Results on the Small Quasi-Kernel
for Anti-claw-free and One-way Split Digraphs

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Outline

1. Small Quasi-kernel Conjecture
2. Anti-Claw-Free Digraphs
3. One-way Split Digraphs

Kernels and Quasi-Kernels

Definition

A kernel is an independent set $K \subseteq V(D)$ such that any vertex $v \in V(D) \setminus K$ has an arc from v to a vertex $u \in K$.

Definition

A quasi-kernel is an independent set $Q \subseteq V(D)$ such that any vertex $v \in V(D) \setminus Q$, there exists a directed path with at most two arcs from v to a vertex $u \in Q$.

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2. Form inductively a quasi-kernel Q in the remaining digraph $D - N^-[v]$.

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3. If v has no out-neighbour in Q, then we add v to Q. Otherwise, Q is a quasi-kernel of D.

□

Conjecture and a equalvalent conjecture

Conjecture 1([P. L. Erdős and L. A. Székely, 1976])

Every sink-free digraph D=(V(D),A(D)) has a quasi-kernel of size at most |V(D)|/2.

- A sink is a vertex which has no out-neighour in D.

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Conjecture 2([A. Kostochka, R. Luo, and S. Shan, 2020])

Let S be the set of sinks of D. Then D has a quasi-kernel Q such that $|Q| \leq \frac{|V(D)|+|S|-|N^{-}(S)|}{2}$.

Our Result I (Anti-Claw-Free Digraphs)

Theorem

Every sink-free digraph with no induced anti-claw has a quasi-kernel of size at most $|V(D)|/2$.

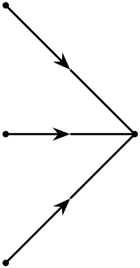

Figure: Anti-Claw($\vec{K}_{3,1}$)

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Every sink-free digraph with no induced $\vec{K}_{4,1}$ and no induced $\vec{K}^{+}_{4,1}$ has a quasi-kernel of size at most $|V(D)|/2$.

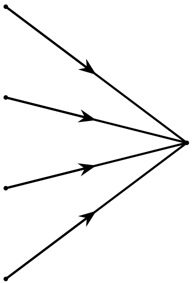


Figure: $\vec{K}_{4,1}$

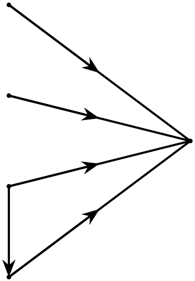
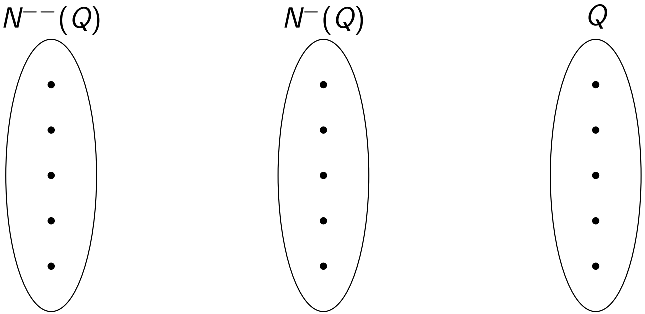
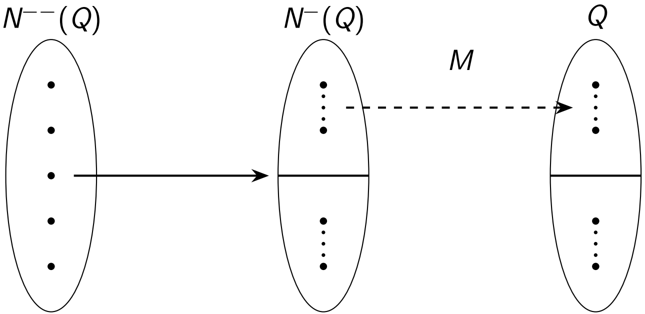


Figure: $\vec{K}^{+}_{4,1}$

Quasi-kernel Q and its second neighbourhood



The Maximal Matching From $N^{-}(Q)$ to Q



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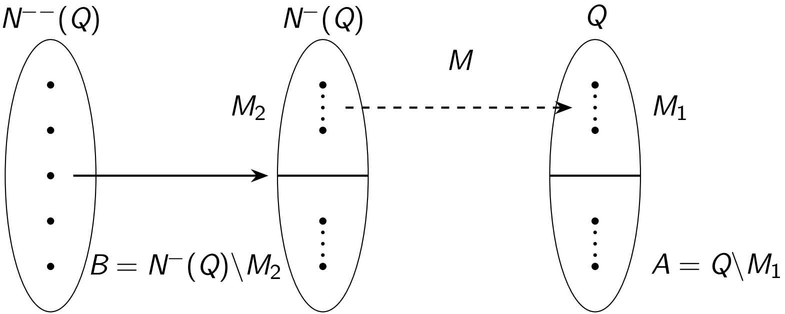

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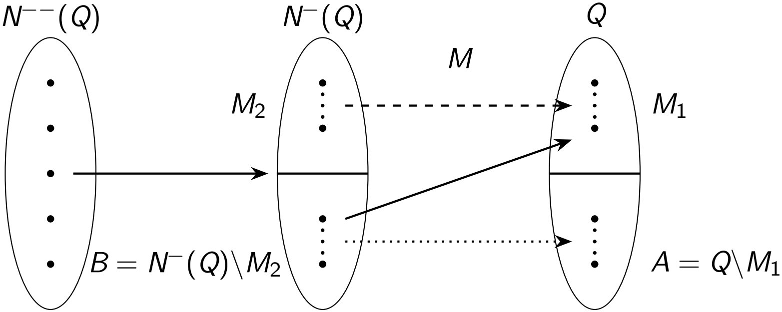

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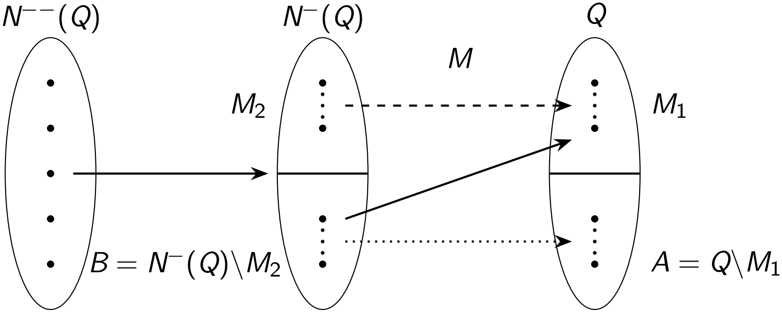
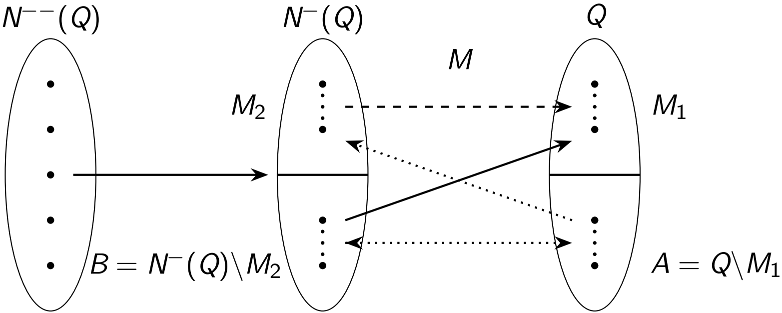


OBERVATION 1

M_1 is a quasi-kernel of $D[V(D) \setminus A]$.

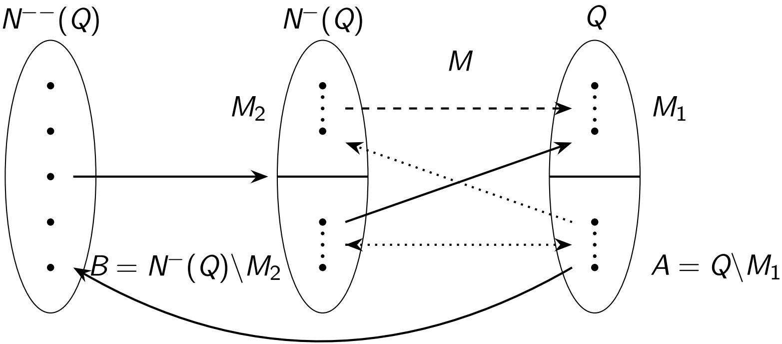


Observation 2

If Q is minimal quasi-kernel then there is no arc from A to $N^{-}(Q)$.

The Maximal Matching From $N^{-}(Q)$ to Q

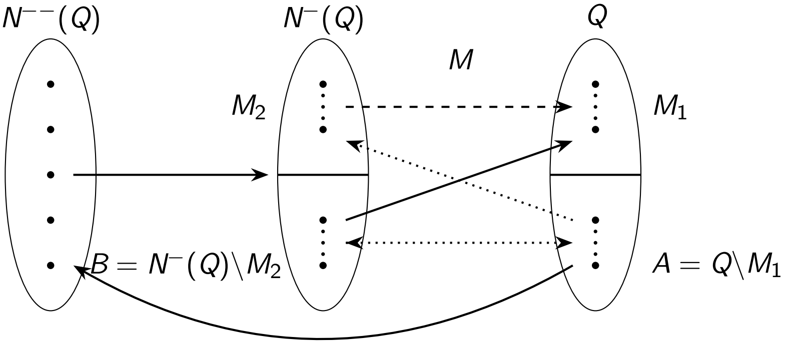
For a minimal quasi-kernel Q in D



Observation 3

If Q is minimal quasi-kernel then for all $v \in A$, $|N^{+}(v) \cap N^{--}(Q)| \geq 1$.

Sketch of the Proof



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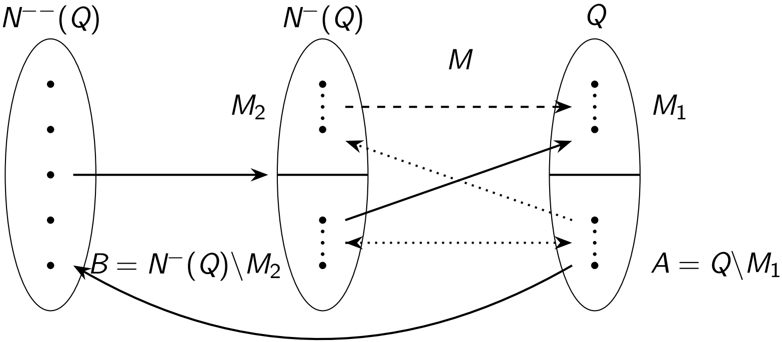

1. Let's assume that Q is not small or equalvalently $|N^{--}(Q)| + |N^{-}(Q)| < |Q|$

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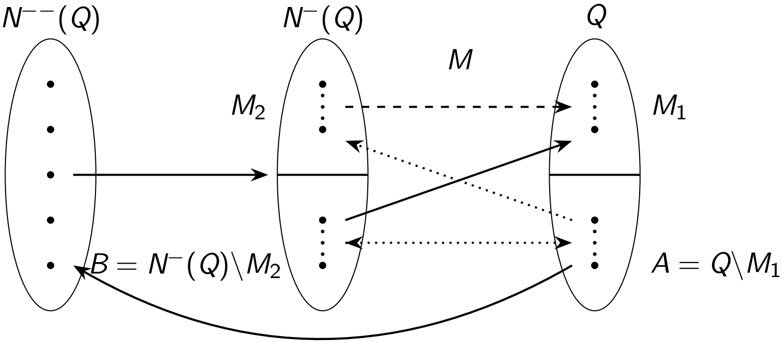

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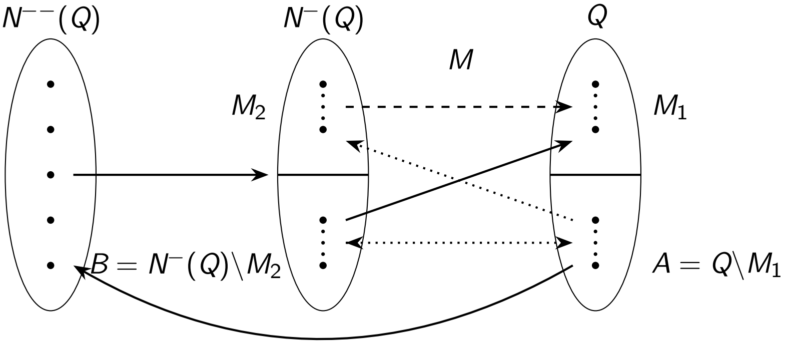

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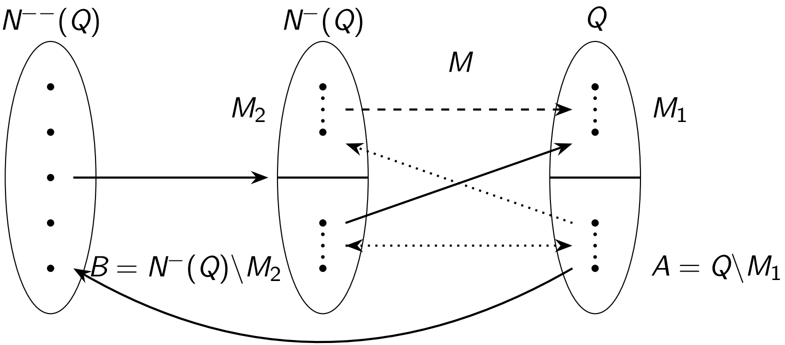

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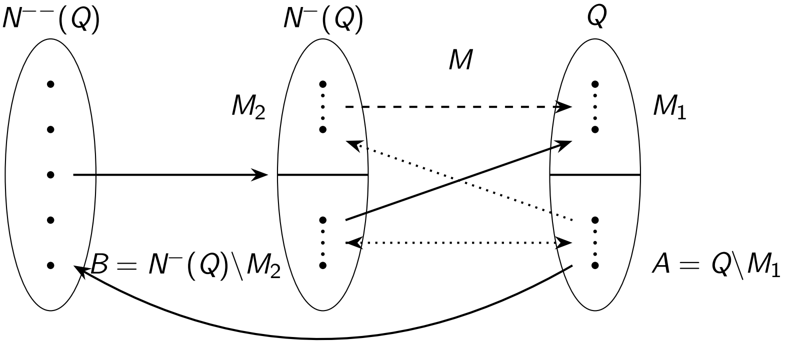

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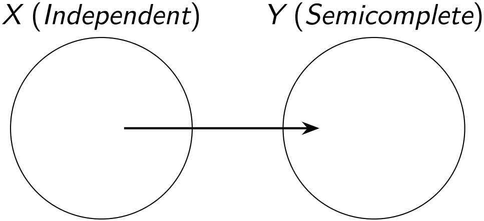
3. v also must have an in-nerghbour in M_1 which togather with two in-neighbours in A forms an induced anti-claw, a contradiction.

The One-way Split Digraphs

Definition

A digraph D is called a one-way split digraph, if its vertex set can be partitioned into X and Y, such that X induces an independent set and Y induces a semicomplete digraph (a digraph in which there is at least one arc between every pair of vertices) and any arcs between X and Y go from X to Y.



Our Result II (One-way Split Digraphs)

Theorem

Let D be a one-way split digraph of order n with no sinks. Then D has a quasi-kernel of size at most $\frac{n+3}{2} - \sqrt{n}$. Furthermore, for infinitely many values of n there exists a one-way split digraph of order n, with no sink, such that the minimum size of quasi-kernels of D is $\frac{n+3}{2} - \sqrt{n}$.

Sketch of the Proof (for the conjecture)

1. For any vertex in $v \in V(X)$, we choose an arbitary vertex in $N^+(v)$ and denote it by $R(v)$. Construct an auxiliary digraph H whose vertex set $V(H) = V(X)$, and $vw \in A(H)$ if and only if $R(v)R(w) \in A(Y)$ or $R(v) = R(w)$.

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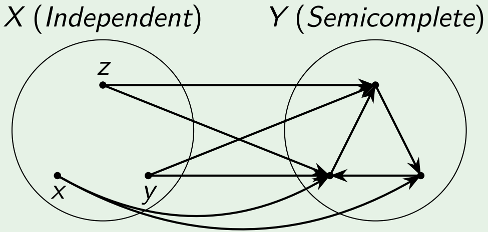

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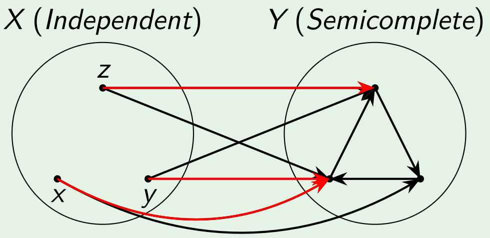

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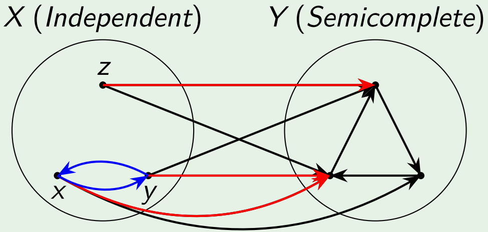

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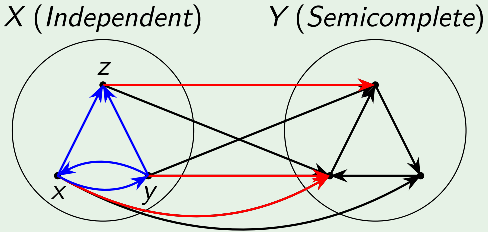

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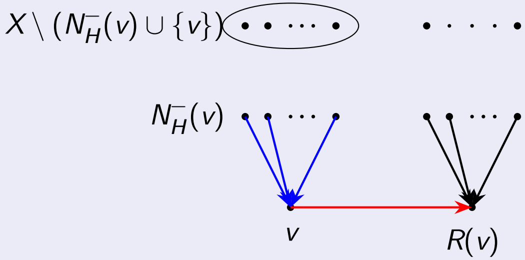
4. How to choose the quasi-kernel? We choose all vertex in X but the closed in-neighbourhood of v and an vertex in Y. The size of this set is $\frac{|X|-1}{2} + 1 \leq |V|/2$ (if we assume $V(Y)$ is not an empty set).

How to Choose the vertex and Why it is a quasi-kernel?

Demonstration



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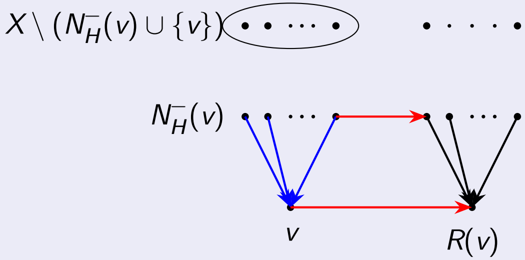

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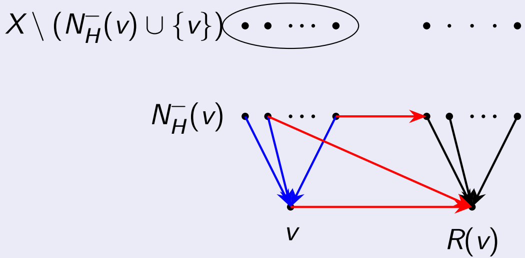

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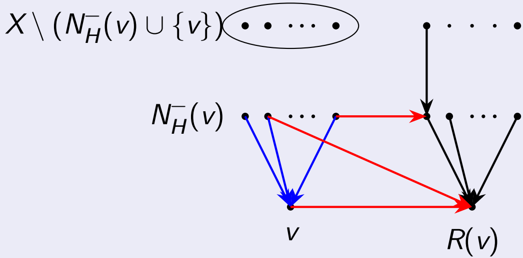
DEMONSTRATION



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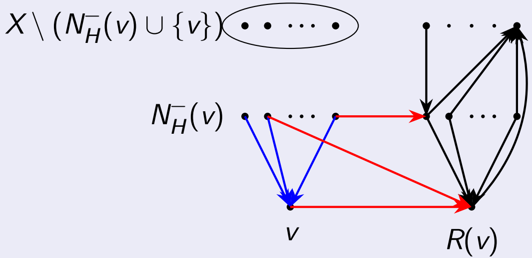

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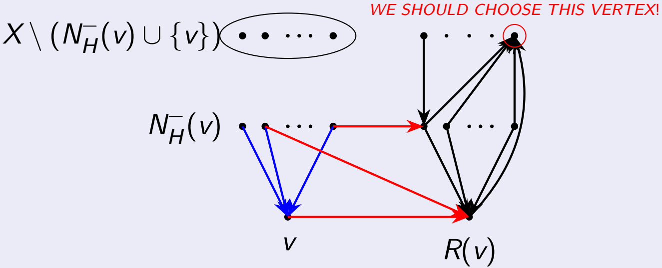

A Extremal Example

Example

We let $k \geq 1$ be any integer and construct the digraph D_k of order $(2k+1)^2$ as follows. Let T be a k-regular tournament of order $2k+1$ and for each vertex, v, of T add $2k$ new vertices, V_v, with arcs into v. The resulting digraph, D_k, has order $(2k+1)^2$ and is a one-way split digraph with partition $V(T)$ (the tournament) and $V(D_k) \setminus V(T)$ (the independent set).

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The size of quasi-kernels in $|Q|$:

$$|Q| \geq 2k^2 + 1 = \frac{4k^2+4k+1}{2} - \frac{4k+2}{2} + \frac{3}{2} = \frac{n}{2} - \sqrt{n} + \frac{3}{2}$$

Thank you for your attention!

V. Chvátal and L. Lovász
Every digraph has a semi-kernel.
In Lecture Notes in Mathematics, 411 (1974), 175-175.

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