The smallest number of vertices in a 2-arc-strong digraph without pair of arc-disjoint in- and out-branchings ⋆

Ran Gu[1], Gregory Gutin[2], Shasha Li[3], Yongtang Shi[4], and Zhenyu Taoqiu[4]

[1] College of Science, Hohai University, Nanjing, Jiangsu Province 210098, P.R. China
rangu@hhu.edu.cn

[2] Department of Computer Science, Royal Holloway, University of London, Egham, Surrey, TW20 0EX, UK g.gutin@rhul.ac.uk

[3] Department of Mathematics, Ningbo University, Ningbo 315211, Zhejiang, China
yezi_pg@163.com

[4] Center for Combinatorics and LPMC, Nankai University, Tianjin 300071, China
shi@nankai.edu.cn, tochy@mail.nankai.edu.cn

Abstract. Branchings play an important role in digraph theory and algorithms. In particular, a chapter in the monograph of Bang-Jensen and Gutin, Digraphs: Theory, Algorithms and Application, Ed. 2, 2009 is wholly devoted to branchings. The well-known Edmonds Branching Theorem provides a characterization for the existence of k arc-disjoint out-branchings rooted at the same vertex. A short proof of the theorem by Lovász (1976) leads to a polynomial-time algorithm for finding such out-branchings. A natural related problem is to characterize digraphs having an out-branching and an in-branching which are arc-disjoint. Such a pair of branchings is called a good pair.

Bang-Jensen, Bessy, Havet and Yeo (2020) pointed out that it is NP-complete to decide if a given digraph has a good pair. They also showed that every digraph of independence number at most 2 and arc-connectivity at least 2 has a good pair, which settled a conjecture of Thomassen for digraphs of independence number 2. Then they asked for the smallest number n_{ngp} of vertices in a 2-arc-strong digraph which has no good pair. They proved that $7 \le n_{ngp} \le 10$. In this paper, we prove that $n_{ngp} = 10$, which solves the open problem.

Keywords: Arc-disjoint branchings · out-branching · in-branching · arc-connectivity.

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1 Introduction

Let $D = (V, A)$ be a digraph. For a non-empty subset $X \subset V$, the *in-degree* (resp. *out-degree*) of the set X, denoted by $d_D^-(X)$ (resp. $d_D^+(X)$), is the number of arcs with head (resp. tail) in X and tail (resp. head) in $V \setminus X$. The *arc-connectivity* of D, denoted by $\lambda(D)$, is the minimum out-degree of a proper subset of vertices. A digraph is *k-arc-strongly connected* (or, just *k-arc-strong*) if $\lambda(D) \geq k$. In particular, a digraph is *strongly connected* (or, just *strong*) if $\lambda(D) \geq 1$.

An *out-branching* (*in-branching*) of a digraph $D = (V, A)$ is a spanning tree in the underlying graph of D whose edges are oriented in D such that every vertex except one, called the *root*, has in-degree (out-degree) one. Branchings play an important role in digraph theory and algorithms. In particular, Chapter 9 in the monograph [5] is wholly devoted to branchings. The well-known Edmonds Branching Theorem (see e.g. [5]) provides a characterization for the existence of k arc-disjoint out-branchings rooted at the same vertex. A short proof of the theorem by Lovász [11] leads to a polynomial-time algorithm for finding such out-branchings. A natural related problem is to characterize digraphs having an out-branching and an in-branching which are arc-disjoint. Such a pair of branchings is called a *good pair*.

Thomassen [12] conjectured the following:

Conjecture 1. There is a constant c, such that every digraph with arc-connectivity at least c has a good pair.

He also proved that it is NP-complete to decide whether a given digraph D has an out-branching and an in-branching both rooted at the same vertex such that these are arc-disjoint. This implies that it is NP-complete to decide if a given digraph has a good pair [2]. Conjecture 1 has been verified for semicomplete digraphs [1] and their genearlizations: locally semicomplete digraphs [7] and semicomplete compositions [6] (it follows from the main result in [6]).

An out-branching and an in-branching of D are *k-distinct* if each of them has at least k arcs, which are absent in the other. Bang-Jensen et al. [8] proved that the problem of deciding whether a strongly connected digraph D has k-distinct out-branching and in-branching is fixed-parameter tractable when parameterized by k. Settling an open problem in [8], Gutin et al. [10] extended this result to arbitrary digraphs.

In [2], Bang-Jensen et al. showed that every digraph of independence number at most 2 and arc-connectivity at least 2 has a good pair, which settles the conjecture for digraphs of independence number 2.

Theorem 1. *If D is a digraph with $\alpha(D) \leq 2 \leq \lambda(D)$, then D has a good pair.*

Moreover, they also proved that every digraph on at most 6 vertices and arc-connectivity at least 2 has a good pair and gave an example of a 2-arc-strong digraph D on 10 vertices with independence number 4 that has no good pair. They posed the following open problem.

Problem 1 ([2]). What is the smallest number n of vertices in a 2-arc-strong digraph which has no good pair?

In this paper, we prove that every digraph on at most 9 vertices and arc-connectivity at least 2 has a good pair, which answers this problem. The main results of the paper are as follows.

Theorem 2. *Every 2-arc-strong digraph on 7 vertices has a good pair.*

Theorem 3. *Every 2-arc-strong digraph on 8 vertices has a good pair.*

Theorem 4. *Every 2-arc-strong digraph on 9 vertices has a good pair.*

This paper is organised as follows. In the rest of this section, we provide further terminology and notation on digraphs. Undefined terms can be found in [4, 5]. In Section 2, we outline the proofs of Theorems 2, 3 and 4 and state some auxiliary lemmas which we use in their proofs. Section 3 contains a number of technical lemmas which will be used in proofs of our main results. Then we respectively devote one section for proofs of each theorem and its relevant auxiliary lemmas. The proofs not given in this paper due to the space limit can be found in [9].

Additional Terminology and Notation. For a positive integer n, $[n]$ denotes the set $\{1, 2, \dots, n\}$. Throughout this paper, we will only consider digraphs without loops and multiple arcs. Let $D = (V, A)$ be a digraph. We denote by uv the arc whose *tail* is u and whose *head* is v. Two vertices u, v are *adjacent* if at least one of uv and vu belongs to A. If u and v are adjacent, then we also say that u is a *neighbour* of v and vice versa. If $uv \in A$, then v is called an *out-neighbour* of u and u is called an *in-neighbour* of v. Moreover, we say uv is an *out-arc* of u and an *in-arc* of v and that u *dominates* v. The *order* $|D|$ of D is $|V|$.

In this paper, we will extensively use *digraph duality*, which is as follows. Let D be a digraph and let D^{rev} be the *reverse* of D, i.e., the digraph obtained from D by reversing every arc xy to yx. Clearly, D contains a subdigraph H if and only if D^{rev} contains H^{rev}. In particular, D contains a good pair if and only if D^{rev} contains a good pair.

Let $N_D^-(X) = \{y : yx \in A, x \in X\}$ and $N_D^+(X) = \{y : xy \in A, x \in X\}$. Note that X may be just a vertex. For two non-empty disjoint subsets $X, Y \subset V$, we use $N_Y^-(X)$ to denote $N_D^-(X) \cap Y$ and $d_Y^-(X) = |N_Y^-(X)|$. Analogously, we can define $N_Y^+(X)$ and $d_Y^+(X)$. For two non-empty subsets $X_1, X_2 \subset V$, define $(X_1, X_2)_D = \{v_1v_2 \in A \colon v_1 \in X_1 \text{ and } v_2 \in X_2\}$ and $[X_1, X_2]_D = (X_1, X_2)_D \cup (X_2, X_1)_D$. We will drop the subscript when the digraph is clear from the context.

We write $D[X]$ to denote the subdigraph of D induced by X. A *clique* in D is an induced subdigraph $D[X]$ such that any two vertices of X are adjacent. We say that D contains K_p if it has a clique on p vertices. A vertex set X of D is *independent* if no pair of vertices in X are adjacent. A dipath (dicycle) of D with t vertices is denoted by P_t (C_t). We drop the subscript when the order is not specified. A dipath P from v_1 to v_2, denoted by $P_{(v_1,v_2)}$, is often called a (v_1, v_2)-*dipath*. A dipath P is a *Hamilton* dipath if $V(P) = V(D)$. We call C_2 a *digon*. A digraph without digons is called an *oriented graph*. If two digons have and only have one common vertex, then we call this structure a *bidigon*. A *semicomplete* digraph is a digraph D that each pair of vertices has an arc between them. A *tournament* is a semicomplete oriented graph.

In- and out-branchings were defined above. An *out-tree* (*in-tree*) is an out-branching (in-branching) of a subdigraph of D. We use B_s^+ (B_t^-) to denote an out-branching rooted at s (an in-branching rooted at t). The root s (t) is called *out-generator* (*in-generator*) of D. We denote by $\mathrm{Out}(D)$ ($\mathrm{In}(D)$) the set of out-generators (in-denerators) of D. If the root is not specified, then we drop the subscripts of B_s^+ and B_t^-. We also use O_D (I_D) to denote an out-branching (in-branching) of a digraph D. If O_D and I_D are arc-disjoint, then we write (O_D, I_D) to denote a good pair in D.

2 Proofs Outline

In this section, we outline constructions we use to prove our main results. We prove each of them by contradiction. We give the statements of some auxiliary lemmas. Some of their proofs are too complicated and we will not give them in the paper due to the length restriction. For simplicity, when outlining the proof of our main results, we assume that $|D_1| = 7$, $|D_2| = 8$ and $|D_3| = 9$.

2.1 Theorem 2

First we show that the largest clique in D_1 is a tournament by Lemma 6, next we prove that D_1 is an oriented graph in Claim 2.1 by Lemma 7. Lemmas 6 and 7 will be given in Section 3. Then we use Proposition 12 to show that D_1 has a Hamilton dipath in Section 4. After that, we prove that D_1 has a good pair by Propositon 10, which is shown in Section 3.

2.2 Theorem 3

Our proof will follow three steps.

Firstly, we prove that the largest clique R in D_2 has 3 vertices by Lemma 6. Then we show that R is a tournament through Claim 3.1, which is proved by Lemmas 6 and 7.

Our second step is to prove that D_2 is an oriented graph in Claim 3.2 by Lemmas 8, 9 and 10, which are given in Section 3.

In the last step, we proceed as follows in Section 5. We use Proposition 15 to show that D_2 has a Hamilton dipath. To prove it, we show Proposition 14 first. After that, we prove that D_2 has a good pair by Propositon 10.

2.3 Theorem 4

Our proof will follow four steps.

Firstly, we show that the largest clique R in D_3 has 3 vertices by Claim 4.1, which is proved using Proposition 5 given in Section 3, and Lemmas 6 and 7.

Next we show that R has no digons by Claim 4.2, which is proved analogously to Claim 3.1 using Lemmas 7, 8, 9 and 10.

Our third step is to show that D_3 is an oriented graph in Claim 4.3. To do this we need Lemmas 11 and 13 given in Section 6. For the first lemma, we give a generalization of Proposition 6 as Proposition 16, and for the second one, we prove Lemma 12 first.

Then we use Lemma 14 to show that D_3 has a Hamilton dipath in Section 6. To prove it, we show Proposition 17 first. After that, we prove that D_3 has a good pair by Proposition 10.

3 Preliminaries and useful lemmas

Proposition 5 *Let D be a digraph with $\lambda(D) \geq 2$ and with a good pair (B_s^+, B_s^-). If there exists a vertex t in D such that $D[\{s,t\}]$ is a digon, then D has a good pair (B_t^+, B_t^-).*

Proof. Let $B_t^+ = ts + B_s^+ - e_1$ and $B_t^- = B_s^- + st - e_2$, where e_1 (e_2) is the only in-arc (out-arc) of t in B_s^+ (B_s^-). Observe that B_t^+ (B_t^-) is an out-branching (in-branching) rooted at t in D. Since the root of any out-branching has in-degree zero, if $ts \in B_s^+ \cup B_s^-$, then ts must be in B_s^- and moreover ts is the only out-arc e_2 of t in B_s^-. Similarly, if $st \in B_s^+ \cup B_s^-$, then st must be in B_s^+ and moreover st is the only in-arc e_1 of t in B_s^+. Thus, B_t^+ and B_t^- are arc-disjoint and so (B_t^+, B_t^-) is a good pair of D.

Proposition 6 *Let D be a digraph with a subdigraph Q that has a good pair (O_Q, I_Q). Let $X = N_D^-(Q)$ and $Y = N_D^+(Q)$ with $X \cap Y = \emptyset$ and $X \cup Y = V - V(Q)$. Let X_i (Y_j) be the initial (terminal) strong components in $D[X]$ ($D[Y]$), $i \in [a]$ ($j \in [b]$). If one of the following holds, then D has a good pair. Meanwhile, we can always get two arc-disjoint $\mathcal{P}_X, \mathcal{P}_Y$ and respectively an out- and an in-forest T_X and T_Y in D.*

1. $d_Y^-(X_1) \geq 1,\ d_Y^-(X_i) \geq 2,\ i \in \{2, \ldots, a\}$ *and* $d_X^+(Y_j) \geq 2,\ j \in [b]$.
2. $d_X^+(Y_1) \geq 1,\ d_X^+(Y_j) \geq 2,\ j \in \{2, \ldots, b\}$ *and* $d_Y^-(X_i) \geq 2,\ i \in [a]$.

Proof. Let B^+ be an out-tree containing O_Q and an in-arc of any vertex in Y from Q. Let B^- be an in-tree containing I_Q and an out-arc of any vertex in X to Q. Set $\mathcal{X} = \{X_i,\ i \in [a]\}$ and $\mathcal{Y} = \{Y_j,\ j \in [b]\}$. By the digraph duality, it suffices to prove that condition 1 implies that D has a good pair.

Now assume that $d_Y^-(X_1) \geq 1$, $d_Y^-(X_i) \geq 2$, $i \in \{2, \ldots, a\}$, and $d_X^+(Y_j) \geq 2$, $j \in [b]$. Then there are at least two arcs from Y_j (for each $j \in [b]$) to X, at least two arcs from Y to X_i (for each $i \in \{2, \ldots, a\}$) and at least one arc from Y to X_1. Set $X'_1 = X_1$. If there is an arc y^1x_1 from Y to X'_1 with y^1 in some Y_j, $j \in [b]$, then we choose such an arc and let $Y'_1 = Y_j$, otherwise we choose an arbitrary arc y^1x_1 from Y to X'_1 and let Y'_1 be an arbitrary strong component in $\mathcal{Y}$. Let $\mathcal{P}_X = \{y^1x_1\}$. There now exists an arc, y_1x^1, out of Y'_1 ($x^1 \in X$) which is different from y^1x_1 (as Y'_1 has at least two arcs out of it). If there is such an arc y_1x^1 with x^1 in some X_i, $i \in \{2, \dots, a\}$, then we choose one of these arcs and let $X'_2 = X_i$, otherwise we choose such an arbitrary arc y_1x^1 out of Y'_1 ($x^1 \in X$) and let X'_2 be an arbitrary strong component in $\mathcal{X} - X'_1$. Let $\mathcal{P}_Y = \{y_1x^1\}$. Likewise, for $t \geq 2$, we get an arc y^tx_t into X'_t ($y^t \in Y$) which is different from $y_{t-1}x^{t-1}$ in $\mathcal{P}_Y$. If there is such an arc y^tx_t with y^t in some $Y_j \in \mathcal{Y} - \{Y'_1, \dots, Y'_{t-1}\}$, then choose one of these arcs and let $Y'_t = Y_j$, otherwise we choose such an arbitrary arc y^tx_t and let Y'_t be an arbitrary strong component in $\mathcal{Y} - \{Y'_1, \dots, Y'_{t-1}\}$. Add y^tx_t to $\mathcal{P}_X$. For $s \geq 2$, we get an arc y_sx^s out of Y'_s ($x^s \in X$) which is different from y^sx_s in $\mathcal{P}_X$. If there is such an arc y_sx^s with x^s in some $X_i \in \mathcal{X} - \{X'_1, \dots, X'_{s-1}\}$, then we choose one of these arcs and let $X'_s = X_i$, otherwise we choose such an arbitrary arc y_sx^s and let X'_s be an arbitrary strong component in $\mathcal{X} - \{X'_1, \dots, X'_{s-1}\}$. Add y_sx^s to $\mathcal{P}_Y$. Hence we get two arc sets $\mathcal{P}_X$ and $\mathcal{P}_Y$ with $\mathcal{P}_X \cap \mathcal{P}_Y = \emptyset$.

We will now show that D has a good pair. Let D_X be the digraph obtained from $D[X]$ by adding one new vertex y^* and arcs from y^* to x_i for $i \in [a]$. Analogously let D_Y be the digraph obtained from $D[Y]$ by adding one new vertex x^* and arcs from y_j to x^* for $j \in [b]$. Since Out(D_X)= $\{y^*\}$ and In(D_Y)= $\{x^*\}$, there exists an out-branching $B^+_{y^*}$ in D_X and an in-branching $B^-_{x^*}$ in D_Y. Set $T_X = B^+_{y^*} - y^*$ and $T_Y = B^-_{x^*} - x^*$.

By construction, (O_D, I_D) is a good pair of D with $O_D = B^+ + \mathcal{P}_X + T_X$ and $I_D = B^- + \mathcal{P}_Y + T_Y$.

Corollary 1. *Let D be a digraph with $\lambda(D) \geq 2$ that contains a subdigraph Q with a good pair. Set $X = N^-_D(Q)$ and $Y = N^+_D(Q)$. If $X \cap Y = \emptyset$ and $X \cup Y = V - V(Q)$, then D has a good pair.*

Proof. Let X_i be the initial strong components in $D[X]$ and Y_j be the terminal strong components in $D[Y]$, $i \in [a]$ and $j \in [b]$. Since $\lambda(D) \geq 2$, $d^-_Y(X_i) \geq 2$ and $d^+_X(Y_j) \geq 2$, for any $i \in [a]$ and $j \in [b]$, which implies that D has a good pair by Proposition 6.

Lemma 1 ([2]). *Let D be a digraph and $X \subset V(D)$ be a set such that every vertex of X has both an in-neighbour and an out-neighbour in $V - X$. If $D - X$ has a good pair, then D has a good pair.*

By Lemma 1, in this paper we will often use the fact that if Q is a maximal subdigraph of D with a good pair and $X = N^-_D(Q), Y = N^+_D(Q)$, then $X \cap Y = \emptyset$.

Lemma 2. *Let D be a 2-arc-strong digraph containing a subdigraph Q with a good pair, $X = N^-_D(Q)$ and $Y = N^+_D(Q)$. If $X \cap Y = \emptyset$ and $X \cup Y = V - V(Q) - \{w\}$, where $w \in V - V(Q)$, then D has a good pair.*

Proof. Assume that Q has a good pair (O_Q, I_Q). Let B^+ be an out-tree containing O_Q with an in-arc of any vertex in Y from Q, while B^- be an in-tree containing I_Q with an out-arc of any vertex in X to Q.

First assume that either $(Y,w)_D \neq \emptyset$ or $(w,X)_D \neq \emptyset$. By the digraph duality, we may assume that $(Y,w)_D \neq \emptyset$, i.e., there exists an arc e from Y to w in D. Let $D' = D - e$. Set $X' = N^-_{D'}(Q) = X$ and $Y' = N^+_{D'}(Q) \cup \{w\} = Y \cup \{w\}$. Let X'_i be the initial strong components in $D'[X']$ and Y'_j be the terminal strong components in $D'[Y']$, $i \in [a]$ and $j \in [b]$. If w has an in-neighbour v in Y with v in some Y'_j, $j \in [b]$, then let $e = vw$ and $Y^*_1 = Y'_j$, otherwise we choose an arbitrary in-neighbour v of w in Y and let $e = vw$ and Y^*_1 be an arbitrary terminal strong component of $D'[Y']$. Since $\lambda(D) \geq 2$, $d^+_{X'}(Y^*_1) \geq 1$, $d^+_{X'}(Y'_j) \geq 2$ and $d^-_{Y'}(X'_i) \geq 2$, for any $Y'_j \neq Y^*_1$, $j \in [b]$ and $i \in [a]$, which implies that we get arc sets $\mathcal{P}_{X'}$ and $\mathcal{P}_{Y'}$ with $\mathcal{P}_{X'} \cap \mathcal{P}_{Y'} = \emptyset$, and digraphs $T_{X'}$ and $T_{Y'}$ by Proposition 6. By construction, D has a good pair $(B^+ + \mathcal{P}_{X'} + T_{X'} + e, B^- + \mathcal{P}_{Y'} + T_{Y'})$.

Now assume that $(Y,w)_D = \emptyset$ and $(w,X)_D = \emptyset$, which implies that $d^-_X(w) \geq 2$ and $d^+_Y(w) \geq 2$. Let X_i be the initial strong components in $D[X]$ and Y_j be the terminal strong components in $D[Y]$, $i \in [a]$ and $j \in [b]$. Since $\lambda(D) \geq 2$ and $(w,X)_D = (Y,w)_D = \emptyset$, $d^-_Y(X_i) \geq 2$ and $d^+_X(Y_j) \geq 2$ for any $i \in [a]$ and $j \in [b]$. By Proposition 6, we get $\mathcal{P}_X, T_X$ and $\mathcal{P}_Y, T_Y$ with $\mathcal{P}_X \cap \mathcal{P}_Y = \emptyset$. It follows that $(B^+ + \mathcal{P}_X + T_X + w^-w, B^- + \mathcal{P}_Y + T_Y + ww^+)$ is a good pair of D, where $w^- \in X$ and $w^+ \in Y$.

Proposition 7 ([2]) *Every digraph on 3 vertices has a good pair if and only if it has at least 4 arcs .*

Following [4], we shall use $\delta_0(D)$ to denote the *minimum semi-degree* of D, which is the minimum over all in- and out-degrees of vertices of D.

Proposition 8 ([2]) *Let D be a digraph on 4 vertices with at least 6 arcs except E_4 (see Fig. 1). If $\delta^0(D) \geq 1$ or D is a semicomplete digraph, then D has a good pair.*

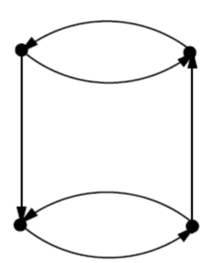

Fig. 1. E_4.

Lemma 3 ([5], p.354). *Let $D = (V, A)$ be a digraph. Then D is k-arc-strong if and only if it contains k arc-disjoint (s, t)-paths for every choice of distinct vertices $s, t \in V$.*

Lemma 4 (Edmonds' branching theorem [4]). *A directed multigraph $D = (V, A)$ with a special vertex z has k arc-disjoint out-branchings rooted at z if and only if $d^-(X) \geq k$ for all $\emptyset \neq X \subseteq V - z$.*

Lemma 5 ([2]). *If D is a 2-arc-strong digraph on n vertices that contains a subdigraph on $n-3$ vertices with a good pair, then D has a good pair.*

Lemma 6. *If D is a 2-arc-strong digraph on n vertices that contains a subdigraph Q on $n-4$ vertices with a good pair, then D has a good pair.*

Lemma 7. *Let D be a 2-arc-strong digraph on n vertices that contains a subdigraph Q on $n-5$ vertices with a good pair, $X = N_D^-(Q)$, $Y = N_D^+(Q)$ and $X \cap Y = \emptyset$. If $|X| \geq 2$ or $|Y| \geq 2$, then D has a good pair.*

Lemma 8. *Let D be a 2-arc-strong digraph on n vertices that contains a subdigraph Q on $n-6$ vertices with a good pair. Let $X = N_D^-(Q)$ and $Y = N_D^+(Q)$ with $X \cap Y = \emptyset$. If $|X| = |Y| = 2$ and at most one of X and Y is an independent set, then D has a good pair.*

Lemma 9. *Let $D = (V, A)$ be a 2-arc-strong digraph on n vertices that contains a subdigraph Q on $n-6$ vertices with a good pair. Set $X = N_D^-(Q) = \{x_1, x_2\}$ and $Y = N_D^+(Q) = \{y_1, y_2\}$ with $X \cap Y = \emptyset$, and $W = V - X - Y - V(Q) = \{w_1, w_2\}$. If X, Y are both independent sets, then D has a good pair except for the case below:*

(∗) *$(Y, X)_D = \{y_j x_i, y_{3-j} x_{3-i}\}$ for some $i, j \in [2]$, $D[W] = C_2$ and $N_W^+(y_j) \cap N_W^+(y_{3-j}) = N_W^-(x_i) \cap N_W^-(x_{3-i}) = \emptyset$ while $N_W^+(y_j) \cap N_W^-(x_i) \neq \emptyset$ and $N_W^+(y_{3-j}) \cap N_W^-(x_{3-i}) \neq \emptyset$.*

We use $D \supseteq E_3$ ($D \not\supseteq E_3$) to denote that D contains an arbitrary orientation (no orientation) of E_3 as a subdigraph. (E_3 is a mixed graph and only the two edges are to be oriented.) E_3 is shown in Fig. 2.

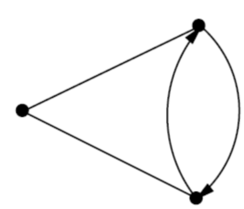

Fig. 2. E_3.

Lemma 10. *Let $D = (V, A)$ be a 2-arc-strong digraph on n vertices that contains a subdigraph Q on $n-6$ vertices with a good pair. Set $X = N_D^-(Q) = \{x_1, x_2\}$ and $Y = N_D^+(Q) = \{y_1, y_2\}$ with $X \cap Y = \emptyset$, and $W = V - X - Y - V(Q) = \{w_1, w_2\}$. If $n = 8$ or 9 and X, Y are both independent, then D has a good pair.*

Proposition 9 ([3]) *A digraph D has an out-branching (resp. in-branching) if and only if it has precisely one initial (resp. terminal) strong component. In that case every vertex of the initial (resp. terminal) strong component can be the root of an out-branching (resp. in-branching) in D.*

We use T_x^+ (resp. T_x^-) to denote an out-tree (resp. in-tree) rooted at x.

Proposition 10 *Let D be an oriented graph on n vertices. Let $P_D = x_1x_2 \ldots x_n$ be the Hamilton dipath of D and $D' = D - A(P)$. Assume that there are exactly two non-adjacent strong components I_1 and I_2 in D'. Set $q \in \{2, 3, n-1, n\}$. If for some q, x_{q-1} and x_q are respectively in I_1 and I_2, then D has a good pair.*

Proof. W.l.o.g., assume that $x_{q-1} \in I_1$ and $x_q \in I_2$. Since I_i is strong, $\delta^0(I_i) \geq 1$, for any $i \in [2]$.

First assume $q \in \{n-1, n\}$. Let x be an in-neighbour of x_q in I_2. We get an out-branching of D as $B_{x_1}^+ = P_D - x_{q-1}x_q + xx_q$. Then we will show that there is an in-branching B_x^- in $D - A(B_{x_1}^+)$. Since I_2 is strong, $I_2 - xx_q$ is connected and has only one terminal srong component which contains x. This implies that there is an in-branching T_x^- in $I_2 - xx_q$. Note that there exists an in-branching $T_{x_{q-1}}^-$ in I_1, as I_1 is strong. Then $B_x^- = T_x^- + x_{q-1}x_q + T_{x_{q-1}}^-$, which implies that $(B_{x_1}^+, B_x^-)$ is a good pair of D.

Now we assume $q \in \{2, 3\}$. Let y be an out-neighbour of x_{q-1} in I_1. We get an in-branching of D as $B_{x_n}^- = P_D - x_{q-1}x_q + x_{q-1}y$. Then we will show that there is an out-branching B_y^+ in $D - A(B_{x_n}^-)$. Since I_1 is strong, $I_1 - x_{q-1}y$ is connected and has only one initial srong component which contains y. This implies that there is an out-branching T_y^+ in $I_1 - x_{q-1}y$. Note that there exists an out-branching $T_{x_q}^+$ in I_2, as I_2 is strong. Then $B_y^+ = T_y^+ + x_{q-1}x_q + T_{x_q}^+$. So, $(B_y^+, B_{x_n}^-)$ is a good pair of D.

Proposition 11 *Let D be a 2-arc-strong oriented graph on at least seven vertices. Then D has a dipath P_6.*

Proof. Suppose that there is no P_6 in D. Assume that P_t is the longest dipath in D, then $t \geq 4$, as there is no digon in D and $\lambda(D) \geq 2$. Observe that there is no C_t in D, otherwise D has a longer dipath P_{t+1}.

First assume that $t = 4$ and set $P_4 = x_1x_2x_3x_4$. Since $d_D^+(x_4) \geq 2$ and D has no digon, the out-neighbourhood of x_4 either contains x_1 or contains a vertex in $V - V(P_4)$. This implies that there is a P_5 in D, a contradiction.

Henceforth we may assume that $t = 5$ and set $P_5 = x_1x_2x_3x_4x_5$. Since $\lambda(D) \geq 2$, $d_D^+(x_5) \geq 2$ and $d_D^-(x_1) \geq 2$. Then we get $N_D^+(x_5) = \{x_2, x_3\}$ and $N_D^-(x_1) = \{x_3, x_4\}$, as P_5 is the longest dipath in D and D has no digon. Observe that there exsits a different 4-length dipath, $x_4x_5x_3x_1x_2$, in D. Likewise, $N_D^+(x_2) = \{x_3, x_5\}$, which implies that $D[\{x_2, x_5\}]$ is a digon, a contradiction.

4 Good pairs in digraphs of order 7

Proposition 12 *A 2-arc-strong oriented graph D on n vertices has a P_7, where $7 \leq n \leq 9$.*

Proof. Suppose to the contraty that P is the longest dipath in D, where $|P| = 6$. Obviously D has no C_6 by Proposition 11. Set $P = x_1x_2x_3x_4x_5x_6$. Since $\lambda(D) = 2$, we have $d^+_D(x_6) \geq 2$ and $d^-_D(x_1) \geq 2$. Note that $N^+_D(x_6) \subseteq \{x_2, x_3, x_4\}$ and $N^-_D(x_1) \subseteq \{x_3, x_4, x_5\}$.

Assume first that $N^+_D(x_6) \cap N^-_D(x_1) = \emptyset$.

If $N^+_D(x_6) = \{x_2, x_3\}$ and $N^-_D(x_1) = \{x_4, x_5\}$, then we can get a new P_6 in D as $x_6x_2x_3x_4x_5x_1$. Likewise, we have $x_3 \in N^+_D(x_1)$ and $x_4 \in N^-_D(x_6)$. Then there exists a good pair $(B^+_{x_4}, B^-_{x_4})$ in $D[P]$ with $B^+_{x_4} = x_4x_1x_2 + x_4x_5 + x_4x_6x_3$ and $B^-_{x_4} = x_5x_6x_2x_3x_4 + x_1x_3$, which implies that D has a good pair by Lemma 5 and Lemma 1.

If $N^+_D(x_6) = \{x_2, x_4\}$ and $N^-_D(x_1) = \{x_3, x_5\}$, then we can get a new P_6 in D as $x_6x_4x_5x_1x_2x_3$. Likewise, we have $x_1 \in N^-_D(x_6)$, which implies that there is a C_6 as $x_6x_2x_3x_4x_5x_1x_6$, a contradiction.

Henceforth, we may assume that there is at least one common vertex in $N^+_D(x_6)$ and $N^-_D(x_1)$. Without loss of generality, assume that x_3 is one of the common vertices. The case when $x_4 \in N^+_D(x_6) \cap N^-_D(x_1)$ can be proved analogously by reversing all arcs of D. Then we can get a new P_6 in D as $x_4x_5x_6x_3x_1x_2$. Likewise, we have

$$N^+_{D-x_3}(x_2) \subseteq \{x_5, x_6\} \text{ and } N^-_{D-x_3}(x_4) \subseteq \{x_1, x_6\}. \tag{1}$$

Note that $x_4 \in N^-_D(x_1)$ and $x_5 \in N^+_D(x_2)$ will not hold at the same time, or there will exist a C_6 as $x_1x_2x_5x_6x_3x_4x_1$, a contradiction.

If $x_2 \in N^+_D(x_6)$, then we have $x_5 \in N^+_D(x_2)$ as D is an oriented graph. This implies that x_5 is an in-neighbour of x_1. Observe a new P_6 as $x_4x_5x_6x_2x_3x_1$, we can get that $x_6 \in N^+_D(x_1)$. Thus there exists a C_6 as $x_6x_2x_3x_4x_5x_1x_6$, a contradiction.

Thus we have $N^+_D(x_6) = \{x_3, x_4\}$. If $x_4 \in N^-_D(x_1)$, then x_6 is an out-neighbour of x_2. Observe a new P_6 as $x_5x_6x_4x_1x_2x_3$, we can get that $x_1 \in N^-_D(x_5)$. Thus there exists a good pair $(B^+_{x_2}, B^-_{x_6})$ in $D[P]$ with $B^+_{x_2} = x_2x_6x_4 + x_6x_3x_1x_5$ and $B^-_{x_6} = P_D$, which implies that D has a good pair by Lemma 5 and Lemma 1. If $x_5 \in N^-_D(x_1)$, then we can get a new P_6 as $x_6x_3x_4x_5x_1x_2$. Likewise, we have $N^+_{D-x_3}(x_2) \subseteq \{x_4, x_5\}$ and $N^-_{D-x_5}(x_6) \subseteq \cup\{x_1, x_4\}$. By (1) and $x_4 \in N^+_D(x_6)$, we can get that $x_5 \in N^+_D(x_2)$ and $x_1 \in N^-_D(x_6)$. Then there exists a good pair $(B^+_{x_2}, B^-_{x_6})$ in $D[P]$ with $B^+_{x_2} = x_2x_5x_1x_6x_3 + x_6x_4$ and $B^-_{x_6} = P_D$, which implies that D has a good pair as $\lambda(D) = 2$, a contradiction.

Now we are ready to prove Theorem 2. For convenience, we restate it here.

Theorem 2. *Every 2-arc-strong digraph on 7 vertices has a good pair.*

Proof. Suppose that D has no good pair. Let R be a largest clique in D. By Lemma 5 and Proposition 8, $|R| = 3$. Moreover, R is a tournament by Lemma 6 and Proposition 7.

Claim 2.1 *D is an oriented graph.*

Proof. Proof. Suppose that there is a digon Q in D with $V(Q) = \{s, t\}$. Observe that Q has a good pair. Since R is a tournament with three vertices, both in- and out-neibourhoods of Q in D have at least two vertices. This implies that D has a good pair by Lemma 7, a contradiction.

Assume that $P_D = x_1x_2 \dots x_7$ is a Hamilton dipath of D by Proposition 12. Set $D' = D - A(P_D)$. Let I_i and T_j respectively be the initial and terminal strong component in D', where $i \in [a]$ and $j \in [b]$. Note that $a, b \geq 2$ by Proposition 9. Since D is an oriented graph and $\lambda(D) \geq 2$, $|I_i|, |T_j| \geq 3$, for any $i \in [a], j \in [b]$. Thus there are only two non-adjacent strong components in D', say I_1 and I_2, with $|I_1| = 3$ and $|I_2| = 4$. Note that $|N^-_{D'}(x_1)| \geq 2$ and $|N^+_{D'}(x_7)| \geq 2$ as $\lambda(D) \geq 2$, which implies that $x_1, x_7 \in I_2$. Moreover, $x_2, x_6 \in I_1$ by Claim 2.1. Then D has a good pair by Proposition 10.

5 Good pairs in digraphs of order 8

The digraph E_3 used in the next proposition is shown in Fig. 2.

Proposition 13 ([2]) *Let D be a 2-arc-strong digraph without any subdigraph on order 4 that has a good pair. If D contains an orientation Q of E_3 as a subdigraph, then $N^+_D(Q) \cap N^-_D(Q) = \emptyset$, $|N^+_D(Q)| \geq 2$ and $|N^-_D(Q)| \geq 2$.*

Proposition 14 *Let D be a 2-arc-strong oriented graph on n vertices without K_4 as a subdigraph, where $8 \leq n \leq 9$. If D has two disjoint cycles C^1 and C^2 which cover exactly 7 vertices, then D contains a P_8.*

Proof. Suppose that P_7 is the longest dipath of D by Proposition 12. In fact there exist arcs between C^1 and C^2 from both directions, otherwise D has a P_8 as $\lambda(D) \geq 2$. W.l.o.g., assume $|C^1| \geq |C^2|$. Then $|C^1| = 4$ and $|C^2| = 3$. Let $C^1 = x_1x_2x_3x_4x_1$, $C^2 = x_5x_6x_7x_5$, $P_7 = x_1x_2 \dots x_7$ and y_j be the vertex in $V - V(C^1 \cup C^2)$, where $j = 1$ when $n = 8$ and $j \in [2]$ when $n = 9$. From the maximality of P_7 in D, we have the following facts.

Fact 14.1. For any j, at least one of $(C^i, y_j)_D$ and $(y_j, C^{3-i})_D$ is empty for any $i \in [2]$.
Fact 14.2. For any j, at least one of arcs x_iy_j and y_jx_{i+1} is not in A for any $i \in [6]$.
Fact 14.3. For $n = 9$, let $y_jy_{3-j} \in A$. If $x_iy_j \in A$, then $y_{3-j}x_{i+1}, y_{3-j}x_{i+2} \notin A$, where $j \in [2]$ and $i \in [5]$.

Since D is oriented, there are at least three arcs between y_j and C^i, for some i, by Fact 14.1. W.l.o.g., assume $i = 1$. Note that $d^+_{C^1}(y_j) \geq 1$ and $d^-_{C^1}(y_j) \geq 1$. Then $N(y_j) \subset \{y_{3-j}\} \cup C^1$ when $n = 9$ and $N(y_j) \subset C^1$ when $n = 8$.

If y_j is not adjacent to y_{3-j} or $n = 8$, then $N^+(y_j) = \{x_1, x_2\}$ and $N^-(y_{3-j}) = \{x_3, x_4\}$ by Fact 14.2, which implies that D has a P_8 as $y_jx_1 \in A$, a contradiction.

Hence $n = 9$ and y_1 is adjacent to y_2. W.l.o.g., assume that $y_1y_2 \in A$. If $x_1y_1 \in A$, then $N^+(y_2) = \{x_1, x_4\}$ by Fact 14.3 and $\lambda(D) \geq 2$, which implies that D has a Hamilton dipath as $y_2x_1 \in A$, a contradiciton. Hence x_1 is not adjacent to y_1. By Fact 14.2, $N^+(y_1) = \{x_2, y_2\}$ and $N^-(y_1) = \{x_3, x_4\}$. By Fact 14.3 and the longestness of P_7, $N^+(y_2) = \{x_2, x_3\}$. It implies that $D[\{x_2, x_3, y_1, y_2\}]$ is a K_4, a contradiction.

Proposition 15 *Let $D = (V, A)$ be a 2-arc-strong digraph on n vertices without good pair, where $8 \leq n \leq 9$. If D is an oriented graph without K_4 as a subdigraph, then D has a P_8.*

Now we are ready to show Theorem 3. For convenience, we restate it here.

Theorem 3. *Every 2-arc-strong digraph on 8 vertices has a good pair.*

Proof. Suppose that D has no good pair. Let R be a largest clique in D. By Lemma 6 and Proposition 8, $|R| = 3$.

Claim 3.1 *No subdigraph of D of order at least 3 has a good pair.*

Proof. By Lemma 6, it suffices to show that there is no $Q \subset D$ on 3 vertices with a good pair. Suppose that Q has a good pair. If Q is an orientation of E_3, then we use Lemma 7 to find a good pair of D by Proposition 13, a contradiction. Now assume that Q is a bidigon. Set $V(Q) = \{x, y, z\}$ with $Q[\{x, y\}] = C_2$ and $Q[\{y, z\}] = C_2$. If there exists a vertex w in $N_D^+(Q) \cap N_D^-(Q)$, then $D[Q \cup \{w\}]$ has a good pair by Lemma 1. Thus $N_D^+(Q) \cap N_D^-(Q) = \emptyset$. If $N_D^-(Q) = \{w\}$, then $D[Q \cup \{w\}]$ has a good pair as $B_w^+ = wzyx$ and $B_z^- = wxyz$. By symmetry, this implies that $|N_D^+(Q)| \geq 2$ and $|N_D^-(Q)| \geq 2$. Thus by Lemma 7, D has a good pair, a contradiction. $\diamondsuit$

By the claim above, R is a tournament.

Claim 3.2 *D is an oriented graph.*

Proof. Suppose that there is a digon Q in D with $V(Q) = \{s, t\}$. Observe that Q has a good pair. Since R is a tournament with 3 vertices, both in- and out-neibourhoods of Q in D have at least two vertices with $N_D^+(Q) \cap N_D^-(Q) = \emptyset$. This implies that D has a good pair by Lemmas 2, 8, 9 and 10, and Corollary 1, a contradiction. $\diamondsuit$

By Proposition 15, assume that $P_D = x_1x_2 \ldots x_8$ is a Hamilton dipath of D. Set $D' = D - A(P_D)$. Let I_i and T_j respectively be the initial and terminal strong component in D', where $i \in [a]$ and $j \in [b]$. Note that $a, b \geq 2$ by Proposition 9. Since D is an oriented graph and $\lambda(D) \geq 2$, $|I_i|, |T_j| \geq 3$ for any $i \in [a], j \in [b]$. Thus there are only two non-adjacent strong components in D', say I_1 and I_2, as $n = 8$. Since $\lambda(D) \geq 2$, x_1 has at least two in-neighbours and one out-neighbour in D', while x_8 has at least two out-neighbours and one in-neighbour in D'. If $|I_1| = 3$ and $|I_2| = 5$, then $x_1, x_8 \in I_2$ and $|A(I_2)| \geq 6$. Note that at least one of x_2 and x_7 is in I_1 as $|R| = 3$. Then we use Proposition 10 to get a good pair of D. Now assume $|I_1| = |I_2| = 4$. If $x_8 \in I_1$ then $x_7 \in I_2$ by Claim 3.2. By Proposition 10, D has a good pair.

6 Good pairs in digraphs of order 9

We have several generalizations of Proposition 6 here, which are easy to check as they satisfy the conditions in Proposition 6.

Proposition 16 *Let $D = (V, A)$ be a digraph and Q be a subdigraph of D with good pair (O_Q, I_Q). Set $X = N_D^-(Q)$ and $Y = N_D^+(Q)$ with $X \cap Y = \emptyset$ and $X \cup Y = V - V(Q) - W$, where $W = \{w_1, w_2\}$. Let e_1 be an arc from w_1 to X and e_2 be an arc from Y to w_2. Set $X' = X \cup w_1$, $Y' = Y \cup w_2$ and $D' = (V, A')$ with $A' = A - \{e_1, e_2\}$. Let $\mathcal{X}$ be the set of initial strong components in $D'[X']$ and $\mathcal{Y}$ be the set of terminal strong components in $D'[Y']$. Assume that there exists X_0 and Y_0 in $\mathcal{X}$ and $\mathcal{Y}$ respectively such that $d_Y^-(X_0) = 1$ and $d_X^+(Y_0) = 1$. Let e_x and e_y be arcs from Y to X_0 and from Y_0 to X respectively. If one of the following holds, then D has a good pair.*

1. *$e_x \neq e_y$, but at least one of $\mathcal{X}$ or $\mathcal{Y}$ has only one element.*
2. *e_x (or e_y) is adjacent to some Y_x (or X_y) in $\mathcal{Y}$ (or $\mathcal{X}$), such that $d_X^+(Y_x) \geq 3$ (or $d_Y^-(X_y) \geq 3$).*
3. *e_x (or e_y) is adjacent to $Y' - V(\mathcal{Y})$ (or $X' - V(\mathcal{X})$).*
4. *e_x (or e_y) is adjacent to some $Y_x \neq Y_0$ (or $X_y \neq X_0$) in $\mathcal{Y}$ (or $\mathcal{X}$), such that there exists an arc from Y_x (or X_y) to $X' - V(\mathcal{X})$ (or $Y' - V(\mathcal{Y})$).*

Lemma 11. *Let D be a 2-arc-strong digraph on 9 vertices that contains a digon Q. Assume that D has no subdigraph with a good pair on 3 or 4 vertices. Set $X = N_D^-(Q)$ and $Y = N_D^+(Q)$ with $X \cap Y = \emptyset$. If $|X| = 3$ and $|Y| = 2$, then D has a good pair.*

Lemma 12. *Let $D = (V, A)$ be a 2-arc-strong digraph on 9 vertices that contains a digon Q. Assume that D has no subdigraph with a good pair on at least 3 vertices. Set $X = N_D^-(Q)$ and $Y = N_D^+(Q)$ with $X \cap Y = \emptyset$ and $W = V - V(Q) - X - Y$. Assume that $|X| = |Y| = 2$ and there is an arc $e = st \in A$ such that $s \in Y$ and $t \in W$ (resp. $s \in W$ and $t \in X$). If there are at least three arcs in $D[Y \cup \{t\}]$ (resp. $D[X \cup \{s\}]$), then D has a good pair.*

Lemma 13. *Let D be a 2-arc-strong digraph on 9 vertices that contains a digon Q. Assume that D has no subdigraph with a good pair on 3 or 4 vertices. Set $X = N_D^-(Q)$ and $Y = N_D^+(Q)$ with $X \cap Y = \emptyset$. If $|X| = 2$ and $|Y| = 2$, then D has a good pair.*

Proposition 17 *Let $D = (V, A)$ be a 2-arc-strong oriented graph on 9 vertices without K_4 as a subdigraph. If D have two cycles C^1 and C^2 with $C^1 \cap C^2 = \emptyset$ which cover 8 vertices, then D contains a Hamilton dipath.*

Lemma 14. *Let $D = (V, A)$ be a 2-arc-strong digraph on 9 vertices without good pair. If D is an oriented graph without K_4 as a subdigraph, then D has a Hamilton dipath.*

Now we are ready to show Theorem 4. For convenience, we restate it here.

Theorem 4. *Every 2-arc-strong digraph on 9 vertices has a good pair.*

Proof. By contradiction, suppose that D has no good pair.

Claim 4.1 *No subdigraph of D of order at least 4 has a good pair.*

Let R be a largest clique in D. Then R has three vertices by Claim 4.1 and Proposition 8.

Claim 4.2 *No subdigraph of D of order at least 3 has a good pair.*

Proof. By Lemma 7, it suffices to show that there is no $Q \subset D$ on 3 vertices with good pair. Suppose to the contrary that Q has a good pair. Analogous to Claim 3.1, $|N_D^+(Q)| \geq 2$ and $|N_D^-(Q)| \geq 2$ with $N_D^+(Q) \cap N_D^-(Q) = \emptyset$. Thus by Lemma 8, D has a good pair, a contradiction. ◊

By the claim above, R is a tournament.

Claim 4.3 *D is an oriented graph.*

Proof. Suppose that D has a digon Q. Set $X = N_D^-(Q)$ and $Y = N_D^+(Q)$. By Claim 4.2, $X \cap Y = \emptyset$. Since $\lambda(D) \geq 2$, both X and Y have at least two vertices. If $|X|+|Y| = 4$, then D has a good pair by Lemma 13, a contradiction. If $|X|+|Y| = 5$, then D has a good pair by Lemma 11 and the digraph duality, a contradiction. If $|X| + |Y| = 6$, then D has a good pair by Lemma 2, a contradiction. If $|X| + |Y| = 7$, then D has a good pair by Corollary 1, a contradiction. ◊

Now we are ready to finish the proof of Theorem 3. By Lemma 14, assume that $P_D = x_1x_2 \ldots x_9$ is a Hamilton dipath of D. Set $D' = D - A(P_D)$. Let $I_i,\ i \in [a]$, be the initial strong components in D' and let $T_j,\ j \in [b]$, be the terminal strong components in D'. Note that $a, b \geq 2$ by Proposition 9. Since D is an oriented graph and $\lambda(D) \geq 2$, $|I_i|, |T_j| \geq 3$, for any $i \in [a], j \in [b]$. Since $\lambda(D) \geq 2$, x_1 has at least two in-neighbours and one out-neighbour in D' and x_9 has at least two out-neighbours and one in-neighbour in D'. Thus there are only two non-adjacent strong components in D', say I_1 and I_2, as $n = 9$ and D is an oriented graph. We distinguish two cases below.

Case 1: $|I_1| = 4$ and $|I_2| = 5$.

If $x_9 \in I_1$, then $x_8 \in I_2$ as $|R| = 3$. Analogously, if $x_1 \in I_1$, then $x_2 \in I_2$. By Proposition 10, D has a good pair for each cases. Henceforth, both x_1 and x_9 are in I_2. Note that at least one of x_2 and x_8 is in I_1 as $|R| = 3$. By Proposition 10, D has a good pair, a contradiction.

Case 2: $|I_1| = 3$ and $|I_2| = 6$.

In this case, $x_1, x_9 \in I_2$ and $|A(I_2)| \geq 7$. If one of x_2 and x_8 is in I_1, then D has a good pair by Proposition 10. Thus both x_2 and x_8 are in I_2. Then $V(I_1) = \{x_3, x_5, x_7\}$, which implies that D has a good pair by Proposition 10, a contradiction.

This completes the proof of Theorem 4.

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