharifi). *Under the above hypotheses, there exists $\Omega \in \mathbb{C}^\times$ such that $\frac{L(\psi_f, i)}{\Omega} \in \mathcal{O}_f$ for all odd i in the range $3 \leq i \leq k-3$, at least one of these numbers is non-zero modulo $\wp$, and there exists ρ such that hypothesis H_k holds for the irregular pair (p, k) and*

$$\frac{L(\psi_f, i)}{\Omega} \equiv \rho(\xi_i) \pmod{\wp}$$

for all odd i in the range $3 \leq i \leq k-3$.

We remark that the computations of McCallum and Sharifi (see [MS03], Theorem 5.1, [MS]) imply that the space $H^+_{k,eis,\{2,3\}}$ defined in the Introduction is at most one-dimensional for all irregular pairs (p, k) with $p < 10{,}000$. In fact, McCallum and Sharifi only compute the equivalent of the eigenvalue of T_2, so in this range, we even have the stronger statement

$$\dim_{\mathbb{F}_p} \left(H^+_{k,eis,\{2\}} \right) \leq 1.$$

In [Sh4-06], the above Conjecture is claimed to be true for any k. However, our methods do not give any information on the L-values corresponding to the points in which we encounter interferences with the boundary symbols (see the proof of Theorem 1.2).

References

[AS86] Ash, A., Stevens, G.: Modular forms in characteristic ℓ and special values of their L-functions, *Duke Mathematical Journal* **53**, No. 3, 1986. MR860675 (88h:11036)

[GS91] Greenberg, R., Stevens, G.: On the conjecture of Mazur, Tate, and Teitelbaum, in *p-adic Monodromy and the Birch and Swinnerton-Dyer Conjecture (Contemporary Mathematics* **165)**, B. Mazur and G. Stevens, eds., 1991. MR1279598 (94m:11006)

[Ma72] Manin, J.: Parabolic Points and Zeta Functions of Modular Curves, *Math. USSR Izvestija* **36**, No. 1, 1972, 19–66. MR0314846 (47:3396)

[Ma73] Manin, J.: Periods of parabolic forms and p-adic Hecke series, *Math. USSR Sbornik* **21**, No. 3, 1973.

[MS03] McCallum, W., Sharifi, R.: A Cup Product in Galois Cohomology, *Duke Mathematical Journal* **120**, No. 2, 2003. MR2019977 (2004j:11136)

[MS] McCallum, W., Sharifi, R.: *Magma routines for computing the table of pairings for $p <$* 1000, http://abel.math.harvard.edu/∼sharifi/computations.html, http://math.arizona.edu/∼wmc 284.

[Mer94] Merel, L.: Universal Fourier expansions of modular forms, *On Artin's conjecture for odd 2-dimensional representations*, Springer, Berlin, 1994, pp. 59-94. MR1322319 (96h:11032)

[Mi71] Milnor, J.: Introduction to Algebraic K-theory, *Annals of Mathematics Studies* **72**, Princeton University Press, 1971. MR0349811 (50:2304)

[Oh03] Ohta, M.: Congruence modules related to Eisenstein series, *Ann. Scient. École Norm. Sup.*, 4^{e} série **36** (2003), 225-269. MR1980312 (2004d:11045)

[Sh1-04] Sharifi, R.: The various faces of a pairing on p-units, slides from a talk at International Univ. Bremen on 5/10/04, http://www.math.mcmaster.ca/∼ sharifi/bremen.pdf.

[Sh2-04] Sharifi, R.: Computations on Milnor's K_2 of Integer Rings, slides from a talk at Max Planck Institute of Mathematics on 5/17/04, http://www.math.mcmaster.ca/∼sharifi/dagslides.pdf.

[Sh3-05] Sharifi, R.: Iwasawa Theory and the Eisenstein Ideal, Duke Math. J. **137** (2007), 63–101. MR2309144

[Sh4-06] Sharifi, R.: Cup Products and L-values of Cusp Forms, preprint.

[St89] Stevens, G.: The Eisenstein measure and real quadratic fields, in *The Proceedings of the International Number Theory Conference (Université Laval, 1987)*, J.-M. De Koninck and C. Levesque, eds., de Gruyter, 1989. MR1024612 (90m:11077)

[W97] Washington, L.C.: Introduction to Cyclotomic Fields, *Graduate Texts in Mathematics* **83**, Springer, New York, 1997. MR1421575 (97h:11130)

Department of Mathematics, Boston University, 111 Cummington Street, Boston, Massachusetts 02215

E-mail address: `celiab@math.bu.edu`