Bieri-Strebel Groups With Irrational Slopes

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Declaration

I confirm that the work presented in this thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the document.

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Abstract

For an algebraic integer β, that is the zero of the irreducible integer polynomial

$$X^n - a_{n-1}X^{n-1} - a_{n-2}X^{n-2} - \cdots - a_1X - a_0$$

with all $a_i \geq 0$, we define the Bieri-Strebel group $G_\beta = G\left([0,1], \mathbb{Z}\left[\frac{1}{\beta}\right], \langle\beta\rangle\right)$. This is the group of piecewise linear homeomorphisms of the unit interval to itself with breakpoints in $\mathbb{Z}\left[\frac{1}{\beta}\right]$ and slopes that are a power of β. The best known example of this is G_2 which is better known as Thompson's Group F. It is well known [1] that elements of F can be expressed as pairs of binary trees, and using these trees it is possible to demonstrate many properties of F. We denote $F_\beta \subset G_\beta$ the set of elements $g \in G_\beta$ for which there exist 'tree-pairs' to represent g. The question arises: For which β is $F_\beta = G_\beta$.

Higman [2] has shown that for $\beta \in \mathbb{N}$, $F_\beta = G_\beta$. In his 1995 [3] and 2000 [4] papers, Cleary was able to show that $F_\beta = G_\beta$ if $\beta = \frac{\sqrt{5}+1}{2}$ or $\beta = \sqrt{2}+1$, and in their 2018 master's thesis Brown [5] was able to show this holds for all β whose associated polynomial is

$$X^2 - a_1X - 1$$

for some $a_1 \in \mathbb{N}$.

In this thesis, we have considered all quadratic integers β, zero of the irreducible integer polynomial

$$X^2 - a_1X - a_0$$

for some $a_1, a_0 \in \mathbb{N}$, and found necessary and sufficient conditions on a_1 and a_0 such that that $F_\beta = G_\beta$. We have also shown that there exists β for which F_β is a proper subset of G_β and conjecture that it is not even a group.

For the cases in which $F_\beta = G_\beta$, we have been able to find a presentation for G_β, with which we have been able to determine a presentation for the abelianisation of G_β. We have been able to find arbitrarily high torsion in these G_β^{ab}.

Contents

Chapter 1

Introduction

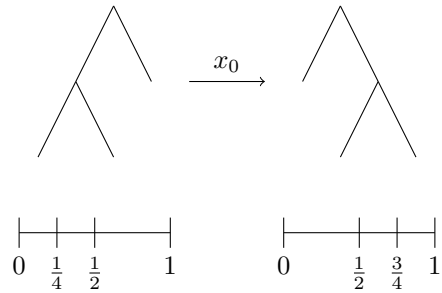
Groups of piecewise-linear homeomorphisms on real intervals have been particularly studied for the past 60 years, as there have been examples found which have been shown to have surprising properties. Perhaps the most infamous of these is Thompson's Group F , named after Richard Thompson [6], the group of continuous piecewise-linear homeomorphisms of the unit interval in which all breakpoints lie in $\mathbb{Z}\left[\frac{1}{2}\right] = \left\{a + \frac{b}{2} : a, b \in \mathbb{Z}\right\}$ and all slopes have gradient which is a power of 2. The group F is an example of an infinite group which is finitely presented and torsion-free. The group F also has infinite cohomological dimension [7], as for all $N \in \mathbb{N}$, there is an injective homomorphism which embeds the free abelian group of rank N into F,

$$\phi_N : \mathbb{Z}^N \hookrightarrow F.$$

The group F has an infinite presentation

$$F = \langle x_0, x_1, x_2, \ldots | x_j x_i = x_i x_{j+1} \text{ for } i < j \rangle .$$

The elements of F are uniquely defined by their breakpoints and so we can describe any such element g as a pair of subdivisions of the unit interval $g = (S_1, S_2)$, where S_1 and S_2 have the same number of breakpoints which all must lie in $\mathbb{Z}\left[\frac{1}{2}\right]$. Then g defines a set of affine transformations which map the ith interval in S_1 to the ith interval in S_2. It is well known that each of the elements of F can be expressed as a pair of binary trees [1], with each leaf in a tree representing a sub-interval of $[0, 1]$ which has length that is a power of 2. In fact we can show the generators found in the presentation of F as tree pairs. The generator x_0 is displayed in tree-pair form atop the following page.



The group F is a sub-group of Thompson's Group V, the group of all left-continuous piecewise-linear homeomorphisms of the unit interval in which all breakpoints lie in $\mathbb{Z}\left[\frac{1}{2}\right]$ and all slopes have gradient which is a power of 2. The group V was one of the first found examples of a finitely presented infinite simple group. In [2], Higman generalised the group V to V_n, and subsequently the group F to F_n, for any $1 < n \in \mathbb{N}$. The group V_n is the group of left-continuous piecewise-linear homeomorphisms of the unit interval in which all breakpoints lie in $\mathbb{Z}\left[\frac{1}{n}\right]$ and all slopes have gradient which is a power of n, and $F_n \subset V_n$ is the sub-group of V_n such that every element is a continuous homeomorphism. Note that the group F is re-defined as F_2. Furthermore the elements of the group F_n can also be expressed as pairs of trees, but now each caret has n legs [8]. Through the use of the tree-representation of F_n, it is possible to find a presentation for F_n, and from this show that F_n is finitely generated, but has infinite presentation. The groups F_n were also shown to be torsion-free, finitely presented groups with infinite cohomological dimension by Brown and Geoghegan [9].

In [3] and [4], Cleary considered variants in which the breakpoints and slopes were irrational. These were F_ω in 1995 and F_τ in 2000 where $\omega = \sqrt{2} + 1$, and $\tau = \dfrac{\sqrt{5}+1}{2}$. Both F_ω and F_τ were shown by Cleary to be finitely presented, torsion-free groups with infinite cohomological dimensional.

But the greatest generalisation of these groups of piecewise-linear homeomorphisms on real intervals came before this, from Bieri and Strebel [10].

1.1 Bieri-Strebel Groups

Bieri and Strebel defined groups of piecewise linear homeomorphisms on real intervals as shown below.

Definition 1.1.1. *The Bieri-Strebel Group*

The Bieri-Strebel Group $G(I, A, P)$ is the group of all piecewise-linear homeomorphisms of the interval I, with breakpoints in A, a subring of the real numbers $\mathbb{R}$, and slopes with gradient in P where P is a group of units contained in A.

The Bieri-Strebel groups encompass almost all of the examples of groups of piecewise-linear homoemorphisms of real intervals that have come before this. These include many which will appear for the first time explicitly in this thesis. We note that Thompson's group F is a Bieri-Strebel group,

$$F = F_2 = G\left([0,1], \mathbb{Z}\left[\frac{1}{2}\right], \langle 2\rangle\right).$$

Definition 1.1.2. An algebraic integer β is the zero of some monic integer polynomial

$$f = X^n + \sum_{i=0}^{n-1} a_i X^i$$

for some $a_i \in \mathbb{Z}$.

For any algebraic integer $\beta \in \mathbb{R}$, we define the Bieri-Strebel group G_β as

$$G_\beta = G\left([0,1], \mathbb{Z}\left[\frac{1}{\beta}\right], \langle \beta\rangle\right).$$

The groups F_ω and F_τ introduced by Cleary, can be expressed here as

$$F_\omega = G_\omega = G\left([0,1], \mathbb{Z}\left[\frac{1}{\omega}\right], \langle \omega\rangle\right), \text{ and}$$
$$F_\tau = G_\tau = G\left([0,1], \mathbb{Z}\left[\frac{1}{\tau}\right], \langle \tau\rangle\right)$$

Proposition 1.1.3. Let β be an algebraic integer and recall the definition of the Bieri-Strebel Group G_β. If $g \in G_\beta$ is of finite order, then $g = id$, the identity homeomorphism.

Proof. The proof for this is well known in the literature, see [5], but I have included this to demonstrate a reason for considering the elements, as defined in the group, as piecewise-linear homeomorphisms.

Let $g \in G_\beta$ be non-trivial. As g is a piecewise-linear homeomorphism, then g is defined by the breakpoints

$$\{(0,0) = (p_0, q_0), (p_1, q_1), \ldots, (p_t, q_t) = (1,1)\}$$

where $g(p_i) = q_i$, and if we let r_i be the gradients of the slope between (p_i, q_i) and (p_{i+1}, q_{i+1}), then $r_i \neq r_{i+1}$. Note that either $r_0 = 1$ or $r_0 = \beta^{s_0}$ for some integer $s_0 \neq 0$. If $r_0 = 1$, then $r_1 \neq r_0$, so either r_0 or r_1 is non-trivial. Let $p \in \{p_0, \ldots, p_t\}$ be the first breakpoint of g such that $g(x) = x$ for all $x \in [0, p]$. Note that if $p = 1$, then g is the identity. Otherwise, as p is a breakpoint in g, the right

gradient at p is not equal to 1. Consider the right derivative of g at p,

$$D_p^+(g) = \lim_{h \to 0^+} \left\{ \frac{g(p+h) - g(p)}{h} \right\}.$$

So $g(p) = p$, but $D_p^+(g) \neq 1$. Consider g^n for some $n \in \mathbb{N}$, and let $x \in [0, p]$. Then

$$g^n(x) = \underbrace{g \circ g \circ \cdots \circ g(x)}_{n} = \underbrace{g \circ g \circ \cdots \circ g(x)}_{n-1} = g(x) = x.$$

Now we consider the right derivative of g^n at p. In composition $g_2(g_1(x))$ of elements $g_1, g_2 \in G_\beta$, the gradient of the intersections of any intersecting intervals in the image of g_1 and the domain of g_2, is found by taking the product of the gradients. Therefore $D_p^+(g^n) = \left(D_p^+(g)\right)^n \neq 1$ for all $n \in \mathbb{N}$. Thus g is not of finite order n for any $n \in \mathbb{N}$. Therefore, $g \in G_\beta$ is of finite order only if there exists no such $p \in [0, 1)$ such that $g(x) = x$ for all $x \in [0, p]$, and $D_p^+(g) \neq 1$. Therefore $g \in G_\beta$ is of finite order only if g is the identity homeomorphism. □

The *Thompson-like* Bieri-Strebel groups, such as F_n[7], F_ω[3], and F_τ[4] have been shown to be finitely presented, and have infinite cohomological dimension over $\mathbb{Z}$. Also, all elements of these groups can be expressed as tree pairs.

1.2 Tree pairs

In the year 2000 Cleary was able to show that G_τ is finitely presented [4], for $\tau = \dfrac{\sqrt{5}+1}{2}$. Burillo, Nucinkis, and Reeves [11] were able to use tree pair representations of the elements of G_τ to find an explicit finite presentation, and hence were able to show that the abelianisation of G_τ contained 2-torsion.

In their Master's thesis [5], Brown extended the work of Burillo, Nucinkis, and Reeves, by considering G_{τ_k} where τ_k is the positive real zero of the irreducible integer polynomial

$$f_{\tau_k} = X^2 - kX - 1.$$

Brown's work focused on finding the tree pair representations of elements of G_{τ_k}, finding a presentation for G_{τ_k}, and subsequently showing that $G_{\tau_k}^{ab}$ contains 2-torsion for all $k \in \mathbb{N}$.

These results rely heavily on being able to express elements of the Bieri-Strebel group as tree pairs.

1.3 Results Within This Thesis

This thesis focuses on the Bieri-Strebel group of the form

$$G_\beta = G\left([0,1], \mathbb{Z}\left[\frac{1}{\beta}\right], \langle\beta\rangle\right)$$

where the algebraic integer β is the positive real root of the irreducible polynomial

$$f_\beta = X^n - a_{n-1}X^{n-1} - a_{n-2}X^{n-2} - \cdots - a_1X - a_0$$

with $a_i \in \mathbb{Z}_{\geq 0}$. In Proposition 2.2.5, we will prove that there is exactly one such positive real root β.

Theorem. For all $0 < p \in \mathbb{Z}\left[\frac{1}{\beta}\right]$, there exists an expression

$$p = \frac{b_0 + b_1\beta + \cdots + b_{n-1}\beta^{n-1}}{\beta^N}$$

for some $b_i, N \in \mathbb{Z}_{\geq 0}$.

In Chapter 2, this is proved as theorem 2.2.19 and the result is shown to hold for all such β the positive real zero of the irreducible integer polynomial

$$f_\beta = X^n - a_{n-1}X^{n-1} - a_{n-2}X^{n-2} - \cdots - a_1X - a_0.$$

We move on to focus solely on the quadratic instances of this polynomial

$$f_\beta = X^2 - a_1X - a_0.$$

We define regular β-subdivisions which correspond to trees. We find properties which arise when there are multiple trees representing the same regular β-subdivisions. In particular we consider the cases in which β is *Pisot*.

Definition 1.3.1. An algebraic integer β is **Pisot** if $1 < \beta \in \mathbb{R}$ and all other zeros of the minimal polynomial of β over $\mathbb{Z}$, have absolute value less than 1. [12]

The following theorem appears as Corollary 2.5.12 to Theorem 2.5.10.

Theorem. If β is Pisot, then every element of G_β can be expressed as a pair of regular β-subdivisions.

This means that every element in G_β can be expressed as a pair of trees, as long as β is Pisot.

In fact, in all previous works connecting Bieri-Strebel groups of the form G_β to tree pair structures, we have always had Pisot β:

- n is Pisot for all $1 < n \in \mathbb{N}$;
- $\omega = \sqrt{2} + 1$ and $\tau = \dfrac{\sqrt{5}+1}{2}$ are both Pisot;
- β, the positive zero of $X^2 - kX - 1$ is Pisot for all $k \in \mathbb{N}$.

In each of these cases, F_n, F_ω, F_τ, and F_{τ_k}, have been shown to have the property that all elements can be expressed as tree pairs. Our Theorem 1.3 extends this to quadratic integers β, the positive zero of the Pisot polynomial $f_\beta = X^2 - a_1X - a_0$. This leads us to ask whether you can find tree pairs for every element in G_β if β is non-Pisot. We have shown this to not be true in Theorem 3.2.30.

Theorem. If β is non-Pisot then there exists $g \in G_\beta$ such that there are no regular subdivisions S_1, S_2 such that $g = (S_1, S_2)$.

Thus if β is non-Pisot there are elements of G_β for which there cannot be a tree pair representation.

Building on our earlier work on regular β-subdivisions of the unit interval in which β is a quadratic integer and Pisot, we construct generators from tree pairs and find an explicit presentation for G_β. The following is the statement for Theorem 4.3.20.

Theorem. Let β be the positive real zero of the Pisot polynomial $f_\beta = X^2 - a_1X - a_0$. Let $K = a_1 + a_0$. Then

$$G_\beta = \langle x_0, x_1, x_2, \ldots, z_0, z_1, z_2, \ldots \,|\, R_1, R_2 \rangle$$

with the relations

$$\begin{aligned} R_1: \quad x_i x_j &= x_{j+K-1} x_i \ \forall\ i < j \\ x_i z_j &= z_{j+K-1} x_i \ \forall\ i < j \\ z_i x_j &= x_{j+K-1} z_i \ \forall\ i < j \\ z_i z_j &= z_{j+K-1} z_i \ \forall\ i < j \\ R_2: \quad x_{i+a_0} x_{i+a_0+1} \cdots x_{i+2a_0-1} x_i &= z_i z_{i+1} \cdots z_{i+a_0-1} z_i \ \forall\ i \geq 0. \end{aligned}$$

Lastly, we will consider a specific Pisot case, β_n the zero of $X^2 - (n+1)X - n$ for even $n \in \mathbb{N}$, and show that the abelianisation of G_{β_n}, $G^{ab}_{\beta_n}$ contains elements with $(n+1)$-torsion. This is the statement for Theorem 4.4.11

Theorem.

$$G^{ab}_{\beta_n} \cong \mathbb{Z}^{2n+1} \oplus \mathbb{Z}/(n+1)\mathbb{Z}$$

Thus, there are Bieri-Strebel groups of the form G_β in which the abelianisation $G^{ab}_{\beta_n}$ contains arbitrarily high torsion.

Chapter 2

Regular Subdivisions of the Unit Interval

2.1 Background

Recall the definition of Bieri-Strebel groups initially introduced in [10].

Definition. *The Bieri-Strebel Group*
The Bieri-Strebel Group $G(I, A, P)$ is the group of all piecewise-linear homeomorphisms of the interval I, with breakpoints in A, a subring of the real numbers $\mathbb{R}$, and slopes with gradient in P where P is a group of units contained in A.

We will consider the family of Bieri-Strebel groups denoted G_β where

$$G_\beta = G\left([0,1], \mathbb{Z}\left[\frac{1}{\beta}\right], \langle\beta\rangle\right)$$

and β is a positive real root of $X^n - a_{n-1}X^{n-1} - a_{n-2}X^{n-2} - \cdots - a_1X - a_0 = 0$, for some $0 \leq a_i \in \mathbb{Z}$, $a_0 \neq 0$, and $\sum_{i=0}^{n-1} a_i > 1$. Here $\langle\beta\rangle = \{\beta^i : i \in \mathbb{Z}\}$.
Given $g \in G_\beta$, $g : [0,1] \to [0,1]$, with breakpoints $\{(0,0) = (p_0, q_0), (p_1, q_1), \ldots, (p_t, q_t) = (1,1)\}$,

$$g(x) = \left(\frac{q_{i+1} - q_i}{p_{i+1} - p_i}\right)(x - p_i) + q_i \text{ for } x \in [p_i, p_{i+1}]$$

for $i \in \{0, \ldots, t-1\}$. As each linear section must have gradient which is a power of β,

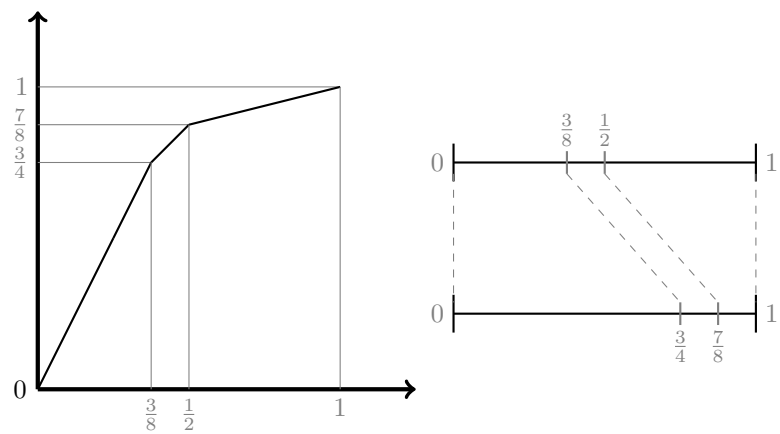
$$\frac{q_{i+1} - q_i}{p_{i+1} - p_i} = \beta^{r_i}$$

for some $r_i \in \mathbb{Z}$.

Example 1. If $\beta = 2$, we get the group G_2, which is better known as Thompson's group F

$$G_\beta = G\left([0,1], \mathbb{Z}\left[\frac{1}{2}\right], \langle 2 \rangle\right)$$

Below is the element g' of G_2.



The pair of β-subdivisions on the right are shown in the form of a *rectangle diagram.* The domain of g' is placed above the co-domain, and straight lines are drawn from the ith breakpoint in the domain to the ith breakpoint in the co-domain. The breakpoints do not need to be labelled if it is clear what they are. Each $g \in G_\beta$ has a corresponding rectangle diagram.

It is well known that each element of Thompson's group F, can be represented as a pair of binary trees [13]. There have been several variants of Thompsons group.

Definition 2.1.1. The set F_β is the set of all maps $g \in G_\beta$ such that g can be represented by a pair of regular trees.

We will define the set F_β more clearly in this chapter in Definition 2.4.8, once we have a better understanding of regular trees.

2.2 Irrational Subdivisions

2.2.1 Positive roots of polynomials

The following Lemma is in fact a consequence of Descartes' rule of signs [14]. A proof has been included to demonstrate our particular requirements.

Lemma 2.2.2. Every polynomial of the form

$$f = a_n X^n - a_{n-1} X^{n-1} - a_{n-2} X^{n-2} - \cdots - a_1 X - a_0$$

with $a_i \geq 0$ for $i \in \{1, \ldots, n-1\}$ and $a_n, a_0 > 0$ has a unique positive real zero β.
I.e., each polynomial of this form, has one and only one positive real zero.

Proof. We prove this by induction on the degree of these polynomials.
Any polynomial of the form $a_1 X - a_0$ has just a single zero $\alpha_1 = \dfrac{a_0}{a_1} \in \mathbb{R}^+$.
Assume that the lemma holds true for all $n \leq k-1$ for some $k \in \mathbb{N}$.
Consider

$$f = a_k X^k - a_{k-1} X^{k-1} - a_{k-2} X^{k-2} - \cdots - a_1 X - a_0,$$

with $a_i \geq 0$ and $a_k > 0$, $a_0 > 0$.
Consider the derivative of f:

$$f' = a_k k X^{k-1} - a_{k-1}(k-1) X^{k-2} - a_{k-2}(k-2) X^{k-3} - \cdots - a_2(2) X - a_1.$$

By the inductive hypothesis, every polynomial of this form of degree less than k has precisely one positive real root, call this α_{k-1}. So there exists exactly one stationary point of f over $\mathbb{R}^+$, consider what this stationary point could be:

Case 1: The stationary point is a local maximum of f.
As $f(\alpha_{k-1})$ is the only stationary point of f over $\mathbb{R}^+$, f must be strictly increasing before α_{k-1} and strictly decreasing afterwards. However f is a polynomial with a positive coefficient of the highest power of X, which means $f(X) \to \infty$ as $X \to \infty$. This is a contradiction, so $f(\alpha_{k-1})$ is not a local maxima of f.

Case 2: The stationary point is a saddle point of f.

As this is the only stationary point in $\mathbb{R}^+$, and as shown above $f(X) \to \infty$ as $X \to \infty$, f must be increasing over $\mathbb{R}^+$ and in fact strictly increasing on $\mathbb{R}^+ \backslash \{\alpha_{k-1}\}$. As $f(0) = -a_0 < 0$ and f is continuous, f increases continuously from a negative value, $-a_0$ at $X = 0$, to ∞ as $X \to \infty$. By the intermediate value theorem, there must exist a unique point $\alpha_k \in \mathbb{R}^+$ such that $f(\alpha_k) = 0$. Thus the lemma is true in case 2.

Case 3: The stationary point is a local minimum of f.

As $f(\alpha_{k-1})$ is the only stationary point of f over $\mathbb{R}^+$, then f must be strictly decreasing on $(0, \alpha_{k-1})$ and strictly increasing on (α_{k-1}, ∞). We also know that $f(0) = -a_0 < 0$ and since $f(\alpha_{k-1})$ is a local minimum of f, $f(\alpha_{k-1}) < f(0) < 0$. We now have that there can be no root of f in $[0, \alpha_{k-1}]$, and that $f(X)$ is a strictly increasing continuous function for $X \in (\alpha_{k-1}, \infty)$ with $f(X) \to \infty$ as $X \to \infty$. By the intermediate value theorem, there must be a unique real root $\alpha_k \in (\alpha_{k-1}, \infty)$. This is the only zero of f over $\mathbb{R}^+$.

In all three cases, we find that either there is a contradiction or f has a unique positive real zero. By induction the result holds true for all polynomials of the form

$$f = a_n X^n - a_{n-1} X^{n-1} - a_{n-2} X^{n-2} - \cdots - a_1 X - a_0$$

with $a_i \geq 0 \; for \; i \in \{0, 1, \ldots, n-1\}$ and $a_n > 0$, $a_0 > 0$. □

2.2.3 Subdivision Polynomials

Definition 2.2.4. A polynomial $f \in \mathbb{Z}[X]$ is a **subdivision polynomial** if it is of the form

$$f = X^n - a_{n-1} X^{n-1} - a_{n-2} X^{n-2} - \cdots - a_1 X - a_0$$

with $a_i \in \mathbb{Z}_{\geq 0}$, $a_0 \neq 0$, $\sum_{i=0}^{n-1} a_i > 1$ for all $j \in \{0, \ldots, n-1\}$, and $f(X) \neq f'(X^d)$ for any other subdivision polynomial f' and some $d \in \mathbb{Z}_{\geq 2}$.

This final condition is best understood by considering the following examples.

Example 2. Whilst $f = X - m$ is a subdivision polynomial for all $m \in \mathbb{Z}_{\geq 2}$, $f(X^d) = X^d - m$ is not

a subdivision polynomial for any $d \in \mathbb{Z}_{\geq 2}$.

Example 3. Higher degree subdivision polynomials examples:

- $f = X^2 - X - 1$
- $f = X^6 - X^3 - X^2 - 1$
- $f = X^{24} - X^{15} - X^{10} - 1$

Higher degree subdivision polynomial non-examples:

- $f = X^4 - X^2 - 1 = (X^2)^2 - (X^2) - 1$
- $f = X^6 - X^4 - X^2 - 1 = (X^2)^3 - (X^2)^2 - (X^2) - 1$
- $f = X^6 - X^3 - 1 = (X^3)^2 - (X^3) - 1$

Clearly $f = X^n - a_{n-1}X^{n-1} - a_{n-2}X^{n-2} - \cdots - a_1X - a_0 \in \mathbb{Z}[X]$ is not a subdivision polynomial if there exists $d \in \mathbb{Z}_{\geq 2}$, such that d is a common factor of all $i > 0$ where a_i is non-zero.

Remark 1. If $f = X^n - a_{n-1}X^{n-1} - a_{n-2}X^{n-2} - \cdots - a_1X - a_0$ is a subdivision polynomial. Define the set $AI_{>0} = \{i \in \{1, \ldots, n-1\} : a_i > 0\}$. At least one of the following must be true:

- $gcd(n, j) = 1$ for some $j \in AI_{>0}$.
- $gcd(n, j_1, \ldots, j_t) = 1$ where $\{j_1, \ldots, j_t\} = AI_{>0}$.

The first condition is clearly a special case of the second condition.

Proposition 2.2.5. Consider a degree n subdivision polynomial $f \in \mathbb{Z}[X]$,

$$f = X^n - a_{n-1}X^{n-1} - a_{n-2}X^{n-2} - \cdots - a_1X - a_0$$

with $a_i \in \mathbb{Z}_{\geq 0}$, $a_0 \neq 0$ and $\sum_{i=0,\, i\neq j}^{n-1} a_i \geq 1$ for all $j \in \{0, \ldots, n-1\}$. Then f has just one positive zero $\beta > 1$.

Proof. By Lemma 2.2.2 a subdivision polynomial

$$f = X^n - a_{n-1}X^{n-1} - a_{n-2}X^{n-2} - \cdots - a_1X - a_0$$

has a positive real zero which we will denote β. Consider

$$f(1) = 1 - a_{n-1} - \cdots - a_1 - a_0 < 0$$

as $\sum_{i=0}^{n-1} a_i > 1$ by our definition of subdivision polynomials. We have that $f(1) < 0$ and $f(X) \to \infty$ as $X \to \infty$ where f has a unique zero greater than 0. Since f is continuous we would clearly have a contradiction if β, the unique positive real zero, was less than 1. □

Each irreducible subdivision polynomial defines a unique $1 < \beta \in \mathbb{R}$.

The subdivision polynomial f defines a subdivision of the unit interval into real sub-intervals, which have lengths equal to powers of τ, where $\tau = \frac{1}{\beta}$:

$$a_0\tau^n + a_1\tau^{n-1} + \cdots + a_{n-1}\tau = 1. \tag{2.1}$$

These sub-intervals are not prescribed an order, so we can assume that the $a_0 + a_1 + \cdots + a_{n-1}$ sub-intervals can be positioned end to end to span the unit interval without overlapping.

It is clear that $\beta \in \mathbb{Z}[\tau]$ and is in fact a unit of the ring $\mathbb{Z}[\tau]$. Dividing both sides of the equation 2.1 by τ demonstrates this.

$$\begin{aligned} 1 &= a_0\tau^n + a_1\tau^{n-1} + \cdots + a_{n-1}\tau \\ 1 &= (a_0\tau^{n-1} + a_1\tau^{n-2} + \cdots + a_{n-1})\tau \\ 1 &= \beta\tau \end{aligned}$$

Therefore, we can express $\mathbb{Z}[\tau]$ as $\mathbb{Z}[\beta]\left[\frac{1}{\beta}\right]$. For every element p in $\mathbb{Z}[\beta]$, p can be expressed as

$$p = b_0 + b_1\beta + \cdots + b_{n-1}\beta^{n-1}$$

for some $b_i \in \mathbb{Z}$ (see[15]). Therefore, for all $p \in \mathbb{Z}[\tau] = \mathbb{Z}[\beta]\left[\frac{1}{\beta}\right]$, we can write an expression for p as

$$p = \frac{b_0 + b_1\beta + \cdots + b_{n-1}\beta^{n-1}}{\beta^m}, \tag{2.2}$$

for some $b_i \in \mathbb{Z}$ and $m \in \mathbb{Z}_{\geq 0}$. It becomes clear that this expression is not unique, in particular by

using $\beta^{n-1} = a_{n-1}\beta^{n-2} + \cdots + a_1 + a_0\beta^{-1}$, we see that

$$\begin{aligned}
p &= \frac{b_0 + b_1\beta + \cdots + b_{n-1}\beta^{n-1}}{\beta^m} \\
p &= \frac{b_0 + b_1\beta + \cdots + b_{n-1}\big(a_{n-1}\beta^{n-2} + \cdots + a_1 + a_0\beta^{-1}\big)}{\beta^m} \\
p &= \frac{b_{n-1}a_0\beta^{-1} + (b_0 + b_{n-1}a_1) + \cdots + (b_{n-2} + b_{n-1}a_{n-1})\beta^{n-2}}{\beta^m} \\
p &= \frac{b_{n-1}a_0 + (b_0 + b_{n-1}a_1)\beta + \cdots + (b_{n-2} + b_{n-1}a_{n-1})\beta^{n-1}}{\beta^{m+1}} \\
p &= \frac{c_0 + c_1\beta + \cdots + c_{n-1}\beta^{n-1}}{\beta^{m+1}}
\end{aligned}$$

where $c_i \in \mathbb{Z}$ and $m \in \mathbb{Z}_{\geq 0}$.

Theorem 2.2.6. For all $0 < p \in \mathbb{Z}[\tau]$, there exists $b_0, \ldots, b_{n-1}, m \in \mathbb{Z}_{\geq 0}$ such that

$$p = \frac{b_0 + b_1\beta + \cdots + b_{n-1}\beta^{n-1}}{\beta^m}$$

We will prove this over the next few pages by showing that repeated use of the substitution

$$\beta^N = a_{n-1}\beta^{N-1} + \cdots + a_1\beta^{N-n+1} + a_0\beta^{N-n}.$$

will eventually give us an expression for p with only positive coefficients for all $m \geq \hat{N}$ for some $\hat{N} \in \mathbb{Z}_{\geq 0}$.

First note the following.

Remark 2. If we can show that for any $p \in \mathbb{Z}[\beta]$ with

$$p = b_0 + b_1\beta + \cdots + b_{n-1}\beta^{n-1}$$

can be written as

$$p = \frac{c_0 + c_1\beta + \cdots + c_{n-1}\beta^{n-1}}{\beta^m},$$

with $c_i \geq 0$ for some $m \in \mathbb{Z}_{\geq 0}$, then we can say the same for all $p \in \mathbb{Z}[\tau]$.

This remark will be justified further in Corollary 2.2.20

2.2.7 Linear system of Coefficients

As previously said $p \in \mathbb{Z}[\tau]$ does not have a unique expression, but does for each choice of m, when written in the form shown earlier on 2.2. I.e. if

$$p = \frac{b_0 + b_1\beta + \cdots + b_{n-1}\beta^{n-1}}{\beta^m}$$

then $b_{n-1}, \ldots, b_1, b_0$ are unique for each choice of $m \in \mathbb{Z}_{\geq 0}$. To make notation easier, for each $t \in \mathbb{Z}_{\geq 0}$ define a unitary function $[\cdot]_t$ which takes vectors from $\mathbb{R}^n$ and maps them onto $\mathbb{R}$ as follows. Let $c_0, \ldots, c_{n-1} \in \mathbb{R}$.

$$\left[\begin{pmatrix} c_{n-1} \\ c_{n-2} \\ \vdots \\ c_1 \\ c_0 \end{pmatrix}\right]_t = \frac{c_0 + c_1\beta + \cdots + c_{n-2}\beta^{n-2} + c_{n-1}\beta^{n-1}}{\beta^t}.$$

We will say that $b_i^{(m)}$ is the coefficient of β^i when the denominator of the expression for p is β^m. This allows us to use a shorthand for this expression of p, in the form of a vector with index m:

$$p = \left[\begin{pmatrix} b_{n-1}^{(m)} \\ b_{n-2}^{(m)} \\ \vdots \\ b_1^{(m)} \\ b_0^{(m)} \end{pmatrix}\right]_m = \frac{b_0^{(m)} + b_1^{(m)}\beta + \cdots + b_{n-2}^{(m)}\beta^{n-2} b_{n-1}^{(m)}\beta^{n-1}}{\beta^m}.$$

This makes it easy to see what happens when we use the substitution

$\beta^{n-1} = a_{n-1}\beta^{n-2} + \cdots + a_1 + a_0\beta^{-1}$.

$$\begin{aligned}
p &= \frac{b_0^{(m)} + b_1^{(m)}\beta + \cdots + b_{n-1}^{(m)}\beta^{n-1}}{\beta^m} \\
&= \frac{b_0^{(m)} + b_1^{(m)}\beta + \cdots + b_{n-1}^{(m)}\left(a_{n-1}\beta^{n-2} + \cdots + a_1 + a_0\beta^{-1}\right)}{\beta^m} \\
&= \frac{b_{n-1}^{(m)}a_0\beta^{-1} + (b_0^{(m)} + b_{n-1}^{(m)}a_1) + \cdots + (b_{n-2}^{(m)} + b_{n-1}^{(m)}a_{n-1})\beta^{n-2}}{\beta^m} \\
&= \frac{b_{n-1}^{(m)}a_0 + (b_0^{(m)} + b_{n-1}^{(m)}a_1)\beta + \cdots + (b_{n-2}^{(m)} + b_{n-1}^{(m)}a_{n-1})\beta^{n-1}}{\beta^{m+1}} \\
&= \frac{b_0^{(m+1)} + b_1^{(m+1)}\beta + \cdots + b_{n-1}^{(m+1)}\beta^{n-1}}{\beta^{m+1}}
\end{aligned}$$

This can be seen as a linear system of equations:

$$p = \left[\begin{pmatrix} b_{n-1}^{(m+1)} \\ b_{n-2}^{(m+1)} \\ \vdots \\ b_1^{(m+1)} \\ b_0^{(m+1)} \end{pmatrix}\right]_{m+1} = \left[\begin{pmatrix} a_{n-1} & 1 & 0 & 0 & \dots & 0 \\ a_{n-2} & 0 & 1 & 0 & \dots & 0 \\ a_{n-3} & 0 & 0 & 1 & & 0 \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ a_1 & 0 & 0 & 0 & \dots & 1 \\ a_0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} b_{n-1}^{(m)} \\ b_{n-2}^{(m)} \\ \vdots \\ b_1^{(m)} \\ b_0^{(m)} \end{pmatrix}\right]_{m+1}.$$

We will denote the matrix in this system by A:

$$A = \begin{pmatrix} a_{n-1} & 1 & 0 & 0 & \dots & 0 \\ a_{n-2} & 0 & 1 & 0 & \dots & 0 \\ a_{n-3} & 0 & 0 & 1 & & 0 \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ a_1 & 0 & 0 & 0 & \dots & 1 \\ a_0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}. \tag{2.3}$$

Note that this matrix is also the companion matrix of the polynomial f. It therefore follows that the characteristic equation of the matrix is precisely $f = 0$. The eigenvalues are therefore the roots of

$$X^n - a_{n-1}X^{n-1} - a_{n-2}X^{n-2} - \cdots - a_1X - a_0 = 0$$

We know then that A has an unique positive real eigenvalue β.

Definition 2.2.8. A **directed graph** $\Gamma(V, E)$ is a pair of sets, one set of vertices, V, and one multiset of directed edges, $E = \{(x, y)|x, y \in V^2\}$. Edges can be repeated.

A **walk** in $\Gamma(V, E)$ is a sequence of vertices $v_1, \ldots, v_r$, such that $(v_i, v_{i+1}) \in E$ for all $i \in \{1, \ldots, r-1\}$.

A **path** in $\Gamma(V, E)$ is a walk in which the vertices do not repeat.

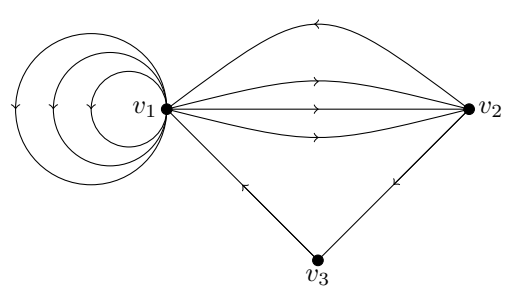
A **cycle** of length k in $\Gamma(V, E)$ is a walk $v_1, \ldots, v_k, v_1$ in which $v_1, \ldots, v_k$ is a path.

Definition 2.2.9. A non-negative real square matrix $A \in M_n(\mathbb{R}_{\geq 0})$ is **irreducible** if the associated directed graph G_A is *strongly connected*. I.e., if v_i, v_j are two distinct vertices in G_A, then there is a path from v_i to v_j.

The same matrix A is **primitive** if there exists $k \in \mathbb{N}$, such that all entries of A^k are strictly positive.

The directed graph $G_A = \Gamma(V_A, E_A)$ associated to our non-negative integer matrix A, G_A, has each vertex v_i associated with the ith in the matrix A. An edge (v_i, v_j) exists in E_A, if the jth entry in the ith row of A is non-zero. If this entry has value, $d > 0$, then there will be d copies of the edge (v_i, v_j).

Example 4. Consider the associated directed graph for $A = \begin{pmatrix} 3 & 1 & 0 \\ 3 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ which is the companion matrix to the subdivision polynomial $f = X^3 - 3X^2 - 3X - 1$.



Lemma 2.2.10. The matrix A in (2.3), is irreducible.

Proof. Consider the associated graph G_A.

Let $v_i \in G_A$, be a vertex in the directed graph associated with A, and let v_i be the vertex associated with the ith row of A.

For $1 \leq i \leq n-1$, $A(i, i+1) = 1$, the $(i+1)^{th}$ entry in the ith row of A is 1. This means that the edge $(v_i, v_{i+1}) \in E_A$ for $1 \leq i \leq n-1$. The entry $A(n, 1) = 1$, so $(v_n, v_1) \in E_A$.

Given two vertices v_i, v_j, we can find a path from v_i to v_j by following the sequence of vertices $v_i, v_{i+1}, \ldots, v_{j-1}v_j$. If $j < i$, then the sequence will be $v_i, v_{i+1}, \ldots, v_n, v_1, \ldots, v_j$. Thus G_A is strongly connected, so A is irreducible by definition. □

Theorem 2.2.11. ***The Perron Frobenius Theorem*** [16][17]
If A is an irreducible non-negative real matrix then the spectral radius, the maximal modulus of any eigenvalue of A, $\rho(A)$, is a positive real number, which is itself an eigenvalue of A.
If A is *primitive* and λ is an eigenvalue of A such that $|\lambda| = \rho(A)$, then $\lambda = \rho(A)$.

We have already seen that A has characteristic equation

$$X^n - a_{n-1}X^{n-1} - a_{n-2}X^{n-2} - \cdots - a_1X - a_0 = 0.$$

Thus if A is a primitive matrix, then the positive real number β is the unique eigenvalue of maximal modulus.

Lemma 2.2.12. The matrix A is primitive.

Proof. We will make use of the fact that a non-negative matrix A is primitive if the associated digraph is strongly connected and contains two cycles $C_1, \ldots, C_k$ with lengths $l_1, \ldots, l_k$ respectively, such that $gdc(l_1, \ldots, l_k) = 1$ [18].
We have already seen in Lemma 2.2.10 that G_A, the associated digraph to A is strongly connected. All this leaves us to do is to show that G_A contains cycles $C_1, \ldots, C_k$ with lengths $l_1, \ldots, l_k$ respectively, such that $gdc(l_1, \ldots, l_k) = 1$.
In remark 1, we noted two conditions for f to be a subdivision polynomial. We defined the set $AI_{>0} = \{i \in \{1, \ldots, n-1\} : a_i > 0\}$. At least one of the following must be true:

- $gcd(n, j) = 1$ for some $j \in AI_{>0}$.
- $gcd(n, j_1, \ldots, j_t) = 1$ where $\{j_1, \ldots, j_t\} = AI_{>0}$.

We will show that if either condition holds we are able to find two cycles in G_A whose lengths are relatively prime.

CASE 1: There exists some some $j \in AI_{>0}$ such that $gcd(n, j) = 1$.
Then the graph G_A must contain an edge from the vertex v_{n-j} to the vertex v_1. This allows us to consider two cycles in the graph $C_1 = v_1, v_2, \ldots, v_n, v_1$ of length n, and $C_2 = v_1, v_2, \ldots, v_{n-j}, v_1$ of

length $n-j$. Note that if $gcd(n,j)=1$, then also $gcd(n,n-j)=1$. Therefore there are two distinct cycles in G_A whose lengths are relatively prime, and thus A must be primitive.

CASE 2: $gcd(n, j_1, \ldots, j_t) = 1$ where $\{j_1, \ldots, j_t\} = AI_{>0}$.

Then the graph G_A must contain the edges $(v_{n-j_1}, v_1), (v_{j_2}, v_1), \ldots, (v_{j_t}, v_1)$. This allows us to define $t+1$ cycles as follows:

$$\begin{aligned} C_0 &= v_1, \ldots, v_n, v_1 \\ C_1 &= v_1, \ldots, v_{n-j_1}, v_1 \\ &\vdots \\ C_t &= v_1, \ldots, v_{n-j_t}, v_1. \end{aligned}$$

The length of cycle C_0 is n, and the lengths of cycles $C_1, \ldots, C_t$ are $n-j_1, \ldots, n-j_t$ respectively. We can note that since $gcd(n, j_1, \ldots, j_t) = 1$ and $j_i < n$ for all i, that

$$gcd(n, n-j_1, \ldots, n-j_t) = 1.$$

Thus we have found cycles in G_A whose lengths are relatively prime, and thus A must be primitive. □

Thus we have the following remark.

Remark 3. The matrix A has exactly one positive real eigenvalue, which is $\rho(A) = \beta$. All other eigenvalues of A have absolute value less than β.

2.2.13 Generalised Fibonacci Sequence

We define n linear recurrences $\{F_N^{(j)}\}_{-n+2}^{\infty}$ for $j \in \{1, \ldots, n\}$, each with characteristic polynomial

$$f = X^n - a_{n-1}X^{n-1} - a_{n-2}X^{n-2} - \cdots - a_1X - a_0$$

and with initial conditions, $F_1^{(j)}, F_0^{(j)}, \ldots, F_{2-n}^{(j)}$ given by the jth row of our matrix A. i.e.

$$F_N^{(j)} = a_{n-1}F_{N-1}^{(j)} + a_{n-2}F_{N-2}^{(j)} + \cdots + a_1F_{N-(n-1)}^{(j)} + a_0F_{N-n}^{(j)}$$

Example 5. Let $f = X^3 - 3X^2 - 3X - 1$, then $A = \begin{pmatrix} 3 & 1 & 0 \\ 3 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ and

$$\begin{array}{lll} F_1^{(1)} = 3 & F_0^{(1)} = 1 & F_{-1}^{(1)} = 0 \\ F_1^{(2)} = 3 & F_0^{(2)} = 0 & F_{-1}^{(2)} = 1 \\ F_1^{(3)} = 1 & F_0^{(3)} = 0 & F_{-1}^{(3)} = 0 \end{array}$$

Here $F_N^{(j)} = 3F_{N-1}^{(j)} + 3F_{N-2}^{(j)} + F_{N-3}^{(j)}$ for $j \in \{1, 2, 3\}$.

Definition 2.2.14. A polynomial f is **asymptotically simple** if the set of all zeros of f with maximal modulus contains a unique zero with greatest multiplicity.

Consider our characteristic polynomial f. By the Perron Frobenius Theorem, as f is the characteristic polynomial of the irreducible matrix A, then f has a unique eigenvalue of maximal modulus, namely β.

Proposition 2.2.15. ***Szczyrba*** **[19]**
Given a linear recurrence $\{F_N\}_{N\in\mathbb{N}}$ with asymptotically simple characteristic polynomial and non-trivial initial conditions, then the ratio limit

$$\lim_{N\to\infty} \left\{ \frac{F_{N+1}}{F_N} \right\}$$

exists and coincides with the unique zero with maximal modulus of the characteristic polynomial maximal multiplicity.

For each of the linear recurrences $\{F_N^{(j)}\}_{-n+2}^{\infty}$ this gives us a direct corollary.

Corollary 2.2.16. For $j \in \{1, \ldots, n\}$ the ratio limit of the sequence

$$\lim_{N\to\infty} \left\{ \frac{F_{N+1}^{(j)}}{F_N^{(j)}} \right\} = \beta.$$

Recall the irreducible matrix A. We define a linear system of recurrences

$$\{G_m^{(j)}\}_{m\in\mathbb{Z}}$$

by taking $G_m^{(j)}$ to be the jth entry in the first column of the matrix A^m. This gives us the recursion relations for all $N \in \mathbb{Z}$:

$$\begin{aligned}
G_N^{(1)} &= a_{n-1}G_{N-1}^{(1)} + G_{N-1}^{(2)} \\
G_N^{(2)} &= a_{n-2}G_{N-1}^{(1)} + G_{N-1}^{(3)} \\
&\vdots \\
G_N^{(j)} &= a_{n-j}G_{N-1}^{(1)} + G_{N-1}^{(j+1)} \\
&\vdots \\
G_N^{(n-1)} &= a_1G_{N-1}^{(1)} + G_{N-1}^{(n)} \\
G_N^{(n)} &= a_0G_{N-1}^{(1)}.
\end{aligned}$$

Lemma 2.2.17. $G_N^{(j)} = F_N^{(j)}$ for $N \geq 2 - n$.

Proof. We need to show that the recursion relations on $\left\{G_N^{(j)}\right\}_{N\in\mathbb{Z}}$ expand to give

$$G_N^{(j)} = a_{n-1}G_{N-1}^{(j)} + a_{n-2}G_{N-2}^{(j)} + \cdots + a_1G_{N-(n-1)}^{(j)} + a_0G_{N-n}^{(j)}$$

for all $j \in \{1, \ldots, n\}$. We will do this by considering 3 cases.

Case 1: $j = 1$.

$$\begin{aligned}
G_N^{(1)} &= a_{n-1}G_{N-1}^{(1)} + G_{N-1}^{(2)} \\
&= a_{n-1}G_{N-1}^{(1)} + a_{n-2}G_{N-2}^{(1)} + G_{N-2}^{(3)} \\
&\vdots \\
&= a_{n-1}G_{N-1}^{(1)} + \cdots + a_1G_{N-n+1}^{(1)} + G_{N-n+1}^{(n)} \\
&= a_{n-1}G_{N-1}^{(1)} + \cdots + a_1G_{N-n+1}^{(1)} + a_0G_{N-n}^{(1)}
\end{aligned}$$

Thus $G_N^{(1)} = F_N^{(1)}$ for all N.

Case 2: $j = n$.

$$
\begin{aligned}
G_N^{(n)} &= a_0 G_{N-1}^{(1)} \\
&= a_0 \left(a_{n-1} G_{N-2}^{(1)} G_{N-2}^{(2)} \right) \\
&= a_{n-1} \left(a_0 G_{N-2}^{(1)} \right) + a_0 \left(a_{n-2} G_{N-3}^{(1)} + G_{N-3}^{(3)} \right) \\
&= a_{n-1} G_{N-1}^{(n)} + a_{n-2} \left(a_0 G_{N-3}^{(1)} \right) + a_0 \left(a_{n-3} G_{N-4}^{(1)} + G_{N-4}^{(4)} \right) \\
&\vdots \\
&= a_{n-1} G_{N-1}^{(n)} + \cdots + a_2 \left(a_0 G_{N-n+1}^{(1)} \right) + a_0 \left(a_1 G_{N-n}^{(1)} + G_{N-n}^{(n)} \right) \\
&= a_{n-1} G_{N-1}^{(n)} + \cdots + a_1 G_{N-n+1}^{(n)} + a_0 G_{N-n}^{(n)}
\end{aligned}
$$

Thus $G_N^{(n)} = F_N^{(n)}$ for all N.

Case 3: $j \in \{2, \ldots, n-1\}$.

$$
\begin{aligned}
G_N^{(j)} &= a_{n-j} G_{N-1}^{(1)} + G_{N-1}^{(j+1)} \\
&= a_{n-j} \left(a_{n-1} G_{N-2}^{(1)} + G_{N-2}^{(2)} \right) + G_{N-1}^{(j+1)} \\
&= a_{n-1} \left(a_{n-j} G_{N-2}^{(1)} \right) + a_{n-j} \left(a_{n-2} G_{N-3}^{(1)} + G_{N-3}^{(3)} \right) + G_{N-1}^{(j+1)} \\
&= a_{n-1} \left(a_{n-j} G_{N-2}^{(1)} \right) + a_{n-2} \left(a_{n-j} G_{N-3}^{(1)} \right) + \\
&\quad + a_{n-j} \left(a_{n-3} G_{N-4}^{(1)} + G_{N-4}^{(4)} \right) + G_{N-1}^{(j+1)} \\
&= a_{n-1} \left(a_{n-j} G_{N-2}^{(1)} \right) + \cdots + a_{n-(j-1)} \left(a_{n-j} G_{N-j}^{(1)} \right) + a_{n-j} G_{N-j}^{(j)} + \\
&\quad + G_{N-1}^{(j+1)} \\
&= a_{n-1} \left(a_{n-j} G_{N-2}^{(1)} \right) + \cdots + a_{n-(j-1)} \left(a_{n-j} G_{N-j}^{(1)} \right) + a_{n-j} G_{N-j}^{(j)} + \\
&\quad + a_{n-1} \left(a_{n-j-1} G_{N-3}^{(1)} \right) + \cdots + a_{n-(j-1)} \left(a_{n-j-1} G_{N-j-1}^{(1)} \right) + \\
&\quad + a_{n-j-1} G_{N-j-1}^{(j)} + G_{N-2}^{(j+2)} \\
&= a_{n-1} \left(a_{n-j} G_{N-2}^{(1)} + a_{n-j-1} G_{N-3}^{(1)} \right) + \\
&\quad + a_{n-2} \left(a_{n-j} G_{N-3}^{(1)} + a_{n-j-1} G_{N-4}^{(1)} \right) + \cdots \\
&\quad \cdots + a_{n-(j-1)} \left(a_{n-j} G_{N-j}^{(1)} + a_{n-j-1} G_{N-j-1}^{(1)} \right) + \\
&\quad + a_{n-j} G_{N-j}^{(j)} + a_{n-j-1} G_{N-j-1}^{(j)} + G_{N-2}^{(j+2)} \\
&= a_{n-1} \left(a_{n-j} G_{N-2}^{(1)} + \cdots + a_0 G_{N-(n-j+2)}^{(1)} \right) + \cdots \\
&\quad \cdots + a_{n-(j-1)} \left(a_{n-j} G_{N-j}^{(1)} + \cdots + a_0 G_{N-n}^{(1)} \right) + \\
&\quad + a_{n-j} G_{N-j}^{(j)} + a_{n-j-1} G_{N-j-1}^{(j)} + \cdots + a_1 G_{N-(n-1)}^{(j)} + a_0 G_{N-n}^{(j)}
\end{aligned}
$$

Note that we have all the correct coefficients for $G_i^{(j)}$ with $i \leq N - j$.

Claim:

$$a_{n-j} G_{N-2}^{(1)} + \cdots + a_0 G_{N-2}^{(1)} = G_{N-1}^{(j)}$$

for all $j \in \{2, \ldots, n-1\}$ and for all $N \in \mathbb{N}$.

Recall the result from case 1.

$$\begin{aligned}
a_{n-j}G^{(1)}_{N-2} + \cdots + a_0 G^{(1)}_{N-2} &= G^{(1)}_{N+j-2} - a_{n-1}G^{(1)}_{N+j-3} - \cdots - a_{n-(j-1)}G^{(1)}_{N-1} \\
&= G^{(2)}_{N+j-3} - a_{n-2}G^{(1)}_{N+j-4} - \cdots - a_{n-(j-1)}G^{(1)}_{N-1} \\
&\vdots \\
&= G^{(j-2)}_{N+1} - a_{n-(j-2)}G^{(1)}_{N} - a_{n-(j-1)}G^{(1)}_{N-1} \\
&= G^{(j-1)}_{N} - a_{n-(j-1)}G^{(1)}_{N-(j-1)} \\
&= G^{(j)}_{N-1}
\end{aligned}$$

With this we have enough to prove case 3, so

$$G^{(j)}_N = a_{n-1}G^{(j)}_{N-1} + a_{n-2}G^{(j)}_{N-2} + \cdots + a_1 G^{(j)}_{N-(n-1)} + a_0 G^{(j)}_{N-n}$$

for all $j \in \{1, \ldots, n\}$ and $N \in \mathbb{Z}$. □

2.2.18 Positive coefficients

Theorem 2.2.19. For all $0 < p \in \mathbb{Z}[\beta]$, there exists $\hat{N} \in \mathbb{N}$ such that for all $N \geq \hat{N}$

$$p = \left[\begin{pmatrix} b^{(N)}_{n-1} \\ b^{(N)}_{n-2} \\ \vdots \\ b^{(N)}_1 \\ b^{(N)}_0 \end{pmatrix}\right]_N = \frac{b^{(N)}_0 + b^{(N)}_1\beta + \cdots + b^{(N)}_{n-2}\beta^{n-2} + b^{(N)}_{n-1}\beta^{n-1}}{\beta^N}$$

with $b^{(N)}_i \in \mathbb{Z}_{\geq 0}$ for all $i \in \{0, \ldots, n-1\}$.

Proof. Let $p \in \mathbb{Z}[\beta]$, such that $p > 0$ and

$$p = \left[\begin{pmatrix} b_{n-1}^{(0)} \\ b_{n-2}^{(0)} \\ \vdots \\ b_1^{(0)} \\ b_0^{(0)} \end{pmatrix}\right]_0 = b_0^{(0)} + b_1^{(0)}\beta + \cdots + b_{n-2}^{(0)}\beta^{n-2} + b_{n-1}^{(0)}\beta^{n-1}.$$

By using the substitution $\beta^{n-1} = a_{n-1}\beta^{n-2} + \cdots a_0\beta^{-1}$, we can find $b_i^{(1)}$ and $b_i^{(2)}$ for $i \in \{0, \ldots, n-1\}$ such that

$$p = \left[\begin{pmatrix} b_{n-1}^{(1)} \\ b_{n-2}^{(1)} \\ \vdots \\ b_1^{(1)} \\ b_0^{(1)} \end{pmatrix}\right]_1 = \left[\begin{pmatrix} a_{n-1} & 1 & 0 & 0 & \ldots & 0 \\ a_{n-2} & 0 & 1 & 0 & \ldots & 0 \\ a_{n-3} & 0 & 0 & 1 & & 0 \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ a_1 & 0 & 0 & 0 & \ldots & 1 \\ a_0 & 0 & 0 & 0 & \ldots & 0 \end{pmatrix} \begin{pmatrix} b_{n-1}^{(0)} \\ b_{n-2}^{(0)} \\ \vdots \\ b_1^{(0)} \\ b_0^{(0)} \end{pmatrix}\right]_1$$

$$p = \left[\begin{pmatrix} b_{n-1}^{(2)} \\ b_{n-2}^{(2)} \\ \vdots \\ b_1^{(2)} \\ b_0^{(2)} \end{pmatrix}\right]_2 = \left[\begin{pmatrix} a_{n-1} & 1 & 0 & 0 & \ldots & 0 \\ a_{n-2} & 0 & 1 & 0 & \ldots & 0 \\ a_{n-3} & 0 & 0 & 1 & & 0 \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ a_1 & 0 & 0 & 0 & \ldots & 1 \\ a_0 & 0 & 0 & 0 & \ldots & 0 \end{pmatrix} \begin{pmatrix} b_{n-1}^{(1)} \\ b_{n-2}^{(1)} \\ \vdots \\ b_1^{(1)} \\ b_0^{(1)} \end{pmatrix}\right]_2$$

$$p = \left[\begin{pmatrix} b_{n-1}^{(2)} \\ b_{n-2}^{(2)} \\ \vdots \\ b_1^{(2)} \\ b_0^{(2)} \end{pmatrix}\right]_2 = \left[A^2 \begin{pmatrix} b_{n-1}^{(0)} \\ b_{n-2}^{(0)} \\ \vdots \\ b_1^{(0)} \\ b_0^{(0)} \end{pmatrix}\right]_2 .$$

By repeating this substitution N times, we see that

$$p = \left[\begin{pmatrix} b_{n-1}^{(N)} \\ b_{n-2}^{(N)} \\ \vdots \\ b_1^{(N)} \\ b_0^{(N)} \end{pmatrix}\right]_N = \left[A^N \begin{pmatrix} b_{n-1}^{(0)} \\ b_{n-2}^{(0)} \\ \vdots \\ b_1^{(0)} \\ b_0^{(0)} \end{pmatrix}\right]_N .$$

By Lemma 2.2.17, $A^N = \begin{pmatrix} F_N^{(1)} & F_{N-1}^{(1)} & \dots & F_{N-(n-1)}^{(1)} \\ F_N^{(2)} & F_{N-1}^{(2)} & \dots & F_{N-(n-1)}^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ F_N^{(n)} & F_{N-1}^{(n)} & \dots & F_{N-(n-1)}^{(n)} \end{pmatrix}$ so we can write $b_i^{(N)}$ as

$$b_i^{(N)} = F_N^{(i)} b_{n-1}^{(0)} + F_{N-1}^{(i)} b_{n-2}^{(0)} + \cdots F_{N-(n-2)}^{(i)} b_1^{(0)} + F_{N-(n-1)}^{(i)} b_0^{(i)}.$$

By Corollary 2.2.16 we have $\lim_{N\to\infty} \frac{F_{N+1}^{(i)}}{F_N^{(i)}} = \beta$. We will use the notation $\approx$ here to imply that for $i \in \{1, \dots, n\}$ and sufficiently large N,

$$\frac{F_{N+1}^{(i)}}{F_N^{(i)}} \approx \beta.$$

Hence, if we take sufficiently large N, $F_N^{(i)} \approx \beta^k F_{N-k}^{(i)}$. Therefore

$$\begin{aligned} b_i^{(N)} &\approx b_{n-1}^{(0)} \beta^{n-1} F_{N-(n-1)}^{(i)} + \cdots + b_1^{(0)} \beta F_{N-(n-1)}^{(i)} + b_0^{(0)} F_{N-(n-1)}^{(i)} \\ b_i^{(N)} &\approx \left(b_{n-1}^{(0)} \beta^{n-1} + \cdots + b_1^{(0)} \beta + b_0^{(0)}\right) F_{N-(n-1)}^{(i)} \\ b_i^{(N)} &\approx p \cdot F_{N-(n-1)}^{(i)}. \end{aligned}$$

As A is a non-negative irreducible real matrix, $F_N^{(i)} = G_N^{(i)} \geq 0$ for all $N \in \mathbb{N}$. Since $p > 0$,s we can conclude for large enough N

$$b_i^{(N)} \approx p \cdot F_{N-(n-1)}^{(i)} \geq 0$$

for all $i \in \{0, \dots, n-1\}$. □

Theorem 2.2.6 is proved in the following corollary to Theorem 2.2.19.

Corollary 2.2.20. Let $p \in \mathbb{Z}[\tau]$ such that $p > 0$. Then there exists an expression for p

$$p = \frac{b_0 + b_1\beta + \cdots + b_{n-1}\beta^{n-1}}{\beta^m}$$

where $b_i, m \in \mathbb{Z}_{\geq 0}$

Proof. Theorem 2.2.19 tells us that for all $0 < p' \in \mathbb{Z}[\beta]$ there exists an expression

$$p' = \frac{b_0 + b_1\beta + \cdots + b_{n-1}\beta^{n-1}}{\beta^N}$$

where $b_i, N \geq 0$.

As we have previously stated, we can write $\mathbb{Z}[\tau]$ as

$$\mathbb{Z}[\tau] = \mathbb{Z}[\beta]\left[\frac{1}{\beta}\right].$$

Therefore all $0 < p \in \mathbb{Z}[\tau]$ can be written as

$$\begin{aligned} p &= \frac{\left(\dfrac{b_0 + b_1\beta + \cdots + b_{n-1}\beta^{n-1}}{\beta^N}\right)}{\beta^m} \\ &= \frac{b_0 + b_1\beta + \cdots + b_{n-1}\beta^{n-1}}{\beta^M} \end{aligned}$$

where $b_i, M \geq 0$, and for some $M = N + m > 0$.

Therefore each element $p > 0$ of the ring $\mathbb{Z}[\tau]$ can be expressed as a sum of powers of τ using only nonnegative coefficients. $\square$

2.3 Subdivisions and Trees

2.3.1 β-Subdivisions

We will begin by formalising our definition of a subdivision.

Definition 2.3.2. A **finite subdivision** of $[0,1]$, $S = S(B[S], I(S))$, of the real interval $[0,1]$, is described by a pair of sets:

- The finite set of breakpoints in S is $B[S] \subset [0,1]$. $B[S] = \{0 = b_0, b_1, \ldots, b_n = 1\}$, where $b_i < b_{i+1}$ for each i.
- The finite set of sub-intervals is $I[S]$. For all $I, J \in I[S]$, $I, J \subset [0,1]$, and $I \cap J = \emptyset$

The sets $B[S] \cup I[s] = [0,1]$ and $B[S] \cap I[S] = \emptyset$.
The **size** of the subdivision $size(S) = |I(S)| = |B(S)| - 1$

Remark 4. A subdivision of $[0,1]$, S, can be defined solely by finding either of $B[S]$ or $I[S]$. Having one set allows us to derive the other.

This remark allows us to refer to a subdivision by a set of breakpoints (respectively a set of sub-intervals) without having to define the corresponding sub-intervals (respectively breakpoints). In certain circumstances it will be more advantageous to think of a subdivision as a set of breakpoints, and in other cases as a set of sub-intervals.

Definition 2.3.3. We denote the length of an interval I to be $L(I)$.

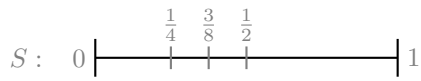
Let β be the unique positive root of the *irreducible* subdivision polynomial

$$f_\beta = X^n - a_{n-1}X^{n-1} - a_{n-2}X^{n-2} - \cdots - a_1X - a_0$$

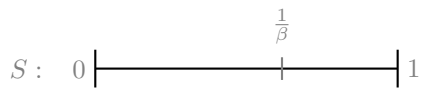
. Note that two important properties of f_β are that f_β is minimal and not equivalent to $g(X^k)$ for any $g \in \mathbb{Z}[X]$ and $k \in \mathbb{Z}_{\geq 2}$.

Definition 2.3.4. A β-subdivision of $[0,1]$, S, is any subdivision of $[0,1]$, S such that for any $\ell_i \in I[S]$ $L(\ell_i) = \beta^{r_i}$, for some $r_i \in \mathbb{Z}$.

Example 6. Let $f_2 = X - 2 \in \mathbb{Z}[X]$, an irreducible polynomial over $\mathbb{Z}$ which has 2 as the only positive root. Then if S is a subdivision of $[0,1]$ with $B[S] = \{0, \frac{1}{4}, \frac{3}{8}, \frac{1}{2}, 1\}$, then S is a 2-subdivision.



Example 7. Let $\beta = \dfrac{1+\sqrt{5}}{2}$. Then, $f_\beta = X^2 - X - 1$, where β is the unique positive root of f_β. We can find a β-subdivision below.



Note that there is not a unique subdivision polynomial with β as a root. Consider $f_2 = X - 2$, and $f_2' = X^2 - X - 2$, both of which are subdivision polynomials for which 2 is a zero. Since f_2 is irreducible, this is the only irreducible subdivision polynomial for which 2 is a zero.

Lemma 2.3.5. Let the set $\{0 = p_0, p_1, \ldots, p_t = 1\}$ partition of $[0,1]$ into t intervals such that $p_i \in \mathbb{Z}[\tau]$ for $i \in \{0, \ldots, t\}$. Then there exists a β-subdivision S such that $\{p_0, p_1, \ldots, p_t\} \subset B[S]$.

Proof. Since $p_i \in Z[\tau] \cap [0,1]$ for $i \in \{0, \ldots, t\}$, define $q_i = p_i - p_{i-1} \in \mathbb{Z}[\tau] \cap [0,1]$ for $i \in \{1, \ldots, t\}$.

We will prove the Lemma by recalling Corollary 2.2.20. As $q_i \in \mathbb{Z}[\tau] \cap [0,1]$, there exists $N \in \mathbb{N}$ such that

$$q_i = \left[\begin{pmatrix} b_{n-1}^{(i)} \\ b_{n-2}^{(i)} \\ \vdots \\ b_1^{(i)} \\ b_0^{(i)} \end{pmatrix} \right]_N = \frac{b_0^{(i)} + b_1^{(i)}\beta + \cdots + b_{n-2}^{(i)}\beta^{n-2} + b_{n-1}^{(i)}\beta^{n-1}}{\beta^N}$$

where $b_j^{(i)} \in \mathbb{Z}_{\geq 0}$ for $j \in \{0, \ldots, n-1\}$. Each sub-interval q_i can be substituted for $b_0^{(i)} + \cdots + b_{n-1}^{(i)}$ sub-intervals which all have length which is some power of β. Since we can convert every interval q_i in this way, there is a β-subdivision which contains the breakpoints $\{p_0, p_1, \ldots, p_t\}$. $\square$

Recall the definition of the Bieri-Strebel group.

Definition. *The Bieri-Strebel Group*

The Bieri-Strebel Group $G(I, A, P)$ is the group of all piecewise-linear homeomorphisms of the unit interval (I), with breakpoints in A, and slopes with gradient in P where P **is a group of units contained in** A**.**

In particular, we have defined G_β below

$$G_\beta = G\left([0,1], \mathbb{Z}\left[\frac{1}{\beta}\right], \langle\beta\rangle\right)$$

for β the unique positive zero of an irreducible subdivision polynomial.

Given $g \in G_\beta$, $g : [0,1] \to [0,1]$, with breakpoints $\{(0,0) = (p_0, q_0), (p_1, q_1), \ldots, (p_t, q_t) = (1,1)\}$,

$$g(x) = \left(\frac{q_{i+1} - q_i}{p_{i+1} - p_i}\right)(x - p_i) + q_i \text{ for } x \in [p_i, p_{i+1}]$$

for $i \in \{0, \ldots, t-1\}$.

Then there exist two subdivisions of $[0,1]$, P and Q, both of size t, where $P = \{p_0, p_1, \ldots, p_t\}$ and $Q = \{q_0, q_1, \ldots, q_t\}$, and we write $g = (P, Q)$, the affine interpolation from the subdivision P to the subdivision Q.

Remark 5. Given any β-subdivisions P, Q, such that $size(P) = size(Q)$, the map $g = (P, Q) \in G_\beta$.

Proposition 2.3.6. Let $g = (P, Q) \in G_\beta$ have breakpoints $\{(0,0) = (p_0, q_0), (p_1, q_1), \ldots, (p_t, q_t) = (1,1)\}$. Then there exists β-subdivisions P', Q' such that $g = (P', Q')$.

Proof. Let $g \in G_\beta$, such that $g = (P, Q)$, where $P = \{0 = p_0, p_1, \ldots, p_t = 1\}$ and $Q = \{0 = q_0, q_1, \ldots, q_t = 1\}$. By Lemma 2.3.5, there exists P_1, a β-subdivision of $[0,1]$ such that $B[P] \subset B[P_1] = \{0 = b_0, b_1, \ldots, b_s = 1\}$. Define Q_1, a subdivision of $[0,1]$ by taking $Q_1 = g(P_1)$ with $B[Q_1] = \{0 = g(b_0), g(b_1), \ldots, g(b_s) = 1\}$. Since $p_i \in B[P_1]$, and $g(p_i) = q_i$ for $i \in \{0, \ldots, t\}$, then $B[Q] \subset B[Q_1]$.

For all $b_j \in B[P_1]$, such that $p_i \leq b_j < p_{i+1}$),

$$g(b_j) = \beta^{r_i}(b_j - p_i) + q_i$$

for some $r_i \in \mathbb{Z}$.

Therefore if b_j and b_{j+1} are adjacent breakpoints in P_1, then the difference between the adjacent breakpoints $g(b_j), g(b_{j+1})$ in Q_1 is,

$$\begin{aligned} g(b_{j+1}) - g(b_j) &= \beta^{r_i}(b_{j+1} - p_i) + q_i - \beta^{r_i}(b_j - p_i) + q_i \\ &= \beta^{r_i}(b_{j+1} - p_i) - \beta^{r_i}(b_j - p_i) \\ &= \beta^{r_i}(b_{j+1} - b_j) \end{aligned}$$

Since b_j, b_{j+1} are adjacent breakpoints in the β-subdivision P_1, $b_{j+1} - b_j = \beta^{r'_j}$ for some $r'_j \in \mathbb{Z}$. Thus,

$$g(b_{j+1}) - g(b_j) = \beta^{r_i}\beta^{r'_j} = \beta^{r}_{i,j}$$

for some $r_{i,j} \in \mathbb{Z}$.

So the difference between any two adjacent breakpoints in Q_1 is a power of β. So Q_1 is also a β-subdivision. Hence, $g = (P_1, Q_1)$ where P_1 and Q_1 are both β-subdivisions. □

We now know that every element of G_β can be expressed as the affine interpolation between two β-subdivisions.

2.3.7 Regular β-subdivisions

We will now define what it means for a subdivision to be *regular*. We will provide a precise definition for regular β-subdivisions for which β is a quadratic integer. This definition will be analogous to a definition for regular β-subdivisions for any β, the unique root of an irreducible subdivision polynomial.

Let the quadratic integer β be the positive real zero of

$$f_\beta = X^2 - a_1 X - a_0$$

which is an irreducible polynomial, i.e. $\beta \notin \mathbb{Z}$. We can deduce from this that

$$\beta^2 = a_1\beta^1 + a_0$$
$$\beta^N = a_1\beta^{N-1} + a_0\beta^{N-2}.$$

Since $1 = \beta^0$,

$$1 = a_1\beta^{-1} + a_0\beta^{-2}$$
$$1 = a_1\tau + a_0\tau^2.$$

The subdivision polynomial f_β defines a β-subdivision of $[0, 1]$ with a_1 sub-intervals of length τ and a_0 sub-intervals of length τ^2. It is useful to let $K = a_1 + a_0$, and $N = a_1 + a_0 - 1$.

Note that the β-subdivision of $[0, 1]$ defined by f_β contains K sub-intervals, and so the number of intervals has increased by N.

Definition 2.3.8. The coefficient vector $\underline{\mathbf{a}}$ of a subdivision polynomial $f_\beta = X^n - a_{n-1}X^{n-1} - a_{n-2}X^{n-2} - \cdots - a_1X - a_0$ is the vector $\underline{\mathbf{a}} = (a_{n-1}, \ldots, a_1, a_0)$.

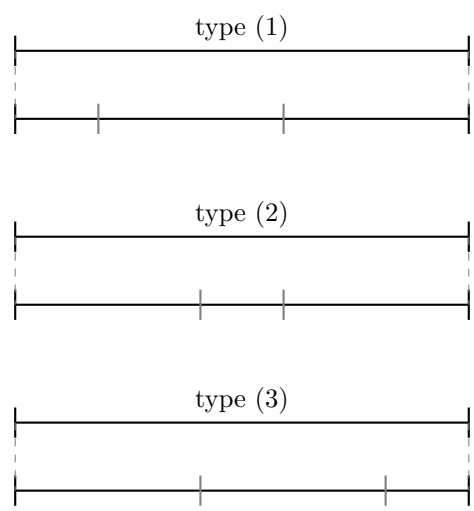
Example 8. Let $f_3 = X - 3$, then the coefficient vector $\underline{\mathbf{a}} = (3)$.

In [5] Brown looked at subdivision polynomials with coefficient vector $(k, 1)$ for $k \geq 1$. Brown gives us the following definition.

Definition 2.3.9. [Brown]
A k-partition of the interval $[0, 1]$ of **type** (i), where $1 \leq i \leq k+1$, is a subdivision of $[0, 1]$ containing $k+1$ sub-intervals k of which have length τ and 1 which has length τ^2 which can be found in the ith position.

In a k-partition of $[0, 1]$ of type (i), there are $i - 1$ longer sub-intervals of length τ preceding the short sub-interval of length τ^2.

Example 9. 2-partitions of type (1), (2), (3).



A k-partition of type (i) can be performed on a general interval $[A, B]$, and is defined to be the image of the intervals of the k-partition of $[0, 1]$ of type (i) under the map

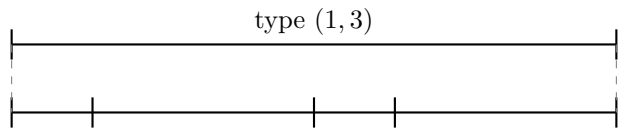
$$x \mapsto A + (B - A)x.$$

We can extend this definition to subdivision polynomials with coefficient vector (a_1, a_0), and here the type of the partition will still depend on the location of the shorter intervals.

Definition 2.3.10. An (a_1, a_0)**-partition** of $[0,1]$ of type $(i_1, i_2 \ldots, i_{a_0})$ with $1 \leq i_1 < i_2 < \cdots$ $\cdots < i_{a_0} \leq K$, is a subdivision of $[0,1]$, P, such that $I(P)$ contains a_1 sub-intervals of length τ and a_0 sub-intervals of length τ^2. The a_0 intervals of length τ^2 are found in positions i_j for $1 \leq j \leq a_0$.

Clearly a k-partition of type (i) can be equivalently described as a $(k,1)$-partition of type (i_1).

Example 10. Below is a $(2,2)$-partition of type $(1,3)$.



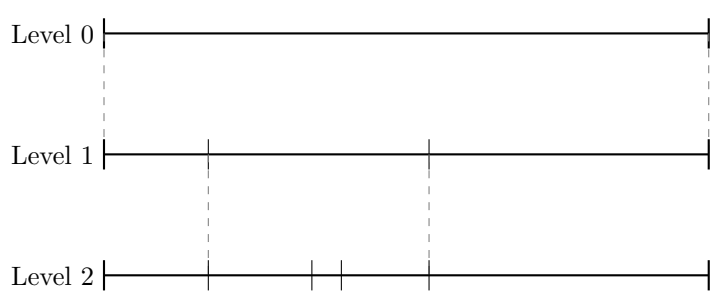
These (a_1, a_0)-partitions can similarly be performed on any interval. An (a_1, a_0)-partition of type $(i_1, \ldots, i_{a_0})$ of an interval $[A, B]$ is defined to be the image of the intervals of the (a_1, a_0)-partition of type $(i_1, \ldots, i_{a_0})$ of $[0,1]$ under the map

$$x \mapsto A + (B - A)x$$

We use (a_1, a_0)-partitions to build up a specific type of subdivision.

Definition 2.3.11. An (a_1, a_0)-subdivision of level 0 is the unit interval $[0,1]$.
An (a_1, a_0)-subdivision of level N is a subdivision of $[0,1]$ obtained by performing an (a_1, a_0)-partition of any type on an interval in an (a_1, a_0)-subdivision of level $N-1$.

Example 11. Below is a $(2,1)$-subdivision of level 2.



Since $L\big([0,1]\big) = 1 = \beta^0$, the unit interval is a β-subdivision. By noting that the substitution $\beta^t = a_1\beta^{t-1} + a_0\beta^{t-2}$ is being used whenever performing an (a_1, a_0)-partition on an interval of length β^t, we have the following remark.

Remark 6. An (a_1, a_0)-subdivision of any level is a β-subdivision.

Definition 2.3.12. A β-subdivision which is equivalent to an (a_1, a_0)-subdivision of some level, is called a **regular** β-subdivision.

Whilst all regular β-subdivisions are β-subdivisions, **not** all β-subdivisions are *regular* β-subdivisions.

2.4 (a_1, a_0)-trees

Recall the definition of a *directed simple graph*

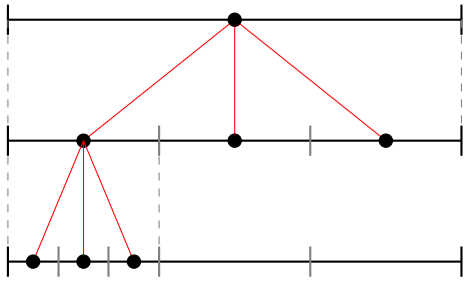
Definition 2.4.1. A **directed simple graph** $\Gamma(V, E)$ is a pair of sets, one set of vertices, V, and one set of directed edges, $E = \{(x, y)|x, y \in V^2, x \neq y\}$. A vertex $v \in V$ has **in-degree** $d_{in}(v) = \big|\{(x, y) \in E\big|y = v\}\big|$ and **out-degree** $d_{out}(v) = \big|\{(x, y) \in E\big|x = v\}\big|$.
The **degree** of v is $d(v) = d_{in}(v) + d_{out}(v)$.

Unlike directed graphs, directed simple graphs do not admit repeated edges and so consist of a set, rather than a multiset, of edges.

Definition 2.4.2. A **(rooted) tree** is a directed simple graph with a root R such that for all $x \in V$, there exists a unique set of vertices $P = \{p_0, p_1, \ldots, p_t\} \subset V$ with $(p_i, p_{i+1}) \in E$, $(p_{i+1}, p_i) \notin E$, where $R = p_0$ and $x = p_t$. In a tree vertices are also called **nodes**.
Any node x in the tree with degree $d(x) = d_{in}(x) = 1$ is called a leaf. A non-leaf node, y in the tree must have $d_{out}(y) \neq 0$ and is the root of a sub-tree called a **caret** and is the *parent* to some number of other nodes called *children*. We use $x(j)$ to denote the jth child of the node x as we read from left to right.

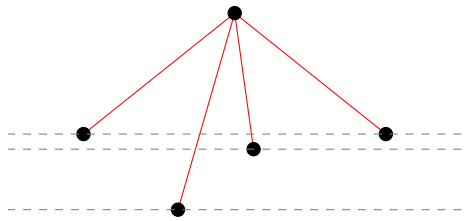
It is worth noting that for a root node R $d_in(R) = 0$, and $d_in(X) = 1$ for all $X \in V \backslash R\}$.

In all of our trees, any edge is directed down the tree and so we will dispense with arrows to highlight this.
The following tree can be seen to represent a regular 3-subdivision of level 2.



For most subdivisions the nodes and leaves of the trees will represent different lengths. To show this, fix the height of a node representing a fixed length. The lower the node, the shorter the interval. Consider the following partition.

We can model this partition with the following caret.

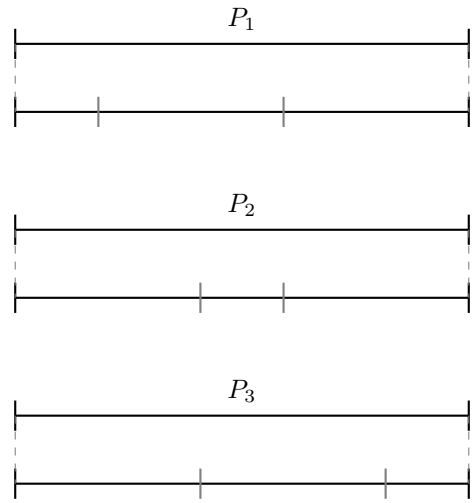


We cannot move any of the nodes vertically, but they have freedom to move horizontally as long as they do not pass each other. If two nodes represent intervals of the same length then they must be at the same height, and if node x represents a shorter interval than node y then x should be lower than y. We will consider (a_1, a_0)-partitions through a tree-representation.

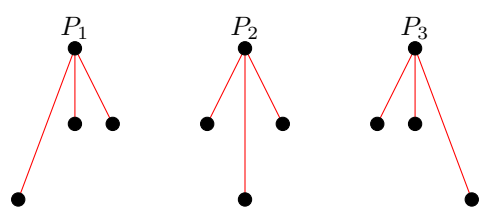
Brown [5] introduces modified trees, called k-trees as a way of representing the k-subdivisions. In a k-tree, the root node represents the interval $[0, 1]$, then each time a k-partition is performed on an interval, $k + 1$ children are added to the leaf representing the partitioned interval. Each child is identified with an interval in the k-partition in left-to-right order, with the short interval drawn twice as far below its parent as the other children.

Brown introduced k-partitions as having $k+1$ different types, and this will need to be the same when defining the associated carets.

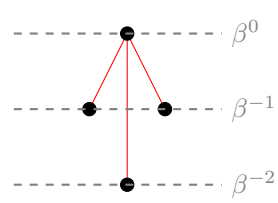
Example 12. Consider the subdivision polynomial $f_\beta = X^2 - 2X - 1$ Then f_β can describe any of the following partitions.



These partitions have corresponding carets:



The root of the polynomial equation $f = X^2 - 2X - 1 = 0$ is $\beta = \sqrt{2} + 1$. The ratio of the lengths represented by the highest node to the lengths represented by the middle nodes is β. In fact, if this change in height is found between any two nodes, the ratio of lengths represented is also β. The ratio of the lengths represented by the highest node to the lengths represented by the lowest node in these carets is β^2.



These k-trees can be readily adapted to correspond to (a_1, a_0)-subdivisions. The root of the (a_1, a_0)-caret is the interval that is being (a_1, a_0)-partitioned. We assign each sub-interval in an (a_1, a_0)-partition of type $(i_1, \ldots, i_{a_0})$ to a child in the (a_1, a_0)-caret with the i_j^{th} child being drawn twice as far below the parent, for $j = 1, \ldots, a_0$.

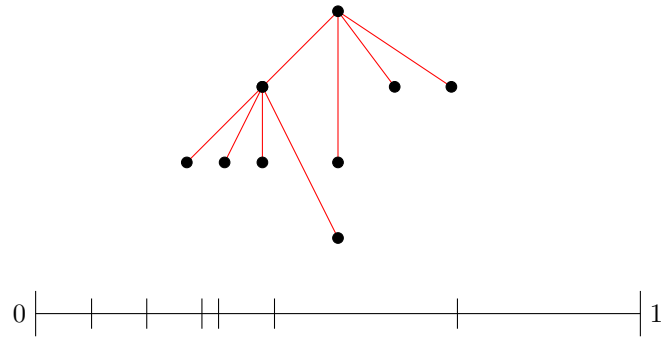
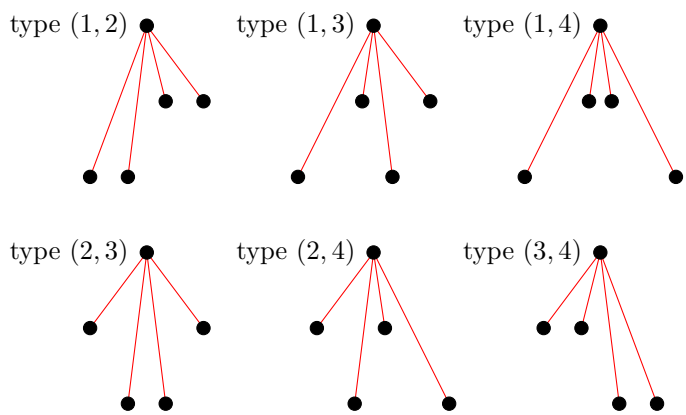


Figure 2.1: Example of 3-tree and associated 3-subdivision of level 2.

Example 13. There are six types of $(2,2)$-partition, shown here as $(2,2)$-carets:



Note, the position of the longer legs in each caret correlate to the shorter sub-intervals in the $(2,2)$-partition.

Definition 2.4.3. An (a_1, a_0)-tree of level N, is a tree with N carets in which every caret is an (a_1, a_0)-caret of some type.

Since (a_1, a_0)-carets correspond to (a_1, a_0)-partitions

Remark 7. Each (a_1, a_0)-tree of level N corresponds to an (a_1, a_0)-subdivision of size N.

Definition 2.4.4. Let X be a node in the (a_1, a_0)-tree $\mathcal{T}$. A **sub-tree** from the node X, is $\mathcal{T}_X$, the (a_1, a_0)-tree found within $\mathcal{T}$ which has X as the root node.
The **absence** of the sub-tree from X is $\mathcal{T} \backslash \mathcal{T}_X$, the (a_1, a_0)-tree identical to $\mathcal{T}$, except that X is now a leaf.

In Figure 2.3, we see a $(2,1)$-tree with root node N. The sub-tree from the first child of N, $\mathcal{T}_{N(1)}$,

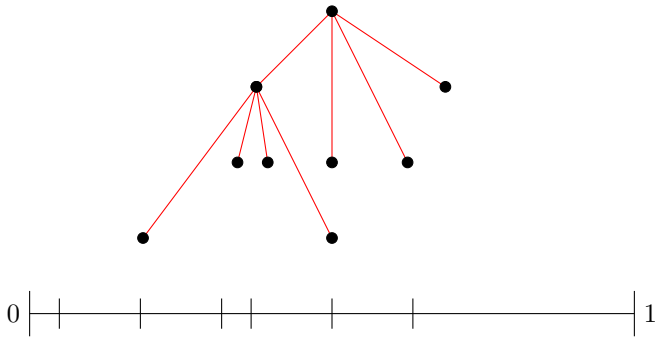


Figure 2.2: Example of $(2, 2)$-tree of level 2 and associated $(2, 2)$-subdivision.

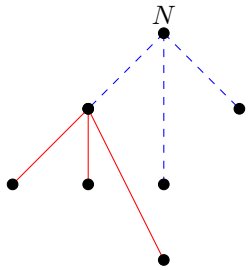


Figure 2.3: The sub-tree $\mathcal{T}_{N(1)}$ of the $(2, 1)$-tree $\mathcal{T}$ highlighted in red.

is highlighted with edges in red. The absence of the sub-tree from $N(1)$, $\mathcal{T}\backslash\mathcal{T}_{N(1)}$ is highlighted with dashed edges in blue.

Definition 2.4.5. The **depth** of an (a_1, a_0)-tree $\mathcal{T}$ is $D(\mathcal{T}) = d - 1$ where β^{-d} is the smallest size of an interval in the corresponding (a_1, a_0)-subdivision.
The **height** of a node $X \in \mathcal{T}$ is $H(X) = h$ where β^{-h} is the length of the corresponding interval in the corresponding (a_1, a_0)-subdivision.

If $\mathcal{T}$ is an (a_1, a_0)-tree, $D(\mathcal{T}) = H(X) + 1$, where X is a non-leaf node of maximal height.

Definition 2.4.6. An **end-caret** in an (a_1, a_0)-tree $\mathcal{T}$, is an (a_1, a_0)-caret in $\mathcal{T}$, such that all the children in the caret are leaves.

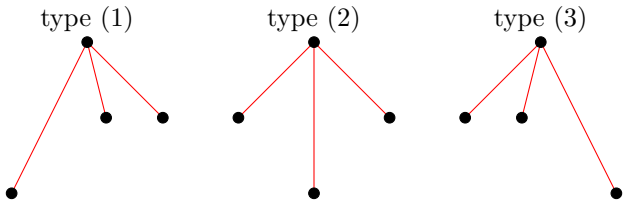
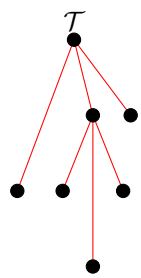


Figure 2.4: Any $(2, 1)$-caret has depth 1

The **root-caret** of an (a_1, a_0)-tree $\mathcal{T}$, is the (a_1, a_0)-caret with the root of $\mathcal{T}$ as the parent.

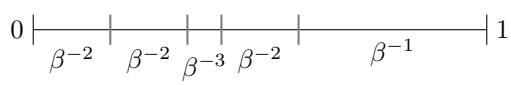
Definition 2.4.7. The leaf sequence of an (a_1, a_0)-tree, $\mathcal{T}$ is a vector with entries equal to the heights of the leaves in $\mathcal{T}$ as read from left to right and is denoted $\mathcal{L}(\mathcal{T})$.

Example 14. Consider the following $(2, 1)$-tree $\mathcal{T}$.



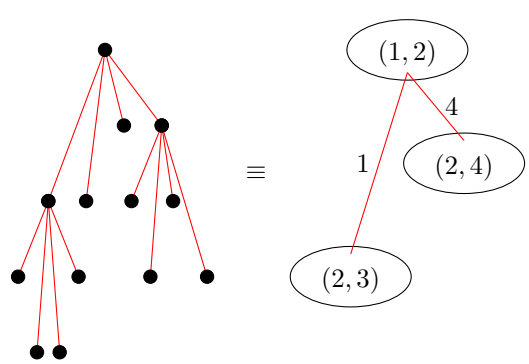
Then the leaf sequence of $\mathcal{T}$ is $\mathcal{L}(\mathcal{T}) = (2, 2, 3, 2, 1)$.

The $(2, 1)$-tree corresponds to the following $(2, 1)$-subdivision of level 2.



Notice, the leaf sequence only tells us the intervals and their order in the $(2, 1)$-subdivision at level 2, and does not tell us how this was obtained.

It is convenient to introduce a reduced (a_1, a_0)-tree notation. Each non-leaf node in an (a_1, a_0)-tree corresponds to an (a_1, a_0)-partition of some type in the (a_1, a_0)-subdivision. Replace each (a_1, a_0)-caret with a node labelled with the type of (a_1, a_0)-caret, and if the ith child of the (a_1, a_0)-caret is a non-leaf node, give the edge joining the two labelled nodes the label i. As an example consider the following $(2, 2)$-tree.



This notation becomes particularly useful when dealing with (a_1, a_2)-trees where a_1 or a_2 are significantly large.

If $g = (S_1, S_2) \in G_\beta$, where S_1, S_2 are regular β-subdivisions, then we can also write

$$g = (\mathcal{T}_1, \mathcal{T}_2)$$

where $\mathcal{T}_1$ and $\mathcal{T}_2$ are corresponding (a_1, a_0)-trees to the (a_1, a_0)-subdivisions S_1 and S_2 respectively.

Now that we have an understanding of what an (a_1, a_0)-tree is, we can provide a proper definition for F_β, for β the positive zero of an irreducible subdivision polynomial $f_\beta = X^2 - a_1 X - a_0$.

Definition 2.4.8.

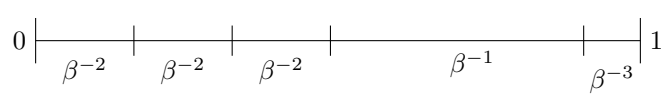
$$F_\beta := \left\{ g \in G_\beta \middle| \text{ There exists } (a_1, a_0)\text{-trees } \mathcal{T}_1 \text{ and } \mathcal{T}_2 \text{ such that } g = (\mathcal{T}_1, \mathcal{T}_2) \right\}$$

2.4.9 (a_1, a_0)-refinements

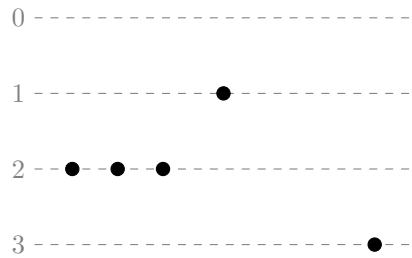
Definition 2.4.10. An (a_1, a_0)**-refinement of size** 0 of a β-subdivision S is S.
An (a_1, a_0)**-refinement of size** i**,** R, of a β-subdivision S is obtained by performing an (a_1, a_0)-partition on a sub-interval in an (a_1, a_0)-refinement of size $i - 1$ of S .
We denote the size of an (a_1, a_0)-refinement of size i a β-subdivision, $[S : R] = i$.

An (a_1, a_0)-refinement of a β-subdivision S can be thought of as hanging (a_1, a_0)-trees from nodes which represent the intervals of S.

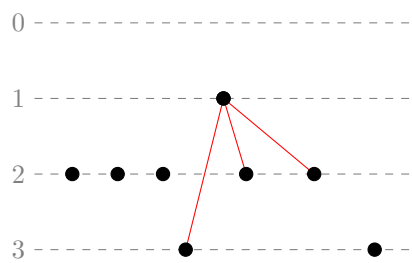
Example 15. Let $f_\beta = X^2 - 2X - 1$, and so $\beta = \sqrt{2} + 1$. The following β-subdivision, S, is not regular.



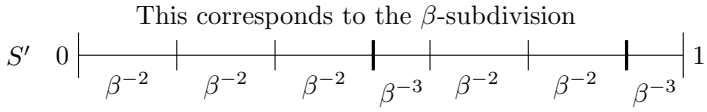
We can model this β-subdivision as a forest of *empty* $(2, 1)$-trees where the ith node represents the ith sub-interval in the β-subdivision as read from left to right.



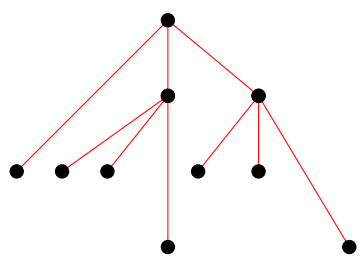
We can demonstrate a $(2,1)$-refinement on S of size 1, by performing an $(2,1)$-partition of type (1) on the fourth sub-interval of S.



This corresponds to the β-subdivision



The β-subdivision S' is regular and has associated $(2,1)$-tree



We have shown an instance when we can find an (a_1, a_0)-refinement of a non-regular β-subdivision, which is a regular β-subdivision.

Lemma 2.4.11. An (a_1, a_0)-refinement of an (a_1, a_0)-subdivision is still an (a_1, a_0)-subdivision.

Proof. An (a_1, a_0)-subdivision S is a regular β-subdivision, and has an associated (a_1, a_0)-tree $\mathcal{T}$. An (a_1, a_0)-refinement of S involves performing (a_1, a_0)-partitions on the sub-intervals of S. Each sub-interval of S corresponds to a leaf in $\mathcal{T}$, and each (a_1, a_0)-partition, corresponds to an (a_1, a_0)-caret.

So an (a_1, a_0)-refinement of S corresponds to hanging (a_1, a_0)-trees from the leaves of $\mathcal{T}$. In doing this, we will still have a tree in which every caret is an (a_1, a_0)-caret, so is still an (a_1, a_0)-tree. Every

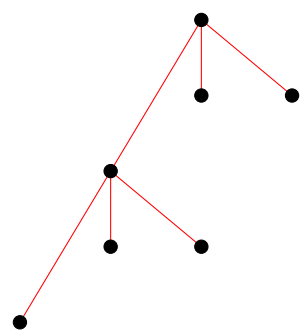
(a_1, a_0)-tree defines an (a_1, a_0)-subdivision, and so every (a_1, a_0)-refinement of an (a_1, a_0)-subdivision is also an (a_1, a_0)-subdivision. □

Remark 8. Let S' be an (a_1, a_0)-refinement of a β-subdivision S. Any (a_1, a_0)-refinement of S' is also an (a_1, a_0)-refinement of S.

Uniform β-Subdivisions

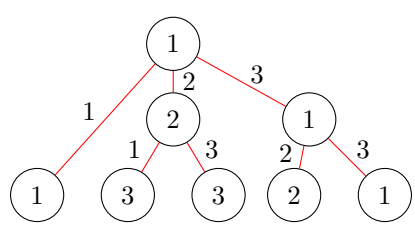
An advantage of using (a_1, a_0)-refinements is that we can avoid dealing with an (a_1, a_0)-tree with incredibly *unbalanced* leaf sequences.

Example 16. The $(2, 1)$-tree, $\mathcal{T}$, with leaf sequence $\mathcal{L}(\mathcal{T}) = (4, 3, 3, 1, 1))$,



which corresponds to the $(2, 1)$-subdivision S

There is a large difference in the lengths of sub-intervals in this $(2, 1)$-subdivision. We can find a refinement of S, S' such that the ratio between any two sub-intervals is at most $\beta = \sqrt{2} + 1$. We will use the reduced $(2, 1)$-tree notation for the corresponding $(2, 1)$-tree to S', $\mathcal{T}'$:



The leaf sequence is $\mathcal{L}(\mathcal{T}') = (4, 3, 3, 3, 3, 4, 3, 3, 3, 4, 4, 3, 4, 3, 4, 3, 3)$

Definition 2.4.12. An **uniform β-subdivision of depth** N is a β-subdivision with only intervals of length $\frac{1}{\beta^N}$ or $\frac{1}{\beta^{N+1}}$ for some $N \in \mathbb{N}$.

Any (a_1, a_0)-tree, $\mathcal{T}$, corresponding to a uniform (a_1, a_0)-subdivision, S, will have leaf sequence $\mathcal{L}(\mathcal{T}) = (\ell_1, \ldots, \ell_r)$, where $\ell_i \in \{N, N+1\}$ for all $i \in \{1, \ldots, r\}$.
In a uniform (a_1, a_0) subdivision, each interval is either **long** or **short**.

Note that a uniform β-subdivision is not necessarily regular, nor is a regular β-subdivision necessarily uniform. A subdivision which is both uniform and regular will be called a uniform (a_1, a_0)-subdivision of depth N. The $(2, 1)$-subdivision, S' in example 16 is a uniform $(2, 1)$-subdivision, whereas S is not.

Lemma 2.4.13. Given a β-subdivision, there exists an (a_1, a_0)-refinement which is a uniform β-subdivision.

Proof. Let S be a β-subdivision, such that the smallest sub-interval in S is of length β^{-D}. Note that $D \geq 2$ and that if $D = 2$, then S is already a uniform β-subdivision. All sub-intervals in $I(S)$ have length β^{-d} for $1 \leq d \leq D$.

If $D = 3$, then all sub-intervals are of length β^{-d} for $1 \leq d \leq 3$. By performing an (a_1, a_0)-partition of some type on all sub-intervals of length β^{-1}, we create an (a_1, a_0)-refinement of S which only has sub-intervals of length β^{-2} and β^{-3}. This (a_1, a_0)-refinement is a uniform β-subdivision of depth 2.

Each (a_1, a_0)-partition on an interval of length β^{-N} results in replacing that sub-interval with new sub-intervals with lengths $\beta^{-(N+1)}$ and $\beta^{-(N+2)}$. So if a sub-interval has length greater than β^{D-1}, we can perform successive (a_1, a_0)-partitions until there is no such sub-interval.

This leaves us with a β-subdivision only containing intervals of length $\beta^{-(D-1)}$ and β^{-D}, which is a uniform β-subdivision of depth $D - 1$. □

Lemma 2.4.14. Let S be a uniform β-subdivision of depth N. Then there exists S_t an (a_1, a_0)-refinement of S, where S_t is a uniform β-subdivision of depth $N + t$ for $t \in \mathbb{N}$.

Proof. As S is a uniform β-subdivision of depth N, all sub-intervals in $I(S)$ are either of length β^{-N} or $\beta^{-(N+1)}$. These are called long and short intervals respectively.

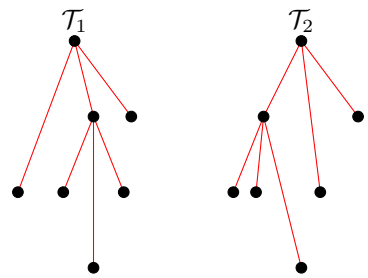
To create S_1 we perform an (a_1, a_0)-partition of some type on every long interval in S simultaneously. All sub-intervals in $I(S_1)$ will have length $\beta^{-(N+1)}$ or $\beta^{-(N+2)}$. Hence, S_1 is a uniform β-subdivision of depth $N + 1$, and S_1 is an (a_1, a_0)-refinement of S.

To create S_t, a uniform β-subdivision of depth t, we perform an (a_1, a_0)-partition of some type on each long interval in S_{t-1} simultaneously. S_t is an (a_1, a_0)-refinement of S_{t-1}, and thus is also an (a_1, a_0)-refinement of S. □

2.4.15 Leaf-Equivalent Trees

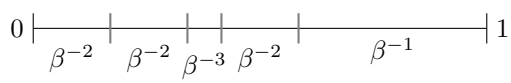
If the subdivision polynomial f has degree, $\delta f = 1$, then the trees associated to the regular β-subdivisions are unique. This is not the case when $\delta f \geq 2$. This means that the each regular β-subdivision, S does not necessarily have a unique (a_1, a_0)-tree that corresponds to it.

Example 17. Consider the following $(2, 1)$-trees,



$$\mathcal{L}(\mathcal{T}_1) = (2, 2, 3, 2, 1) = \mathcal{L}(\mathcal{T}_2)$$

Both $(2, 1)$-trees correspond to the following $(2, 1)$-subdivision.



Definition 2.4.16. Two (a_1, a_0)-trees $\mathcal{T}_1$ and $\mathcal{T}_2$ are said to be **leaf-equivalent** if

$$\mathcal{L}(\mathcal{T}_1) = \mathcal{L}(\mathcal{T}_2)$$

We say that $\mathcal{T}_1 \sim \mathcal{T}_2$.

Whenever a subdivision has more than one corresponding (a_1, a_0)-tree, we are able to choose any corresponding (a_1, a_0)-tree we like.

Common (a_1, a_0)-refinements

Definition 2.4.17. Let S_1, S_2 be β-subdivisions. S' is a **common refinement** of S_1 and S_2 if S' is an (a_1, a_0)-refinement of both S_1 and S_2.

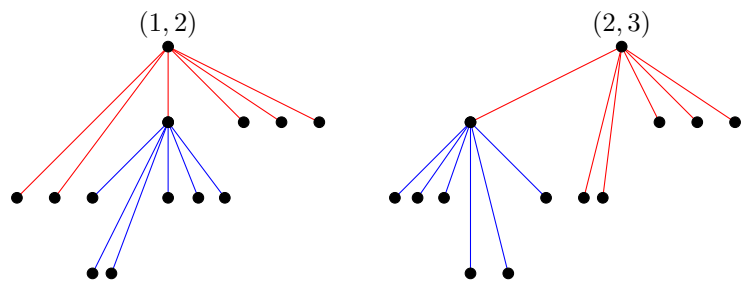
We can also define common refinements on regular β-subdivisions and (a_1, a_0)-trees.

Definition 2.4.18. Two (a_1, a_0)-subdivisions S_1 and S_2 have a **common refinement** if there exists $S_1' = S_2'$, where S_1' and S_2' (a_1, a_0)-refinements of S_1 and S_2 respectively.

Similarly, two (a_1, a_0)-trees $\mathcal{T}_1$ and $\mathcal{T}_2$ have a **common refinement** if there exists $\mathcal{T}_1'$ and $\mathcal{T}_2'$, (a_1, a_0)-refinements of $\mathcal{T}_1$ and $\mathcal{T}_2$ respectively, such that

$$\mathcal{T}_1' \sim \mathcal{T}_2'.$$

Example 18. The $(4, 2)$-carets of type $(1, 2)$ and type $(2, 3)$ have a common refinement. The following $(4, 2)$-trees have root-carets of type $(1, 2)$ and $(2, 3)$ respectively, and are leaf-equivalent.



2.4.19 Grafting

Grafting is the process of finding a common refinement between two β-subdivisions. We can then choose whichever original β-subdivision we wish to use.

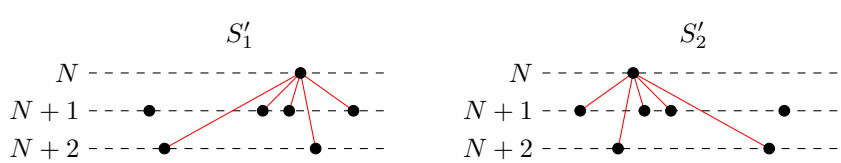
Definition 2.4.20.

An (a_1, a_0)-caret is of **minimal type** if it has type $(1, i_2, \ldots, i_{a_0})$.

An (a_1, a_0)-caret is of **maximal type** if it is of type $(i_1, i_2, \ldots, i_{a_0-1}, K)$, where $K = a_1 + a_0$. If an (a_1, a_0)-caret is not minimal, respectively maximal, it is **non-minimal**, respectively **non-maximal**.

Definition 2.4.21. *Grafting*

Let S_1 be a uniform β-subdivision of depth N such that the jth interval, I_j is a short sub-interval, and the $j + 1^{th}$ sub-interval , I_{j+1} is a long sub-interval. We construct S_2, a uniform β-subdivision of depth N identical to S_1, except the jth and $j + 1^{th}$ sub-intervals have been swapped. Then S_1 and S_2 have a common refinement. This common refinement is found by performing an (a_1, a_0)-partition of type $(i_1, \ldots, i_{a_0})$ on the sub-interval I_{j+1}, where $i_{a_0} < k = a_1 + a_0$. This is shown by hanging a non-maximal (a_1, a_0)-caret from the node representing I_{j+1} below using $(3, 2)$-carets as an example.

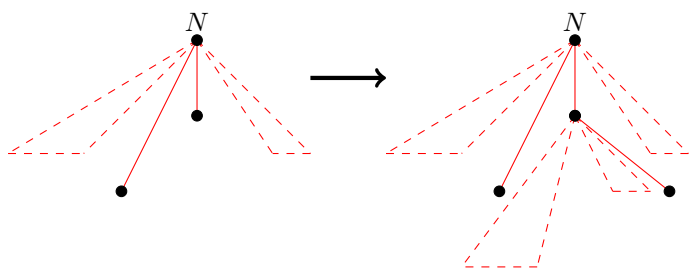


The resulting (a_1, a_0)-refinement S_1', is equivalent to an (a_1, a_0)-refinement of S_2, S_2', in which we perform an (a_1, a_0)-partition of type $(i_1 + 1, \ldots, i_{a_0} + 1)$ on the jth sub-interval in S_2.
We have moved a node a node of depth $N + 1$ to the right of a node of depth N by going from S_1' to S_2'. This is called a **right graft on S_1 at $j + 1$.**

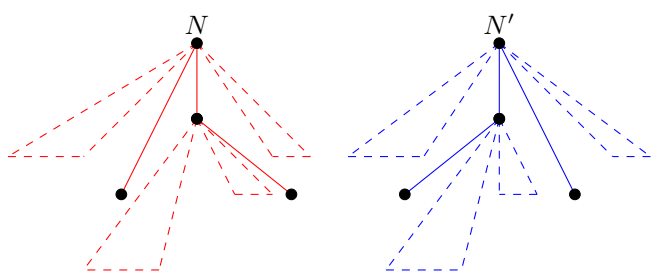
Conversely, if we go from S_2 to S_2' and replace with S_1', we have performed a **left graft on S_2 at j.**

We can use grafting to change the type of an (a_1, a_0)-caret X, when X is an end caret.

Definition 2.4.22. Let N be the root node of an (a_1, a_0)-caret such that for some $2 \leq j \leq K = a_1 + a_0$, $N(j-1)$ is a long leg, and $N(j)$ is a short child. A **right graft on N at $j + 1$**, is performed by hanging a non-maximal caret, from $N(j)$.



There is now a leaf-equivalent tree with root node N'.



We can now substitute the (a_1, a_0)-tree with root node N with the leaf-equivalent (a_1, a_0)-tree with root node N'. A **left graft on N' at j**, is defined analogously.

Note that a left graft moves a long leg to the left passing a short leg in an (a_1, a_0)-caret, and a right graft moves a long leg to the right passing a short leg in an (a_1, a_0)-caret.

The process of grafting finds a common refinement between the sub-tree $\mathcal{T}_N$ and $\mathcal{T}_N'$, which means they correspond to the same (a_1, a_0)-subdivision. As we have more than one (a_1, a_0)-tree corre-

sponding to the same (a_1, a_0)-subdivision, we have the freedom to choose either of our corresponding (a_1, a_0)-trees.

Remark 9. If it is possible to perform a right graft on a node N to get to the node N', then it is possible to perform a left graft on N' to get to N.

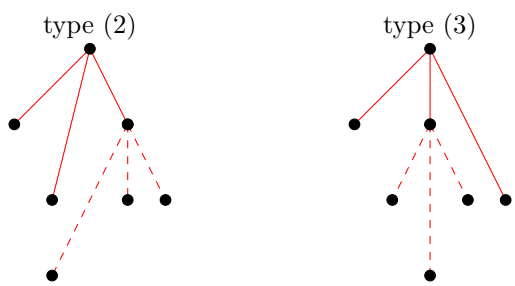
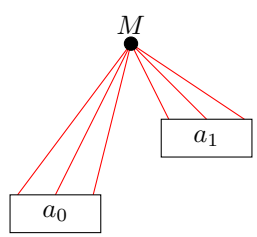


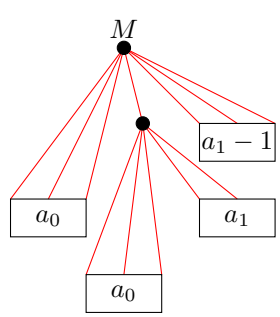
Figure 2.5: A right graft on a $(2, 1)$-caret at $N(3)$

Lemma 2.4.23. Let M be an end (a_1, a_0)-caret of minimal (respectively maximal) type. If $a_1 \geq a_0$, we can right (respectively left) graft M to be of non-minimal (respectively non-maximal) type

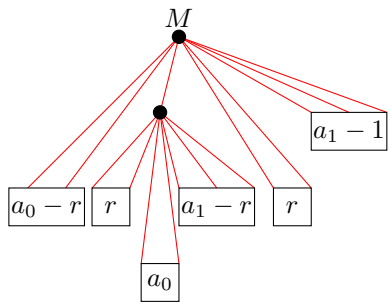
Proof. Consider the following (a_1, a_0)-caret M of type $(1, 2, \ldots, a_0)$, clearly of minimal type.



Take the $(a_0 + 1)^{th}$ child of M, which will be a short leg/long interval, then hang an (a_1, a_0)-caret of type $(1, 2, \ldots, a_0)$.



As we have assumed that $a_1 \geq a_0$, we can find a_0 distinct equivalent trees $\bar{\mathcal{T}}_r$ such that M is of type $(1, \ldots, a_0 - r, a_0 - r + 2, \ldots, a_0 + 1)$ for $r \in \{1, \ldots, a_0\}$.

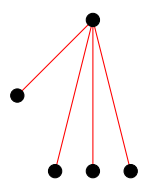


By taking $r = a_0$, M is of type $(2, 3, \ldots, a_0 + 1)$ in $\bar{\mathcal{T}}_{a_0}$, therefore is non-minimal.

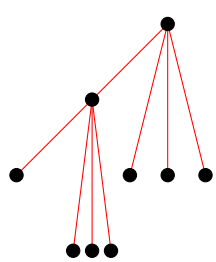
□

It is not clear whether this Lemma holds for $a_1 < a_0$.

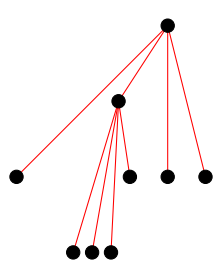
Example 19. Consider the following $(1, 3)$-caret M of type $(2, 3, 4)$. Then M is of non-minimal type. For this choice of $(a_1, a_0) = (1, 3)$, there is only one non-minimal caret type.



We want to know if we can graft M to be of non-maximal type, so we need to graft M to be of type $(1, 2, 3)$. We need to perform a left graft at position 2, by hanging a caret of non-minimal type.



This is equivalent to the following tree in which M is of type $(1, 3, 4)$.



At this point we would try left graft on position 3, but the caret hanging from $M(2)$ is minimal. Thus we need to graft this caret until it is non-minimal. However, in order to do this we would need to be able to graft from type $(1, 2, 3)$ to type $(2, 3, 4)$ which is the equivalent of grafting from $(2, 3, 4)$ to $(1, 2, 3)$. This is the task we started with, so we have formed a cycle.

Note that this does not prove that there is no common refinement, but it means we cannot use the same algorithm. This certainly suggests that the graft would not be possible.

2.5 Pisot β-subdivisions

We will prove that having the condition $a_1 \geq a_0$ is equivalent to saying that β is Pisot. i.e. If the two zeros of $f = X^2 - a_1X - a_0$ are β and β^*, then $|\beta^*| < 1 < \beta$.

Definition 2.5.1. An algebraic integer β is **Pisot** if $1 < \beta \in \mathbb{R}$ and all other zeros of the minimal polynomial of β over $\mathbb{Z}$, have absolute value less than 1. [12]

Lemma 2.5.2. If β is the zero of an irreducible subdivision polynomial of the form

$$f_\beta = X^2 - a_1X - a_0$$

then

$$a_1 \geq a_0 \text{ if and only if } \beta \text{ is Pisot.}$$

Proof. We know that for an irreducible subdivision polynomial $f_\beta = X^2 - a_1X - a_0$, there is a unique positive real zero β, and $\beta > 1$. Now f_β has one other root, β^*, which must also be real.

$$\beta^* = \frac{a_1 - \sqrt{a_1^2 + 4a_0}}{2}$$

If β is Pisot, then $|\beta^*| < 1$.

As $0 < a_1, a_0 \in \mathbb{Z}$, $\beta^* = \dfrac{a_1 - \sqrt{a_1^2 + 4a_0}}{2} < 0$, so if β is Pisot, we have $-1 < \beta^*$.

$$\begin{aligned}
-1 &< \frac{a_1 - \sqrt{a_1^2 + 4a_0}}{2} \\
-2 &< a_1 - \sqrt{a_1^2 + 4a_0} \\
\sqrt{a_1^2 + 4a_0} &< a_1 + 2 \\
a_1^2 + 4a_0 &< a_1^2 + 4a_1 + 4 \\
4a_0 &< 4a_1 + 4 \\
a_0 &< a_1 + 1 \\
a_0 &\leq a_1
\end{aligned}$$

Every step in this series of inequalities, is reversible, and so by working upwards, $a_0 \leq a_1$ implies that $-1 < \beta^* < 0$, and thus β is Pisot.

Hence, $a_1 \geq a_0$ is a necessary and sufficient condition for β to be Pisot. □

Unless otherwise stated, all lemmas, propositions, and theorems will be true for β Pisot.

Definition 2.5.3. The **connected** (a_1, a_0)**-caret** C_i is the (a_1, a_0)-caret of type $(i+1, \ldots, i+a_0-1)$.

We see that in a connected (a_1, a_0)-caret, there are no short legs between any two long legs. In C_i there are i short legs to the left of the first long leg.

Proposition 2.5.4. If $a_1 \geq a_0$, and X and Y are (a_1, a_0)-carets of type $(i_1, \ldots, i_{a_0})$ and $(j_1, \ldots, j_{a_0})$ respectively. Then there is a common refinement between X and Y.

Proof. Let X be the root node of an end caret of type $(i_1, \ldots, i_{a_0})$, with only leaves for children. We will add a_1 new (a_1, a_0)-carets, one to each of the short children of X. If $X(j)$ is a short leg, then hang the connected (a_1, a_0)-caret C_t from $X(j)$ where t is the number of long legs to the right of $X(j)$. i.e. If $i_s < j < i_{s+1}$ hang the connected (a_1, a_0)-caret C_{a_0-s} from $X(j)$ for $s \in \{1, \ldots, a_0 - 1\}$. If $j < i_1$ then hang the connected (a_1, a_0)-caret C_0 from $X(j)$. If $j > i_{a_0}$, then hang the connected (a_1, a_0)-caret C_{a_0} from $X(j)$.

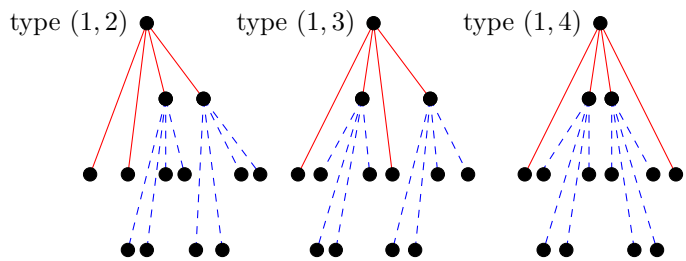
The following sub-tree $\mathcal{T}_X$ will have leaf sequence

$$\mathcal{L}(\mathcal{T}_X) = \big(\underbrace{2, \ldots, 2}_{a_0}, \underbrace{3, \ldots, 3}_{a_0}, \underbrace{2, \ldots, 2}_{a_1}, \cdots, \underbrace{3, \ldots, 3}_{a_0}, \underbrace{2, \ldots, 2}_{a_1}\big)$$

Since this leaf sequence can be obtained from an (a_1, a_0)-caret of any type, there is a common refinement between any two (a_1, a_0)-carets. $\square$

Any such substitution of (a_1, a_0)-caret types is called a **basic move**.

Example 20. Consider the following $(2, 2)$-carets. By following the algorithm described in the proof of Proposition 2.5.4, we see that there is a common refinement between all three of these $(2, 2)$-carets.



The leaf sequence of each of these (a_1, a_0)-trees is $(2, 2, 3, 3, 2, 2, 3, 3, 2, 2)$.

Lemma 2.5.5. Let $\mathcal{T}$ be an (a_1, a_0)-tree of depth 2 where $a_1 \geq a_0$. Let X be an (a_1, a_0)-caret of some type. Then there exists a common refinement between $\mathcal{T}$ and X.

Proof. Let N be the root of $\mathcal{T}$. If $type(N) = type(X)$, then by hanging $\mathcal{T}_{N(i)}$ from the ith child of X, we will get an exact copy of $\mathcal{T}$. Thus if $type(N) = type(X)$, there is a common refinement between $\mathcal{T}$ and X.

Suppose then that $type(N) = (j_1, \ldots, j_{a_0}) \neq (i_1, \ldots, i_{a_0}) = type(X)$. If we are able to perform a basic move, to make $type(N) = type(X)$, then we can repeat the earlier process and hang the sub-tree $\mathcal{T}_{N(i)}$ from $X(i)$ to find our common refinement.

In order to perform our basic move, we need to have the correct type of (a_1, a_0)-caret hanging from each of the short children of N. Let $N(j)$ be a short child, so $H(N(j)) = 1$, and suppose $j_s < j < j_{s+1}$ for $s \in \{1, \ldots, a_0 - 1\}$.

If $N(j)$ is a leaf, then we hang the connected (a_1, a_0)-caret $C_{a_0 - s}$. Otherwise, we have an end (a_1, a_0)-caret hanging from $N(j)$. If $type(N(j)) = C_{a_0 - s}$, then we are done. If this is not the case, then we must perform a basic move on $N(j)$ to make $type(N(j)) = type(C_{a_0 - s})$. This is possible, as $N(j)$ is an end-caret, and by Proposition 2.5.4, we can find a common refinement between two (a_1, a_0)-carets of any different types.

If $j < j_1$ (or $j > j_{a_0}$), then we can perform the same process to ensure that $N(j)$ is the parent of a connected (a_1, a_0)-caret of type C_{a_0} (or C_0 respectively).

By doing this for all $N(j)$ with $H(N(j)) = 1$, we are able to perform basic moves to make $type(N) = type(X)$. Then we can find a common refinement between the (a_1, a_0)-tree $\mathcal{T}$ of depth 2 and the (a_1, a_0)-caret X. $\square$

Thus if $\mathcal{T}$ is an (a_1, a_0)-tree of depth 2, we can find an (a_1, a_0)-refinement of $\mathcal{T}$ which is leaf equivalent to an (a_1, a_0)-tree $\mathcal{T}'$, which has root-caret X.

Proposition 2.5.6. Let $\mathcal{T}$ be an (a_1, a_0)-tree, and X an (a_1, a_0)-caret of some type, where $a_1 \geq a_0$. There exists a common refinement between $\mathcal{T}$ and X.

Proof. We know from Proposition 2.5.4 Lemma 2.5.5, that if $D(\mathcal{T}) \leq 2$ and $a_1 \geq a_0$, then there is a common refinement between $\mathcal{T}$ and X. Assume that for $d \leq D \in \mathbb{N}$ that any (a_1, a_0)-tree of depth d has a common refinement with the (a_1, a_0)-caret X.

Now consider an (a_1, a_0)-tree $\mathcal{T}$ of depth $D+1$. Let N be the root node of $\mathcal{T}$. If $type(N) = type(X)$, then we can find a common refinement by hanging the sub-tree $\mathcal{T}_{N(j)}$ from $X(j)$.

If $type(N) = (i_1, \ldots, i_{a_0}) \neq type(X)$, then we consider each short child of N, $N(j)$, where $H(N(j)) = 1$. If $N(j)$ is a leaf, we can hang the appropriate (a_1, a_0)-caret to perform a basic move on N.

- If $j < i_1$, we hang the connected (a_1, a_0)-caret C_{a_0}.
- If $i_s < j < i_{s+1}$ for $s \in \{1, \ldots, a_0 - 1\}$, we hang the connected (a_1, a_0)-caret $C_{a_0 - s}$.
- If $j > i_{a_0}$ we hang the connected (a_1, a_0)-caret C_0.

If $N(j)$ is not a leaf, but is of the type that we would choose were it a leaf, then we do not need to make any changes to the sub-tree $\mathcal{T}_{N(j)}$.

Otherwise, we want to change the type of $N(j)$. Since the sub-tree $\mathcal{T}_{N(j)}$ has depth D, we know that there is an (a_1, a_0)-refinement which is leaf-equivalent to an (a_1, a_0)-refinement of any (a_1, a_0)-caret, by assumption.

- If $j < i_1$, we substitute $\mathcal{T}_{N(j)}$ for an (a_1, a_0)-refinement of C_{a_0} which is leaf-equivalent to an (a_1, a_0)-refinement of $\mathcal{T}_{N(j)}$.
- If $i_s < j < i_{s+1}$ for $s \in \{1, \ldots, a_0 - 1\}$, we substitute $\mathcal{T}_{N(j)}$ for an (a_1, a_0)-refinement of $C_{a_0 - s}$ which is leaf-equivalent to an (a_1, a_0)-refinement of $\mathcal{T}_{N(j)}$.

- If $j > i_{a_0}$ we substitute $\mathcal{T}_{N(j)}$ for an (a_1, a_0)-refinement of C_0 which is leaf-equivalent to an (a_1, a_0)-refinement of $\mathcal{T}_{N(j)}$.

We are now able to perform a basic move on the root node N to be of $type(X)$. Call the subsequent (a_1, a_0)-tree $\mathcal{T}'$ with root node N'. Now by hanging the sub-tree $\mathcal{T}'_{N(i)}$ from $X(i)$ for $1 \leq i \leq K = a_1 + a_0$. The resulting (a_1, a_0)-tree will be an (a_1, a_0)-refinement of X which is leaf-equivalent to an (a_1, a_0)-refinement of $\mathcal{T}$.

By induction, any (a_1, a_0)-tree $\mathcal{T}$ has a common refinement with an (a_1, a_0)-caret X of any type.

□

Remark 10. Given $1 \leq i_1 < \cdots < i_{a_0} \leq K = a_1 + a_0$, there is always a process to find an (a_1, a_0)-refinement of an (a_1, a_0)-tree which is leaf equivalent to some (a_1, a_0)-tree $\mathcal{T}'$ which has a root-caret of type $(i_1, \ldots, i_{a_0})$, as long as $a_1 \geq a_0$.

This allows us to substitute an (a_1, a_0)-tree $\mathcal{T}$ for an (a_1, a_0)-tree $\mathcal{T}'$ which has any root-caret we want it to have.

As $f_\beta = X^2 - a_1 X - a_0$ is an irreducible integer polynomial, then there are no integer roots to $f_\beta = 0$. The rational root theorem [20], tells us that if $\frac{p}{q} \in \mathbb{Q}$ is a root of $f_\beta = 0$, then p divides a_0 and q divides 1. The only such solution can be an integer solution, which gives us the following remark.

Remark 11. As $f = X^2 - a_1 X - a_0$ is irreducible over $\mathbb{Z}$, then β is irrational.

Lemma 2.5.7. Let β be the unique positive zero of the irreducible integer subdivision polynomial $f = X^2 - a_1 X - a_0$.

The number of long intervals in a uniform β-subdivision of depth N is fixed, as is the number of short intervals.

Proof. Suppose for contradiction that there exists two uniform β-subdivisions of depth N, S, S' where

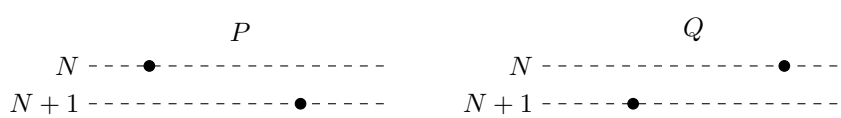
S has m longs and n shorts, and S' contains m' longs and n' shorts. Then

$$\begin{aligned}
\frac{m}{\beta^N} + \frac{n}{\beta^{N+1}} =& 1 = \frac{m'}{\beta^N} + \frac{n'}{\beta^{N+1}} \\
m + \frac{n}{\beta} =& m' + \frac{n'}{\beta} \\
m - m' =& \frac{n'}{\beta} - \frac{n}{\beta} \\
(m - m')\beta =& n' - n \\
\beta =& \frac{n' - n}{m - m'}
\end{aligned}$$

Thus we have a contradiction as β is irrational. □

Lemma 2.5.8. Let P and Q be uniform β-subdivisions of depth N, such that Q can be obtained by swapping a long interval with an adjacent short interval in P. If P' is an (a_1, a_0)-refinements of P, then there exists a β-subdivision S, which is a common refinement between P' and Q.

Proof. Let I_i denote the ith interval in P and I'_i denote the ith interval in Q. Without loss of generality, let $L(I_j) = \beta^{-N} = L(I'_{j+1})$ and $L(I_{j+1}) = \beta^{-(N+1)} = L(I'_j)$.



Let P' be an (a_1, a_0)-refinement of P and let T_i be the (a_1, a_0)-tree representing the sequence of (a_1, a_0)-partitions performed on the interval I_i to get from P to P'. The (a_1, a_0)-tree $\mathcal{T}_j$ will also be referred to as $\mathcal{T}$, to make notation simpler.

We construct Q', a β-subdivision, that is an (a_1, a_0)-refinement of Q such that $P' = Q'$. If $k \notin \{j, j+1\}$, hang the (a_1, a_0)-tree $\mathcal{T}_k$ from I'_k in Q. As the same (a_1, a_0)-partition is performed on these intervals, any sub-interval $I \subset I_k = I'_k$ will be identical in P' as in Q'.

Let R_j be the root node of the (a_1, a_0)-tree T_j. Let R_j be of type $(i_1, \ldots, i_{a_0})$. If $i_1 > 1$, then R_j is non-minimal, and so we can perform a left graft on P at j. Therefore we can find our common refinement Q', by hanging the sub-tree $\mathcal{T}_{R_j(1)}$ from I'_j and an (a_1, a_0)-tree $\mathcal{T}'_{j+1}$ from I'_{j+1}. The root node of $\mathcal{T}'_{j+1}$, R'_{j+1}, is of type $(i_1 - 1, \ldots, i_{a_0} - 1)$, and the sub-tree $\mathcal{T}'_{R'_{j+1}(t)} = \mathcal{T}_{R_j(t+1)}$ for $t \in \{1, \ldots, K - 1 = a_1 + a_0 - 1\}$. The sub-tree $\mathcal{T}'_{R'_{j+1}(K)} = \mathcal{T}_{j+1}$. This process has been partially shown below in reduced (a_1, a_0)-tree notation.

The β-subdivisions P' and Q' are identical, and so P' and Q have a common refinement.

If R_j is minimal, so $i_1 = 1$, then we know that there is an (a_1, a_0)-refinement of P', P^*, in which R_j is non-minimal. The (a_1, a_0)-tree $\mathcal{T}_j$ has an (a_1, a_0)-refinement which is leaf-equivalent to $\mathcal{T}_j^*$, and the root node of $\mathcal{T}_j^*$ is non-minimal. We know this by Proposition 2.5.6.

We can then perform the same process as before to find an (a_1, a_0)-refinement of Q, Q^*, such that P^* and Q^* are identical β-subdivisions. In this case P' and Q have a common refinement in $P^* = Q^*$.

□

The following remark comes as a result of the fact that the symmetric group on n elements is generated by adjacent permutations [21].

Remark 12. Let P, Q be uniform β-subdivisions of depth N. There exist uniform β-subdivisions of depth N, say $P = P_0, P_1, \ldots, P_n = Q$, such that P_{i+1} can be obtained by swapping a long interval with a short interval in P_i.

Lemma 2.5.9. Let S_1 and S_2 be uniform β-subdivisions of depth N. There exists a common refinement between S_1 and S_2.

Proof. By Remark 12, we know that there is a sequence of uniform β-subdivisions $S_1 = P_0, P_1, \ldots, P_n = S_2$, such that P_{i+1} can be obtained by swapping a long interval with a short interval in P_i.

Lemma 2.5.8 tells us that there is a common refinement between P_1 and any (a_1, a_0)-refinement of P_0. In particular, P_0 is the trivial (a_1, a_0)-refinement of P_0, so there is a common refinement of P_0 and P_1. We shall call this P_1'.

Lemma 2.5.8, can be used again here as P_2 has a common refinement with any (a_1, a_0)-refinement of P_1. Since P_1' is an (a_1, a_0)-refinement of P_1, there is a β-subdivision, P_2', a common refinement of P_1' and P_2. Since P_1' is an (a_1, a_0)-refinement of P_0, so is P_2' an (a_1, a_0)-refinement of P_0.

We continue in this vain, by finding the β-subdivision P_i', a common refinement between P_i and P_{i-1}'. In each case P_i' is an (a_1, a_0)-refinement of both P_i and P_0.

By constructing the β-subdivision P'_n, we have found a common refinement between P_0 and P_n. Thus S_1 and S_2 have a common refinement.

□

Theorem 2.5.10. If β, the zero of an irreducible subdivision polynomial $f_\beta = X^2 - a_1X - a_0$, is Pisot, then any two β-subdivisions have a common refinement.

Proof. Let P and Q be β-subdivisions. By Lemma 2.4.13 there exists uniform β-subdivisions P_1 and Q_1 of depths D_1 and D_2 respectively, which are (a_1, a_0)-refinements of P and Q respectively.

Lemma 2.4.14 tells us that there exists two uniform β-subdivisions of depth $N_1 + N_2$, P_2 and Q_2, such that P_2 is an (a_1, a_0)-refinement of P_1, and Q_2 is an (a_1, a_0)-refinement of Q_2.

By Lemma 2.5.9, any two uniform β-subdivisions of the same depth have a common refinement if $a_1 \geq a_0$. If β is Pisot and the zero of an irreducible subdivision polynomial $f = X^2 - a_1X - a_0$, then $a_1 \geq a_0$. Therefore, there exists S, a β-subdivision which is an (a_1, a_0)-refinement of both P_2 and Q_2.

Therefore S is an (a_1, a_0)-refinement of both P and Q, and so there is a common refinement between P and Q. □

Corollary 2.5.11. If β is Pisot, then any β-subdivision has an (a_1, a_0)-refinement which is a regular β-subdivision.

Proof. Let S_1 be a β-subdivision, and let S_2 be the trivial subdivision of $[0, 1]$, i.e. $B[S_2] = \{0, 1\}$. Note that S_2 is a regular β-subdivision of level 0, and so is also known as an (a_1, a_0)-subdivision.

Our Theorem 2.5.10 tells us that there must exist a common refinement between S_1 and S_2. Let S^* be a β-subdivision which is an (a_1, a_0)-refinement of both S_1 and S_2.

Lemma 2.4.11, any (a_1, a_0)-refinement of an (a_1, a_0)-subdivision, is an (a_1, a_0)-subdivision. This means that S^* is a regular β-subdivision.

As we have found S^* an (a_1, a_0)-refinement of S_1, where S_1 could be any β-subdivision, then any β-subdivision has an (a_1, a_0)-refinement which is a regular β-subdivision. □

We now note the important corollary to this theorem.

Corollary 2.5.12. Let β be the positive zero of the irreducible polynomial $f = X^2 - a_1X - a_0$, with $a_1 \geq a_0 \geq 1$. Let $g \in G_\beta$. There exists (a_1, a_0)-trees, $\mathcal{T}_1, \mathcal{T}_2$, such that

$$g = (\mathcal{T}_1, \mathcal{T}_2)$$

and thus

$$F_\beta = G_\beta.$$

Proof. We have already seen that $F_\beta \subset G_\beta$ for all β.

Let $g \in G_\beta$. Then by Proposition 2.3.6, $g = (P, Q)$, where P and Q are β-subdivisions.

We know from Corollary 2.5.11 that P has an (a_1, a_0)-refinement, P_1, which is a regular β-subdivision. For the jth interval in P, $I_j \in I(P)$, the (a_1, a_0)-refinement to get from P to P_1 subdivides I_j. As this subdivision must be made up of a series of (a_1, a_0)-partitions, we can think of this as hanging an (a_1, a_0)-tree, $\mathcal{T}_j$ from the a node which represents the interval I_j.

We can construct Q_1, a β-subdivision which is an (a_1, a_0)-refinement of Q, by hanging the (a_1, a_0)-tree $\mathcal{T}_j$ from the jth interval in Q. Then

$$g = (P, Q) = (P_1, Q_1).$$

As Q_1 is a β-subdivision, there exists an (a_1, a_0)-refinement of Q_1 which is a regular β-subdivision. We will call this (a_1, a_0)-refinement Q_2. For the kth interval in P, $I'_k \in I(P)$, the (a_1, a_0)-refinement to get from Q_1 to Q_2 subdivides I'_k. Again, this subdivision is akin to hanging an (a_1, a_0)-tree $\mathcal{T}'_k$ from a node representing the interval I'_k.

We can similarly hang the (a_1, a_0)-tree $\mathcal{T}'_k$ from the kth sub-interval of P_1 to find an (a_1, a_0)-refinement of P_1 which we will call P_2. Since P_2 is an (a_1, a_0)-refinement of a regular β-subdivision, then P_2 is also a regular β-subdivision. Thus

$$g = (P, Q) = (P_1, Q_1) = (P_2, Q_2).$$

Here P_2, Q_2 are regular β-subdivisions, so have associated (a_1, a_0)-trees $\mathcal{T}_1$ and $\mathcal{T}_2$ respectively.

Thus $g = (\mathcal{T}_1, \mathcal{T}_2) \in F_\beta$. So $G_\beta \subset F_\beta$ when β is Pisot. Since $F_\beta \subset G_\beta$,

$$F_\beta = G_\beta$$

when β is Pisot.

□

Of all the Bieri-Strebel Groups of type G_β, the cases in which we have been able to find expressions for all elements as tree-pairs, have consistently held the property that β is Pisot. Thus we make the following conjecture.

Conjecture 2.5.13. Let β be the unique positive real zero of the irreducible integer polynomial $f = X^n - a_{n-1}X^{n-1} - a_{n-2}X^{n-2} - \cdots - a_1X - a_0$, with $a_i > 0$ for $i \in \{0, \ldots, n-1\}$.
If β is Pisot, then

$$F_\beta = G_\beta.$$

In this chapter we have been able to show that if β is a Perron number that the matrix associated to the subdivision polynomial is primitive. In fact, every Pisot number is a Perron number, but the converse is not true. We will explore this further in the next chapter.

Chapter 3

Non-Pisot β-Subdivisions

In the last chapter we were able to show that any element of G_β, for Pisot β the zero of $f_\beta = X^2 - a_1X - a_0$, can be expressed as a pair of (a_1, a_0)-trees. In this chapter we will aim to show that if β is non-Pisot, than there are elements $g_i \in G_\beta$ such that there exists no (a_1, a_0)-trees $\mathcal{T}_1, \mathcal{T}_2$ such that

$$g_i = (\mathcal{T}_1, \mathcal{T}_2).$$

This means that if β is non-Pisot, then

$$F_\beta \subset G_\beta.$$

I.e., F_β is a proper subset of G_β. To do this, we will find find points in $\mathbb{Z}[\tau] \cap [0, 1]$ which can never be found as a breakpoint in a regular β-subdivision.

First, we will prove that for every point $p \in \mathbb{Z}[\tau] \cap [0, 1]$, there exists an element $g_p = (S_1, S_2) \in G_\beta$ for some β-subdivisions S_1, S_2, such that $p \in B[S_1]$.

3.1 Breakpoints

3.1.1 The ring $\mathbb{Z}[\tau]$

Recall, $\tau = \dfrac{1}{\beta}$ where β is the unique positive real zero of the subdivision polynomial

$$f = X^n - a_{n-1}X^{n-1} - a_{n-2}X^{n-2} - \cdots - a_1X - a_0.$$

We can see that $\beta \in \mathbb{Z}[\tau]$ and is in fact a unit of the ring $\mathbb{Z}[\tau]$.

$$\begin{aligned}
1 &= a_0\tau^n + a_1\tau^{n-1} + \cdots + a_{n-1}\tau \\
1 &= (a_0\tau^{n-1} + a_1\tau^{n-2} + \cdots + a_{n-1})\tau \\
1 &= \beta\tau
\end{aligned}$$

For every element p in $\mathbb{Z}[\beta]$, p can be expressed as

$$p = b_0 + b_1\beta + \cdots + b_{n-1}\beta^{n-1}$$

for some $b_i \in \mathbb{Z}$. [15] Therefore, for all $p \in \mathbb{Z}[\tau] = \mathbb{Z}[\beta]\left[\frac{1}{\beta}\right]$, we can write an expression for p as

$$p = \frac{b_0 + b_1\beta + \cdots + b_{n-1}\beta^{n-1}}{\beta^m}$$

for some $b_i \in \mathbb{Z}$ and $m \in \mathbb{Z}_{\geq 0}$. It becomes clear that this expression is not unique, in particular by using $\beta^{n-1} = a_{n-1}\beta^{n-2} + \cdots + a_1 + a_0\beta^{-1}$, we see that

$$\begin{aligned}
p &= \frac{b_0 + b_1\beta + \cdots + b_{n-1}\beta^{n-1}}{\beta^m} \\
p &= \frac{b_0 + b_1\beta + \cdots + b_{n-1}\left(a_{n-1}\beta^{n-2} + \cdots + a_1 + a_0\beta^{-1}\right)}{\beta^m} \\
p &= \frac{b_{n-1}a_0\beta^{-1} + (b_0 + b_{n-1}a_1) + \cdots + (b_{n-2} + b_{n-1}a_{n-1})\beta^{n-2}}{\beta^m} \\
p &= \frac{b_{n-1}a_0 + (b_0 + b_{n-1}a_1)\beta + \cdots + (b_{n-2} + b_{n-1}a_{n-1})\beta^{n-1}}{\beta^{m+1}} \\
p &= \frac{b'_0 + b'_1\beta + \cdots + b'_{n-1}\beta^{n-1}}{\beta^{m+1}}
\end{aligned}$$

where $b'_i \in \mathbb{Z}$ and $m \in \mathbb{Z}_{\geq 0}$.

Proposition 3.1.2. Let $p \in \mathbb{Z}[\tau] \cap [0,1]$. There exists a β-subdivision of $[0,1]$ which contains p as a breakpoint.

Proof. Clearly p and $1-p$ are in $\mathbb{Z}[\tau] \cap [0,1]$, so by Theorem 2.2.19 can be expressed as

$$p = \frac{b_0 + b_1\beta + \cdots + b_{n-2}\beta^{n-2} b_{n-1}\beta^{n-1}}{\beta^m}$$
$$1-p = \frac{c_0 + c_1\beta + \cdots + c_{n-2}\beta^{n-2} c_{n-1}\beta^{n-1}}{\beta^{m'}}$$

for some $b_0, \ldots, b_{n-1}, c_0, \ldots, c_{n-1}, m, m' \in \mathbb{Z}_{\geq 0}$. We can use these expressions for p and $1-p$ to construct a β-subdivision of $[0,1]$ in which p is a breakpoint. We will make S, our subdivision of $[0,1]$, by taking the initial $b_0 + \cdots + b_{n-1}$ sub-intervals in S contain b_i sub-intervals of length β^{i-m} for $0 \leq i \leq n-1$. From this, we see that $p \in B[S]$. However, S is not yet a β-subdivision. To make this so, we need to split the remainder of the unit interval, which must have length $1-p$, into $c_0 + \cdots + c_{n-1}$ sub-intervals of which precisely c_j are of length $\beta^{j-m'}$ for $0 \leq j \leq n-1$.
S is now a subdivision of $[0,1]$, in which all intervals in $I(S)$ have length which is a power of β. So S is a β-subdivision containing p as a breakpoint. □

Corollary 3.1.3. For all $p \in \mathbb{Z}[\tau] \cap [0,1]$, there exists $g \in G_\beta$ such that (p,p) is a breakpoint of g.

Proof. Let $p \in \mathbb{Z}[\tau] \cap [0,1]$. By Proposition 3.1.2, we know that there exists a β-subdivision S such that $p \in B[S]$. S can be thought of as the union of two subdivsions $S_{\leq p}$ and $S_{\geq p}$ which are β-subdivisions of $[0,p]$ and $[p,1]$ respectively. Let the intervals in $I(S_{\leq p})$ be labelled $I_1, \ldots, I_k$ where $k = size(S_{\leq p})$. Let I_k have length β^m. We consider 2 cases:

Case 1:

First consider the case where at least one of the subdivisions $S_{\leq p}$ or $S_{\geq p}$ contains at least two intervals of different lengths. Without loss of generality, suppose $S_{\leq p}$ contains at least two intervals whose length is not equal. Then there exists I_j which has length $\beta^{m'}$ for some $1 \leq j \leq k-1$ where $m' \neq m$. So I_j and I_k must be sub-intervals of different lengths.
Take $\bar{S}$ to be the β-subdivision of $[0,1]$ which is identical to S except the intervals I_j and I_k have been swapped. We will construct g by taking $g = (S, \bar{S})$, the affine interpolation from S to $\bar{S}$. Since S and $\bar{S}$ are both β-subdivisions of the same size, it is clear that $(S, \bar{S}) \in G_\beta$. The gradient of $y = g(x)$ for $x \in [p,1]$ is 1. The gradient of $y = g(x)$ for $x \in I_k$ is $\beta^{m'-m} \neq 1$. Thus (p,p) is a breakpoint in the map $g \in G_\beta$.

Case 2:

Suppose that all lengths of intervals in $S_{\leq p}$ are the same, and also all lengths of intervals in $S_{\geq p}$ are

the same. Then by making use of the substitution

$$\beta^m = a_{n-1}\beta^{m-1} + \cdots + a_1\beta^{m-(n-1)} + a_0\beta^{m-n}$$

to subdivide the interval I_k into smaller powers of β. For $n \geq 2$, we now have the same conditions as in case 1, and so can follow that procedure. In the case $n = 1$ and $size(S_{\leq p}) = 1$, then all sub-intervals before p will be the same in which case we can again subdivide the sub-interval immediately before p and we will then have the conditions to follow the instructions set out in case 1. $\square$

Corollary 3.1.3 tells us that every $p \in \mathbb{Z}[\tau]$ is a breakpoint in the domain of some some $g \in G_\beta$. If we are able to show that for some β, there exists $p \in \mathbb{Z}[\tau]$, such that p is not a breakpoint in any regular β-subdivision, then we can definitively say that

$$F_\beta \neq G_\beta.$$

We will show that some β do have this property, and we will aim to find all such β which are the unique positive zero of the irreducible quadratic subdivision polynomial

$$X^2 - a_1X - a_0.$$

To do this we will have to work out when a point $p \in \mathbb{Z}[\tau]$ is a breakpoint of a regular β-subdivision.

3.1.4 Obtainable points

Let β be the unique positive zero of the irreducible subdivision polynomial $f = X^2 - a_1X - a_0$. The origin of the following definition is from the work of Cleary [3], and was reintroduced in his later work [4].

Definition 3.1.5. A point $p \in \mathbb{Z}[\tau] \cap [0,1]$ is said to be **obtainable** if there exists a regular β-subdivision, S, such that $p \in B[S]$.

We say that p is obtained as a breakpoint in S, or more simply p is obtained in S.

Recall that a regular β-subdivision is also called an (a_1, a_0)-subdivision, and these can always be represented as (a_1, a_0)-trees.

Lemma 3.1.6. Let the point p be obtainable in an (a_1, a_0)-subdivision, S. Then p is obtainable in any (a_1, a_0)-refinement of S.

Proof. For all (a_1, a_0)-refinements of an (a_1, a_0)-subdivision S, $\bar{S}$, the set of breakpoints $B[S] \subset B[\bar{S}]$. So if $p \in B[S]$, then $p \in B[\bar{S}]$. □

Recall from Lemma 2.4.13 that any (a_1, a_0)-subdivision can be refined to a uniform (a_1, a_0)-subdivision.

Remark 13. If $p \in \mathbb{Z}[\tau] \cap [0,1]$ is obtainable in an (a_1, a_0)-subdivision S of depth D, then p is obtainable in a uniform (a_1, a_0)-subdivision of depth D.

This means that we can find all obtainable points by considering uniform (a_1, a_0)-subdivisions. This was initially considered by Cleary for the cases $(a_1, a_0) = (2, 1)$[3] and $(a_1, a_0) = (1, 1)$[4].

Definition 3.1.7. $p \in \mathbb{Z}[\tau] \cap [0,1]$ is **obtainable at depth** N if there exists an (a_1, a_0)-subdivision S of depth N such that $p \in B[S]$.

The following Lemma is clear.

Lemma 3.1.8. If $P \in \mathbb{Z}[\tau] \cap [0,1]$ is obtainable at depth N, then P is obtainable at depth $N + 1$.

Recall from Remark 11, that β is irrational.

Lemma 3.1.9. For all $p \in \mathbb{Z}[\tau], 1 \leq N \in \mathbb{Z}$ there exists a unique integer pair m_1, m_2 such that $p = \dfrac{m_1}{\beta^N} + \dfrac{m_2}{\beta^{N+1}}$

Proof. First we need to show existence. For any $p \in \mathbb{Z}[\tau]$, we can write an expression for p in the form

$$p = b_1 + b_2\tau = \frac{b_1}{\beta^0} + \frac{b_2}{\beta^1},$$

for some $b_1, b_2 \in \mathbb{Z}$. By using the substitution $\frac{1}{\beta^0} = \frac{a_1}{\beta^1} + \frac{a_0}{\beta^2}$, we can rewrite this expression as

$$p = \frac{b_1}{\beta^0} + \frac{b_2}{\beta^1} = \frac{a_1 b_1 + b_2}{\beta^1} + \frac{a_0 b_1}{\beta^2}.$$

Since $a_0, a_1, b_1, b_2 \in \mathbb{Z}$, there exists an integer pair c_1, c_2 such that we have an expression for p in the form

$$p = \frac{c_1}{\beta^1} + \frac{c_2}{\beta^2}.$$

In fact, we can always use the substitution $\frac{1}{\beta^N} = \frac{a_1}{\beta^{N+1}} + \frac{a_0}{\beta^{N+2}}$, to take an expression for p at *depth* N,

$$p = \frac{m_1}{\beta^N} + \frac{m_2}{\beta^{N+1}}$$

to attain a similar expression at *depth* $N+1$,

$$p = \frac{a_1 m_1 + m_2}{\beta^{N+1}} + \frac{a_1 m_1}{\beta^{N+2}}.$$

As there clearly exists an integer pair to satisfy such an expression when $N = 1$, there must also exist an integer pair to satisfy such an expression when $N \in \mathbb{Z}_{\geq 1}$.

To show uniqueness, we will revisit the ideas presented in Lemma 2.5.7. Suppose for contradiction that for some $p \in \mathbb{Z}[\tau]$ and for some $N \in \mathbb{Z}$,

$$\frac{m_1}{\beta^N} + \frac{m_2}{\beta^{N+1}} = p = \frac{n_1}{\beta^N} + \frac{n_2}{\beta^{N+1}}$$

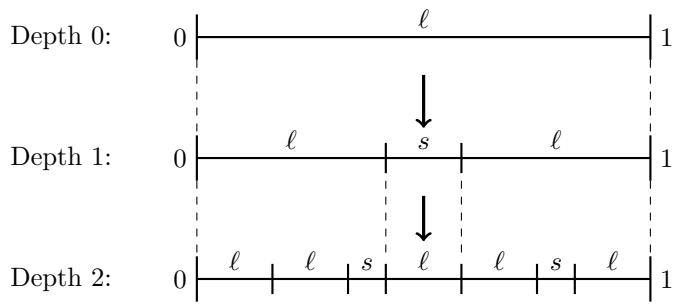


Figure 3.1: Obtainable points in a specific $(2,1)$-subdivision

for some $m_1, m_2, n_1, n_2 \in \mathbb{Z}$, $(m_1, m_2) \neq (n_1, n_2)$.

$$\begin{aligned}
\frac{m_1}{\beta^N} + \frac{m_2}{\beta^{N+1}} &= \frac{n_1}{\beta^N} + \frac{n_2}{\beta^{N+1}} \\
m_1 + \frac{m_2}{\beta} &= n_1 + \frac{n_2}{\beta} \\
m_1 - n_1 &= \frac{n_2 - m_2}{\beta} \\
\frac{m_1 - n_1}{n_2 - m_2} &= \frac{1}{\beta} \\
\frac{n_2 - m_1}{m_1 - n_1} &= \beta.
\end{aligned}$$

However, we know that β is irrational. Hence we have a contradiction, so there is in fact a unique integer pair m_1, m_2 for each $N \in \mathbb{Z}$ such that $p = \frac{m_1}{\beta^N} + \frac{m_2}{\beta^{N+1}}$. □

Note that Lemma 2.5.7, is a corollary of the Lemma above.

Remark 14. If $p \in \mathbb{Z}[\tau] \cap [0,1]$ is a breakpoint in a uniform (a_1, a_0)-subdivision of depth N, then the number of longer and shorter intervals preceding p is uniquely defined.

Definition 3.1.10. A $(long, short)$-pair (m_1, m_2) is obtainable at depth N if there is a uniform (a_1, a_0)-subdivision of depth N with an initial segment containing $m_1 + m_2$ intervals, m_1 of which are of length $\frac{1}{\beta^N}$ and m_2 of which are of length $\frac{1}{\beta^{N+1}}$.

The longer intervals described in the $(long, short)$-pair are *longs*, ℓ, and the shorter intervals are *shorts*, s. In Figure 3.1 we can see that the only $(long, short)$-pairs obtainable at depth 0 are $(0,0)$ and $(1,0)$, as you can either take the entire interval or none of it.

We recall the notation used in the previous chapter,

$$\begin{bmatrix} m_1 \\ m_2 \end{bmatrix}_N = \frac{m_1}{\beta^N} + \frac{m_2}{\beta^{N+1}}.$$

Remark 15. For each $p \in \mathbb{Z}[\tau] \cap [0,1]$ obtainable at depth N, there exists a unique $(long, short)$-pair (m_1, m_2) that is obtainable at depth N such that

$$p = \begin{bmatrix} m_1 \\ m_2 \end{bmatrix}_N = \frac{m_1}{\beta^N} + \frac{m_2}{\beta^{N+1}}.$$

This remark suggests that each $(long, short)$-pair is representative of an unique obtainable point in $\mathbb{Z}[\tau] \cap [0,1]$. Therefore by considering the set of all $(long, short)$-pairs obtainable at depth N, we can find all $p \in \mathbb{Z}[\tau] \cap [0,1]$ such that p is obtainable at depth N. We introduce a visual representation of this idea in the following section.

3.2 (a_1, a_0)-Tiles

In this section we consider the set of obtainable $(long, short)$-pairs in a uniform (a_1, a_0)-subdivision of level N, and plot them as lattice points in $\mathbb{Z}^2$.
Let β be the positive real zero of the irreducible subdivision polynomial $f = X^2 - a_1 X - a_0$.

Definition 3.2.1. The (a_1, a_0)-tile of level 0, T_0, is the set $\{(0,0), (1,0)\}$.
The (a_1, a_0)-tile of level $N \in \mathbb{Z}$, T_N, is the set of points $(p,q) \in \mathbb{Z}^2$, $p, q \geq 0$ such that there exists a uniform (a_1, a_0)-subdivision of depth N which contains $\frac{p}{\beta^N} + \frac{q}{\beta^{N+1}}$ as a breakpoint.

The $(2,1)$-tiles of level $0, 1$, and 2 can be seen in figure 3.2, and the $(1,3)$-tiles of level $0, 1$, and 2 can be seen in figure 3.3.

Remark 16. If $(p,q) \in T_N$, and $x = \begin{bmatrix} p \\ q \end{bmatrix}_N = \frac{p}{\beta^N} + \frac{q}{\beta^{N+1}}$, then x is obtainable in a uniform (a_1, a_0)-subdivision.

As the (a_1, a_0)-tile of level N is defined as the set of positive integer pairs (p,q) such that $P = \frac{p}{\beta^N} + \frac{q}{\beta^{N+1}}$ is obtainable at depth N. The contrapositive of this statement gives us the following Lemma.

Lemma 3.2.2. If $(p,q) \notin T_N$, and $x = \begin{bmatrix} p \\ q \end{bmatrix}_N = \frac{p}{\beta^N} + \frac{q}{\beta^{N+1}}$, then x is **not** obtainable at depth N.

The (a_1, a_0)-tile of level N can be considered to represent the set of all *long,short*-pairs that can be found in an initial segment of some uniform (a_1, a_0)-subdivision of depth N. The $(long, short)$-pair is unique in each tile because β is irrational.

Remark 17. If $(p,q) \in T_N$, the (a_1, a_0)-tile of level N, then there exists an (a_1, a_0)-tree of depth N which contains $\frac{p}{\beta^N} + \frac{q}{\beta^{N+1}}$ as a breakpoint.

Note that whilst a uniform (a_1, a_0)-subdivision of depth 0 considers the interval $[0,1]$ as a long interval, a uniform (a_1, a_0)-subdivision of depth -1 considers the interval $[0,1]$ as a short interval. We consider the (a_1, a_0)-tile of level -1, T_{-1}, to be the set of points $(p,q) \in \mathbb{Z}^2$, with $p, q \geq 0$ such that $p\beta + q$ is a breakpoint in some (a_1, a_0)-subdivision of $[0,1]$ of depth -1. Since $\beta > 1$, this set consists of just two points, $T_{-1} = \{(0,0), (0,1)\}$, and $A(T_{-1}) = (0,1)$.

Remark 18. Let $N \in \mathbb{Z}$, such that $N \leq -2$. Then the (a_1, a_0)-tile of level N is

$$T_N = \{(0,0)\}.$$

Lemma 3.2.3. If $(p,q) \in T_N$, then

$$\begin{pmatrix} a_1 & 1 \\ a_0 & 0 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} p' \\ q' \end{pmatrix} \in T_{N+1}$$

Proof. Let $(p,q) \in T_N$, such that $P = \frac{p}{\beta^N} + \frac{q}{\beta^{N+1}}$. Then P is obtainable at depth N. Therefore P is also obtainable at depth $N+1$, by Lemma 3.1.8.

$$\begin{aligned} P &= \frac{p}{\beta^N} + \frac{q}{\beta^{N+1}} \\ &= p\left(\frac{a_1}{\beta^{N+1}} + \frac{a_0}{\beta^{N+2}}\right) + \frac{q}{\beta^{N+1}} \\ &= \frac{a_1 p + q}{\beta^{N+1}} + \frac{a_0 p}{\beta^{N+2}} \\ &= \left[\begin{pmatrix} a_1 & 1 \\ a_0 & 0 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}\right]_{N+1} \\ &= \left[\begin{pmatrix} p' \\ q' \end{pmatrix}\right]_{N+1}. \end{aligned}$$

Lemma 3.1.9 tells us that this is in fact the unique expression for P in terms of $\beta^{-(N+1)}$ and $\beta^{-(N+2)}$. Since P is obtainable at depth $N+1$, the $(long, short)$-pair (p', q') is obtainable at depth $N+1$. Therefore, $\begin{pmatrix} a_1 & 1 \\ a_0 & 0 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} p' \\ q' \end{pmatrix} \in T_{N+1}$. □

Lemma 3.2.4. Let T_N be the (a_1, a_0)-tile of level N. Then there is some $(p, q) \in T_N$ such that

$$\frac{p}{\beta^N} + \frac{q}{\beta^{N+1}} = 1.$$

Proof. If S is a subdivision of $[0, 1]$, then $1 \in B[S]$. Therefore any $((a_1, a_0))$-subdivision of $[0, 1]$ of depth N, contains 1 as a breakpoint. By Remark 15, there must be a $(long, short)$-pair (p, q) which is obtainable at depth N such that

$$\frac{p}{\beta^N} + \frac{q}{\beta^{N+1}} = 1.$$

□

Definition 3.2.5. The **Apex** of an (a_1, a_0)-tile of the level N , T_N, is $A(T_N) = (p, q)$ where

$$\frac{p}{\beta^N} + \frac{q}{\beta^{N+1}} = 1.$$

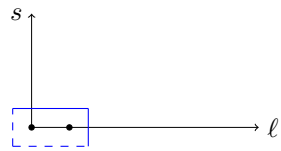
Remark 19. The Apex of any (a_1, a_0)-tile of level 0 is $A(T_0) = (1, 0)$.
The Apex of any (a_1, a_0)-tile of level 1 is $A(T_1) = (a_1, a_0)$.

Lemma 3.2.6. Let T_N be the (a_1, a_0)-tile of level N, and let $A(T_N) = (p_N, q_N)$. Then for all $(p, q) \in T_N$, $0 \leq p \leq p_N$ and $0 \leq q \leq q_N$.

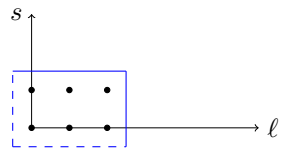
Proof. Clearly if $(p, q) \in T_N$, the (a_1, a_0)-tile of level N, then $0 \leq p$ and $0 \leq q$. From the previous chapter, Lemma 2.5.7 tells us that the number of long (respectively short) sub-intervals in a uniform (a_1, a_0)-subdivision of depth N is uniquely defined. This means that there are precisely α_1 long sub-intervals and α short sub-intervals in any uniform (a_1, a_0)-subdivision of depth N for some $\alpha_1, \alpha_2 \in \mathbb{Z}_{\geq 0}$.

We know from Lemma 3.2.4, that the Apex of the tile, $A(T_N)$ exists in T_N. Let $(p_N, q_N) = A(T_N)$, then $\frac{p_N}{\beta^N} + \frac{q_N}{\beta^{N+1}} = 1$. Therefore $\alpha_1 = p_N$ and $\alpha_2 = q_N$ as the unit interval must be spanned by

The level 0 $(2,1)$-tile, T_0 :



The level 1 $(2,1)$-tile, T_1 :



The level 2 $(2,1)$-tile, T_2 :

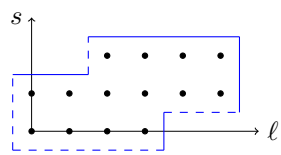
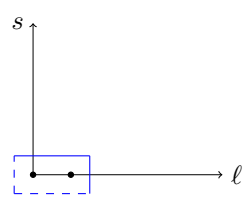
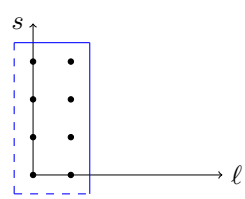


Figure 3.2: The $(2,1)$-tiles of levels $0, 1$, and 2

The level 0 $(1,3)$-tile, T_0 :



The level 1 $(1,3)$-tile, T_1 :



The level 2 $(1,3)$-tile, T_2 :

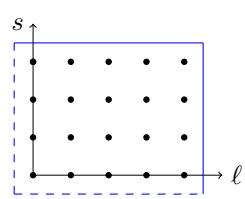


Figure 3.3: The $(1,3)$-tiles of levels $0, 1, 2$

the intervals counted in $A(T_N)$. As there cannot be any more than p_N long (respectively q_N short) sub-intervals in any uniform (a_1, a_0)-subdivision, if $(p, q) \in T_N$, $p \leq p_N$ and $q \leq q_N$. $\square$

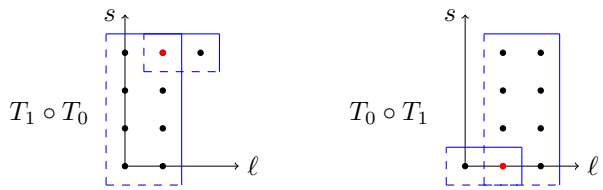


Figure 3.4: Composing the $(1, 3)$-tiles of level $0, 1$

Lemma 3.2.7. For $N \geq 0$, let T_N, T_{N+1} be the (a_1, a_0)-tiles of level N and $N+1$ respectively.

$$\text{Then } \begin{pmatrix} a_1 & 1 \\ a_0 & 0 \end{pmatrix} A(T_N) = A(T_{N+1}).$$

Proof. Let $A(T_N) = (p_N, q_N)$, so $1 = \frac{p_N}{\beta^N} + \frac{q_N}{\beta^{N+1}}$. By using the substitution $\frac{1}{\beta^N} = \frac{a_1}{\beta^{N+1}} + \frac{a_0}{\beta^{N+2}}$, we can find the expression

$$\begin{aligned} 1 &= p_N \left(\frac{a_1}{\beta^{N+1}} + \frac{a_0}{\beta^{N+2}} \right) + \frac{q_N}{\beta^{N+1}} \\ &= \frac{a_1 p_N + q_N}{\beta^{N+1}} + \frac{a_0 p_N}{\beta^{N+2}} \end{aligned}$$

□

Definition 3.2.8. We **compose** two (a_1, a_0)-tiles X, Y to get

$$X \circ Y := X \cup \{A(X) + y \big| y \in Y\} = X \cup \{A(X) + Y\}.$$

We extend this definition to composition of tiles

Note that here we must take the apex of a tile X to be the pair (p_X, q_X) such that for all $(p, q) \in X$, $p \leq p_X$ and $q \leq q_X$. This

Remark 20. For two well defined (a_1, a_0)-tile X and Y,

$$A(X \circ Y) = A(X) + A(Y).$$

$T_1 \circ T_0$

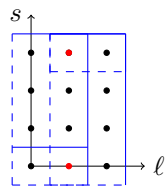


Figure 3.5: $\bigcirc(T_0, T_1)$

We can see in figure 3.4 that composition of tiles is not commutative.

Lemma 3.2.9. Composition of (a_1, a_0)-tiles is an associative operation.

Proof. Let X, Y, Z be (a_1, a_0)-tiles such that $A(X), A(Y)$, and $A(Z)$ are well defined. We will first consider $(X \circ Y) \circ Z$.

$$\begin{aligned}(X \circ Y) \circ Z &= (X \cup \{A(X) + Y\}) \circ Z \\ &= (X \cup \{A(X) + Y\}) \cup \{A(X \circ Y) + Z\} \\ &= X \cup \{A(X) + Y\} \cup \{A(X \circ Y) + Z\}.\end{aligned}$$

Next consider $X \circ (Y \circ Z)$:

$$\begin{aligned}X \circ (Y \circ Z) &= X \cup \{A(X) + (Y \circ Z)\} \\ &= X \cup \{A(X) + Y \cup \{A(Y) + Z\}\} \\ &= X \cup (\{A(X) + Y\} \cup \{A(X) + A(Y) + Z\}) \\ &= X \cup \{A(X) + Y\} \cup \{A(X) + A(Y) + Z\}.\end{aligned}$$

These expressions are the same as in remark 20 we noted that $A(X \circ Y) = A(X) + A(Y)$. $\square$

It makes sense that the composition is associative as it can be described visually by overlaying a series of tiles only overlapping the origin of one tile with the apex of the tile that comes before it.

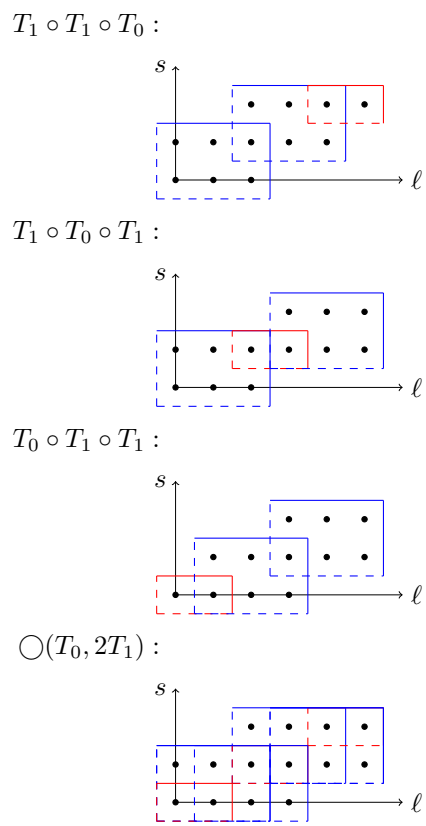


Figure 3.6: Possible composition of more than two $(2,1)$-tiles, $\bigcirc(T_0, 2T_1)$

Definition 3.2.10. The set of all points that can be found in some composition of the (a_1, a_0)-tiles X and Y in any order is

$$\bigcirc(X, Y) = \{X \circ Y\} \cup \{Y \circ X\}.$$

As composition is associative, we can define this set for more than two (a_1, a_0)-tiles, X, Y, Z:

$$\bigcirc(X, Y, Z) = \bigcirc\left(\bigcirc(X, Y), Z\right) = \bigcirc\left(X, \bigcirc(Y, Z)\right).$$

If an (a_1, a_0)-tile is repeated in composition we can write it using the following shorthand.

$$\bigcirc(\mu_1 X, \mu_2 Y) = \bigcirc(\underbrace{X, \ldots, X}_{\mu_1}, \underbrace{Y, \ldots, Y}_{\mu_2}).$$

Lemma 3.2.11. For $N \geq 0$, let $(p,q) \in T_N$ the (a_1,a_0)-tile of level N. Then $(p,q) \in T_{N+1}$.

Proof. Let $(p,q) \in T_N$. Then there exists a uniform (a_1,a_0)-subdivision, S, that contains the breakpoint

$$\begin{bmatrix} p \\ q \end{bmatrix}_N = \frac{p}{\beta^N} + \frac{q}{\beta^{N+1}}.$$

Let $\mathcal{T}$ be the corresponding uniform (a_1,a_0)-tree to S. Then initial $p+q$ leaves of $\mathcal{T}$ contain p leaves of height N and q leaves of height $N+1$.

Consider an (a_1,a_0)-tree T of size 1, i.e. an (a_1,a_0)-caret of type $(i_1,\ldots,i_{a_0}) = \mathcal{L}(T)$. Choose the type of this $i_1 > 1$, so the first leaf in T has height 1.

We hang the (a_1,a_0)-tree $\mathcal{T}$ from the first leaf of T to get $T(1)_{\mathcal{T}}$. The first $p+q$ leaves of will contain p leaves with height $N+1$, and q leaves of height $N+2$.

Thus $\begin{bmatrix} p \\ q \end{bmatrix}_{N+1} = \frac{p}{\beta^{N+1}} + \frac{q}{\beta^{N+2}}$ is a breakpoint in an (a_1,a_0)-subdivision of $[0,1]$. Therefore (p,q) belongs to the (a_1,a_0)-tile of level $N+1$, T_{N+1}. □

Lemma 3.2.11 is true for all $(p,q) \in T_N$, so we reach the following remark.

Remark 21. For $N \geq 0$, let T_N, T_{N+1} be the (a_1,a_0)-tiles of levels N and $N+1$ respectively. Then

$$T_N \subset T_{N+1}.$$

Note, if T_{-1} and T_0 are the (a_1,a_0)-tiles of level -1 and level 0 respectively.

$$T_{-1} \not\subset T_0.$$

Lemma 3.2.12. Let $(p,q) \in T_N$. If $P = \begin{bmatrix} p \\ q \end{bmatrix}_N \in \mathbb{Z}[\tau] \cap [0,1]$ is obtainable at depth N, then $\frac{P}{\beta}$ is obtainable at depth $N+1$.

Proof. If $(p,q) \in T_N$, then $P = \begin{bmatrix} p \\ q \end{bmatrix}_N \in \mathbb{Z}[\tau] \cap [0,1]$ is obtainable at depth N. By Lemma 3.2.11,

$(p,q) \in T_{N+1}$, so $P' = \begin{bmatrix} p \\ q \end{bmatrix}_{N+1} \in \mathbb{Z}[\tau] \cap [0,1]$ is obtainable at depth $N+1$.

$$P' = \begin{bmatrix} p \\ q \end{bmatrix}_{N+1} = \frac{p}{\beta^{N+1}} + \frac{q}{\beta^{N+2}} = \frac{1}{\beta}\left(\frac{p}{\beta^N} + \frac{q}{\beta^{N+1}}\right) = \frac{1}{\beta}\begin{bmatrix} p \\ q \end{bmatrix}_N = \frac{P}{\beta}.$$

□

Remark 22. $\Big[A(T)_N\Big]_{N+t} = \dfrac{1}{\beta^t}$

We consider the (a_1, a_0)-tile of level -1, T_{-1}, to be the set of points $(p,q) \in \mathbb{Z}^2$, with $p, q \geq 0$ such that $p\beta + q$ is a breakpoint in some (a_1, a_0)-subdivision of $[0,1]$ of depth -1. Since $\beta > 1$, the set consists of just two points, $T_{-1} = \{(0,0),(0,1)\}$, and $A(T_{-1}) = (0,1)$.

Note that whilst a uniform (a_1, a_0)-subdivision of depth 0 considers the interval $[0,1]$ as a long interval, a uniform (a_1, a_0)-subdivision of depth -1 considers the interval $[0,1]$ as a short interval.

Remark 23. Let T_{-1} and T_0 be the (a_1, a_0)-tiles of level -1 and level 0 respectively. Then

$$T_{-1} \not\subset T_0.$$

Proposition 3.2.13. For $N \geq 2$, let T_{N-2}, T_{N-1}, T_N be the (a_1, a_0)-tile of level $N-2, N-1, N$ respectively. Then

$$T_N = \bigcirc(a_1 T_{N-1}, a_0 T_{N-2}).$$

Proof. We must first show that each point in T_N can be found in the composition $\bigcirc(a_1 T_{N-1}, a_0 T_{N-2})$.

Suppose $(p,q) \in T_N$, the (a_1, a_0)-tile of level N. Let $\mathcal{T}$ be an (a_1, a_0)-tree corresponding to a uniform (a_1, a_0)-subdivision S, with $P = \dfrac{p}{\beta^N} + \dfrac{q}{\beta^{N+1}} \in B[S]$. Let R be the root node of $\mathcal{T}$, which has $k = a_1 + a_0$ children, $R(1), \ldots, R(k)$. Let the type of R be $(i_1, \ldots, i_{a_0})$. The breakpoint P_N is contained in exactly one of the sub-trees $\mathcal{T}_{R(1)}, \ldots, \mathcal{T}_{R(k)}$. Note that each of these sub-trees are uniform of depth $N-1$ or depth $N-2$.

Suppose without loss of generality that P as a breakpoint is found in the sub-tree $R(j)$, with $1 \leq i_\alpha < j \leq i_{\alpha+1} \leq k$. Then for $m < j$, all leaves of $\mathcal{T}_{R(m)}$ must be included in the $p+q$ leaves that come before the breakpoint P_N. As $i_\alpha < j \leq i_{\alpha+1} < k$, we know that α of the first children of R will have height 2, and $j - 1 - \alpha$ with height 1.

If $H(R(m)) = 1$, then $R(m)$ represents the interval $\frac{1}{\beta}$, and if $H(R(m)) = 2$, then $R(m)$ represents the interval $\frac{1}{\beta^2}$. There exists $0 \leq p', q' \in Z$ such that

$$\begin{aligned} P &= \frac{j-1-\alpha}{\beta} + \frac{\alpha}{\beta^2} + \frac{p'}{\beta^N} + \frac{q'}{\beta^{N+1}} \\ &= \frac{j-1-\alpha}{\beta} + \frac{\alpha}{\beta^2} + \frac{P'}{\beta^{H(R(j))}} \\ &= (j-1-\alpha)\left[A(T_{N-1})\right]_N + \alpha\left[A(T_{N-2})\right]_N + \frac{P'}{\beta^{H(R(j))}}. \end{aligned}$$

Note P as a breakpoint in $\mathcal{T}$ corresponds to the breakpoint P' in $\mathcal{T}_{R(j)}$, and $\frac{P'}{\beta^{H(R(j)))}} = \frac{p'}{\beta^N} + \frac{q'}{\beta^{N+1}}$. Here p' and q' represent the number of long, and short intervals respectively, that add up to P' at level $N - H(r(j))$. Clearly then, $(p', q') \in T_{N-H(R(j))}$. This means we are able to describe any point $(p, q) \in T_N$ as some $\alpha_1 A(T_{N-1}) + \alpha_0 A(T_{N-2}) + (p', q')$, where $0 \leq \alpha_1 \leq a_1$, $0 \leq \alpha_0 \leq a_0$, and $(p', q') \in T_{N-1} \cup T_{N-2}$. Therefore

$$T_N \subset \bigcirc(a_1 T_{N-1}, a_0 T_{N-2}).$$

Conversely, suppose $(p, q) \in \bigcirc(a_1 T_{N-1}, a_0 T_{N-2})$. We can construct a uniform (a_1, a_0)-tree $\mathcal{T}$ of depth N in which $P = \frac{p}{\beta^N} + \frac{q}{\beta^{N+1}}$ is a breakpoint.

There exists and expression for (p, q) in terms of $A(T_{N-1}), A(T_N - 2)$, namely

$$(p, q) = \gamma_1 A(T_{N-1}) + \gamma_2 A(T_{N-2}) + (p', q')$$

with $0 \leq \gamma_1 \leq a_1$, $0 \leq \gamma_2 \leq a_0$, $(p', q') \in T_{N-1} \cup T_{N-2}$.

We construct an (a_1, a_0)-tree $\mathcal{T}$ by taking the root (a_1, a_0)-caret to be of type $(i_1, \ldots, i_{\gamma_2}, i_{\gamma_2+1}, \ldots, i_{a_0})$ with $i_{\gamma_2} < \gamma_1 + \gamma_2 \leq i_{\gamma_2+1}$. For each child of R, $R(i)$, $i \neq \gamma_1 + \gamma_2 + 1$, we will hang a uniform (a_1, a_0)-tree of depth $N - H(R(i))$. From $R(\gamma_1 + \gamma_2 + 1)$ we will hang the uniform (a_1, a_0)-tree which has p' leaves of height $N - H(R(\gamma_1 + \gamma_2 + 1))$, q' leaves of height $N + 1 - H(R(\gamma_1 + \gamma_2 + 1))$ within the first $p' + q'$ leaves.

The resulting (a_1, a_0)-tree is uniform and contains $P = \frac{p}{\beta^N} + \frac{q}{\beta^{N+1}}$ as a breakpoint. Therefore

$$\bigcirc(a_1 T_{N-1}, a_0 T_{N-2}) \subset T_N.$$
$$\therefore T_N = \bigcirc(a_1 T_{N-1}, a_0 T_{N-2}).$$

$\square$

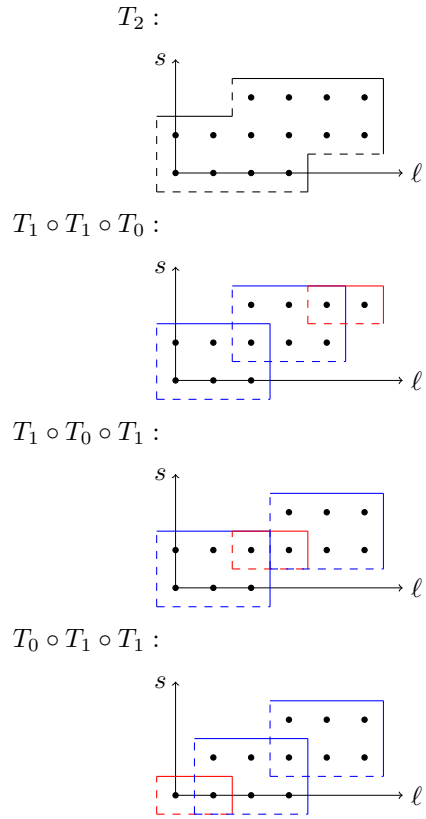


Figure 3.7: The possible combinations in $\bigcirc(2T_1, T_0)$

This can be seen in a construction of the $(2,1)$-tile of level 2 from the $(2,1)$-tiles of level 0 and 1, seen in figure 3.7.

We can see the process of composing (a_1, a_0)-tiles through the associated vectors.

Definition 3.2.14. Let T_N be the (a_1, a_0)-tile of level N. The **associated vector** $V(T_N)$ is

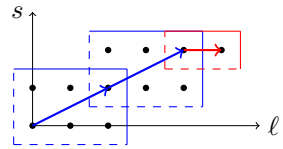
$$V(T_N) := \overrightarrow{O(T_N)A(T_N)}.$$

This is shown for $(2,1)$-tiles in figure 3.8 and for $(1,3)$-tiles in figure 3.9.

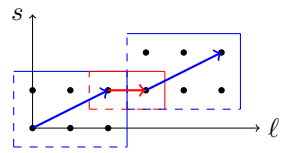
Definition 3.2.15. Let C be an (a_1, a_0)-caret of type $(i_1, \ldots, i_{a_0})$. The **reverse of** C is C^r which is an (a_1, a_0)-caret of type $(k - i_{a_0}, \ldots, k-)$, where $k = a_1 + a_0$.

We can similarly define the **reverse** of an (a_1, a_0)-tree.

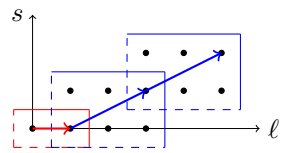
$T_1 \circ T_1 \circ T_0$:



$T_1 \circ T_0 \circ T_1$:



$T_0 \circ T_1 \circ T_1$:



The tile T_2 :

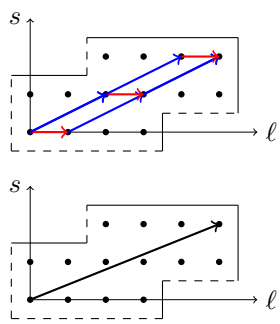


Figure 3.8: The associated vectors for the possible combinations in $\bigcirc(2T_1, T_0)$

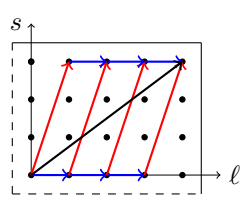


Figure 3.9: T_2 as composed by $V(T_1)$ and $V(T_0)$

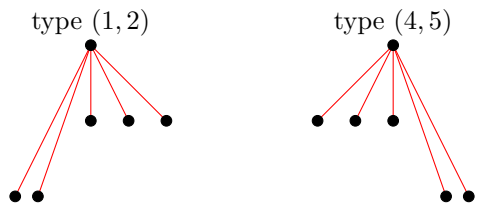


Figure 3.10: The reverse of the $(3,2)$-caret of type $(1,2)$ is type $(4,5)$

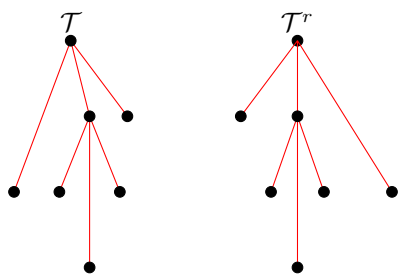


Figure 3.11: The reverse of an $(2,1)$-tree

Definition 3.2.16. If $\mathcal{T}$ is an (a_1, a_0)-tree of depth 1 with root (a_1, a_0)-caret R, the **reverse tree** $\mathcal{T}$ is $\mathcal{T}^r = R^r$.

If $\mathcal{T}$ is an (a_1, a_0)-tree is of depth N, with root (a_1, a_0)-caret R, then the **reverse of** $\mathcal{T}$ is $\mathcal{T}^r$, an (a_1, a_0)-tree with root (a_1, a_0)-caret R^r, and each sub-tree $\mathcal{T}_{R(j)} = \mathcal{T}^r_{R^r(k-j)}$.

The visualization of composing (a_1, a_0)-tiles as seen through associated vectors suggests that each (a_1, a_0)-tile of any given level is rotationally symmetrical. This is better understood in the following Lemma.

Lemma 3.2.17. Let $(p, q) \in T_N$, the (a_1, a_0)-tile of level N. If $A(T_N) = (\alpha_N, \alpha'_N)$, then

$$(\alpha_N - p, \alpha'_N - q) \in T_N.$$

Proof. Let $\mathcal{T}$ be a uniform (a_1, a_0)-tree with $P = \begin{bmatrix} p \\ q \end{bmatrix}_N$ as a breakpoint.

First recall that $\begin{bmatrix} \alpha_N \\ \alpha'_N \end{bmatrix} = 1$ by Lemma 3.2.4. Therefore

$$\begin{aligned}
\begin{bmatrix} \alpha_N - p \\ \alpha'_N - q \end{bmatrix}_N &= \frac{\alpha_N - p}{\beta^N} + \frac{\alpha'_N - q}{\beta^{N+1}} \\
&= \frac{\alpha_N}{\beta^N} + \frac{\alpha'_N}{\beta^{N+1}} - \left(\frac{p}{\beta^N} + \frac{q}{\beta^{N+1}} \right) \\
&= \begin{bmatrix} \alpha_N \\ \alpha'_N \end{bmatrix}_N - \begin{bmatrix} p \\ q \end{bmatrix}_N \\
&= 1 - P.
\end{aligned}$$

We need to show that if $P \in \mathbb{Z}[\beta] \cap [0,1]$ is a breakpoint in some uniform (a_1, a_0)-subdivision S, then there exists a uniform (a_1, a_0)-subdivision S' such that $1 - P \in B[S]$.

If $\mathcal{T}$ be the uniform (a_1, a_0)-tree which contains P as a breakpoint, then the reverse (a_1, a_0)-tree $\mathcal{T}^r$ must contain the breakpoint $1 - P$. □

3.2.18 Tile Width

The matrix $A = \begin{pmatrix} a_1 & 1 \\ a_0 & 0 \end{pmatrix}$ has eigenvector $v_\beta = \begin{pmatrix} x \\ y \end{pmatrix}$ associated to the eigenvalue β.

$$\begin{aligned}
Av_\beta &= \beta v_\beta \\
\begin{pmatrix} a_1 & 1 \\ a_0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= \beta \begin{pmatrix} x \\ y \end{pmatrix} \\
\begin{pmatrix} a_1 x + y \\ a_0 x \end{pmatrix} &= \begin{pmatrix} \beta x \\ \beta y \end{pmatrix}.
\end{aligned}$$

The line $L := \{rv_\beta | r \in \mathbb{R}\}$ is the extension of the eigenvector through the origin. This has equation $L: \ y = \frac{a_0}{\beta} x = (\beta - a_1)x = |\beta^*|x$, where β^* is the Gaussian conjugate of β.

Remark 24. Since $\beta \notin \mathbb{Q}$, if $(x, y) \in \mathbb{Z}^2$ is on the line L, then $x = y = 0$.

We now define a semi-norm on $\mathbb{R}^2$, with respect to the line L.

Definition 3.2.19. For all $(p, q) \in \mathbb{R}^2$ define $\overline{(p, q)}$ to be the minimal Euclidean distance from (p, q) to the line $L: \ y = \frac{a_0}{\beta} x = (\beta - a_0)x$.

Furthermore, we describe points below L, i.e. adding some positive value to the y-coordinate is necessary to get to L, **negative**. Similarly, points above this line are **positive**. We will use the following notation.

$$T_N^+ := \{P \in T_N \big| P \text{ is positive}\}$$
$$T_N^- := \{P \in T_N \big| P \text{ is negative}\}.$$

The function $\overline{\cdot} : \mathbb{R}^2 \to \mathbb{R}$ satisfies the properties of a semi-norm. For $(p, q), (p', q') \in \mathbb{R}^2$ and $\alpha \in \mathbb{R}$,

1. $\overline{(p,q) + (p',q')} \leq \overline{(p,q)} + \overline{(p',q')}$
2. $\overline{\alpha(p,q)} = |\alpha|\overline{(p,q)}$

Remark 25.

Remark 26. If $(x, y), (p, q) \in \mathbb{R}^2$ are *both* positive **or** *both* negative, then

$$\overline{(x,y) + (p,q)} = \overline{(x,y)} + \overline{(p,q)}$$

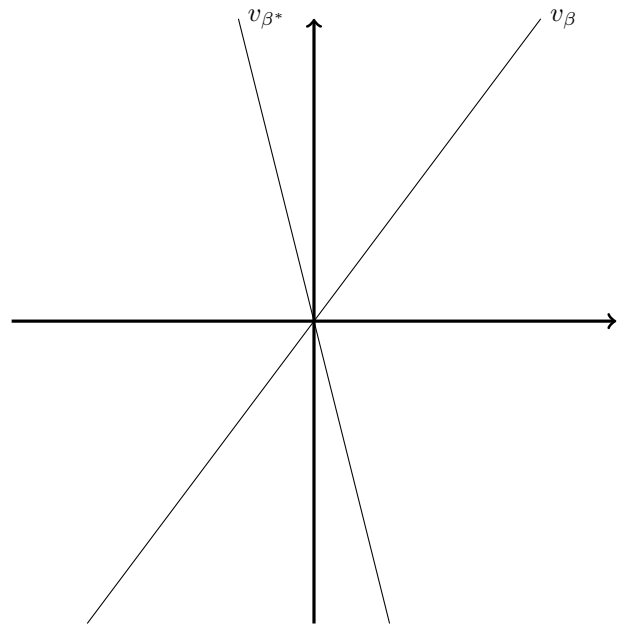
Lemma 3.2.20. Let $(p, q) \in T_N$ such that (p, q) is positive *(respectively negative)*, then

$$(p', q') = \begin{pmatrix} a_1 & 1 \\ a_0 & 0 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} \in T_{N+1}$$

and (p', q') is negative *(respectively positive).*

Proof. Let M be a square real matrix of size n. It is well known [22] that if all of the eigenvalues of M are distinct then their corresponding eigenvectors are linearly independent and thus form a basis of $\mathbb{R}^n$.

Our matrix A has two distinct eigenvalues, β and β^*, and so their corresponding eigenvectors form a basis for $\mathbb{R}^2$. Note that $\beta^* = \dfrac{-a_0}{\beta}$, which makes β^* a negative number (notice that we say negative number to highlight the difference between the commonly understood meaning of negative with respect to the real numbers, and the negative coordinates in $\mathbb{R}^2$ with respect to our semi-norm). Let v_β and v_{β^*} be the normalised eigenvectors of A associated with β and β^* respectively.



Therefore every point $P \in \mathbb{R}^2$ can be expressed as

$$P = r_1 v_\beta + r_2 v_{\beta^*}$$

for some $r_1, r_2 \in \mathbb{R}$. As v_β and v_{β^*} are eigenvectors of A,

$$Av_\beta = \beta v_\beta$$

$$Av_{\beta^*} = \beta^* v_{\beta^*}$$

Then letting P $= r_1 v_\beta + r_2 v_{\beta^*}$

$$\begin{aligned}
A * P &= \begin{pmatrix} a_1 & 1 \\ a_0 & 0 \end{pmatrix} (r_1 v_\beta + r_2 v_{\beta^*}) \\
&= r_1 \begin{pmatrix} a_1 & 1 \\ a_0 & 0 \end{pmatrix} v_\beta + r_2 \begin{pmatrix} a_1 & 1 \\ a_0 & 0 \end{pmatrix} v_{\beta^*} \\
&= r_1 \beta v_\beta + r_2 \beta^* v_{\beta^*} \\
&= r_1 \beta v_\beta - r_2 |\beta^*| v_{\beta^*} \\
&= r_1' v_\beta + r_2' v_{\beta^*}
\end{aligned}$$

The sign of the coefficient of the eigenvector v_{β^*} after multiplication by A is the opposite of the sign beforehand. Therefore multiplication by A takes positive (respectively negative) coordinates in $\mathbb{R}^2$ and maps them to negative (respectively positive) coordinates in $\mathbb{R}^2$ with respect to our semi-norm. $\square$

$A(T_0)$ is clearly a negative point, as it lies on the x-axis.

Remark 27.

$$\begin{aligned} \overline{A(T_N)} \text{ is positive} &\quad \text{if } N \text{ is odd} \\ \overline{A(T_N)} \text{ is negative} &\quad \text{if } N \text{ is even.} \end{aligned}$$

Lemma 3.2.21. Let $(p, q) \in \mathbb{R}^2$. Then

$$\overline{\begin{pmatrix} a_1 & 1 \\ a_0 & 0 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}} = |\beta^*| \overline{(p, q)}$$

where $\beta^* = -\dfrac{a_0}{\beta}$ is the Galois conjugate of β.

Proof. We have already shown that for all $P \in \mathbb{R}^2$, we can write $P = r_1 v_\beta + r_2 v_{\beta^*}$ for some $r_1, r_2 \in \mathbb{R}$.

$$\overline{P'} = \overline{r_1 v_\beta + r_2 v_{\beta^*}} = |r_2| \overline{v_{\beta^*}}.$$

If we map the point P by A, then

$$\begin{aligned} P' = A * P &= \begin{pmatrix} a_1 & 1 \\ a_0 & 0 \end{pmatrix} (r_1 v_\beta + r_2 v_{\beta^*}) \\ &= r_1 \begin{pmatrix} a_1 & 1 \\ a_0 & 0 \end{pmatrix} v_\beta + r_2 \begin{pmatrix} a_1 & 1 \\ a_0 & 0 \end{pmatrix} v_{\beta^*} \\ &= r_1 \beta v_\beta + r_2 \beta^* v_{\beta^*} \\ &= r_1 \beta v_\beta - r_2 |\beta^*| v_{\beta^*}. \end{aligned}$$

Then $\overline{P'}$ can be calculated

$$\begin{aligned}\overline{P'} &= \overline{r_1\beta v_\beta + r_2\beta^* v_{\beta^*}} \\ &= |r_2\beta^*|\overline{v_{\beta^*}} \\ &= |r_2||\beta^*|\overline{v_{\beta^*}} \\ &= |\beta^*|\overline{P}.\end{aligned}$$

□

Since $A(T_{N+1}) = \begin{pmatrix} a_1 & 1 \\ a_0 & 0 \end{pmatrix} A(T_N)$, we reach the following remark.

Remark 28. For $N \geq 0$, let T_N be the (a_1, a_0)-tile of level N. Then

$$\overline{A(T_{N+1})} = |\beta^*|\overline{A(T_N)}$$

Remark 29. If β is Pisot, $\overline{A(T_{N+1})} < \overline{A(T_N)}$. If β is Non-Pisot, then $\overline{A(T_{N+1})} > \overline{A(T_N)}$.

Note that $|\beta^*| = 1$ implies that either 1 or -1 is a solution to f_β, the irreducible integer polynomial, so there is no case such that $\overline{A(T_{N+1})} = \overline{A(T_N)}$. Recall that the associated vector of the (a_1, a_0)-tile of level N, T_N, is $V(T_N)$.

Remark 30. If β is Pisot then the associated vector of the (a_1, a_0)-tile of level N, $V(T_N)$, aligns more closely to the line L spanned by v_β as N increases. I.e., considering $V(T_N)$ as a position vector,

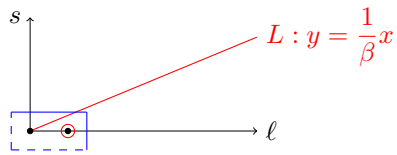
$$\overline{V(T_{N+1})} < \overline{V(T_N)}$$

Conversely note if β is non-Pisot, then

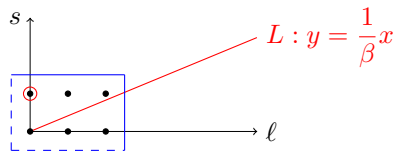
$$\overline{V(T_{N+1})} > \overline{V(T_N)}$$

For a given T_N there exists some $(p', q') \in \mathbb{Z}^2$ $p, q \geq 0$, such that $\overline{(p', q')} > \overline{(p, q)}$ for all $(p, q) \in T_N$. In this case, we might say that (p', q') is too positive, or too negative. We want to define the points in T_N which are the most positive and the most negative. These points are the further from the line L than all other points within T_N, and hence have maximal value under the semi-norm.

The level 0 $(2,1)$-tile, T_0 :



The level 1 $(2,1)$-tile, T_1 :



The level 2 $(2,1)$-tile, T_2 :

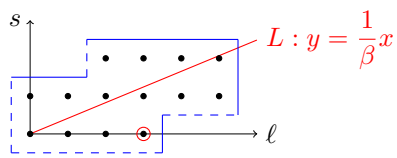


Figure 3.12: Maximal distances highlighted in $(2,1)$-tiles of level $0, 1$ and 2

Definition 3.2.22. For $0 \leq N \in \mathbb{N}$

$$D_m(T_N) = \left\{ (p,q) \in T_N \,\middle|\, \overline{(p,q)} = \max_{(x,y)\in T_N} \left\{ \overline{(x,y)} \right\} \in \mathbb{R} \right\}$$
$$D_m^+(T_N) = \left\{ (p,q) \in T_N^+ \,\middle|\, \overline{(p,q)} = \max_{(x,y)\in T_N^+} \left\{ \overline{(x,y)} \right\} \in \mathbb{R} \right\}$$
$$D_m^-(T_N) = \left\{ (p,q) \in T_N^- \,\middle|\, \overline{(p,q)} = \max_{(x,y)\in T_N^-} \left\{ \overline{(x,y)} \right\} \in \mathbb{R} \right\}.$$

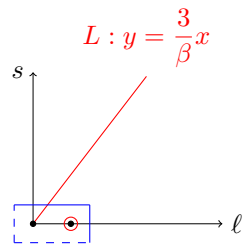
Thus $D_m(T_N)$ is the set of points in T_N which are the furthest distance from L. In fact we can show these to be singletons.

Lemma 3.2.23. $D_m(T_N)$, $D_m^+(T_N)$ and $D_m^-(T_N)$ are singletons for all $N \geq 0$.

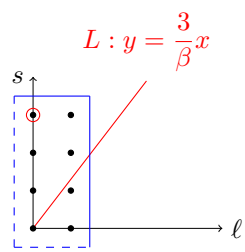
Proof. Suppose for contradiction that $P, Q \in D_m(T_N) \subset \mathbb{Z}^2$, with $P = (p_1, p_2) \neq Q = (q_1, q_2)$. Then $\overrightarrow{PQ}$ is parallel to L as the points P and Q are equally far from L. Then

$$P = Q + \lambda \begin{pmatrix} p_1 \\ \beta - a_1 \end{pmatrix}$$

The level 0 $(1,3)$-tile, T_0 :



The level 1 $(1,3)$-tile, T_1 :



The level 2 $(1,3)$-tile, T_2 :

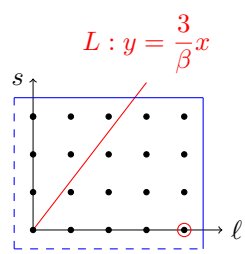


Figure 3.13: Maximal distances highlighted in $(1,3)$-tiles of levels $0,1,2$

$$\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ \beta - a_1 \end{pmatrix}.$$

This gives us the following two equations:

$$p_1 = q_1 + \lambda$$
$$p_2 = q_2 + \lambda(\beta - a_1).$$

This allows us to solve for λ in terms of p_1, q_1 to get $\lambda = p_1 - q_1$. We can rearrange the second equation to give

$$\begin{aligned} p_2 &= q_2 + (p_1 - q_1)(\beta - a_1) \\ p_2 - q_2 + a_1(p_1 - q_1) &= (p_1 - q_1)\beta \\ \frac{p_2 - q_2 + a_1(p_1 - q_1)}{p_1 - q_1} &= \beta. \end{aligned}$$

We know that $p_1, p_2, q_1, q_2, a_1 \in \mathbb{Z}$, so this implies that $\beta \in \mathbb{Q}$. However we know that β is irrational, so we have a contradiction. In this case $D_m(T_N)$ is a singleton. This argument also applies to $D_m^+(T_N)$ and $D_m^-(T_N)$.

□

We have proved that $D_m^{\pm}(T_N)$ are singletons for all $N \in \mathbb{N}$.

Remark 31. $\overline{D_m^-(T_N)} > 0$ for all $N \geq 0$. $\overline{D_m^+(T_N)} > 0$ for all $N \geq 1$.

This is easily seen if we recall that $T_N \subset T_{N+1}$, and so $T_0 \subset T_N$ for all $N \in \mathbb{N}$. As $D_m(T_0) > 0$, then $D_m(T_N) > 0$ for all $N \in \mathbb{Z}_{\geq 0}$.

Lemma 3.2.24. For all $N \in \mathbb{N}$

$$\overline{D_m^{\pm}(T_N)} \leq \overline{D_m^{\pm}(T_{N+1})}.$$

Proof. Remark 21 tells us that $T_N \subset T_{N+1}$ for all $N \in \mathbb{N}_0$. Therefore $D_m(T_N) \in T_{N+1}$, and so $\overline{D_m(T_N)} \leq \overline{D_m(T_{N+1})}$. □

Let the notation $[\cdot]_N$ extend to $D_m^{\pm}(T_N)$, so that if $D_m^{\pm}(T_N) = (x, y)$,

$$[D_m^{\pm}(T_N)]_N = \begin{bmatrix} x \\ y \end{bmatrix}_N = \frac{x}{\beta^N} + \frac{y}{\beta^{N+1}}.$$

The point $D_m^+(T_N)$ (respectively $D_m^-(T_N)$) is defined such that the interval $[0, [D_m^+(T_N)]_N]$ (respectively the interval $[0, [D_m^-(T_N)]_N]$) has the greatest (respectively least) ratio of short sub-intervals to long sub-intervals.

Lemma 3.2.25. Let $N \geq 0$, and T_N be the (a_1, a_0)-tile of level N. Then

$$D_m^+(T_N) + D_m^-(T_N) = A(T_N)$$

and as such $[D_m^+(T_N)]_N + [D_m^-(T_N)]_N = [A(T_N)]_N = 1$.

Proof. Recall that the (a_1, a_0)-tile of level N is rotationally symmetrical as shown in Lemma 3.2.17. Then to make the interval $[0, [D_m^+(T_N)]_M]$ contain the greatest ratio of short sub-intervals to long sub-intervals, you must ensure the interval $[[D_m^+(T_N)]_N, 1]$ contains the least possible ratio of short sub-intervals to long intervals. This means that $[[D_m^+(T_N)]_N, 1] = [1 - [D_m^-(T_N)]_N, 1]$, and so we conclude that

$$[D_m^+(T_N)]_N + [D_m^-(T_N)]_N = [A(T_N)]_N = 1$$

and thus $D_m^+(T_N) + D_m^-(T_N) = A(T_N)$. □

Proposition 3.2.26. For $N \geq 1$

$$D_m^+(T_N) = \begin{cases} a_0 A(T_{N-2}) + D_m^+(T_{N-1}) & N \text{ odd} \\ (a_1 - 1)A(T_{N-1}) + D_m^+(T_{N-1}) & N \text{ even} \end{cases}$$

$$D_m^-(T_N) = \begin{cases} (a_1 - 1)A(T_{N-1}) + D_m^-(T_{N-1}) & N \text{ odd} \\ a_0 A(T_{N-2}) + D_m^-(T_{N-1}) & N \text{ even} \end{cases}.$$

Proof. We will prove this by induction. Consider the (a_1, a_0)-tiles of level $-1, 0$ and 1, T_{-1}, T_0 and T_1.

$$
\begin{aligned}
D_m^+(T_{-1}) &= (0,1) \\
D_m^-(T_{-1}) &= (0,0) \\
D_m^+(T_0) &= (0,0) \\
D_m^-(T_0) &= (1,0) \\
D_m^+(T_1) &= (0,a_0) \\
D_m^-(T_1) &= (a_1,0).
\end{aligned}
$$

We can check the hypothesis for the base case $N = 1$,

$$
\begin{aligned}
D_m^+(T_1) &= (0,a_0) = a_0(0,1) + (0,0) \\
&= a_0 A(T_{-1}) + D_m^+(T_0) \\
D_m^-(T_1) &= (0,a_0) = (a_1 - 1)(1,0) + (1,0) \\
&= (a_1 - 1)A(T_0) + D_m^-(T_0).
\end{aligned}
$$

Notice that $D_m^+(T_1) + D_m^-(T_1) = A(T_1)$:

$$
\begin{aligned}
D_m^+(T_2) &= (a_1^2 - a_1, a_0^2) = (a_1 - 1)(a_1, a_0) + (0, a_0) \\
&= (a_1 - 1)A(T_1) + D_m^+(T_1) \\
, D_m^-(T_2) &= (a_0 + a_1, 0) = a_0(1,0) + (a_1, 0) \\
&= a_0 A(T_0) + D_m^-(T_1).
\end{aligned}
$$

Suppose that the Proposition is true for all $1 \leq k < N$.

$$
D_m^+(T_k) = \begin{cases} a_0 A(T_{k-2}) + D_m^+(T_{k-1}) & k \text{ odd} \\ (a_1 - 1)A(T_{k-1}) + D_m^+(T_{k-1}) & k \text{ even} \end{cases}
$$

$$
D_m^-(T_k) = \begin{cases} (a_1 - 1)A(T_{k-1}) + D_m^-(T_{k-1}) & k \text{ odd} \\ a_0 A(T_{k-2}) + D_m^-(T_{k-1}) & k \text{ even} \end{cases}
$$

Proposition 3.2.13 tells us that $T_N = \bigcirc(a_1 T_{N-1}, a_0 T_{N-2})$. We will split this question into four cases, $D_m^+(T_N)$, $D_m^+(T_N)$ with N odd, and $D_m^+(T_N)$, $D_m^+(T_N)$ with N even.

Case 1, N is odd.

If N is odd, we have remark 27 which tells us that $\overline{A(T_N)} > 0$. Since $D_m^+(T_N) + D_m^-(T_N) = A(T_N) > 0$ by Lemma 3.2.25, we see that $\overline{D_m^+(T_N)} > \overline{D_m^-(T_N)}$, and so $D_m(T_N) = D_m^+(T_N)$.

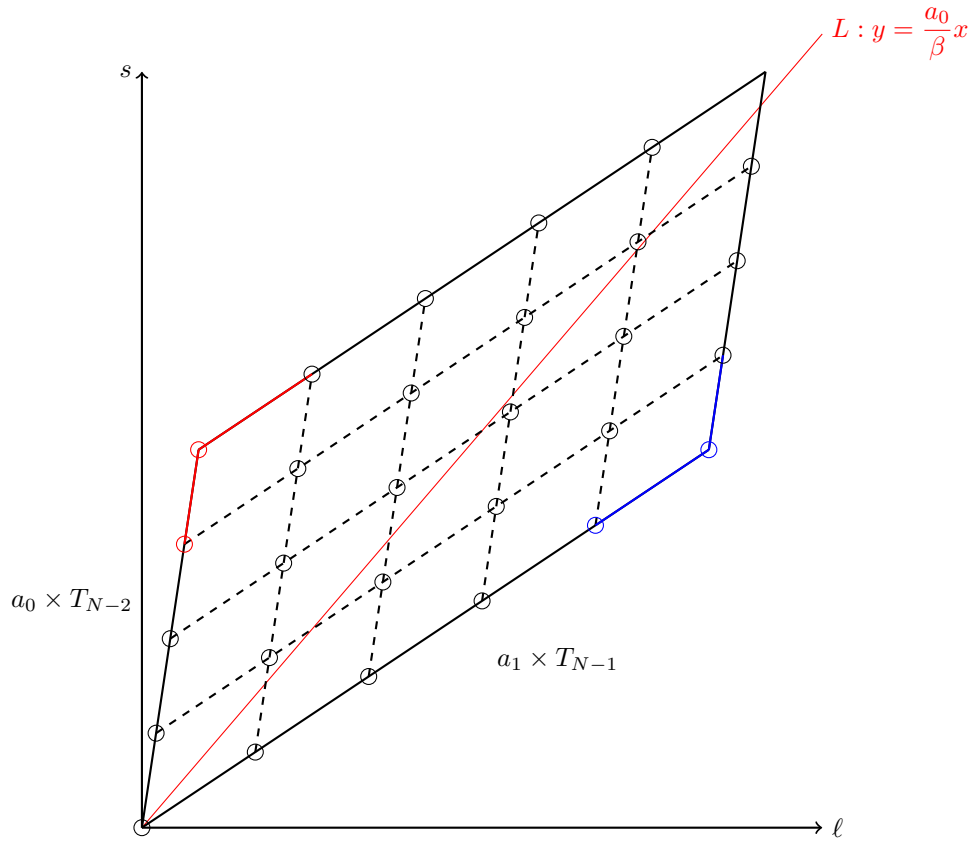


Figure 3.14: The associated vectors in the (a_1, a_0)-tile composition of some T_N

In figure 3.14, we can see the associated vectors of a tile composition of the (a_1, a_0)-tile of level N, T_N, in terms of T_{N-1} and T_{N-2}. Each line in the figure is the associated vector of either T_{N-1} or T_{N-2}, and each circle represents the origin of one of these (a_1, a_0)-tiles. Clearly the apex of T_N is seen to be a positive point.

By looking at figure 3.14, we can justify two possible tiles which could contain $D_m(T_N) = D_m^+(T_N)$, these being where the origin and associated vector has been coloured red. We can therefore assume that $D_m^+(T_N) \in \{(x, y) + (a_0 - 1)A(T_{N-2})|$ for some $(x, y) \in \bigcirc(T_{N-1}, T_{N-2})\}$.

The maximal distance in the tile must be equivalent to one of the maximal distances in either the tile of level T_{N-1} or the tile of level T_{N-2}. Thus $D_m^+(T_N) = (a_0 - 1)A(T_{N-2}) + D_m^+(T_{N-2})$, or $D_m^+(T_N) = a_0 A(T_{N-2}) + D_m^+(T_{N-1})$. We will now compare the two to see which must have greater maximal distance. Making use of Lemma 3.2.24, we find that,

$$\begin{aligned}
\overline{a_0 A(T_{N-2}) + D_m^+(T_{N-1})} &= \overline{a_0 A(T_{N-2})} + \overline{D_m^+(T_{N-1})} \\
&> \overline{a_0 A(T_{N-2})} + \overline{D_m^+(T_{N-2})} \\
&> \overline{(a_0 - 1)A(T_{N-2})} + \overline{D_m^+(T_{N-2})} \\
&= \overline{(a_0 - 1)A(T_{N-2}) + D_m^+(T_{N-2})}.
\end{aligned}$$

Therefore $D_m^+(T_N) = a_0 A(T_{N-2}) + D_m^+(T_{N-1})$. From this we can deduce an iterative formula for $D_m^-(T_N)$. Recall Lemma 3.2.25 tells us that $D_m^+(T_N) + D_m^-(T_N) = A(T_N)$. By rearranging this we see that

$$\begin{aligned}
D_m^-(T_N) &= A(T_N) - D_m^+(T_N) \\
&= (a_1 A(T_{N-1}) + a_0 A(T_{N-2})) - \big(a_0 A(T_{N-2}) + D_m^+(T_{N-1})\big) \\
&= a_1 A(T_{N-1}) - D_m^+(T_{N-1}) \\
&= (a_1 - 1)A(T_{N-1}) + \big(A(T_{N-1}) - D_m^+(T_{N-1})\big) \\
&= (a_1 - 1)A(T_{N-1}) + D_m^-(T_{N-1}).
\end{aligned}$$

Therefore $D_m^-(T_N) = (a_1 - 1)A(T_{N-1}) + D_m^-(T_{N-1})$ when N is odd.

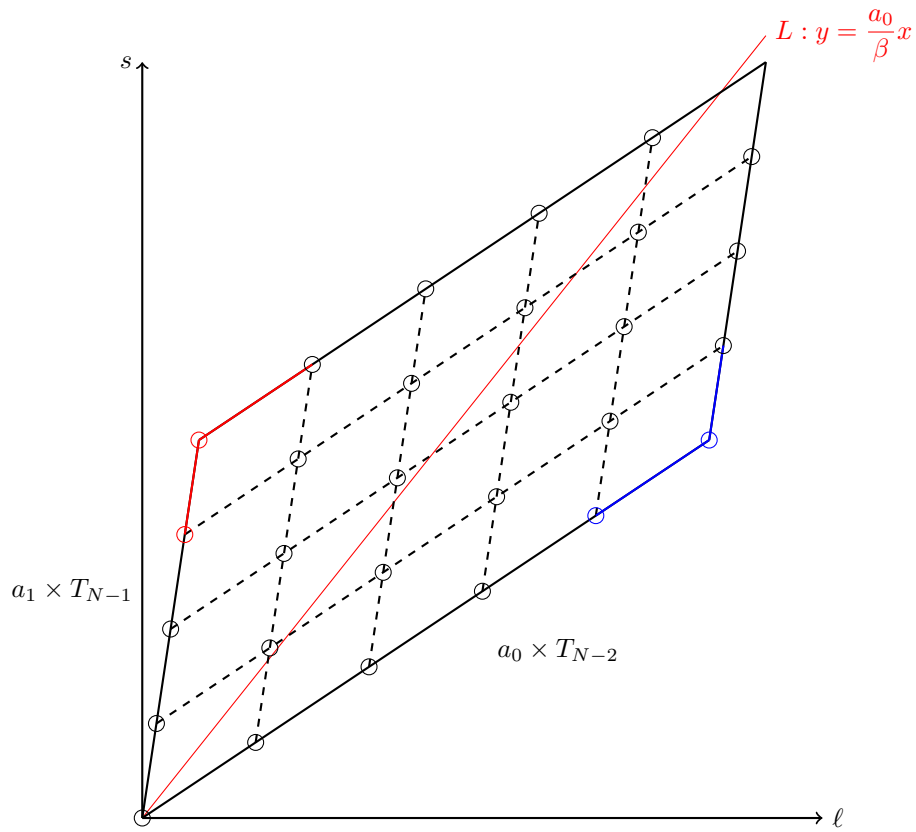


Figure 3.15: The associated vectors in the (a_1, a_0)-tile composition of T_N, N even

Case 2, N is even. In figure 3.15 we see the (a_1, a_0)-tile composition of T_N where N is even. Remark 27 tells us that $A(T_N)$ is negative, and so $D_m(T_N) = D_m^-(T_N)$. The point in T_N of maximal negative distance must lie in one of the (a_1, a_0)-tiles indicated by the blue edges, which have their bases highlighted by blue circles. This means that either $D_m^-(T_N) = (a_0 - 1)A(T_{N-2}) + D_m^-(T_{N-2})$, or $D_m^-(T_N) = a_0 A(T_{N-2}) + D_m^-(T_{N-1})$.

Thus $D_m^+(T_N) = (a_0 - 1)A(T_{N-2}) + D_m^+(T_{N-2})$, or $D_m^+(T_N) = a_0 A(T_{N-2}) + D_m^+(T_{N-1})$. We will now compare the two to see which must have greater maximal distance. Making use of Lemma 3.2.24, we find that

$$\begin{aligned}
\overline{a_0 A(T_{N-2}) + D_m^-(T_{N-1})} &= \overline{a_0 A(T_{N-2})} + \overline{D_m^-(T_{N-1})} \\
&> \overline{a_0 A(T_{N-2})} + \overline{D_m^-(T_{N-2})} \\
&> \overline{(a_0 - 1)A(T_{N-2})} + \overline{D_m^-(T_{N-2})} \\
&= \overline{(a_0 - 1)A(T_{N-2}) + D_m^-(T_{N-2})}.
\end{aligned}$$

Therefore $D_m^-(T_N) = a_0 A(T_{N-2}) + D_m^-(T_{N-1})$.

From this we can deduce an iterative formula for $D_m^+(T_N)$. Recall Lemma 3.2.25 tells us that

$$D_m^+(T_N) + D_m^-(T_N) = A(T_N).$$

By rearranging this we see that

$$\begin{aligned} D_m^+(T_N) &= A(T_N) - D_m^-(T_N) \\ &= (a_1 A(T_{N-1}) + a_0 A(T_{N-2})) - \big(a_0 A(T_{N-2}) + D_m^-(T_{N-1})\big) \\ &= a_1 A(T_{N-1}) - D_m^-(T_{N-1}) \\ &= (a_1 - 1)A(T_{N-1}) + \big(A(T_{N-1}) - D_m^-(T_{N-1})\big) \\ &= (a_1 - 1)A(T_{N-1}) + D_m^+(T_{N-1}). \end{aligned}$$

Therefore $D_m^+(T_N) = (a_1 - 1)A(T_{N-1}) + D_m^+(T_{N-1})$ when N is odd.

By considering the two cases we have shown that for $N \geq 1$, we reach our intended result:

$$D_m^+(T_N) = \begin{cases} a_0 A(T_{N-2}) + D_m^+(T_{N-1}) & N \text{ odd} \\ (a_1 - 1)A(T_{N-1}) + D_m^+(T_{N-1}) & N \text{ even} \end{cases},$$

$$D_m^-(T_N) = \begin{cases} (a_1 - 1)A(T_{N-1}) + D_m^-(T_{N-1}) & N \text{ odd} \\ a_0 A(T_{N-2}) + D_m^-(T_{N-1}) & N \text{ even} \end{cases}.$$

□

It will become extremely useful to create a shorthand for the apex of a tile. Whenever T_N appears in an equation, we will take this to mean $A(T_N)$ unless specified otherwise. The following is a statement about the apexes of the (a_1, a_0)-tiles of level $N-2, N-1$, and N

$$T_N = a_1 T_{N-1} + a_0 T_{N-2}$$

The context will usually make it clear which definition is being used.

Corollary 3.2.27. For $N-2 \geq 0$:

$$D_m^+(T_N) = \begin{cases} (a_1+a_0-1)\big(T_{N-2}+\cdots+T_3+T_1\big) + \begin{pmatrix} 0 \\ a_0 \end{pmatrix} & N \text{ odd} \\ (a_1-1)T_{N-1} + (a_1+a_0-1)\big(T_{N-3}+\cdots+T_3+T_1\big) + \begin{pmatrix} 0 \\ a_0 \end{pmatrix} & N \text{ even} \end{cases},$$

$$D_m^-(T_N) = \begin{cases} (a_1-1)T_{N-1} + (a_1+a_0-1)\big(T_{N-3}+\cdots+T_2+T_0\big) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} & N \text{ odd} \\ (a_1+a_0-1)\big(T_{N-2}+\cdots+T_2+T_0\big) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} & N \text{ even} \end{cases}.$$

Proof. For each of $D_m^+(T_N)$ and $D_m^-(T_N)$ we consider the cases, N odd and N even.

Case 1, $D_m^+(T_N)$ with N odd

By Proposition 3.2.26, we know that we can write $D_m^+(T_N) = a_0T_{N-2} + D_m^+(T_{N-1})$. We are then able to reuse Proposition 3.2.26 to expand $D_m^+(T_{N-1})$:

$$\begin{aligned} D_m^+(T_N) &= a_0T_{N-2} + D_m^+(T_{N-1}) \\ &= a_0T_{N-2} + (a_1-1)T_{N-2} + D_m^+(T_{N-2}) \\ &= (a_1+a_0-1)T_{N-2} + D_m^+(T_{N-2}) \\ &= (a_1+a_0-1)T_{N-2} + a_0T_{N-4} + D_m^+(T_{N-3}) \\ &= (a_1+a_0-1)T_{N-2} + a_0T_{N-4} + (a_1-1)T_{N-4} + D_m^+(T_{N-4}) \\ &= (a_1+a_0-1)\left(T_{N-2}+T_{N-4}\right) + D_m^+(T_{N-4}) \\ &\vdots \\ &= (a_1+a_0-1)\left(T_{N-2}+T_{N-4}+\cdots+T_1\right) + D_m^+(T_1) \\ &= (a_1+a_0-1)\left(T_{N-2}+T_{N-4}+\cdots+T_1\right) + \begin{pmatrix} 0 \\ a_0 \end{pmatrix}. \end{aligned}$$

Case 2, $D_m^+(T_N)$ with N odd

Again by Proposition 3.2.26, we know that we can write $D_m^+(T_N) = (a_1-1)T_{N-1} + D_m^+(T_{N-1})$. We

are then able to use our previous result to expand $D_m^+(T_{N-1})$, as $N-1$ is odd.

$$\begin{aligned} D_m^+(T_N) &= (a_1-1)T_{N-1} + D_m^+(T_{N-1}) \\ &= (a_1-1)T_{N-1} + (a_1+a_0-1)\left(T_{N-3}+T_{N-5}+\cdots+T_1\right) + \begin{pmatrix} 0 \\ a_0 \end{pmatrix}. \end{aligned}$$

Case 3, $D_m^-(T_N)$ with N even

By Proposition 3.2.26, we know that we can write $D_m^-(T_N) = a_0T_{N-2} + D_m^-(T_{N-1})$. We are then able to reuse Proposition 3.2.26 to expand $D_m^-(T_{N-1})$:

$$\begin{aligned} D_m^-(T_N) &= a_0T_{N-2} + D_m^-(T_{N-1}) \\ &= a_0T_{N-2} + (a_1-1)T_{N-2} + D_m^-(T_{N-2}) \\ &= (a_1+a_0-1)T_{N-2} + D_m^-(T_{N-2}) \\ &= (a_1+a_0-1)T_{N-2} + a_0T_{N-4} + D_m^-(T_{N-3}) \\ &= (a_1+a_0-1)T_{N-2} + a_0T_{N-4} + (a_1-1)T_{N-4} + D_m^-(T_{N-4}) \\ &= (a_1+a_0-1)\left(T_{N-2}+T_{N-4}\right) + D_m^-(T_{N-4}) \\ &\vdots \\ &= (a_1+a_0-1)\left(T_{N-2}+T_{N-4}+\cdots+T_2+T_0\right) + D_m^+(T_0)) \\ &= (a_1+a_0-1)\left(T_{N-2}+T_{N-4}+\cdots+T_2+T_0\right) + \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \end{aligned}$$

Case 4, $D_m^+(T_N)$ with N even

Again by Proposition 3.2.26, we know that we can write $D_m^+(T_N) = (a_1-1)T_{N-1} + D_m^+(T_{N-1})$. We are then able to use our previous result to expand $D_m^+(T_{N-1})$, as $N-1$ is even.

$$\begin{aligned} D_m^+(T_N) &= (a_1-1)T_{N-1} + D_m^-(T_{N-1}) \\ &= (a_1-1)T_{N-1} + (a_1+a_0-1)\left(T_{N-3}+T_{N-5}+\cdots+T_2+T_0\right) + \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \end{aligned}$$

Combing the four cases, we have proved that for $N \geq 2$:

$$D_m^+(T_N) = \begin{cases} (a_1 + a_0 - 1)\big(T_{N-2} + \cdots + T_3 + T_1\big) + \begin{pmatrix} 0 \\ a_0 \end{pmatrix} & N \text{ odd} \\ (a_1 - 1)T_{N-1} + (a_1 + a_0 - 1)\big(T_{N-3} + \cdots + T_3 + T_1\big) + \begin{pmatrix} 0 \\ a_0 \end{pmatrix} & N \text{ even} \end{cases},$$

$$D_m^-(T_N) = \begin{cases} (a_1 - 1)T_{N-1} + (a_1 + a_0 - 1)\big(T_{N-3} + \cdots + T_2 + T_0\big) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} & N \text{ odd} \\ (a_1 + a_0 - 1)\big(T_{N-2} + \cdots + T_2 + T_0\big) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} & N \text{ even} \end{cases}.$$

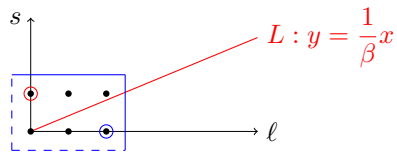
□

Remark 32. For $N \geq 2$,

$$D_m(T_N) = (a_1 + a_0 - 1)A(T_{N-2}) + D_m(T_{N-2}).$$

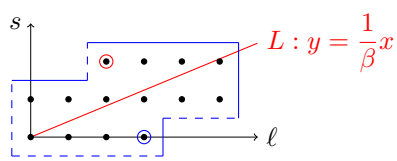
We can see in Figure 3.16 and Figure 3.17, the maximal distances within the tiles as described in Corollary 3.2.27.

The level 1 $(2,1)$-tile, T_1 :



$$D_m^+(T_1) = \begin{pmatrix} 0 \\ a_0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
$$D_m^-(T_1) = (2-1)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

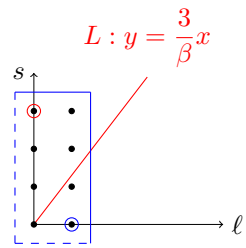
The level 2 $(2,1)$-tile, T_2 :



$$D_m^+(T_2) = (2-1)\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ a_0 \end{pmatrix}$$
$$= \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$
$$D_m^-(T_2) = (2+1-1)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

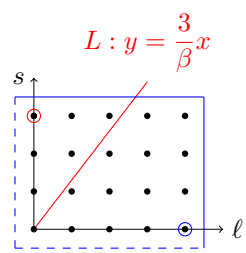
Figure 3.16: The maximal distances in $(2,1)$-tiles as found using their formulae

The level 1 $(1,3)$-tile, T_1 :



$$D_m^+(T_1) = \begin{pmatrix} 0 \\ a_0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$
$$D_m^-(T_1) = (1-1)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

The level 2 $(1,3)$-tile, T_2 :



$$D_m^+(T_2) = (1-1)\begin{pmatrix} 1 \\ 3 \end{pmatrix} + \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$
$$= \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$
$$D_m^-(T_2) = (1+3-1)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} 4 \\ 0 \end{pmatrix}$$

Figure 3.17: The maximal distances in $(1,3)$-tiles as found using their formulae

Lemma 3.2.28. For $N \geq 2$,

$$\overline{D_m(T_N)} = (a_1 + a_0 - 1)|\beta^*|^{N-2}\overline{A(T_0)} + \overline{D_m(T_{N-2})}.$$

In fact we can derive the following:

$$\overline{D_m(T_N)} = \begin{cases} (a_1 + a_0 - 1)\left(|\beta^*|^{N-2} + \cdots + |\beta^*|^3 + |\beta^*|\right)\overline{A(T_0)} + \overline{(0, a_0)} & N \text{ odd} \\ (a_1 + a_0 - 1)\left(|\beta^*|^{N-2} + \cdots + |\beta^*|^2 + 1\right)\overline{A(T_0)} + \overline{(1, 0)} & N \text{ even} \end{cases}.$$

Proof. By combining Remark 32, and Remark 28, we see that

$$\overline{D_m(T_N)} = (a_1 + a_0 - 1)|\beta^*|^{N-2}\overline{A(T_0)} + \overline{D_m(T_{N-2})}.$$

Now recall from Corollary 3.2.27, that

$$D_m^+(T_N) = \begin{cases} (a_1 + a_0 - 1)\big(T_{N-2} + \cdots + T_3 + T_1\big) + \begin{pmatrix} 0 \\ a_0 \end{pmatrix} & N \text{ odd} \\ (a_1 - 1)T_{N-1} + (a_1 + a_0 - 1)\big(T_{N-3} + \cdots + T_3 + T_1\big) + \begin{pmatrix} 0 \\ a_0 \end{pmatrix} & N \text{ even} \end{cases},$$

$$D_m^-(T_N) = \begin{cases} (a_1 - 1)T_{N-1} + (a_1 + a_0 - 1)\big(T_{N-3} + \cdots + T_2 + T_0\big) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} & N \text{ odd} \\ (a_1 + a_0 - 1)\big(T_{N-2} + \cdots + T_2 + T_0\big) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} & N \text{ even} \end{cases}.$$

Also recall that $D_m(T_N) = D_m^+(T_N)$ if N even, and $D_m(T_N) = D_m^-(T_N)$ if N odd. Since all of the apexes added will share the same parity, we can see that

$$\overline{T_{N-2} + \cdots + T_2 + T_0} = \overline{T_{N-2}} + \cdots + \overline{T_2} + \overline{T_0} \text{ if } N \text{ even}$$
$$\overline{T_{N-1} + \cdots + T_3 + T_1} = \overline{T_{N-1}} + \cdots + \overline{T_3} + \overline{T_1} \text{ if } N \text{ odd.}$$

□

Lemma 3.2.29. If β is non-Pisot, then there exists $(p_0, q_0) \in \mathbb{Z}^2$ where $\begin{bmatrix} p_0 \\ q_0 \end{bmatrix}_0 = P \in \mathbb{Z}[\tau] \cap [0, 1]$,

such that

$$\overline{\begin{pmatrix} a_1 & 1 \\ a_0 & 0 \end{pmatrix}^N \begin{pmatrix} p_0 \\ q_0 \end{pmatrix}} > \overline{D_m(T_N)} \text{ for all } N \geq 0.$$

Proof. Let $(p_0, q_0) \in \mathbb{Z}^2$ such that $P = \begin{bmatrix} p_0 \\ q_0 \end{bmatrix}_0 = p_0 + \frac{q_0}{\beta} \in \mathbb{Z}[\tau] \cap [0,1]$.

Let $\begin{pmatrix} p_N \\ q_N \end{pmatrix} = \begin{pmatrix} a_1 & 1 \\ a_0 & 0 \end{pmatrix}^N \begin{pmatrix} p_0 \\ q_0 \end{pmatrix}$ and recall that $\begin{bmatrix} p_N \\ q_N \end{bmatrix}_N = P$.

If we let $\overline{(p_0, q_0)} = d$, then $\overline{(p_N, q_N)} = |\beta^*|^N \times d$. In Lemma 3.2.28, we have a formula for $\overline{D_m(T_N)}$, so suppose for contradiction that

$$\overline{(p_N, q_N)} = |\beta^*|^N \times d \leq \begin{cases} (a_1 + a_0 - 1)\left(|\beta^*|^{N-2} + \cdots + |\beta^*|^3 + |\beta^*|\right)\overline{A(T_0)} + \overline{(0, a_0)} & N \text{ odd} \\ (a_1 + a_0 - 1)\left(|\beta^*|^{N-2} + \cdots + |\beta^*|^2 + 1\right)\overline{A(T_0)} + \overline{(1,0)} & N \text{ even} \end{cases}.$$

Without loss of generality, suppose N is even, and rearrange the inequality to find the expression,

$$\begin{aligned} d &\leq \frac{1}{|\beta^*|^N}\left((a_1 + a_0 - 1)\left(|\beta^*|^{N-2} + \cdots + |\beta^*|^2 + 1\right)\overline{A(T_0)} + \overline{(1,0)}\right) \\ &\leq (a_1 + a_0 - 1)\left(\frac{|\beta^*|^{N-2}}{|\beta^*|^N} + \cdots + \frac{|\beta^*|^2}{|\beta^*|^N} + \frac{1}{|\beta^*|^N}\right)\overline{A(T_0)} + \frac{\overline{(1,0)}}{|\beta^*|^N} \\ &\leq (a_1 + a_0 - 1)\left(|\beta^*|^{-2} + \cdots + |\beta^*|^{2-N} + |\beta^*|^{-N}\right)\overline{A(T_0)} + \overline{(1,0)}|\beta^*|^{-N} \\ &< (a_1 + a_0 - 1)\overline{A(T_0)}\sum_{i=1}^{N}\frac{1}{|\beta^*|^i} + \overline{(1,0)}|\beta^*|^{-N} \\ &< (a_1 + a_0 - 1)\overline{A(T_0)}\sum_{i=1}^{\infty}\frac{1}{|\beta^*|^i} + \overline{(1,0)}|\beta^*|^{-N}. \end{aligned}$$

As $\frac{1}{|\beta^*|} < 1$, the geometric series $\sum_{i=1}^{\infty}\frac{1}{|\beta^*|^i}$ converges to some positive constant [23], and so the value for d is bounded above.

Since $P = (p_0, q_0)$ can be any integer pair such that $\begin{bmatrix} p_0 \\ q_0 \end{bmatrix}_0 \in \mathbb{Z}[\tau] \cap [0,1]$, we can find P such that $\overline{P} = d$ is arbitrarily large. Hence there can be no such upper bound for d, and so we have reached a contradiction. Therefore there must exist $P = (p_0, q_0)$ satisfying the conclusion of the lemma.

□

This Lemma directly proves our theorem.

Theorem 3.2.30. If β is non-Pisot, then there exist breakpoints $P \in \mathbb{Z}[\tau] \cap [0,1]$ that cannot be found in a regular β-subdivision.

Corollary 3.2.31. Let β be the positive root of the irreducible polynomial $X^2 - a_1X - a_0 \in \mathbb{Z}[X]$ where $0 < a_1 < a_0$. Then

$$F_\beta \subsetneq G_\beta.$$

I.e., F_β is a proper subset of G_β.

Proof. Let β be the positive root of the irreducible polynomial $X^2 - a_1X - a_0 \in \mathbb{Z}[X]$ where $0 < a_1 < a_0$. Then β is non-Pisot and by Theorem 3.2.30, there exists $P \in \mathbb{Z}[\tau] \cap [0,1]$ such that P is not a breakpoint in any regular β-subdivision.
Corollary 3.1.3 tells us that for every $p \in \mathbb{Z}[\tau] \cap [0,1]$, there exists $g \in G_\beta$ which contains (p,p) as a breakpoint. In particular this means that there exists $g_P \in G_\beta$ such that (P,P) is a breakpoint of g_P. If we assume for contradiction that $g_P = (\mathcal{T}_1, \mathcal{T}_2) \in F_\beta$ where $\mathcal{T}_1, \mathcal{T}_2$ are (a_1, a_0)-trees then this implies that P is a breakpoint of both $\mathcal{T}_1$ and $\mathcal{T}_2$, but Theorem 3.2.30 tells us that in fact P cannot be a breakpoint in either $\mathcal{T}_1$ or $\mathcal{T}_2$.
Thus we have an element $g_P \in G_\beta$ but $g_P \notin F_\beta$. So F_β is a proper subset of G_β.

$$F_\beta \subsetneq G_\beta.$$

□

3.2.32 Example $f = X^2 - X - 3$

We now know that F_β is sometimes a proper subset of G_β, which leads us to ask is F_β even a subgroup of G_β. Throughout this section we will use the example $\beta = \dfrac{1+\sqrt{13}}{2}$, the zero of $f_\beta = X^2 - X - 3$. We will begin by finding an explicit element of G_β which can not be found in F_β. This is found by looking at properties of the points of maximal distance.

Lemma 3.2.33. For $N \geq 1$,

$$\begin{pmatrix} a_1 & 1 \\ a_0 & 0 \end{pmatrix} D_m^+(T_N) = D_m^-(T_{N+1}) - \begin{pmatrix} a_1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} a_1 & 1 \\ a_0 & 0 \end{pmatrix} D_m^-(T_N) = D_m^+(T_{N+1}) + \begin{pmatrix} a_1 \\ 0 \end{pmatrix}.$$

Proof. We prove this in 4 cases, $D_m^+(T_N)$ with N even and odd, and $D_m^-(T_N)$ with N even and odd.

Case 1, $D_m^+(T_N)$, N odd:

We take the definition for $D_m^+(T_N)$ for odd N from Corollary 3.2.27 and then pre-multiply by the matrix $\begin{pmatrix} a_1 & 1 \\ a_0 & 0 \end{pmatrix}$:

$$\begin{aligned}
\begin{pmatrix} a_1 & 1 \\ a_0 & 0 \end{pmatrix} D_m^+(T_N) &= \begin{pmatrix} a_1 & 1 \\ a_0 & 0 \end{pmatrix} \left((a_1 + a_0 - 1)\big(T_{N-2} + \cdots + T_3 + T_1\big) + \begin{pmatrix} 0 \\ a_0 \end{pmatrix} \right) \\
&= (a_1 + a_0 - 1) \begin{pmatrix} a_1 & 1 \\ a_0 & 0 \end{pmatrix} \big(T_{N-2} + \cdots + T_3 + T_1\big) + \\
&\quad + \begin{pmatrix} a_1 & 1 \\ a_0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ a_0 \end{pmatrix} \\
&= (a_1 + a_0 - 1) \left(\begin{pmatrix} a_1 & 1 \\ a_0 & 0 \end{pmatrix} T_{N-2} + \cdots + \begin{pmatrix} a_1 & 1 \\ a_0 & 0 \end{pmatrix} T_1 \right) + \begin{pmatrix} a_0 \\ 0 \end{pmatrix} \\
&= (a_1 + a_0 - 1)\left(T_{N-1} + \cdots + T_2\right) + \begin{pmatrix} a_0 \\ 0 \end{pmatrix}.
\end{aligned}$$

Note that $\begin{pmatrix} a_0 \\ 0 \end{pmatrix} = \begin{pmatrix} a_1 + a_0 - 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} a_1 \\ 0 \end{pmatrix} = (a_1 + a_0 - 1)T_0 + \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} a_1 \\ 0 \end{pmatrix}$. Thus

$$\begin{aligned}
\begin{pmatrix} a_1 & 1 \\ a_0 & 0 \end{pmatrix} D_m^+(T_N) &= (a_1 + a_0 - 1)\Big(T_{N-1} + \cdots + T_4 + T_2\Big) + \begin{pmatrix} a_1 + a_0 - 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} a_1 \\ 0 \end{pmatrix} \\
&= (a_1 + a_0 - 1)\Big(T_{N-1} + \cdots + T_4 + T_2 + T_0\Big) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} a_1 \\ 0 \end{pmatrix} \\
&= D_m^-(T_{N+1}) - \begin{pmatrix} a_1 \\ 0 \end{pmatrix}.
\end{aligned}$$

Case 2, $D_m^+(T_N)$, N even:

We take the definition for $D_m^+(T_N)$ for even N from Proposition 3.2.26, and then pre-multiply by the

matrix $\begin{pmatrix} a_1 & 1 \\ a_0 & 0 \end{pmatrix}$. So

$$\begin{aligned}\begin{pmatrix} a_1 & 1 \\ a_0 & 0 \end{pmatrix} D_m^+(T_N) &= \begin{pmatrix} a_1 & 1 \\ a_0 & 0 \end{pmatrix} \big((a_1-1)T_{N-1} + D_m^+(T_{N-1})\big) \\ &= (a_1-1)\begin{pmatrix} a_1 & 1 \\ a_0 & 0 \end{pmatrix} T_{N-1} + \begin{pmatrix} a_1 & 1 \\ a_0 & 0 \end{pmatrix} D_m^+(T_{N-1}).\end{aligned}$$

In case 1, we showed that $\begin{pmatrix} a_1 & 1 \\ a_0 & 0 \end{pmatrix} D_m^+(T_{N-1}) = D_m^-(T_N) - \begin{pmatrix} a_1 \\ 0 \end{pmatrix}$ if N is even.

$$\begin{aligned}\begin{pmatrix} a_1 & 1 \\ a_0 & 0 \end{pmatrix} D_m^+(T_N) &= (a_1-1)\begin{pmatrix} a_1 & 1 \\ a_0 & 0 \end{pmatrix} T_{N-1} + D_m^-(T_N) - \begin{pmatrix} a_1 \\ 0 \end{pmatrix} \\ &= (a_1-1)T_N + D_m^-(T_N) - \begin{pmatrix} a_1 \\ 0 \end{pmatrix} \\ &= D_m^-(T_{N+1}) - \begin{pmatrix} a_1 \\ 0 \end{pmatrix}.\end{aligned}$$

Case 3, $D_m^-(T_N)$, N even:

We take the definition for $D_m^+(T_N)$ for even N from Corollary 3.2.27, and then pre-multiply by the

matrix $\begin{pmatrix} a_1 & 1 \\ a_0 & 0 \end{pmatrix}$. So

$$\begin{aligned}
\begin{pmatrix} a_1 & 1 \\ a_0 & 0 \end{pmatrix} D_m^-(T_N) &= \begin{pmatrix} a_1 & 1 \\ a_0 & 0 \end{pmatrix} \left((a_1 + a_0 - 1)\big(T_{N-2} + \cdots + T_0\big) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) \\
&= (a_1 + a_0 - 1) \begin{pmatrix} a_1 & 1 \\ a_0 & 0 \end{pmatrix} \big(T_{N-2} + \cdots + T_0\big) \; + \begin{pmatrix} a_1 & 1 \\ a_0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= (a_1 + a_0 - 1) \left(\begin{pmatrix} a_1 & 1 \\ a_0 & 0 \end{pmatrix} T_{N-2} + \cdots + \begin{pmatrix} a_1 & 1 \\ a_0 & 0 \end{pmatrix} T_0 \right) + \begin{pmatrix} a_1 \\ a_0 \end{pmatrix} \\
&= (a_1 + a_0 - 1)\,(T_{N-1} + \cdots + T_3 + T_1) + \begin{pmatrix} 0 \\ a_0 \end{pmatrix} + \begin{pmatrix} a_1 \\ 0 \end{pmatrix} \\
&= D_m^+(T_{N+1}) + \begin{pmatrix} a_1 \\ 0 \end{pmatrix} \\
&= D_m(T_{N+1}) + \begin{pmatrix} a_1 \\ 0 \end{pmatrix}.
\end{aligned}$$

Case 4, $D_m^-(T_N)$, N odd:

We take the definition for $D_m^-(T_N)$ for odd N from Proposition 3.2.26, and then pre-multiply by the matrix $\begin{pmatrix} a_1 & 1 \\ a_0 & 0 \end{pmatrix}$. So

$$\begin{aligned}
\begin{pmatrix} a_1 & 1 \\ a_0 & 0 \end{pmatrix} D_m^-(T_N) &= \begin{pmatrix} a_1 & 1 \\ a_0 & 0 \end{pmatrix} \big((a_1 - 1)T_{N-1} + D_m^-(T_{N-1}) \big) \\
&= (a_1 - 1) \begin{pmatrix} a_1 & 1 \\ a_0 & 0 \end{pmatrix} T_{N-1} + \begin{pmatrix} a_1 & 1 \\ a_0 & 0 \end{pmatrix} D_m^-(T_{N-1}).
\end{aligned}$$

In case 3, we showed that $\begin{pmatrix} a_1 & 1 \\ a_0 & 0 \end{pmatrix} D_m^-(T_{N-1}) = D_m^+(T_N) + \begin{pmatrix} a_1 \\ 0 \end{pmatrix}$ if N is odd.

$$\begin{aligned}
\begin{pmatrix} a_1 & 1 \\ a_0 & 0 \end{pmatrix} D_m^-(T_N) &= (a_1 - 1) \begin{pmatrix} a_1 & 1 \\ a_0 & 0 \end{pmatrix} T_{N-1} + D_m^+(T_N) + \begin{pmatrix} a_1 \\ 0 \end{pmatrix} \\
&= (a_1 - 1) T_N + D_m^+(T_N) + \begin{pmatrix} a_1 \\ 0 \end{pmatrix} \\
&= D_m^+(T_{N+1}) + \begin{pmatrix} a_1 \\ 0 \end{pmatrix}.
\end{aligned}$$

Putting all four cases together, we have proved that for $N \geq 0$

$$\begin{aligned}
\begin{pmatrix} a_1 & 1 \\ a_0 & 0 \end{pmatrix} D_m^+(T_N) &= D_m^-(T_{N+1}) - \begin{pmatrix} a_1 \\ 0 \end{pmatrix} \\
\begin{pmatrix} a_1 & 1 \\ a_0 & 0 \end{pmatrix} D_m^-(T_N) &= D_m^+(T_{N+1}) + \begin{pmatrix} a_1 \\ 0 \end{pmatrix}.
\end{aligned}$$

□

Lemma 3.2.34. For $a_1 \neq a_0 + 1$, there exists a family of points $\begin{pmatrix} x_N \\ y_N \end{pmatrix} \in \mathbb{R}^2$ with

$$\begin{pmatrix} a_1 & 1 \\ a_0 & 0 \end{pmatrix} \begin{pmatrix} x_N \\ y_N \end{pmatrix} = \begin{pmatrix} x_{N+1} \\ y_{N+1} \end{pmatrix}$$

and a fixed real vector $\begin{pmatrix} X \\ Y \end{pmatrix} \in \mathbb{R}^2$ such that

$$\begin{pmatrix} x_N \\ y_N \end{pmatrix} = \begin{cases} D_m(T_N) + \begin{pmatrix} X \\ Y \end{pmatrix} & \text{if } N \text{ odd} \\ D_m(T_N) - \begin{pmatrix} X \\ Y \end{pmatrix} & if N even \end{cases}.$$

We can determine the values of X and Y to be

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \frac{a_1}{a_1 - a_0 + 1} \begin{pmatrix} 1 \\ -a_0 \end{pmatrix}.$$

Proof. Let $a_1, a_0 \geq 0$, such that $a_1 + 1 \neq a_0$. Suppose there exists $X, Y \in \mathbb{R}$ such that for some even N,

$$\begin{aligned}
\begin{pmatrix} a_1 & 1 \\ a_0 & 0 \end{pmatrix} \begin{pmatrix} x_N \\ y_N \end{pmatrix} &= \begin{pmatrix} x_{N+1} \\ y_{N+1} \end{pmatrix} \\
\begin{pmatrix} a_1 & 1 \\ a_0 & 0 \end{pmatrix} \left(D_m(T_N) - \begin{pmatrix} X \\ Y \end{pmatrix} \right) &= D_m(T_{N+1}) + \begin{pmatrix} X \\ Y \end{pmatrix} \\
\begin{pmatrix} a_1 & 1 \\ a_0 & 0 \end{pmatrix} D_m(T_N) - \begin{pmatrix} a_1 & 1 \\ a_0 & 0 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} &= D_m(T_{N+1}) + \begin{pmatrix} X \\ Y \end{pmatrix}.
\end{aligned}$$

Since N is even, $D_m(T_N) = D_m^-(T_N)$, and $D_m(T_{N+1}) = D_m^+(T_{N+1})$. We can then use Lemma 3.2.33 to expand the left hand side of this system of equations.

$$\begin{aligned}
\begin{pmatrix} a_1 & 1 \\ a_0 & 0 \end{pmatrix} D_m^-(T_N) - \begin{pmatrix} a_1 & 1 \\ a_0 & 0 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} &= D_m^+(T_{N+1}) + \begin{pmatrix} X \\ Y \end{pmatrix} \\
D_m^+(T_{N+1}) + \begin{pmatrix} a_1 \\ 0 \end{pmatrix} - \begin{pmatrix} a_1 X + Y \\ a_0 X \end{pmatrix} &= D_m^+(T_{N+1}) + \begin{pmatrix} X \\ Y \end{pmatrix} \\
\begin{pmatrix} a_1 \\ 0 \end{pmatrix} - \begin{pmatrix} a_1 X + Y \\ a_0 X \end{pmatrix} &= \begin{pmatrix} X \\ Y \end{pmatrix}.
\end{aligned}$$

From the second components, we see that $Y = -a_0 X$. We can substitute this into the first components to find

$$\begin{aligned}
a_1 - a_1 X + a_0 X &= X \\
a_1 - (a_1 - a_0 + 1) X &= 0 \\
X &= \frac{a_1}{a_1 - a_0 + 1}.
\end{aligned}$$

Hence we have

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \frac{a_1}{a_1 - a_0 + 1} \begin{pmatrix} 1 \\ -a_0 \end{pmatrix}.$$

We now need to check that this is still true when N is odd.

We have worked out what values these would take.

Now let $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = D_m(T_0) - \frac{a_1}{a_1 - a_0 + 1} \begin{pmatrix} 1 \\ -a_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \frac{a_1}{a_1 - a_0 + 1} \begin{pmatrix} 1 \\ -a_0 \end{pmatrix}$ Then

$$\begin{pmatrix} x_N \\ y_N \end{pmatrix} = \begin{pmatrix} a_1 & 1 \\ a_0 & 0 \end{pmatrix}^N \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

□

Note that the case $a_1 + 1 = a_0$, can already be discounted, as the subdivision polynomial $f = X^2 - a_1 X - (a_1 + 1)$ is reducible over $\mathbb{Z}$,

$$X^2 - a_1 X - (a_1 + 1) = (X - (a_1 + 1))(X + 1).$$

In Figure 3.18, we see the $(2, 1)$-tiles of level 1 and 2. The points of maximal distance within the tiles have been highlighted. When $(a_1, a_0) = (2, 1)$,

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \frac{2}{2 - 1 + 1} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Thus

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = D_m(T_1) + \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = D_m(T_2) - \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

In Figure 3.19, we see the $(1, 3)$-tiles of level 1 and 2. The points of maximal distance within the

tiles have been highlighted. When $(a_1, a_0) = (1, 3)$,

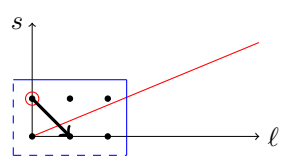
$$\begin{pmatrix} X \\ Y \end{pmatrix} = \frac{1}{1-3+1} \begin{pmatrix} 1 \\ -3 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}.$$

Thus

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = D_m(T_1) + \begin{pmatrix} -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix} + \begin{pmatrix} -1 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 6 \end{pmatrix}$$
$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = D_m(T_2) - \begin{pmatrix} -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} - \begin{pmatrix} -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ -3 \end{pmatrix}.$$

Notice that when $(a_1, a_0) = (2, 1)$, $\overline{(x_1, y_1)} < D_m(T_1)$, but when $(a_1, a_0) = (1, 3)$, $\overline{(x_1, y_1)} > D_m(T_1)$.

The level 1 $(2, 1)$-tile, T_1 :



The level 2 $(2, 1)$-tile, T_2 :

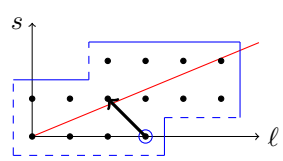
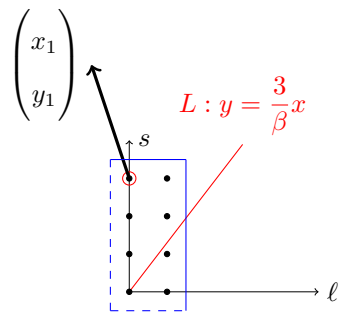


Figure 3.18: The points $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$, $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ inside the $(2, 1)$-tiles of level 1 and 2

The level 1 $(1,3)$-tile, T_1 :



The level 2 $(1,3)$-tile, T_2 :

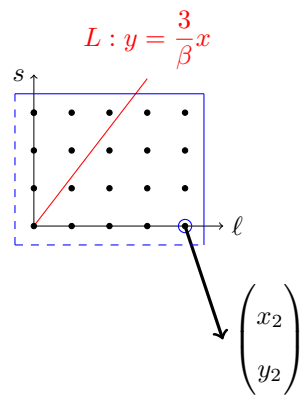


Figure 3.19: The point $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$, $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ outside the $(1,3)$-tiles of level 1 and 2

Whether $\begin{pmatrix} X \\ Y \end{pmatrix}$ is positive or negative with respect to the semi-norm $\bar{\cdot}$ is dependent on the sign of $\dfrac{a_1}{a_1 - a_0 + 1} = \gamma$.

If $a_0 \leq a_1$, then $\dfrac{a_1}{a_1 - a_0 + 1} = \gamma > 0$, and so $\begin{pmatrix} X \\ Y \end{pmatrix}$ is directed *South East*.

If $a_0 \geq a_1 + 2$, then $\dfrac{a_1}{a_1 - a_0 + 1} = \gamma < 0$ and so $\begin{pmatrix} X \\ Y \end{pmatrix}$ is directed *North West*.

The case $\gamma > 0$ is represented in Figure 3.20.

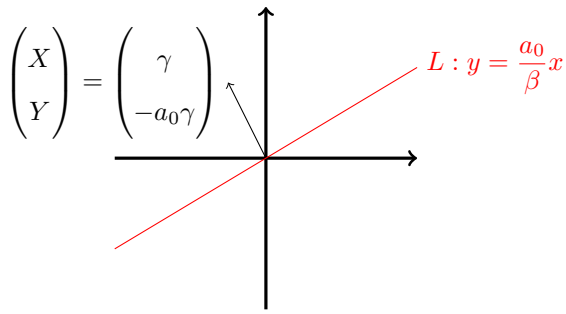


Figure 3.20: $\begin{pmatrix} X \\ Y \end{pmatrix}$ when $a_1 + 2 \leq a_0$

Lemma 3.2.35. If $a_1 + 2 \leq a_0$, for all $N \geq 0$

$$\overline{(x_N, y_N)} > \overline{D_m(T_N)}.$$

Proof. If $a_1 \leq a_0 + 2$, then $\begin{pmatrix} X \\ Y \end{pmatrix} = \gamma \begin{pmatrix} 1 \\ -a_0 \end{pmatrix} = \begin{pmatrix} \gamma \\ -a_0\gamma \end{pmatrix}$ where $\gamma < 0$. In Figure 3.20, we can clearly see that $\begin{pmatrix} \gamma \\ -a_0\gamma \end{pmatrix}$ is positive with respect to the semi-norm as long as $\gamma < 0$, so $\overline{X, Y} > 0$. If we take N to be odd, we see that

$$\begin{aligned}
\overline{(x_N, y_N)} &= \overline{D_m(T_N) + (X, Y)} \\
&= \overline{D_m^+(T_N) + (X, Y)} \\
&= \overline{D_m^+(T_N)} + \overline{(X, Y)} \\
&> \overline{D_m^+(T_N)} \\
&> \overline{D_m(T_N)}.
\end{aligned}$$

Conversely if N is even

$$\begin{aligned}
\overline{(x_N, y_N)} &= \overline{D_m(T_N) - (X, Y)} \\
&= \overline{D_m^-(T_N) - (X, Y)} \\
&= \overline{D_m^-(T_N)} + \overline{(X, Y)} \\
&> \overline{D_m^-(T_N)} \\
&> \overline{D_m(T_N)}.
\end{aligned}$$

□

Note that if $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in \mathbb{Z}^2$ and $a_1 + 2 \leq a_0$, then $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \notin T_0$.

Remark 33. If $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in \mathbb{Z}^2$, then $\begin{pmatrix} x_N \\ y_N \end{pmatrix} \in \mathbb{Z}^2$ for all $N \geq 0$.

This leads us to the following Lemma.

Lemma 3.2.36. If $a_1 + 2 \leq a_0$, and $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in \mathbb{Z}^2$. Then for $N \geq 0$,

$$\begin{pmatrix} x_N \\ y_N \end{pmatrix} \in \mathbb{Z}^2, \text{ and } \begin{pmatrix} x_N \\ y_N \end{pmatrix} \notin T_N.$$

Proof. If $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in \mathbb{Z}^2$ then $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} a_1 & 1 \\ a_0 & 0 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} a_1 x_0 + y_0 \\ a_0 x_0 \end{pmatrix}$. Since $a_0, a_1, x_0, y_0 \in \mathbb{Z}$, then it must also be the case that $a_1 x_0 + y_0 \in \mathbb{Z}$ and $a_0 x_0 \in \mathbb{Z}$. So $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \in \mathbb{Z}^2$. As $A = \begin{pmatrix} a_1 & 1 \\ a_0 & 0 \end{pmatrix} \in M_2(\mathbb{Z})$, we can extend this to

$$\begin{pmatrix} x_N \\ y_N \end{pmatrix} = \begin{pmatrix} a_1 & 1 \\ a_0 & 0 \end{pmatrix}^N \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in \mathbb{Z}^2.$$

If $a_1 + 2 \leq a_0$, then by Lemma 3.2.35,

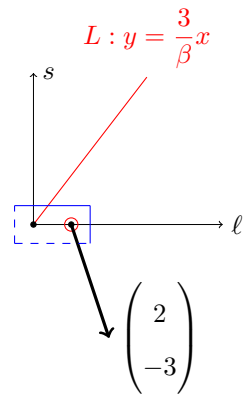
$$\overline{(x_N, y_N)} > \overline{D_m(T_N)}.$$

Thus by definition of $D_m(T_N)$, $(x_N, y_N) \notin T_N$. □

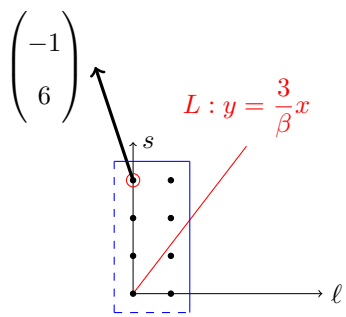
Therefore if $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in \mathbb{Z}^2$ and the corresponding β is non-Pisot, then the real value $P = \begin{bmatrix} x_N \\ y_N \end{bmatrix}_N$ is not obtainable at depth N for all $N \geq 0$.

We will now return to our specific case, $(a_1, a_0) = (1, 3)$. Here $\beta = \dfrac{1 + \sqrt{13}}{2}$ is the positive root of the irreducible polynomial $f = X^2 - X - 3$ and is non-Pisot, as $|\beta^*| \approx 1.3028 > 1$.

The level 0 $(1,3)$-tile, T_0 :



The level 1 $(1,3)$-tile, T_1 :



The level 2 $(1,3)$-tile, T_2 :

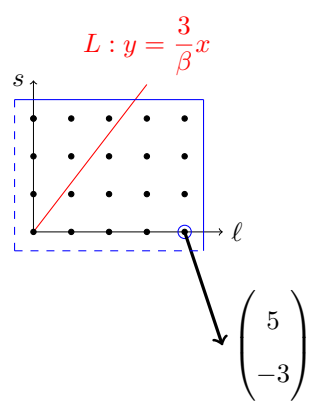


Figure 3.21: Points of fixed distance outside the $(1,3)$-tiles of level 0, 1 and 2

By definition of the fixed points $\begin{pmatrix} x_N \\ y_N \end{pmatrix}$, we can see that

$$\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}_0 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}_1 = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}_2 = \cdots = \begin{bmatrix} x_N \\ y_N \end{bmatrix}_N = \cdots .$$

As $a_1 + 2 \leq a_0$, and $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in \mathbb{Z}^2$ we know that $\begin{pmatrix} x_N \\ y_N \end{pmatrix} \notin T_N$ for all $N \geq 0$. This means that

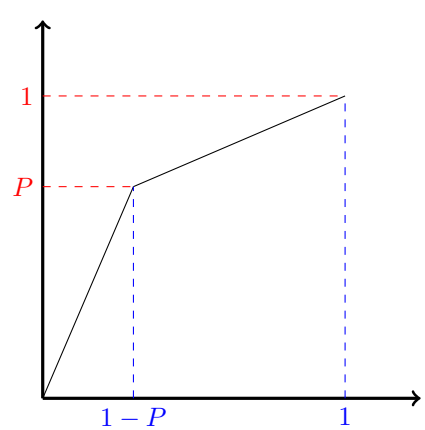
$$P = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}_0 = \begin{bmatrix} x_N \\ y_N \end{bmatrix}_N$$

is **not** obtainable. In our case $P = \begin{bmatrix} 2 \\ -3 \end{bmatrix}_0 = 2 - \frac{3}{\beta} \approx 0.6972$.

As P is not obtainable, we know that $1 - P = -1 + \frac{3}{\beta} = \begin{bmatrix} -1 & 3 \end{bmatrix}_0$ is also not obtainable. However, we notice that

$$1 - P = \begin{bmatrix} -1 \\ 3 \end{bmatrix}_0 = \left[\begin{pmatrix} 1 & 1 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 3 \end{pmatrix} \right]_1 = \begin{bmatrix} 2 \\ -3 \end{bmatrix}_1 = \frac{1}{\beta} \begin{bmatrix} 2 \\ -3 \end{bmatrix}_0 = \frac{P}{\beta}.$$

Consider then the map g as shown below.



The map g is not in F_β as neither P, nor P are obtainable points. The two slopes have gradients

$\dfrac{P}{1-P} = \beta$ and $\dfrac{1-P}{P} = \dfrac{1}{\beta}$ respectively, and so we have an explicit example of an element $g \in G_\beta$, but $g \notin \beta$.

Remark 34. When $(a_1, a_0) = (1, 3)$,

$$P = \begin{bmatrix} x_N \\ y_N \end{bmatrix}_{N+1} \in \mathbb{Z}[\tau] \cap [0,1] \text{ is not obtainable.}$$

3.3 Conjecture

3.3.1 Higher degree algebraic integers

It seems reasonable to hypothesise that for a cubic irreducible subdivision polynomial $f = X^3 - a_2X^2 - a_1X - a_0$ we could construct something akin to (a_2, a_1, a_0)-tiles, T_N, although they might be best described as (a_2, a_1, a_0)-staircases. The corresponding matrix

$$A = \begin{pmatrix} a_2 & 1 & 0 \\ a_1 & 0 & 1 \\ a_0 & 0 & 0 \end{pmatrix}$$

Consider two cases, first where β is the only real root of $f = X^3 - a_2X^2 - a_1X - a_0 = 0$. This matrix has one real eigenvalue, β with corresponding eigenvector v_β. The other eigenvalues are the complex conjugate roots of $f = 0$. The associated eigenvectors to the complex eigenvalues span a plane and spiralling towards the origin if β is Pisot, and spiralling away from the origin if β is non-Pisot.

If β is non-Pisot, the eigenspace of A, will map points in $\mathbb{R}^3$ by way of a parabolic curve, similar to that shown in figure 3.22. This parabolic curve will be skewed, but effectively centered on the eigenvector v_β, so we could still define our semi-norm $\overline{\cdot}$.

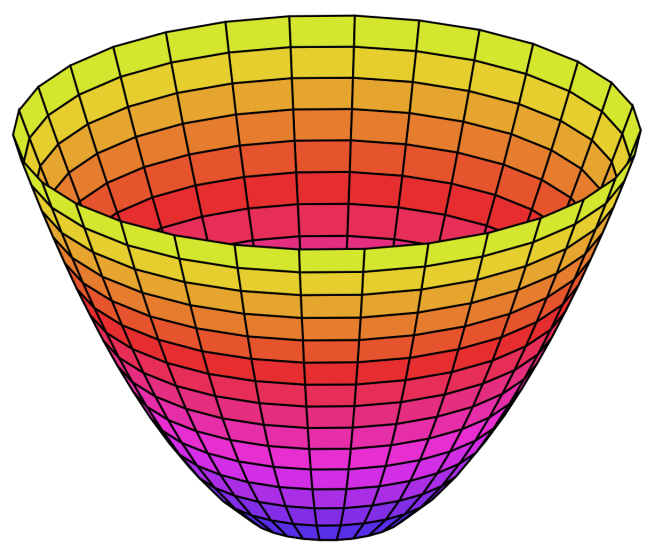


Figure 3.22: Parabolic curve

If we were able to similarly find a formula for the maximal distance of a point in the (a_2, a_1, a_0)-tile of level N, $D_m(T_N)$, then we should be able to show that for some $P = \begin{bmatrix} p_0 \\ q_0 \end{bmatrix}_0 \in \mathbb{Z}\left[\frac{1}{\beta}\right] \cap [0,1]$. Then

$$\overline{A^N \begin{pmatrix} p_0 \\ q_0 \end{pmatrix}} > \overline{D_m(T_N)}$$

for all $N \geq 0$. If we are able to prove this then we can state that there exists $P \in \mathbb{Z}\left[\frac{1}{\beta}\right] \cap [0,1]$ such that P is not obtainable in a β-regular subdivision of any depth.

Alternatively consider the case where $f = X^3 - a_2X^2 - a_1X - a_0 = 0$ has 3 real roots $\beta, \alpha_1, \alpha_2$. Two of these must be negative by Lemma 2.2.2, so $\alpha_1, \alpha_2 < 0$. Also since β is a Perron number but is non-Pisot, we can say that for at least one of these negative roots, say α_1,

$$-|\beta| < \alpha_1 \leq -1.$$

As $|\alpha_1| > 1$, each subdivision level takes the unique triple representing any real value p to unique triple which is further from the eigenvector corresponding to β, v_β. Since all coordinates that are obtainable tend to stay close to the line spanned by v_β, there is enough justification to make the following conjecture.

Conjecture 3.3.2. Let β be the positive real zero of an irreducible integer polynomial $f = X^3 - a_2X^2 - a_1X - a_0$ and let β be Non-Pisot. Then there exists $P \in \mathbb{Z}\left[\frac{1}{\beta}\right] \cap [0,1]$, such that P is not a breakpoint in any regular β-subdivision.

In fact, this argument could theoretically extend to any non-Pisot β the root of any irreducible subdivision polynomial

$$f = X^n - a_{n-1}X^{n-1} - a_{n-2}X^{n-2} - \cdots - a_1X - a_0$$

Therefore we can make the further conjecture:

Conjecture 3.3.3. Let β be the positive real zero of an irreducible integer polynomial $f = X^n -$

$a_{n-1}X^{n-1} - a_{n-2}X^{n-2} - \cdots - a_1X - a_0$ and let β be Non-Pisot. Then there exists $P \in \mathbb{Z}\left[\frac{1}{\beta}\right] \cap [0,1]$, such that P is not a breakpoint in any regular β-subdivision.

This would imply that if β, the positive real zero of $X^n - a_{n-1}X^{n-1} - a_{n-2}X^{n-2} - \cdots - a_1X - a_0$, is non-Pisot, then

$$F_\beta \subsetneq G_\beta$$

i.e. F_β is a proper subset of G_β for all such non-Pisot β.

3.3.4 Is F_β a group?

Let β be the positive real zero of the irreducible subdivision polynomial $f = X^2 - a_1X - a_0$. When $a_1 \geq a_0$, we know that β is Pisot, and by Corollary 2.5.12 that

$$F_\beta = G_\beta.$$

So if β is the root of the Pisot polynomial $f = X^2 - a_1X - a_0$, then F_β is a group.

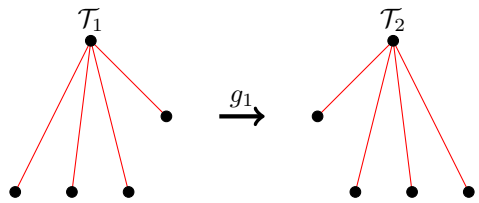
Conversely, if β is the zero of a non-Pisot irreducible integer polynomial $f = X^2 - a_1X - a_0$, we do not know if F_β is a sub-group of G_β. Recall that F_β is a non-empty set consisting of the maps in G_β which can be expressed as pairs of (a_1, a_0)-trees. The operation under which F_β could form a group is composition of maps.

We should already note the following is clear.

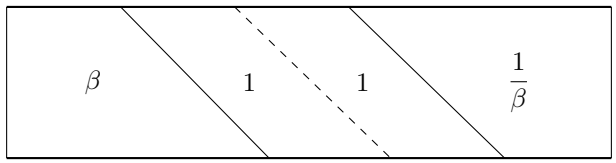
- Composition of maps is associative over the elements of F_β
- The identity map, $id \in F_\beta$
  - For any (a_1, a_0)-tree $\mathcal{T}$, $id = (\mathcal{T}, \mathcal{T})$
- Every element in F_β has an inverse
  - For every pair of (a_1, a_0)-trees $(\mathcal{T}_1, \mathcal{T}_2)$, $(\mathcal{T}_1, \mathcal{T}_2)^{-1} = (\mathcal{T}_2, \mathcal{T}_1)$

Thus, if we are able to show that F_β is closed under composition, then we will have shown that F_β is a group.

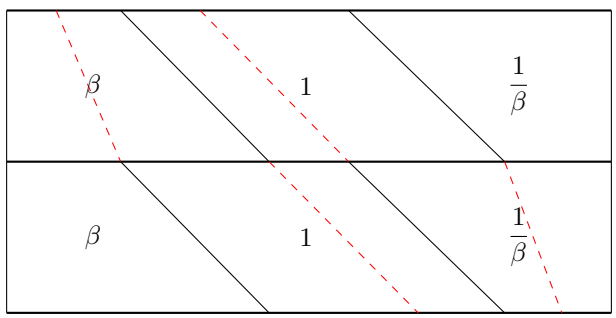
Let us return to the non-Pisot case where $(a_1, a_0) = (1, 3)$, and $\beta = \dfrac{1+\sqrt{13}}{2}$. Then g_1, as shown below, is certainly a map contained in F_β.



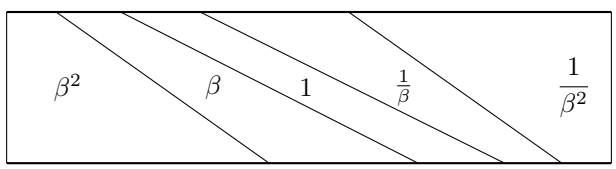
Consider the rectangle diagram for g_1. The gradient of each linear segment of g is highlighted in the middle section of the diagram.



We construct the rectangle diagram for g_1^2, and remove any lines which do not denote a change in gradient. A dashed line does not change the gradient but tracks where a breakpoint has been mapped to.



If we simplify this we have the simplified rectangle diagram for g_1^2.



So g_1^2 has breakpoints $\big\{(0,0)(\tau^3,\tau),(\tau^2,2\tau-\tau^2),(4\tau^2-\tau,\tau+2\tau^2),(3\tau^2,4\tau^2+2\tau^3)(1,1)\big\}$.

In particular, note that $4\tau^2-\tau$ is a breakpoint in the domain of g_1^2, and that $(-1,4)\in L_1$ where

$$L_i : \begin{pmatrix} x_i \\ y_i \end{pmatrix} + \lambda \left(\begin{pmatrix} x_{i+2} \\ y_{i+2} \end{pmatrix} - \begin{pmatrix} x_i \\ y_i \end{pmatrix} \right) \text{ for } \lambda \in [0,1].$$

Lemma 3.3.5. Let (p,q) be a point on the straight line between $\begin{pmatrix} x_i \\ y_i \end{pmatrix}$ and $\begin{pmatrix} x_{i+2} \\ y_{i+2} \end{pmatrix}$ for some $i \geq 0$. i.e. $(p,q) \in L_i$ where

$$L_i : \begin{pmatrix} x_i \\ y_i \end{pmatrix} + \lambda \left(\begin{pmatrix} x_{i+2} \\ y_{i+2} \end{pmatrix} - \begin{pmatrix} x_i \\ y_i \end{pmatrix} \right) \text{ for } \lambda \in [0,1].$$

Then

$$\begin{pmatrix} a_1 & 1 \\ a_0 & 0 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} \in L_{i+1}.$$

Proof. Let $\lambda^* \in [0,1]$ such that

$$\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} x_i \\ y_i \end{pmatrix} + \lambda^* \left(\begin{pmatrix} x_{i+1} \\ y_{i+1} \end{pmatrix} - \begin{pmatrix} x_i \\ y_i \end{pmatrix} \right) \in L_i.$$

Then

$$\begin{aligned}
\begin{pmatrix} a_1 & 1 \\ a_0 & 0 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} &= \begin{pmatrix} a_1 & 1 \\ a_0 & 0 \end{pmatrix} \left[\begin{pmatrix} x_i \\ y_i \end{pmatrix} + \lambda^* \left(\begin{pmatrix} x_{i+2} \\ y_{i+2} \end{pmatrix} - \begin{pmatrix} x_i \\ y_i \end{pmatrix} \right) \right] \\
&= \begin{pmatrix} a_1 & 1 \\ a_0 & 0 \end{pmatrix} \begin{pmatrix} x_i \\ y_i \end{pmatrix} + \lambda^* \left(\begin{pmatrix} a_1 & 1 \\ a_0 & 0 \end{pmatrix} \begin{pmatrix} x_{i+1} \\ y_{i+1} \end{pmatrix} - \begin{pmatrix} a_1 & 1 \\ a_0 & 0 \end{pmatrix} \begin{pmatrix} x_i \\ y_i \end{pmatrix} \right) \\
&= \begin{pmatrix} x_{i+1} \\ y_{i+1} \end{pmatrix} + \lambda^* \left(\begin{pmatrix} x_{i+3} \\ y_{i+3} \end{pmatrix} - \begin{pmatrix} x_{i+1} \\ y_{i+1} \end{pmatrix} \right) \in L_{i+1}.
\end{aligned}$$

□

We have all points on L_i are mapped to L_{i+1}. However we do not yet know that whether an integer point that lies on a line L_i can ever be obtained in some (a_1, a_0)-tile. For this reason, the following is left as a conjecture.

Conjecture 3.3.6. Let $P = \begin{bmatrix} p \\ q \end{bmatrix}_N \in \mathbb{Z}[\tau] \cap [0,1]$ for some $N \in \mathbb{Z}_{\geq 0}$, such that (p,q) lies on the line L_i for some $i \in \mathbb{Z}_{\geq 0}$. Then P is not obtainable at any depth.

In particular, this would mean that the in our earlier example, $\begin{bmatrix} -1 \\ 4 \end{bmatrix}_1 = 4\tau^2 - \tau$ is not obtainable at any depth. Therefore, there exists no regular $(1,3)$-subdivision which contains $4\tau^2 - \tau$ as a breakpoint.

In particular this means that the map g^2 cannot be expressed as a $(1,3)$-tree pair. i.e. F_β is not closed under composition, and thus F_β is not a group.

We conjecture that this extends to all non-Pisot β.

Conjecture 3.3.7. If β, the positive real zero of the irreducible subdivision polynomial $f_\beta = X^2 - a_1X - a_0$ is non-Pisot, then F_β is not a group.

Chapter 4

A Presentation of G_β

4.1 Background

In a previous chapter concerning regular subdivisions of the unit interval, we found Theorem 2.5.11 has the Corollary 2.5.12 which stated that if β the positive real zero of the irreducible subdivision polynomial $f_\beta = X^2 - a_1X - a_0$ is Pisot, then

$$F_\beta = G_\beta.$$

We want to use this information to find a presentation for the group G_β.

If β is the positive zero of the subdivision polynomial $X^2 - X - 1$ then Cleary first showed that F_β was FP_∞ and hence finitely generated in [4]. In [11], Burillo, Nucinkis, and Reeves found an explicit finite presentation for F_β using $(1,1)$-tree pairs, and in particular used this to show that the abelianisation F_τ^{ab} contained 2-torsion.

We will be looking at some examples for our polynomials of the form $X^2 - a_1X - a_0$ to find similar results where possible. An infinite presentation has been found in the work of Brown [5] who in turn found 2-torsion in the abelianisations for these groups.

In this chapter we will find a presentation for F_β where β is Pisot and the zero of the irreducible Pisot polynomial $f = X^2 - a_1X - a_0$, with $a_1, a_0 > 0$. We will then attempt to find properties of the abelianisations for particular choices of β.

Much of the work on (a_1, a_0)-tree pairs is already well known in the irrational Thompsons group

canon, but I have attempted to include as much background as is necessary to understand the notations and proofs. I am indebted to the work of Bieri [10] Brown [5], Burillo [13], Nucinkis and Reeves [11].

4.2 Tree pair Multiplication

Let β be the positive real zero of the irreducible subdivision polynomial $f_\beta = X^2 - a_1X - a_0$, and let β be Pisot. From Corollary 2.5.12, every element of G_β can be expressed as a pair of (a_1, a_0)-trees. We will say that $(\mathcal{T}_1, \mathcal{T}_2)$ is an (a_1, a_0)**-tree pair** if $(\mathcal{T}_1, \mathcal{T}_2) \in F_\beta$, i.e. $size(\mathcal{T}_1) = size(\mathcal{T}_2)$. The size of an (a_1, a_0)-tree pair, is $size(\mathcal{T}_1, \mathcal{T}_2) = size(\mathcal{T}_1) = size(\mathcal{T}_2)$.

In notation, we will only refer to F_β as the (a_1, a_0)-tree pair description of elements will be more useful for us.

4.2.1 Simultaneous refinements

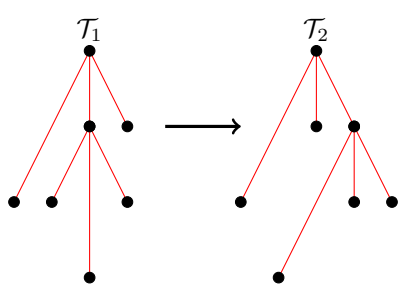
Definition 4.2.2. Let $g = (\mathcal{T}_1, \mathcal{T}_2) \in F_\beta$, for some (a_1, a_0)-trees $\mathcal{T}_1, \mathcal{T}_2$. A **simultaneous refinement of** $(\mathcal{T}_1, \mathcal{T}_2)$ is an (a_1, a_0)-tree pair $(\mathcal{T}_1', \mathcal{T}_2')$ where $\mathcal{T}_1'$ and $\mathcal{T}_2'$ are (a_1, a_0)-refinements of $\mathcal{T}_1$ and $\mathcal{T}_2$ respectively, such that

$$g = (\mathcal{T}_1', \mathcal{T}_2').$$

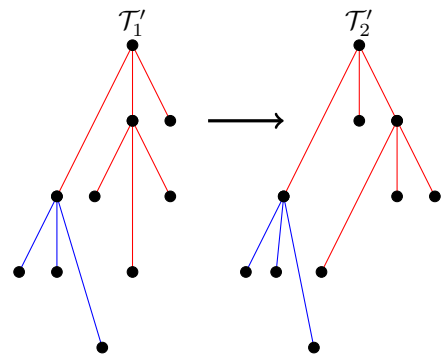
.

A simultaneous refinement of $(\mathcal{T}_1, \mathcal{T}_2)$ is found by hanging an (a_1, a_0)-tree T_i from the ith leaf of both T_1 and T_2.

Example 21. Consider the following $(2, 1)$-tree pair



We can find a simultaneous refinement of $(\mathcal{T}_1, \mathcal{T}_2)$, by performing the same (a_1, a_0)-refinement on each sub-interval represented by a leaf in $\mathcal{T}_1$ as on the sub-interval represented by the corresponding leaf in $\mathcal{T}_2$. I.e, we need to hang the same (a_1, a_0)-tree from corresponding leaves in $\mathcal{T}_1$ and $\mathcal{T}_2$.



Here, $(\mathcal{T}_1', \mathcal{T}_2')$ is a simultaneous refinement of $(\mathcal{T}_1, \mathcal{T}_2)$.

Lemma 4.2.3. Let $g = (\mathcal{T}_1, \mathcal{T}_2) \in F_\beta$, and let $(\mathcal{T}_1', \mathcal{T}_2')$ be a simultaneous refinement of $(\mathcal{T}_1, \mathcal{T}_2)$. If $g' = (\mathcal{T}_1', \mathcal{T}_2')$ then $g = g'$.

Proof. Let I_t and J_t be the intervals corresponding to the tth leaves in $\mathcal{T}_1$ and $\mathcal{T}_2$ respectively. Then $g(I_t) = (J_t)$, for all t. Let $(\mathcal{T}_1', \mathcal{T}_2')$ be a simultaneous refinement of $(\mathcal{T}_1, \mathcal{T}_2)$. Let T_t be the (a_1, a_0)-tree representing the (a_1, a_0)-refinement of I_t that takes $\mathcal{T}_1$ to $\mathcal{T}_2'$. As $(\mathcal{T}_1', \mathcal{T}_2')$ is a simultaneous refinement, T_t is also the (a_1, a_0)-refinement of J_t that takes $\mathcal{T}_2$ to $\mathcal{T}_2'$.

Then if l_k is the kth leaf of T_t, and I_{t_k} and J_{t_k} the corresponding intervals $\mathcal{T}_1'$ and $\mathcal{T}_2'$ respectively. Then

$$H(J_t) - H(I_t) = H(J_{t_k}) - H(I_{t_k})$$

for each of the leaves in T_t . Therefore the gradient of the slope in $(\mathcal{T}_1, \mathcal{T}_2)$ as in $(\mathcal{T}_1', \mathcal{T}_2')$, and thus $g = (\mathcal{T}_1', \mathcal{T}_2')$. □

4.2.4 Composition of (a_1, a_0)-tree pairs

Given $g \in F_\beta$, there is not a unique (a_1, a_0)-tree pair $(\mathcal{T}_1, \mathcal{T}_2)$ such that $g = (\mathcal{T}_1, \mathcal{T}_2)$. This means that in order to define composition of elements in F_β, we must find a well-defined multiplication of any two (a_1, a_0)-tree pairs.

For the purposes of this, we need to address the direction of our composition, and the direction in which $g \in F_\beta$ acts on $[0, 1]$. We have previously shown g as a left action, $g(x)$ for $x \in [0, 1]$ and we will keep to this convention. We will also follow the convention that $(g_2 \circ g_1)(x) = g_2(g_1(x))$.

Definition 4.2.5. (a_1, a_0)*-tree multiplication*

Let $(\mathcal{T}_1, \mathcal{T}_2)$ and $(\mathcal{T}_1', \mathcal{T}_2')$ be (a_1, a_0)-tree pairs. Define (not uniquely)

$$(\mathcal{T}_1, \mathcal{T}_2) \star (\mathcal{T}_1', \mathcal{T}_2') = (R_1, S_2)$$

where (R_1, R_2) and (S_1, S_2) are simultaneous refinements of $(\mathcal{T}_1, \mathcal{T}_2)$ and $(\mathcal{T}_1', \mathcal{T}_2')$ respectively, and $R_2 \sim S_1$.

We note that there must exist (a_1, a_0)-trees R_2 and S_1, refinements of $\mathcal{T}_2$ and $\mathcal{T}_1'$ respectively, with $R_2 \sim S_1$, as we have shown in Chapter 1.

The choice of output of (a_1, a_0)-tree pair multiplication is not unique. Thus in certain cases, as shown in the following remark, we will make a natural choice.

Remark 35. Let $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3$ be (a_1, a_0)-trees. We make the choice

$$(\mathcal{T}_1, \mathcal{T}_2) \star (\mathcal{T}_2, \mathcal{T}_3) = (\mathcal{T}_1, \mathcal{T}_3).$$

Lemma 4.2.6. Let $g_1, g_2 \in F_\beta$ and let $g_1 = (\mathcal{T}_1, \mathcal{T}_2)$ and $g_2 = (\mathcal{T}_1', \mathcal{T}_2')$. Then

$$g_2 \circ g_1 = (\mathcal{T}_1, \mathcal{T}_2) \star (\mathcal{T}_1', \mathcal{T}_2').$$

Proof. Let $g_1, g_2 \in F_\beta$, represented by some (a_1, a_0)-tree pairs $g_1 = (\mathcal{T}_1, \mathcal{T}_2)$ and $g_2 = (S_1, S_2)$. Let $(R_1, R_2) = (\mathcal{T}_1, \mathcal{T}_2) \star (S_1, S_2)$. Note that (R_1, R_2) is not unique, so can be any (a_1, a_0)-tree pair such that $(R_1, \mathcal{T}_2')$ and (S_1', R_2) are simultaneous refinements of $(\mathcal{T}_1, \mathcal{T}_2)$ and (S_1, S_2) respectively.

As $(R_1, \mathcal{T}_2')$ and (S_1', R_2) are simultaneous refinements of $(\mathcal{T}_1, \mathcal{T}_2)$ and (S_1, S_2) respectively, then $g_1 = (R_1, \mathcal{T}_2')$ and $g_2 = (S_1', R_2)$. We now have

$$(R_1, R_2) = (R_1, \mathcal{T}_2') \star (S_1', R_2)$$

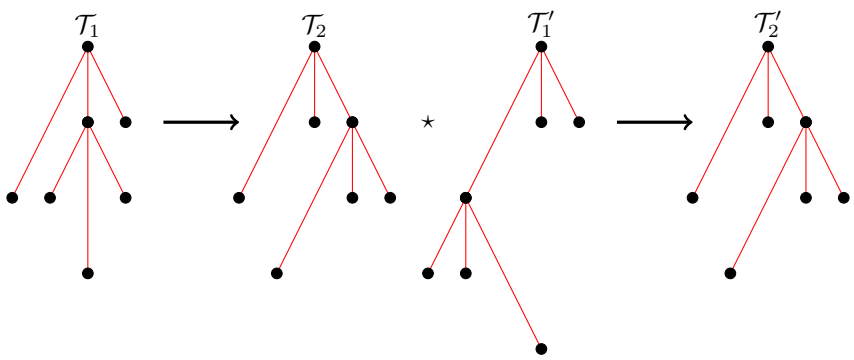
Let $g_3 = (R_1, R_2)$. Let P_1, P_2, P_3 represent the regular β-subdivisions of $[0, 1]$ that are represented by R_1, $\mathcal{T}_2' \sim S_1'$, and R_2 respectively. Then $g_3 = (P_1, P_3)$, which is the same as performing the affine linear interpolation (P_1, P_2) followed by (P_2, P_3). We have already seen that $(P_1, P_2) = g_1$ and $(P_2, P_3) = g_2$, so this means that

$$g_3 = (R_1, R_2) = \mathcal{T} \star (S_1, S_2) = g_2 \circ g_1$$

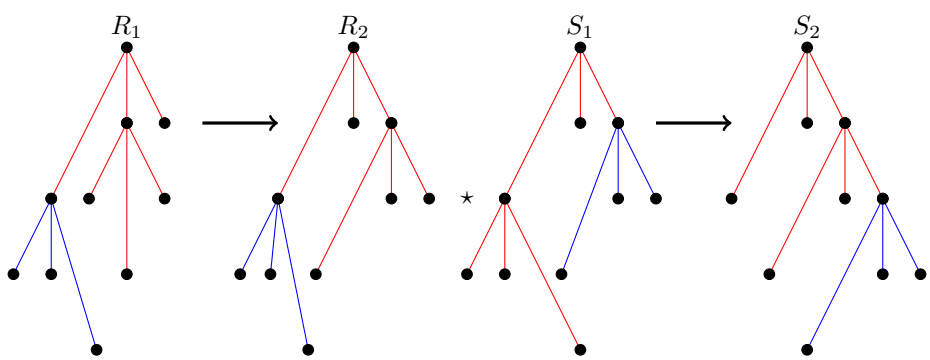
as required. □

In contrast to the order of composition of maps, the multiplication of (a_1, a_0)-tree pairs is written $g_2 \circ g_1 = (\mathcal{T}_1, \mathcal{T}_2) \star (\mathcal{T}_1', \mathcal{T}_2')$. To remind us that this is strange multiplication, we have used the symbol $\star$.

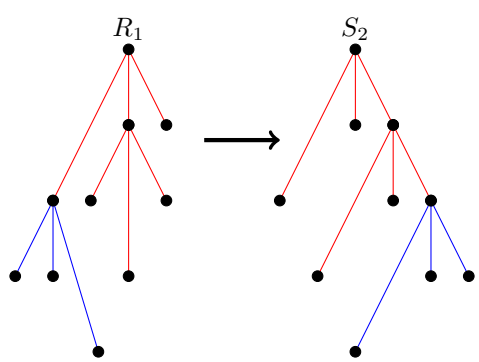
Example 22. Consider the following multiplication of two $(2, 1)$-tree pairs.



We perform simultaneous refinements on $(\mathcal{T}_1, \mathcal{T}_2)$ and $(\mathcal{T}_1', \mathcal{T}_2')$ to get



We see that $R_2 \sim S_1$, so we can find a $(2, 1)$-tree pair $(R_1, S_2) = (\mathcal{T}_1, \mathcal{T}_2) \star (\mathcal{T}_1', \mathcal{T}_2')$.



4.2.7 Right aligned (a_1, a_0)-trees

(a_1, a_0)-spines

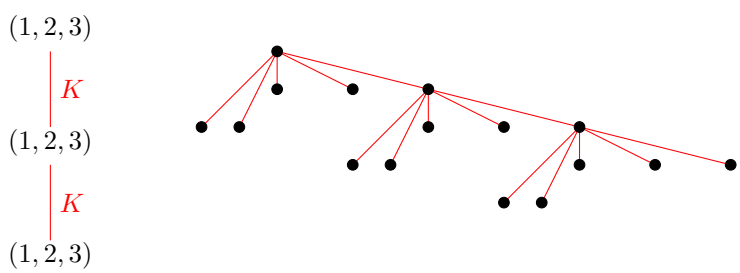
Definition 4.2.8. The **spine** of an (a_1, a_0)-tree $\mathcal{T}$ with root node R, is the (a_1, a_0)-tree S contained in $\mathcal{T}$ that features the (a_1, a_0)-carets of the form $R(K)(K)(K)\cdots(K)$ where $K = a_1 + a_0$.

To be clear, the notation $X(i)(j)$, is the jth child of the ith child of X.

The (a_1, a_0)-carets in the spine of $\mathcal{T}$ are all of the (a_1, a_0)-carets which contain an edge in the unique path from the R to the right-most leaf in $\mathcal{T}$.

Definition 4.2.9. An (a_1, a_0)-spine of size S, $Sp(S)$ is an (a_1, a_0)-tree of size S with only (a_1, a_0)-carets of type $(1, \ldots, a_0)$. Each (a_1, a_0)-caret in an (a_1, a_0)-spine except from the lowest caret has a single child from the right-most leg.

Example 23. The $(3, 2)$-spine of size 3, demonstrated in the reduced $(3, 2)$-tree notation, and standard $(3, 2)$-notation.



It is useful to use the notation $Sp(\mathcal{T})$ to refer to an (a_1, a_0)-spine which is the same size as the (a_1, a_0)-tree $\mathcal{T}$.

Right alignment

Definition 4.2.10. A right aligned (a_1, a_0)-tree is a simplified (a_1, a_0)-tree with a spine of (a_1, a_0)-carets of type $(1, \cdots, a_0)$.

A right aligned (a_1, a_0)-tree $\mathcal{T}$ has a spine of connected (a_1, a_0)-carets of type C_0. Recall the definition of connected (a_1, a_0)-carets from Chapter 1.

Definition 4.2.11. The **connected** (a_1, a_0)**-caret** C_i is the (a_1, a_0)-caret of type $(i, i+1, \ldots, i + a_0 - 1)$.

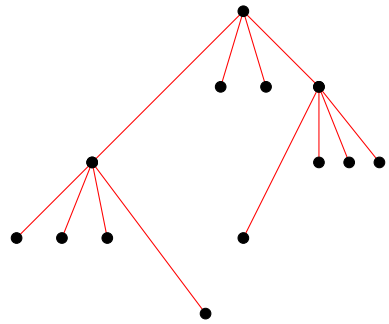


Figure 4.1: A right-aligned $(3,1)$-tree

In a connected (a_1, a_0)-caret, there are no short legs between any two long legs. In C_i there are i short legs to the left of the first long leg. We can use connected (a_1, a_0)-carets to perform *basic moves*.

Lemma 4.2.12. Let X and Y be (a_1, a_0)-carets of any type. There is a common refinement between X and Y.

Proof. Let X be the root node of an (a_1, a_0)-caret of type $(i_1, \ldots, i_{a_0})$, with only leaves for children. We will add a_1 new (a_1, a_0)-carets, one to each of the short children of X. If $X(j)$ is a short leg, then hang the connected (a_1, a_0)-caret C_t from $X(j)$ where t is the number of long legs to the right of $X(j)$. I.e., If $i_s < j < i_{s+1}$ hang the connected (a_1, a_0)-caret $C_{a_0 - s}$ from $X(j)$ for $s \in \{1, \ldots, a_0\}$. If $j < i_1$, hang C_{a_0}, and if $i_{a_0} < j$.

The resulting (a_1, a_0)-tree $\mathcal{T}_X$ will have leaf sequence

$$\mathcal{L}(\mathcal{T}_X) = \big(\underbrace{2, \ldots, 2}_{a_0}, \underbrace{3, \ldots, 3}_{a_0}, \underbrace{2, \ldots, 2}_{a_1}, \cdots, \underbrace{3, \ldots, 3}_{a_0}, \underbrace{2, \ldots, 2}_{a_1}\big)).$$

We can find an (a_1, a_0)-refinement of Y, with the same leaf sequence by following the same rules. Since this leaf sequence can be obtained from an (a_1, a_0)-caret of any type, there is a common refinement between any two (a_1, a_0)-carets. $\square$

Recall the basic move defined in 4.2.12 in Chapter 1.

Definition 4.2.13. A **basic move** is any graft which can be described by the above process defined in Lemma 4.2.12.

Note that for connected (a_1, a_0)-carets of type C_0, and C_{a_0}, the basic move will add only connected (a_1, a_0)-carets of type C_0 and C_{a_0}.

Lemma 4.2.14. There exists a common (a_1, a_0)-refinement between an (a_1, a_0)-tree $\mathcal{T}$ with root node R, and the connected (a_1, a_0)-caret C_0 in which the sub-tree $\mathcal{T}_{R(K)}$ is unchanged.

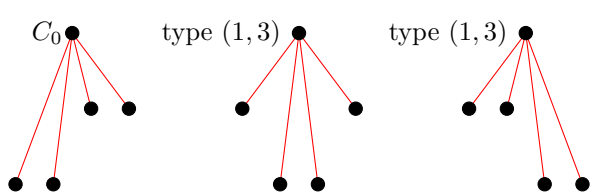
Proof. In 4.2.12, we showed that there is a common (a_1, a_0)-refinement between any two (a_1, a_0)-carets X and Y of type $(i_1, \ldots, i_{a_0})$ and type $(i'_1, \ldots, i'_{a_0})$ respectively, by following the basic move method. However, if there exists $1 \leq s \leq K$ such that $i_t < s < i_{t+1}$ and $i'_t < s < i'_{t+1}$ $s \in \{1, \ldots, K\}$ and $t \in \{1, \ldots a_0 - 1\}$, then the (a_1, a_0)-caret hung from $X(s)$ is the same as that from $Y(s)$. This is redundant, and so to find a common refinement between X and Y we can avoid hanging anything from $X(s)$ and $Y(s)$.

Now let $\mathcal{T}$ be an (a_1, a_0)-tree with root-caret R of type $(i_1, \ldots, i_{a_0})$. If $R(K)$ is a short leg, i.e. $i_{a_0} < K$, then in order to find a common refinement between R and the connected (a_1, a_0)-caret C_0 without hanging anything from either $X(K)$ or $C_0(K)$.

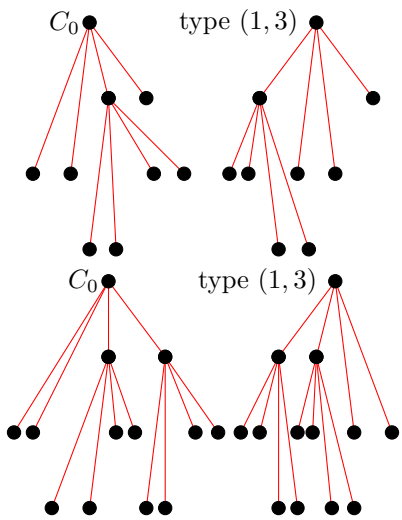
For each $R(j)$ for $j \notin \{i_1, \ldots, i_{a_0}\}$, with $j < K$ the sub-tree $\mathcal{T}_{R(j)}$ has a common refinement with the (a_1, a_0)-caret $C_{a_0 - s}$ where $i_s < j < i_{s+1}$. Call this (a_1, a_0)-refinement of C_{a_0-s}, $\mathcal{T}'_{R(j)}$. Then we can construct an (a_1, a_0)-tree $\mathcal{T}'$ with root caret R, with $\mathcal{T}'_{R(i_t)} = \mathcal{T}_{R(i_t)}$, $R(j) = \mathcal{T}'_j$ for $j \notin \{i_1, \ldots, i_{a_0}\}$, with $j < K$, and $\mathcal{T}'_{R(K)} = \mathcal{T}_{R(K)}$. Then we are able to perform a basic move to change the root-caret R to be the connected (a_1, a_0)-caret of type C_0, with the sub-tree $\mathcal{T}'_{R(K)} = \mathcal{T}_{R(K)}$ unchanged.

Conversely if $R(K) = i_{a_0}$, then $R(K)$ is a long leg. Now for all $j \notin \{i_1, \ldots, i_{a_0}\}$, we can similarly find a common refinement of the sub-tree $\mathcal{T}_{R(j)}$ with an (a_1, a_0)-tree $\mathcal{T}'_j$ with root-caret of type C_{a_0-s}, where $i_s < j < i_{s+1}$. By swapping out all of the sub-trees $\mathcal{T}_{R(j)}$ for $\mathcal{T}'_j$, we construct the (a_1, a_0)-tree $\mathcal{T}'$. This (a_1, a_0)-tree $\mathcal{T}'$ is a sub-tree of the (a_1, a_0)-tree constructed in Lemma 4.2.12, and so we are able to swap out this for the leaf-equivalent refinement of C_0. Note that this means the (a_1, a_0)-tree $\mathcal{T}_{R(K)}$ is unchanged. □

Example 24. Consider the $(2, 2)$-carets of type $(2, 3)$ and $(3, 4)$, and .



There exists a common refinement between the $(2, 2)$-caret of type $(1, 2)$, the connected $(2, 2)$-caret C_0, and any $(2, 2)$-caret X of type (i_1, i_2) where $X_{X(4)}$ is untouched.



Proposition 4.2.15. Any (a_1, a_0)-tree has an (a_1, a_0)-refinement which is leaf equivalent to a right aligned (a_1, a_0)-tree.

Proof. Suppose that the (a_1, a_0)-tree $\mathcal{T}$ has a spine containing n (a_1, a_0)-carets, $R = X_1, \ldots, X_n$, where X_{i+1} is the Kth child of X_i.

Consider the last (a_1, a_0)-caret in the spine of $\mathcal{T}$, X_n. We know that $X_n(K)$ is a leaf. If X_n is of type C_0, then we will leave the sub-tree $\mathcal{T}_{X_n}$ alone. Otherwise, suppose that $X_n(K)$ is a short leg. We know from Lemma 4.2.14, that there is an (a_1, a_0)-refinement of $\mathcal{T}_{X_n}$ which is leaf-equivalent to an (a_1, a_0)-refinement of C_0 in which there is no (a_1, a_0)-refinement of $X_n(K)$. Thus there exists $\mathcal{T}'_{X_n}$ which has a root-caret R' of type C_0, with $R'(K)$ a leaf.

If $X_n(K)$ is a long leg, then we can similarly find a common refinement between $\mathcal{T}_{X_n}$ and C_0, in which we do not refine the node $X_n(K)$. Therefore, there exists $\mathcal{T}'_{X_n}$ which has root-caret R' of type C_0, and so $R'(K)(K)$ is a short leg, and is a leaf. This must be so as there must be a node $Y \in \mathcal{T}'_{X_n}$ of height 2 which is a leaf, and corresponds to $X_n(K)$ in $\mathcal{T}_{X_n}$. Now, $R'(K)$ is the root node of the last (a_1, a_0)-caret in the spine of $\mathcal{T}'_{X_n}$, and the Kth leg is short. We can therefore replace $\mathcal{T}'_{R'(K)}$ with a leaf-equivalent (a_1, a_0)-tree with root-caret of type C_0.

In both cases, we are able to replace $\mathcal{T}_{X_n}$ with a right aligned sub-tree $\mathcal{T}'_{X_n}$.

Now consider the (a_1, a_0)-caret in the spine X_j such that $\mathcal{T}_{X_j}$ is a right aligned (a_1, a_0)-tree. I.e., $X_{j+1}, \ldots, X_n$ are all of type C_0. If X_j is of type C_0, then $\mathcal{T}_{X_j}$ is a right-aligned (a_1, a_0)-tree. Otherwise, we wish to find an (a_1, a_0)-refinement of $\mathcal{T}_{X_j}$ which is leaf-equivalent to a right-aligned (a_1, a_0)-tree.

Suppose $X_j(K)$ is a short leg. By Lemma 4.2.14, we can find an (a_1, a_0)-refinement of $\mathcal{T}_{X_j}$ which is leaf-equivalent to an (a_1, a_0)-tree $\mathcal{T}'_{X_j}$ with root-caret R'_j of type C_0, where $\mathcal{T}_{X_j(K)} = \mathcal{T}'_{R'(K)}$. Therefore the sub-tree $\mathcal{T}'_{R'_j}$ is a right-aligned (a_1, a_0)-tree.

If $X_j(K)$ is a long leg, we can similarly find an (a_1, a_0)-refinement which is leaf-equivalent to an (a_1, a_0)-refinement of the (a_1, a_0)-caret C_0. Call the (a_1, a_0)-refinement of C_0 $\mathcal{T}'$, with root node R'_j. As each sub-interval represented by a node in the sub-tree $\mathcal{T}_{X_j(K)}$ must have also be represented by a node found in $\mathcal{T}'$, $R'_j(K)(K)$ is a short leg, and corresponds to the node $X_j(K)$. Therefore there is an (a_1, a_0)-refinement of the (a_1, a_0)-tree $\mathcal{T}_{R'_j(K)}$ which is leaf-equivalent to a right aligned (a_1, a_0)-tree, as shown in the case where $X_j(K)$ was a short leg.

By repeating this process for each X_i, we can find an (a_1, a_0)-refinement of $\mathcal{T}$ which is leaf-equivalent to a right aligned (a_1, a_0)-tree. □

Corollary 4.2.16. Given an (a_1, a_0)-tree pair $(\mathcal{T}_1, \mathcal{T}_2)$, we can find an equivalent (a_1, a_0)-tree pair $(\mathcal{T}'_1, \mathcal{T}'_2)$ such that T'_1 and T'_2 are right aligned.

Proof. Let $(\mathcal{T}_1, \mathcal{T}_2)$ be an (a_1, a_0)-tree pair. There exists an (a_1, a_0)-refinement of $\mathcal{T}_1$ which is a right aligned (a_1, a_0)-tree. Let this (a_1, a_0)-refinement be $\mathcal{T}'$, and let $(\mathcal{T}'_1, \mathcal{T}'_2)$ be a simultaneous refinement of $(\mathcal{T}_1, \mathcal{T}_2)$.

If $\mathcal{T}'_2$ is not right aligned, we can certainly find an (a_1, a_0)-refinement of $\mathcal{T}'_2$ which is right-aligned, by following the process outlined in Proposition 4.2.15.

Consider the right-most leaf of $\mathcal{T}'_2$ throughout this process. At no point is there a refinement of the corresponding interval to this leaf. Therefore, there is an (a_1, a_0)-refinement of $\mathcal{T}'_2$, R_2, such that R_2 is leaf-equivalent to a right-aligned (a_1, a_0)-tree R'_2 and if (R_1, R_2) is the simultaneous refinement of $(\mathcal{T}'_1, \mathcal{T}'_2)$, then R_1 is a right-aligned (a_1, a_0)-tree. This is because, in the process of refining $\mathcal{T}'_2$ to R_2, we will not affect the spine of $\mathcal{T}'_1$ in the simultaneous refinement to R_1.

Therefore $g = (R_1, R'_2)$ where both R_1 and R'_2 are right-aligned (a_1, a_0)-trees. □

Definition 4.2.17. An (a_1, a_0)-tree pair $(\mathcal{T}_1, \mathcal{T}_2)$ in which both $\mathcal{T}_1$ and $\mathcal{T}_2$ are right aligned, is called a **right aligned** (a_1, a_0)**-tree pair**.

Lemma 4.2.18. Let $(\mathcal{T}_1, \mathcal{T}_2)$ and $(\mathcal{T}'_1, \mathcal{T}'_2)$ be right aligned (a_1, a_0)-tree pairs. Then there exists $(T_1, T_2) = (\mathcal{T}_1, \mathcal{T}_2) \star (\mathcal{T}'_1, \mathcal{T}'_2)$ such that (T_1, T_2) is also a right aligned (a_1, a_0)-tree pair.

Proof. Let $(\mathcal{T}_1, \mathcal{T}_2)$ and $(\mathcal{T}_1', \mathcal{T}_2')$ be right-aligned (a_1, a_0)-tree pairs. Then

$$(\mathcal{T}_1, \mathcal{T}_2) \star (\mathcal{T}_1', \mathcal{T}_2') = (R_1, S_2)$$

where (R_1, R_2) and (S_1, S_2) are simultaneous refinements of $(\mathcal{T}_1, \mathcal{T}_2)$ and $(\mathcal{T}_1', \mathcal{T}_2')$ respectively, and $R_2 \sim S_1$.

Therefore R_2 and S_1 are (a_1, a_0)-refinements of $\mathcal{T}_2$ and $\mathcal{T}_1'$ respectively. Now R_2 and S_1 represent a common refinement of $\mathcal{T}_2$ and $\mathcal{T}_1'$. Since $\mathcal{T}_2$ and $\mathcal{T}_1'$ are right aligned, they will have spines consisting of only connected (a_1, a_0)-carets of type C_0. Suppose the pines of $\mathcal{T}_2$ and $\mathcal{T}_1'$ are $S_X = \{X_1, \ldots, X_n\}$ and $S_Y = \{Y_1, \ldots, Y_m\}$ for some $n, m \in \mathbb{N}$.

If $n = m$, then we can find a common refinement between $\mathcal{T}_2$ and $\mathcal{T}_1'$ without hanging anything from the right-most leaf of either tree. In this case the simultaneous refinements (R_1, R_2) and (S_1, S_2) will still be right-aligned (a_1, a_0)-tree pairs, and thus so will (R_1, S_2).

If $n \neq m$, without loss of generality, we can assume that $n < m$. Therefore in the spine of $\mathcal{T}_2$, $X_n(K)$ is a leaf, whereas $Y_n(K)$ is not a leaf. In fact the sub-tree hanging from $Y_n(K)$ is a right-aligned (a_1, a_0)-tree. We can hang this from $X_n(K)$ to achieve a right-aligned (a_1, a_0)-tree which has a spine, the same size as that of $\mathcal{T}_1'$. As the spines of these right-aligned (a_1, a_0)-trees are now the same size, we have previously shown that we can find a common refinement without adding to the spine.

Therefore, for any right-aligned (a_1, a_0)-tree pairs $(\mathcal{T}_1, \mathcal{T}_2)$ and $(\mathcal{T}_1', \mathcal{T}_2')$, there exists a right aligned (a_1, a_0)-tree pair (T_1, T_2), such that

$$(T_1, T_2) = (\mathcal{T}_1, \mathcal{T}_2) \star (\mathcal{T}_1', \mathcal{T}_2').$$

□

4.2.19 Positive (a_1, a_0)-tree pairs

For every $g \in F_\beta$, there is an (a_1, a_0)-tree pair $(\mathcal{T}_1, \mathcal{T}_2)$. This (a_1, a_0)-tree pair has a simultaneous refinement $(\mathcal{T}_1', \mathcal{T}_2')$, which is a right aligned (a_1, a_0)-tree pair. We also know that the set of right aligned (a_1, a_0)-tree pairs is closed under (a_1, a_0)-tree pair multiplication.

We can therefore consider only right aligned (a_1, a_0)-tree pairs to represent every element $g \in F_\beta$.

Definition 4.2.20.

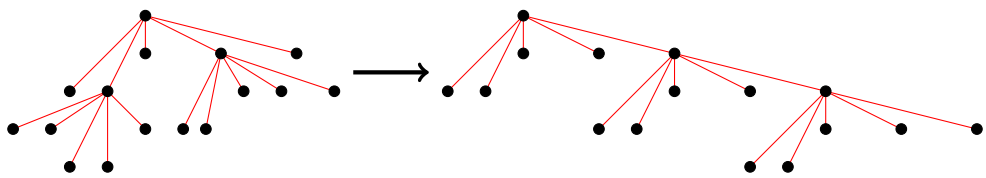
A right aligned (a_1, a_0)-tree pair $(\mathcal{T}_1, \mathcal{T}_2)$ is said to be **positive** if $\mathcal{T}_2$ is an (a_1, a_0)-spine.

A right aligned (a_1, a_0)-tree pair $(\mathcal{T}_1, \mathcal{T}_2)$ is said to be **negative** if $\mathcal{T}_1$ is an (a_1, a_0)-spine.

Remark 36. The right aligned (a_1, a_0)-tree pair $(\mathcal{T}_1, \mathcal{T}_2)$ is negative if $(\mathcal{T}_1, \mathcal{T}_2)^{-1} = (\mathcal{T}_2, \mathcal{T}_1)$ is positive.

In example 22, the $(2, 1)$-tree pairs $(\mathcal{T}_1, \mathcal{T}_2)$, $(\mathcal{T}_1', \mathcal{T}_2')$ and (R_1, S_2) are all positive.

Example 25. Below is a positive $(3, 2)$-tree pair.



Lemma 4.2.21. For each $g \in F_\beta$, there exists positive (a_1, a_0)-tree pairs P, Q, such that $g = P \star Q^{-1}$.

Proof. For all $g \in F_\beta$, there is a right aligned (a_1, a_0)-tree pair $(\mathcal{T}_1, \mathcal{T}_2)$ such that $g = (\mathcal{T}_1, \mathcal{T}_2)$. Since $size(\mathcal{T}_1) = size(\mathcal{T}_2)$, $Sp(\mathcal{T}_1) = Sp(\mathcal{T}_2)$.

$$g = (\mathcal{T}_1, \mathcal{T}_2) = (\mathcal{T}_1, Sp(\mathcal{T}_1)) \star (Sp(\mathcal{T}_2), \mathcal{T}_2) = P \star Q^{-1}$$

where $P = (\mathcal{T}_1, Sp(\mathcal{T}_1))$ and $Q = (\mathcal{T}_2, Sp(\mathcal{T}_2))$. Since both $\mathcal{T}_1$ and $\mathcal{T}_2$ are right aligned (a_1, a_0)-trees, P and Q are both positive right aligned (a_1, a_0)-tree pairs. □

Definition 4.2.22. The element $g \in F_\beta$ is **positive** if $g = (\mathcal{T}, Sp(\mathcal{T}))$, for some right aligned (a_1, a_0)-tree $\mathcal{T}$.

As the element g is not represented by a unique $\underline{\mathbf{a}}$-tree pair, it is not trivial to state whether a given g is positive, even when given $g = (\mathcal{T}_1, \mathcal{T}_2)$ for some (a_1, a_0)-tree pair $(\mathcal{T}_1, \mathcal{T}_2)$.

Lemma 4.2.23. The set of positive elements of F_β is closed under composition.

Proof. Let $g_1 = (\mathcal{T}_1, Sp(\mathcal{T}_1))$ and $g_2 = (\mathcal{T}_2, Sp(T_2))$ be positive elements of F_β for some right aligned (a_1, a_0)-trees $\mathcal{T}_1, \mathcal{T}_2$. Then to find $g_2 \circ g_1 = (\mathcal{T}_1, Sp(\mathcal{T}_1)) \star (\mathcal{T}_2, Sp(T_2))$ we must find (a_1, a_0)-tree pairs (R_1, R_2) and (S_1, S_2) that are simultaneous refinements of $(\mathcal{T}_1, Sp(T_1))$ and $(\mathcal{T}_2, Sp(T_2))$ respectively such that $R_2 \sim S_1$.

This equates to finding a common refinement between $Sp(\mathcal{T}_1)$ and $\mathcal{T}_2$. Since $\mathcal{T}_2$ is a right-aligned (a_1, a_0)-tree, then the spine of $\mathcal{T}_2$ contains only connected (a_1, a_0)-carets of type C_0. Suppose the

spine of $\mathcal{T}_2$ consists of the (a_1, a_0)-carets $X_1, \ldots, X_n$ where $X_i(K) = X_{i+1}$, and X_i is of type C_0 for all i. Let the (a_1, a_0)-spine $SP(\mathcal{T}_1)$ be of size m.

If $m = n$, then $\mathcal{T}_2$ is an (a_1, a_0)-refinement of $Sp(\mathcal{T}_1)$, and so we do not need to refine $\mathcal{T}_2$ to find a common refinement between them.

If $m < n$, then we can first extend the (a_1, a_0)-spine $Sp(\mathcal{T}_1)$ by hanging an (a_1, a_0)-spine of size $n-m$ from the right-most leaf. We now have an (a_1, a_0)-spine of size n, so $\mathcal{T}_2$ is an (a_1, a_0)-refinement of this (a_1, a_0)-spine, and we can therefore find a common refinement without having to refine $\mathcal{T}_2$.

Since we do not need to refine $\mathcal{T}_2$ in these cases, to find a common refinement with $Sp(\mathcal{T}_1)$ then we do not need to simultaneously refine $Sp(\mathcal{T}_2)$.

If $m > n$, then we can hang an (a_1, a_0)-spine of size $m - n$ from the right-most leaf of $\mathcal{T}_2$ to get $\mathcal{T}_2'$. Notice that in the simultaneous refinement of $(\mathcal{T}_2, SP(\mathcal{T}_2))$ to $(\mathcal{T}_2', S)$ the (a_1, a_0)-tree S, is an (a_1, a_0)-refinement of $Sp(\mathcal{T}_2)$, obtained by hanging an (a_1, a_0)-spine of size $m - n$ from the right most leaf of $Sp(\mathcal{T}_2)$. Note that this remains an (a_1, a_0)-spine, and so $S = Sp(\mathcal{T}_2')$. Therefore, we have $g_2 = (\mathcal{T}_2', Sp(\mathcal{T}_2'))$, and $\mathcal{T}_2^p rime$ is an (a_1, a_0)-refinement of $Sp(MT_1)$. We can therefore find a common refinement of $\mathcal{T}_2'$ and $Sp(\mathcal{T}_1)$ in which we do not refine $\mathcal{T}_2'$.

Since we do not need to refine $\mathcal{T}_2'$ to find a common refinement with $Sp(\mathcal{T}_1)$, then we do not need to simultaneously refine $Sp(\mathcal{T}_2')$.

In any case,

$$g_2 \circ g_1 = (\mathcal{T}_1, Sp(\mathcal{T}_1)) \star (\mathcal{T}_2, Sp(\mathcal{T}_2)) = (R_1, R_2) \star (R_2, Sp(R_2))$$

where (R_1, R_2) is a simultaneous refinement of $(\mathcal{T}_1, Sp(\mathcal{T}_1))$ and R_2 is a common refinement of $Sp(\mathcal{T}_1)$ and $\mathcal{T}_2$.

Therefore, there is an expression for $g_2 \circ g_1$ of the form $(R_1, Sp(R_1))$, therefore $g_2 \circ g_1$ is also positive. $\square$

4.3 A presentation for F_β

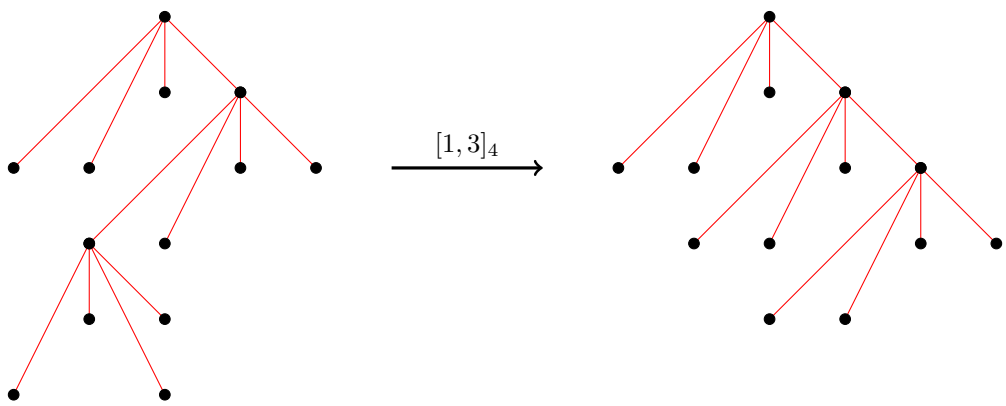
4.3.1 Generating set of F_β

(a_1, a_0)-Generators

Definition 4.3.2. The (a_1, a_0)-generator $[i_1, \ldots, i_{a_0}]_j \in F_\beta$, $1 \leq i_1 < \cdots < i_{a_0} \leq K, j \in \mathbb{Z}_{\geq 0}$, is the map represented by the right aligned (a_1, a_0)-tree pair $(\mathcal{T}, Sp(\mathcal{T}))$. Let S be a spine of size $\left\lfloor \frac{j-1}{K-1} \right\rfloor + 1$. Then S has at least $j+1$ leaves. The (a_1, a_0)-tree $\mathcal{T}$ is obtained by (a_1, a_0)-refining S by hanging the (a_1, a_0)-caret of type $(i_1, \ldots, i_{a_0})$ from the $j+1^{th}$ leaf of S.

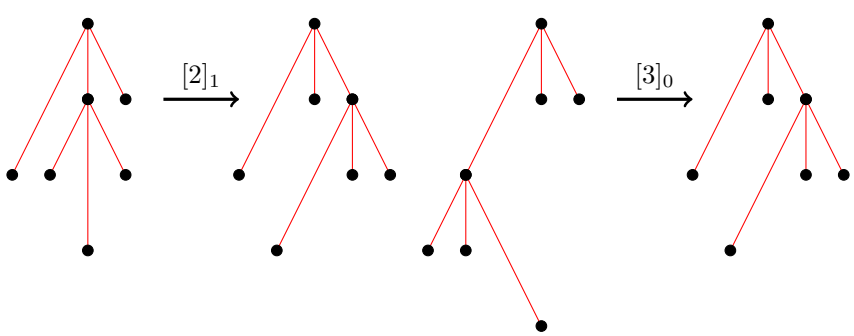
Example 26. Consider the $(2, 2)$-generator $[1, 3]_4$. We need the (a_1, a_0)-spine S to have at least 5 leaves.

Let S be an (a_1, a_0)-spine of size $\left\lfloor \frac{4-1}{2+2-1} \right\rfloor + 1 = \left\lfloor \frac{3}{3} \right\rfloor + 1 = 2$.

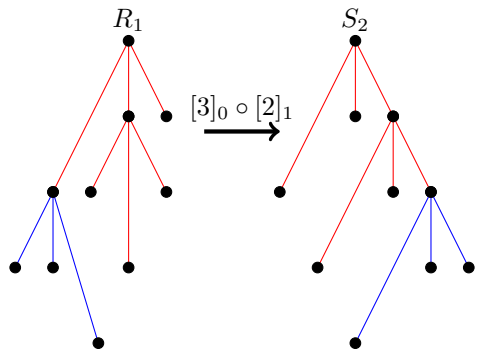


In the left tree of the positive (a_1, a_0)-tree pair representing the (a_1, a_0)-generator $[i_1, \ldots, i_{a_0}]_j$, the (a_1, a_0)-caret $(i_1, \ldots, i_{a_0})$ has j leaves preceding it.

Example 27. Below are the positive (a_1, a_0)-tree pairs representing the $(2, 1)$-generators, $[2]_1$ and $[3]_0$.



These $(2, 1)$-generators were initially used in example 22. There it was seen that $[3]_0 \circ [2]_1 = (R_1, S_1)$.



You can see that the $(2,1)$-tree R_1 has 0 leaves to the left of the $(2,1)$-caret of type (3).

Multiplication of (a_1, a_0)-generators

Remark 37. Each (a_1, a_0)-generator is represented by a positive (a_1, a_0)-tree pair, and is therefore positive. Therefore the product of two (a_1, a_0)-generators is also positive.

Lemma 4.3.3. Let $(\mathcal{T}, Sp(\mathcal{T}))$ be a positive (a_1, a_0)-tree pair, and let $[i_1, \ldots, i_{a_0}]_j = (\mathcal{T}_1, Sp(\mathcal{T}_1))$. Then there exists an (a_1, a_0)-tree $\mathcal{T}'$, such that

$$(\mathcal{T}', Sp(\mathcal{T}')) = (\mathcal{T}, Sp(\mathcal{T})) \star (\mathcal{T}_1, Sp(\mathcal{T}_1))$$

and $\mathcal{T}'$ is an (a_1, a_0)-refinement of $\mathcal{T}$ such that the $j+1^{th}, \ldots, j+K^{th}$ leaves of $\mathcal{T}'$ are the children of an (a_1, a_0)-caret of type $(i_1, \ldots, i_{a_0})$.

Proof. If $(\mathcal{T}', Sp(\mathcal{T}')) = (\mathcal{T}, Sp(\mathcal{T})) \star (\mathcal{T}_1, Sp(\mathcal{T}_1))$, then there exists $\mathcal{T}^*$, a common refinement of $Sp(\mathcal{T})$ and $\mathcal{T}_1$, such that $(\mathcal{T}', Sp(\mathcal{T}')) = (\mathcal{T}', \mathcal{T}^*)) \star (\mathcal{T}^*, Sp(\mathcal{T}'))$ where $(\mathcal{T}', \mathcal{T}^*)$ and $(\mathcal{T}^*, Sp(\mathcal{T}'))$ are simultaneous refinements of $(\mathcal{T}, Sp(\mathcal{T}))$ and $(\mathcal{T}_1, Sp(\mathcal{T}_1))$ respectively.

As $(\mathcal{T}_1, Sp(\mathcal{T}_1))$ is a positive (a_1, a_0)-tree pair, $\mathcal{T}_1$ is a right aligned (a_1, a_0)-tree and has a spine of (a_1, a_0)-carets of type C_0. Let the (a_1, a_0)-carets in the spine of $\mathcal{T}_1$ be $X_1, \ldots, X_n$ such that $X_{j+1} = X_j(K)$.

If $size(Sp(\mathcal{T})) = n$, then $\mathcal{T}_1$ is an (a_1, a_0)-refinement of $Sp(\mathcal{T})$, and $[Sp(\mathcal{T}) : \mathcal{T}_1]$. Our common refinement $\mathcal{T}^* = \mathcal{T}_1$ is obtained by hanging an (a_1, a_0)-caret of type $(i_1, \ldots, i_{a_0})$ from the $j+1$ leaf of $Sp(\mathcal{T})$.

If $size(Sp(\mathcal{T})) = s > n$, then we can find an (a_1, a_0)-refinement of $\mathcal{T}_1$, $\mathcal{T}^*$, by hanging an (a_1, a_0)-spine of size $s-n$ from $X_n(K)$. This (a_1, a_0)-tree $\mathcal{T}^*$ is also an (a_1, a_0)-refinement of $Sp(\mathcal{T})$, obtained

by hanging the (a_1, a_0)-caret of type $(i_1, \ldots, i_{a_0})$ from the $j + 1^{th}$ leaf of $Sp(\mathcal{T})$.

In each of these cases, the corresponding simultaneous refinement of $(\mathcal{T}_1, Sp(\mathcal{T}_1))$ is $(\mathcal{T}^*, Sp(\mathcal{T}^*))$. The corresponding simultaneous refinement of $(\mathcal{T}, Sp(\mathcal{T}))$ is $(\mathcal{T}', \mathcal{T}^*)$, where $\mathcal{T}'$ is an (a_1, a_0)-refinement of $\mathcal{T}$, with $[\mathcal{T}, \mathcal{T}'] = 1$. The only (a_1, a_0)-caret added in this (a_1, a_0)-refinement of $\mathcal{T}$ is the (a_1, a_0)-caret of type $(i_1, \ldots, i_{a_0})$, which is hung from the $j + 1^{th}$ leaf of $\mathcal{T}$.

Alternatively, if $size(Sp(\mathcal{T})) = s < n$, then $\mathcal{T}^* = \mathcal{T}_1$ is an (a_1, a_0)-refinement of $Sp(\mathcal{T})$, with $[Sp(\mathcal{T}) : \mathcal{T}^*] = n - s + 1$. We can obtain $\mathcal{T}_1$ from $Sp(\mathcal{T})$ by hanging an (a_1, a_0)-spine of size $n - s$ from the right-most leaf of $Sp(\mathcal{T})$, and then by hanging an (a_1, a_0)-caret of type $(i_1, \ldots, i_{a_0})$ from the $j + 1^{th}$ leaf of the resulting (a_1, a_0)-spine.

We see that $(\mathcal{T}_1, Sp(\mathcal{T}_1)) = (\mathcal{T}^*, Sp(\mathcal{T}^*))$, and that $(\mathcal{T}', \mathcal{T}^*)$ is a simultaneous refinement of $(\mathcal{T}, Sp(\mathcal{T}))$ and the $j + 1^{th}, \ldots, j + K^{th}$ leaves of $\mathcal{T}'$ are the children of the (a_1, a_0)-caret of type $(i_1, \ldots, i_{a_0})$. □

If $(\mathcal{T}, Sp(\mathcal{T}))$ is a positive (a_1, a_0)-tree pair, then multiplying by the (a_1, a_0)-tree pair representing $[i_1, \ldots, i_{a_0}]_j$, hangs the (a_1, a_0)-caret of type $(i_1, \ldots, i_{a_0})$ from the $j + 1^{th}$ leaf of $\mathcal{T}$. If $\mathcal{T}$ has fewer than $j + 1$ leaves, then the spine is extended until there are sufficient leaves.

In example 22, we see that post-multiplying by the $(2, 1)$-tree pair representing $[3]_0$ hangs an $(2, 1)$-caret from the $(0 + 1)^{th}$ leaf of the (a_1, a_0)-tree $\mathcal{T}_1$.

Lemma 4.3.4. Every positive (a_1, a_0)-tree pair $(\mathcal{T}, Sp(\mathcal{T}))$ can be expressed as

$$(\mathcal{T}, Sp(\mathcal{T}))) = (\mathcal{T}_1, Sp(\mathcal{T}_1)) \star \cdots \star (\mathcal{T}_n, Sp(\mathcal{T}_n))$$

where $(\mathcal{T}_r, Sp(\mathcal{T}_r))$ is the positive (a_1, a_0)-tree pair representing the (a_1, a_0)-generator $[i_1^{(r)}, \ldots, i_{a_0}^{(r)}]_{j_n}$ for each $r \in \{1 \ldots, n\}$.

Proof. Let $(\mathcal{T}, Sp(\mathcal{T}))$ be a positive (a_1, a_0)-tree pair, such that the spine of $\mathcal{T}$ has s carets. Let $id \in F_\beta$ be the identity map. Then $id = (S, S)$ where S is an (a_1, a_0)-spine of size s.

Then $\mathcal{T}$ is an (a_1, a_0)-refinement of S. Let $[S : \mathcal{T}] = n$, and let $S = T_0, T_1, \ldots, T_n = \mathcal{T}$ be (a_1, a_0)-trees such that T_{r+1} is an (a_1, a_0)-refinement of T_r, and $[T_r : T_{r+1}] = 1$ for all r.

If the (a_1, a_0)-refinement of T_{r-1} to get to T_r requires hanging an (a_1, a_0)-caret of type $(i_1^{(r)}, \ldots, i_{a_0}^{(r)})$ from the $j_r + 1^th$ leaf of T_{r-1}, then $(T_r, Sp(T_r)) = (T_{r-1}, Sp(T_{r-1})) \star (\mathcal{T}_r, Sp(\mathcal{T}_r))$ where $(\mathcal{T}_i, Sp(\mathcal{T}_i))$ is the positive (a_1, a_0)-tree pair representing the (a_1, a_0)-generator $[i_1^{(r)}, \ldots, i_{a_0}^{(r)}]_{j_n}$.

This being true for all $r \in \{1, \ldots, n\}$ means that

$$(\mathcal{T}, Sp(\mathcal{T}))) = (S, S) \star (\mathcal{T}_1, Sp(\mathcal{T}_1)) \star \cdots \star (\mathcal{T}_n, Sp(\mathcal{T}_n)).$$

Since (S, S) is the identity (a_1, a_0)-tree pair, we can drop the (S, S) from our tree-pair multiplication.

□

Proposition 4.3.5. The set of positive elements of F_β is generated by (a_1, a_0)-generators.

Proof. If $g \in F_\beta$ is positive, then $g = (\mathcal{T}, Sp(\mathcal{T}))$, where $\mathcal{T}$ is a right aligned (a_1, a_0)-tree.

Thus

$$(\mathcal{T}, Sp(\mathcal{T}))) = (\mathcal{T}_1, Sp(\mathcal{T}_1)) \star \cdots \star (\mathcal{T}_n, Sp(\mathcal{T}_n))$$

where $(\mathcal{T}_r, Sp(\mathcal{T}_r))$ is the positive (a_1, a_0)-tree pair representing to an (a_1, a_0)-generator $[i_1^{(r)}, \ldots, i_{a_0}^{(r)}]_{j_n}$ for all $r \in \{1, \ldots, n\}$.

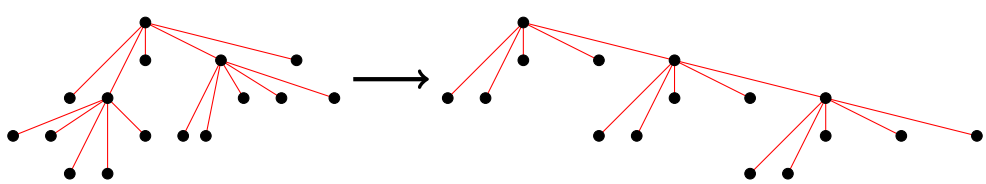
Thus,

$$g = [i_1^{(n)}, \ldots, i_{a_0}^{(n)}]_{j_n} \circ \cdots \circ [i_1^{(1)}, \ldots, i_{a_0}^{(1)}]_{j_1}.$$

□

Any (a_1, a_0)-tree $\mathcal{T}$ can be thought of as a product of (a_1, a_0)-generators, one for each (a_1, a_0)-caret not in the spine.

Example 28. Recall the example of a positive $(3, 2)$-tree pair.



This represents $[3, 4]_1 \circ [1, 2]_3$ but also represents $[1, 2]_7 \circ [3, 4]_1$.

There are clearly some relations between our (a_1, a_0)-generators. This first kind come about if two (a_1, a_0)-carets X, Y, are added (a_1, a_0)-tree to make the (a_1, a_0)-tree $\mathcal{T}$ such that $\mathcal{T}_X$ and $\mathcal{T}_Y$ have no shared nodes. In this case the order in which you add X and Y does not matter.

In fact this means that if $i < j$,

$$[i_1 \ldots, i_{a_0}]_i \circ [j_1 \ldots, j_{a_0}]_j = [j_1 \ldots, j_{a_0}]_{j+K-1} \circ [i_1 \ldots, i_{a_0}]_i$$

Reducing the generating set

Recall the definition of a connected (a_1, a_0)-caret from Chapter 1, and recall the process described as a basic move.

Definition 4.3.6. *Connected (a_1, a_0)-generators*
The ***positive*** **connected** (a_1, a_0)**-generator** $[C_i]_j$ is the (a_1, a_0)-generator

$$[C_i]_j = [i+1, i+2 \ldots, i+a_0]_j.$$

The ***negative*** **connected** (a_1, a_0)**-generators** are of the form $[C_i]_j^{-1}$.

It is convenient to drop the $\circ$ composition notation, and instead write

$$[C_{i_1}]_{j_1} \circ [C_{i_2}]_{j_2} = [C_{i_1}]_{j_1} [C_{i_2}]_{j_2}.$$

Lemma 4.3.7. Let $[i_1, \ldots, i_{a_0}]_j$ be an (a_1, a_0)-generator. Then

$$\begin{aligned}
&[C_{a_0}]_j \cdots [C_{a_0}]_{j+i_1-2}[C_{a_0-1}]_{j+i_1} \cdots [C_{a_0-1}]_{j+i_2-2}[C_{a_0-2}]_{j+i_2} \cdots \\
&\qquad \cdots [C_2]_{j+i_{a_0-1}-2}[C_1]_{j+i_{a_0-1}} \cdots [C_1]_{j+i_{a_0}-2}[C_0]_{j+i_{a_0}} \cdots \\
&\qquad\qquad \cdots [C_0]_{j+K-2}[C_0]_{j+K-1}[i_1, \ldots, i_{a_0}]_j = [C_0]_{j+a_0}[C_0]_{j+a_0+1} \cdots \\
&\qquad\qquad\qquad \cdots [C_0]_{j+K-1}[C_0]_j.
\end{aligned}$$

Proof. Let X be the $(j+1)^{th}$ leaf of some (a_1, a_0)-tree $\mathcal{T}$, and let $g = (\mathcal{T}, Sp(\mathcal{T}))$ be a positive element of F_β.

Consider the (a_1, a_0)-tree pair representation of

$$\begin{aligned}
&[C_{a_0}]_j \cdots [C_{a_0}]_{j+i_1-2}[C_{a_0-1}]_{j+i_1} \cdots [C_{a_0-1}]_{j+i_2-2}[C_{a_0-2}]_{j+i_2} \cdots \\
&\qquad \cdots [C_2]_{j+i_{a_0-1}-2}[C_1]_{j+i_{a_0-1}} \cdots [C_1]_{j+i_{a_0}-2}[C_0]_{j+i_{a_0}} \cdots \\
&\qquad\qquad \cdots [C_0]_{j+K-2}[C_0]_{j+K-1}[i_1, \ldots, i_{a_0}]_j \circ g.
\end{aligned}$$

and then consider the sub-tree $\mathcal{T}_X$. The root node, R, of $\mathcal{T}_X$ is an (a_1, a_0)-caret of type $[i_1, \ldots, i_{a_0}]$. Recall that composition by the connected (a_1, a_0)-generator $[C_i]_k$ hangs the connected (a_1, a_0)-caret C_i from the $(k+1)^{th}$ leaf.

- From each of the short legs $R(1), \ldots, R(i_1 - 1)$, we hang a connected (a_1, a_0)-caret of type C_{a_0}.
- From each of the short legs $R(i_1 + 1), \ldots, R(i_2 - 1)$, we hang a connected (a_1, a_0)-caret of type $C_{a_0 - 1}$.
- From each of the short legs $R(i_s + 1), \ldots, R(i_{s+1} - 1)$, we hang a connected (a_1, a_0)-caret of type $C_{a_0 - s}$.
- From each of the short legs $R(i_{a_0}), \ldots, R(K)$, we hang a connected (a_1, a_0)-caret of type C_0.

In fact, we have constructed the (a_1, a_0)-tree defined in our basic move from Lemma 4.2.12. This means that we can find an (a_1, a_0)-tree $\mathcal{T}'_X$ which is leaf-equivalent to $\mathcal{T}_X$ and consists of only connected (a_1, a_0)-carets of type C_0,

$$\begin{aligned}
&[C_{a_0}]_j \cdots [C_{a_0}]_{j+i_1-2}[C_{a_0-1}]_{j+i_1} \cdots [C_{a_0-1}]_{j+i_2-2}[C_{a_0-2}]_{j+i_2} \cdots \\
&\qquad \cdots [C_2]_{j+i_{a_0-1}-2}[C_1]_{j+i_{a_0-1}} \cdots [C_1]_{j+i_{a_0}-2}[C_0]_{j+i_{a_0}} \cdots \\
&\qquad\qquad \cdots [C_0]_{j+K-2}[C_0]_{j+K-1}[i_1, \ldots, i_{a_0}]_j = [C_0]_{j+a_0}[C_0]_{j+a_0+1} \cdots \\
&\qquad\qquad\qquad \cdots [C_0]_{j+K-1}[C_0]_j.
\end{aligned}$$

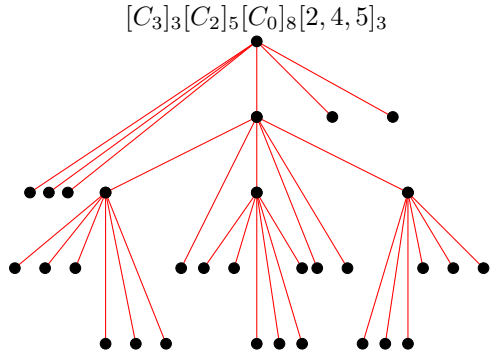
□

Example 29. Consider the $(3,3)$-generator $[2,4,5]_3$. By Lemma 4.3.7

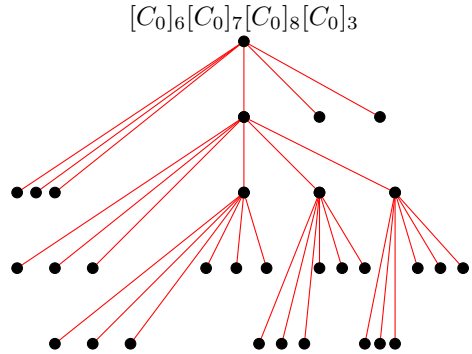
$$[C_3]_3[C_2]_5[C_0]_8[2,4,5]_3 = [C_0]_6[C_0]_7[C_0]_8[C_0]_3.$$

The $(3,3)$-tree representation of these outcomes are shown below. Since these are both positive, they have a $(3,3)$-spine as the right-hand tree which is not included in the diagram below.

$[C_3]_3[C_2]_5[C_0]_8[2,4,5]_3$



$[C_0]_6[C_0]_7[C_0]_8[C_0]_3$



Both of these $(3,3)$-trees have leaf-sequence

$$(2,2,2,3,3,3,4,4,4,3,3,3,4,4,4,3,3,3,4,4,4,3,3,3,1,1).$$

Therefore

$$[2,4,5]_3 = [C_3]_3^{-1}[C_2]_5^{-1}[C_0]_8^{-1}[C_0]_6[C_0]_7[C_0]_8[C_0]_3.$$

Lemma 4.3.7 holds for all (a_1,a_0)-generators $[i_1,\ldots,i_{a_0}]$ and only requires adding (a_1,a_0)-generators of type $[C_0]_{j_0},[C_1]_{j_1},\ldots,[C_{a_0}]_{j_{a_0}}$.

Remark 38. For any (a_1,a_0)-generator $[i_1,\ldots,i_{a_0}]_j$, there exists maps P,Q such that P,Q are products of connected (a_1,a_0)-generators of type $[C_0]_{j_0},\ldots,[C_{a_0}]_{j_{a_0}}$, and $Q\circ[i_1,\ldots,i_{a_0}]_j = P$. Therefore

$$[i_1,\ldots,i_{a_0}]_j = Q^{-1}P$$

Therefore F_β is generated by the connected (a_1,a_0)-generators

$$\langle C_0, C_1,\ldots,C_{a_0},\ldots,C_{a_1}\rangle$$

4.3.8 Relations in the Presentation

4.3.9 Connected (a_1,a_0)-generators

We already know two relations on the set of connected (a_1,a_0)-generators.

The first kind comes from seeing that two independent sub-trees can be added in either order.

$$R_1\colon\ [C_r]_i[C_s]_j = [C_s]_{j+K-1}[C_r]_i \text{ for } i<j.$$

The second kind of relation comes from our basic moves.

$$R_2\colon$$
$$[C_{a_0}]_j\cdots[C_{a_0}]_{j+r-1}[C_0]_{j+a_0+r}\cdots[C_0]_{j+K-1}[C_r]_j = [C_{a_0}]_j\cdots[C_{a_0}]_{j+s-1}[C_0]_{j+a_0+s}\cdots[C_0]_{j+K-1}[C_s]_j$$
$$\text{for all } j\geq 0.$$

Of course, $[C_i]_j^{-1}[C_i]_j$ is the identity map, but we must ask if there are any other relations that can be found between the (a_1,a_0)-generators and their inverses. We want to be able to say that $g = Q^{-1}\circ P$ where P,Q are compositions of connected (a_1,a_0)-generators, to avoid having to find such relations.

Lemma 4.3.10. Let $g \in F_\beta$. Then there exists $0 \leq i_1, \ldots, i_m, i_{m+1}, \ldots, i_r \leq a_1$ such that

$$g = [C_{i_1}]_{j_1}^{-1} \cdots [C_{i_m}]_{j_m}^{-1} [C_{i_{m+1}}]_{j_{m+1}} \cdots [C_{i_r}]_{j_r}.$$

Proof. In Remark 38, we saw that g can be written as the composition of connected (a_1, a_0)-generators and their inverses. There is no restriction on the location on the inverses in this remark, and so the generators and their inverses can appear in any order. Our goal is to show that, by using the relations R_1 and R_2, we can *move* all inverses to the left of this list.

Firstly, consider R_1. For $0 \leq i < j$, and for $0 \leq r, s \leq a_1$,

$$\begin{aligned}
[C_r]_i [C_s]_j &= [C_s]_{j+K-1} [C_r]_i \\
[C_r]_i^{-1} [C_r]_i [C_s]_j [C_r]_i^{-1} &= [C_r]_i^{-1} [C_s]_{j+K-1} [C_r]_i [C_r]_i^{-1} \\
[C_s]_j [C_r]_i^{-1} &= [C_r]_i^{-1} [C_s]_{j+K-1}.
\end{aligned}$$

So if $i < j$, $[C_s]_j [C_r]_i^{-1} = [C_r]_i^{-1} [C_s]_{j+K-1}$.
We can gain more information by considering R_1 again:

$$\begin{aligned}
[C_r]_i [C_s]_j &= [C_s]_{j+K-1} [C_r]_i \\
[C_s]_{j+K-1}^{-1} [C_r]_i [C_s]_j [C_s]_j^{-1} &= [C_s]_{j+K-1}^{-1} [C_s]_{j+K-1} [C_r]_i [C_s]_j^{-1} \\
[C_s]_{j+K-1}^{-1} [C_r]_i &= [C_r]_i [C_s]_j^{-1}.
\end{aligned}$$

So if $i > j$, $C_r]_i [C_s]_j^{-1} = [C_s]_{j+K-1}^{-1} [C_r]_i$.

Now, given $[C_r]_i [C_s]_j^{-1}$ we can find some way to move the inverted (a_1, a_0)-generator to the left provided $i \neq j$. If $i = j$, we need to consider the second kind of relation. For $0 \leq r, s \leq a_1$, and for

$j \geq 0$,

$$\begin{aligned}
[C_{a_0}]j \cdots [C_{a_0}]_{j+r-1}[C_0]_{j+a_0+r} \cdots [C_0]_{j+K-1}[C_r]_j &= \\
&= [C_{a_0}]_j \cdots [C_{a_0}]_{j+s-1}[C_0]_{j+a_0+s} \cdots [C_0]_{j+K-1}[C_s]_j \\
[C_{a_0}]j \cdots [C_{a_0}]_{j+r-1}[C_0]_{j+a_0+r} \cdots [C_0]_{j+K-1}[C_r]_j[C_s]_j^{-1} & \\
&= [C_{a_0}]_j \cdots [C_{a_0}]_{j+s-1}[C_0]_{j+a_0+s} \cdots [C_0]_{j+K-1}[C_s]_j[C_s]_j^{-1} \\
[C_r]_j[C_s]_j^{-1} &= [C_0]_{j+K-1}^{-1} \cdots [C_0]_{j+a_0+r}^{-1}[C_{a_0}]_{j+r-1}^{-1} \cdots [C_{a_0}]_j^{-1}[C_{a_0}]_j \cdots \\
&\quad \cdots [C_{a_0}]_{j+s-1}[C_0]_{j+a_0+s} \cdots [C_0]_{j+K-1}.
\end{aligned}$$

Here we have swapped $[C_r]_j[C_s]_j^{-1}$ for some composition of connected (a_1, a_0)-generators and their inverses, in which all of the inverses are written to the left.

We have devised three methods to move the negative (a_1, a_0)-generators to the left of the positive (a_1, a_0)-generators, which cover all possible combinations of positive and negative (a_1, a_0)-generators.

□

Now we can consider all of the relations on the *positive* connected (a_1, a_0)-generators.

There is a third kind of relation which comes from the following remark.

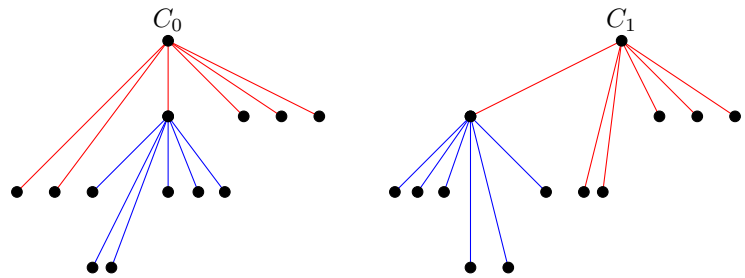
Remark 39. Given a connected (a_1, a_0)-caret C_i, there is a common refinement of C_i and C_{i+1}, obtained by hanging a connected (a_1, a_0)-caret C_r from $C_i(i + a_0 + 1)$ and hanging C_{r+a_0} from $C_{i+1}(i + 1)$, for $0 \leq r \leq a_1 - a_0$. This *increases* C_i to C_{i+1}.

Similarly, we can *decrease* $C_{i+1}$to C_i by hanging $C_{r'}$ from $C_{i+1}(i + 1)$, for $a_0 \leq r' \leq a_1$. This is leaf-equivalent to hanging $C_{r'-a_0}$ from $C_i(i + a_0 + 1)$.

This method of *increasing* C_i works because C_r has at least a_0 short legs on the right hand side, which are matched to the a_0 long children of C_i. This gives us the third kind of relation:

$$R_3\colon\ [C_r]_{j+i+a_0}[C_i]_j = [C_{r+a_0}]_{j+i}[C_{i+1}]_j \text{ for } 0 \leq r \leq a_1 - a_0.$$

Example 30. Consider the connected $(4, 2)$-caret C_0. We can increase C_0 to C_1 by hanging a connected $(4, 2)$-caret C_r from $C_0(3)$, where $0 \leq r \leq 2$. Below we have chosen $C_r = C_1$.



$$[C_1]_{j+2}[C_0]_j = [C_3]_j[C_1]_j.$$

Notice that we can repeat this process and increase C_1 to C_2 by hanging some $C_{r'}$ from $C_1(4)$, where $0 \leq r' \leq 2$. Below we choose $r' = 0$.

$$[C_0]_{j+3}[C_1]_j = [C_2]_{j+1}[C_2]_j$$
$$[C_1]_{j+2}[C_0]_{j+3}[C_0]_j = [C_3]_j[C_2]_{j+1}[C_2]_j$$
$$[C_0]_{j+8}[C_1]_{j+2}[C_0]_j = [C_3]_j[C_2]_{j+1}[C_2]_j.$$

We can move the short leg that is to the right of the 2 long legs, to the left of the long legs by increasing the type.

The maximum number times that we can increase a connected (a_1, a_0)-caret is a_1 times, increasing from C_0 to C_{a_1}.

$$[C_{r_1}]_{j+a_0}[C_{r_2}]_{j+a_0+1}\cdots[C_{r_{a_1}}]_{j+K-1}[C_0]_j = [C_{r_1+a_0}]_j[C_{r_2+a_0}]_{j+1}\cdots[C_{r_{a_1}+a_0}]_{j+a_1-1}[C_{a_1}]_j.$$

Suppose then that $\mathcal{T}$ is a connected (a_1, a_0)-tree with root node of type C_i, for $0 \leq i \leq K-1$.

We can construct algorithms to find a common refinement between $\mathcal{T}$ and C_{i+1}. Let R be the root node of $\mathcal{T}$.

Increase type:

- Consider $R(i + a_0 + 1)$.
  - If $R(i + a_0 + 1)$ is a leaf, hang C_j for $0 \leq j \leq a_1 - a - 0$ from $R(i + a_0 + 1)$. We are done.
  - If $R(i + a_0 + 1)$ is of type C_j for $0 \leq j \leq a_1 - a - 0$ from $R(i + a_0 + 1)$, then we are done.
- Otherwise $R(i + a_0 + 1)$ is a caret of type C_t for $a_1 - a_0 + 1 \leq t \leq K$.
  - Perform the decrease type algorithm on $R_(i + a_0 + 1)$

- Repeat the increase type algorithm on R.

Decrease type :

- Consider $R(i)$.
  - If $R(i)$ is a leaf, hang C_j for $0 \leq j \leq a_1 - a - 0$ from $R(i)$. We are done.
  - If $R(i)$ is of type C_j for $0 \leq j \leq a_1 - a - 0$ from $R(i)$, then we are done.
- Otherwise $R(i)$ is a caret of type C_t for $a_1 - a_0 + 1 \leq t \leq K$.
  - Perform the increase type algorithm on $R(i)$.
  - Repeat the decrease type algorithm on R.

Lemma 4.3.11. Let $\mathcal{T}, \mathcal{T}'$ be (a_1, a_0)-trees with connected root-carets of type C_i and C_{i+1} respectively. If we have to add (a_1, a_0)-carets to increase the type of C_i to C_{i+1}, then $\mathcal{T} \not\sim \mathcal{T}'$.

Proof. Let $\mathcal{T}, \mathcal{T}'$ be (a_1, a_0)-trees such that $\mathcal{T} \sim \mathcal{T}'$ and the root-carets are of type C_i and C_{i+1} respectively. Let S_1 and S_2 be the (a_1, a_0)-subdivisions corresponding to $\mathcal{T}$ and $\mathcal{T}'$ respectively. Then $B[S_1] = B[S_2]$. Here we will use $\tau = \beta^{-1}$ to make notation more convenient.

Let R be the root node of $\mathcal{T}'$. The first i children of R are short legs and represent sub-intervals of length τ. The next a_0 children of R, $R(i+1), \ldots, R(i+a_0)$ are all long legs, and represent sub-intervals of length τ^2 in S_1. Then $R(i + a_0 + 1)$ is a short leg.

If $R(i + a_0 + 1)$ is a leaf for $1 \leq i \leq a_0$, then there is no breakpoint in S_1 in the real interval $\big(a_0\tau^2 + (i-1)\tau, a_0\tau^2 + i\tau\big)$. However $(i+1)\tau$ is a breakpoint of S_2, and

$$i\tau + a_0\tau^2 \leq (i+1)\tau \leq (i+1)\tau + a_0\tau^2.$$

Therefore $R(i + a_0 + 1)$ cannot be a leaf if $\mathcal{T} \sim \mathcal{T}'$. Therefore $R(i + a_0 + 1)$ must be the parent of an (a_1, a_0)-caret of type C_{r_1}. If $0 \leq r_1 \leq a_1 - a_0$, then we can *increase* the type of C_i to C_{i+1} without adding any (a_1, a_0)-carets which would be a contradiction. So then $a_1 - a_0 + 1 \leq r_1 \leq a_1$.

Denote $R(i + a_0 + 1) = R_1$, and consider $R_1(a_1 - a_0 + 1)$. As R_1 is a connected (a_1, a_0)-caret C_{r_1}, for some $a_1 - a_0 + 1 \leq r_1 \leq a_1$, at least the first $a_1 - a_0 + 1$ children of R_1 are short legs and represent sub-intervals of length τ^2 in S_1. Consider the node $R_2 = R_1(a_1 - a_0 + 1)$, and the interval that it

represents, $[i\tau + a_1\tau^2, i\tau + (a_1+1)\tau^2]$. Note that

$$\tau^2 = a_1\tau^3 + a_0\tau^4 > a_1\tau^3 \geq a_0\tau^3$$

Since $(i+1)\tau = i\tau + a_1\tau^2 + a_0\tau^3$,

$$i\tau + a_1\tau < (i+1)\tau < i\tau + (a_1+1)\tau^2$$

Therefore R_2 is not a leaf in $\mathcal{T}$, and is therefore a parent of an (a_1, a_0)-caret of type C_{r_2}, for some $0 \leq r_2 \leq a_1$.

Therefore $0 \leq r_2 \leq a_0 - 1$, and for all $j \geq 2a_0$ $R_2(j)$ is a short leg in the (a_1, a_0)-caret of type C_{r_2}. Let $R_2(2a_0) = R_3$, which represents the sub-interval

$$[i\tau + a_1\tau^2 + (a_0-1)\tau^3 + a_0\tau^4, i\tau + a_1\tau^2 + a_0\tau^3 + a_0\tau^4]$$

Recall that $(i+1)\tau = i\tau + a_1\tau^2 + a_0\tau^3$. Also since $\tau^2 > a_0\tau^3$, we can deduce that $\tau^N > a_0\tau^{N+1}$. Therefore

$$i\tau + a_1\tau^2 + (a_0-1)\tau^3 + a_0\tau^4 < (i+1)\tau < i\tau + a_1\tau^2 + a_0\tau^3 + a_0\tau^4$$

Once again, R_3 is not a leaf and so must be the parent in an (a_1, a_0)-caret of type C_{r_3} for $0 \leq r_3 \leq a_1$.

Therefore $a_1 - a_0 + 1 \leq r_3 \leq a_1$. Let $R_3(a_1 - a_0 + 1) = R_4$, a node representing the interval

$$[i\tau + a_1\tau^2 + (a_0-1)\tau^3 + a_1\tau^4, i\tau + a_1\tau^2 + a_0\tau^3 + (a_1+1)\tau^4].$$

Again, we can conclude that

$$i\tau + a_1\tau^2 + (a_0-1)\tau^3 + a_1\tau^4 < (i+1)\tau < i\tau + a_1\tau^2 + a_0\tau^3 + (a_1+1)\tau^4.$$

So R_4 is not a leaf, and is the parent in an (a_1, a_0)-caret of type C_{r_4}.

Continuing this process, we construct an (a_1, a_0)-tree $\mathcal{T}^*$, whose root-caret is of type C_i and cannot be increased without adding more (a_1, a_0)-carets

- For odd n, $R_n(a_1 - a_0 + 1) = R_{n+1}$,

- R_{n+1} is the parent of an (a_1, a_0)-caret of type $C_{r_{n+1}}$ where $0 < r_{n+1} < a_0 - 1$.

- For even n, $R_n(2a_0) = R_{n+1}$,

  - R_{n+1} is the parent of an (a_1, a_0)-caret of type $C_{r_{n+1}}$ where $a_1 - a_0 + 1 < n + 1 < a_1$.

This gives us the (a_1, a_0)-tree shown in Figure 4.2. Let $t = a_1 - a_0$, as a shorthand. In Figure 4.2, all nodes not labelled are left as leaves. Writing indicates the number of nodes which would be present there, and if no writing is present, then it is possible to deduce the number of nodes.

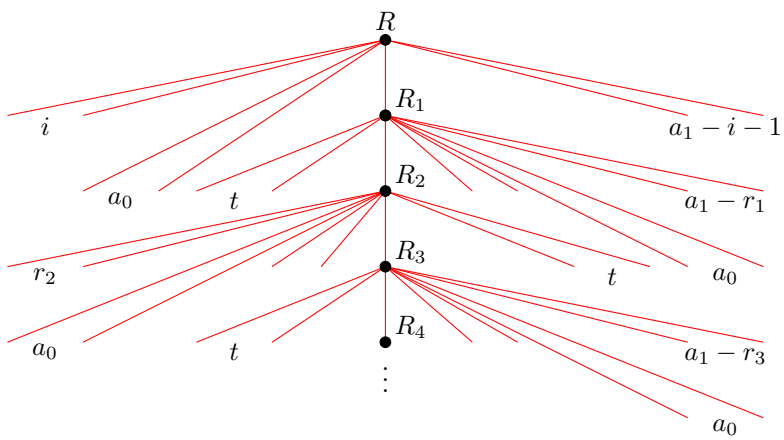


Figure 4.2: The (a_1, a_0)-tree $\mathcal{T}^*[H]$, whose root-caret cannot be increased without adding (a_1, a_0)-carets

Let J_0 be the unit interval, and let J_i be the interval represented by the node R_i.

$$
\begin{aligned}
J_0 :&[0, 1] \\
J_1 :&[i\tau + a_0\tau^2, i\tau + a_0\tau^2] \\
J_2 :&[i\tau + a_1\tau^2, i\tau + (a_1 + 1)\tau^2] \\
J_3 :&[i\tau + a_1\tau^2 + (a_0 - 1)\tau^3 + a_0\tau^4, i\tau + a_1\tau^2 + a_0\tau^3 + a_0\tau^4] \\
J_4 :&[i\tau + a_1\tau^2 + (a_0 - 1)\tau^3 + a_1\tau^4, i\tau + a_1\tau^2 + (a_0 - 1)\tau^3 + (a_1 + 1)\tau^4] \\
&\vdots
\end{aligned}
$$

Notice that each if the intervals J_i are of length τ^i. Let J_∞ be the result of infinitely repeating the process.

Let $\mathcal{L}(J_r)$ be the lower bound of J_r, and $\mathcal{U}(J_r)$ be the upper bound of the interval J_r.

$$\begin{aligned}
\mathcal{L}(J_0) &= 0 \\
\mathcal{L}(J_1) &= i\tau + a_0\tau^2 \\
\mathcal{L}(J_2) &= i\tau + a_1\tau^2 \\
\mathcal{L}(J_3) &= i\tau + a_1\tau^2 + (a_0 - 1)\tau^3 + a_0\tau^4 \\
\mathcal{L}(J_4) &= i\tau + a_1\tau^2 + (a_0 - 1)\tau^3 + a_1\tau^4 \\
\mathcal{L}(J_5) &= i\tau + a_1\tau^2 + (a_0 - 1)\tau^3 + a_1\tau^4 + (a_0 - 1)\tau^5 + a_0\tau^6 \\
\mathcal{L}(J_6) &= i\tau + a_1\tau^2 + (a_0 - 1)\tau^3 + a_1\tau^4 + (a_0 - 1)\tau^5 + a_1\tau^6 \\
&\vdots
\end{aligned}$$

Now consider $\mathcal{L}(J_\infty)$.

$$\begin{aligned}
\mathcal{L}(J_\infty) &= i\tau + a_1\tau^2 + \sum_{k=1}^{\infty} \left((a_0 - 1)\tau^{2k+1} + a_1\tau^{2k+2}\right) \\
&= i\tau + a_1\tau^2 + \sum_{k=1}^{\infty} \tau^{2k+1} \left(a_0 - 1 + a_1\tau\right).
\end{aligned}$$

We can rearrange $1 = a_1\tau + a_0\tau^2$ to get,

$$\begin{aligned}
1 &= a_1\tau + a_0\tau^2 \\
a_0 &= a_0 - 1 + a_1\tau + a_0\tau^2 \\
a_0 - a_0\tau^2 &= a_0 - 1 + a_1\tau.
\end{aligned}$$

Substituting this into $\mathcal{L}(J_\infty)$, gives us

$$\begin{aligned}
\mathcal{L}(J_\infty) &= i\tau + a_1\tau^2 + \sum_{k=1}^{\infty} \tau^{2k+1}\left(a_0 - 1 + a_1\tau\right) \\
&= i\tau + a_1\tau^2 + \sum_{k=1}^{\infty} \tau^{2k+1}\left(a_0 - a_0\tau^2\right) \\
&= i\tau + a_1\tau^2 + \sum_{k=1}^{\infty} \left(a_0\tau^{2k+1} - a_0\tau^{2k+3}\right) \\
&= i\tau + a_1\tau^2 + \left(a_0\tau^3 - a_0\tau^5\right) + \left(a_0\tau^5 - a_0\tau^7\right) + \left(a_0\tau^7 - a_0\tau^9\right) + \cdots \\
&= i\tau + a_1\tau^2 + a_0\tau^3 \\
&= i\tau + \tau = (i+1)\tau.
\end{aligned}$$

Thus $\mathcal{L}(J_r) < (i+1)\tau$ for all $r \in \mathbb{N}$.

Now consider the upper bounds $\mathcal{U}(J-r)$:

$$\begin{aligned}
\mathcal{U}(J_0) &= 1 \\
\mathcal{U}(J_1) &= 1 - (a_1 - i - 1)\tau \\
\mathcal{U}(J_2) &= 1 - \left[(a_1 - i - 1)\tau + (a_0 - 1)\tau^2 + a_0\tau^3\right] \\
\mathcal{U}(J_3) &= 1 - \left[(a_1 - i - 1)\tau + (a_0 - 1)\tau^2 + a_1\tau^3\right] \\
\mathcal{U}(J_4) &= 1 - \left[(a_1 - i - 1)\tau + (a_0 - 1)\tau^2 + a_1\tau^3 + (a_0 - 1)\tau^4 + a_0\tau^5\right] \\
\mathcal{U}(J_5) &= 1 - \left[(a_1 - i - 1)\tau + (a_0 - 1)\tau^2 + a_1\tau^3 + (a_0 - 1)\tau^4 + a_1\tau^5\right] \\
\mathcal{U}(J_6) &= 1 - \left[(a_1 - i - 1)\tau + (a_0 - 1)\tau^2 + a_1\tau^3 + (a_0 - 1)\tau^4 + a_0\tau^5 + (a_0 - 1)\tau^6 + a_0\tau^7\right] \\
&\vdots \\
\mathcal{U}(J_\infty) &= 1 - (a_1 - i - 1)\tau - \left[(a_0 - 1)\tau^2 + a_1\tau^3 + \sum_{k=1}^{\infty}(a_0 - 1)\tau^{2k+2} + a_1\tau^{2k+3}\right] \\
&= (i+1)\tau + a_0\tau^2 - \left[(a_0 - 1)\tau^2 + a_1\tau^3 + \sum_{k=1}^{\infty}(a_0 - 1)\tau^{2k+2} + a_1\tau^{2k+3}\right] \\
&= (i+1)\tau + \tau^2 - \left[a_1\tau^3 + \sum_{k=1}^{\infty}(a_0 - 1)\tau^{2k+2} + a_1\tau^{2k+3}\right].
\end{aligned}$$

Consider the sum,

$$\sum_{k=1}^{\infty}(a_0-1)\tau^{2k+2}+a_1\tau^{2k+3}=\sum_{k=1}^{\infty}\tau^{2k+2}\left((a_0-1)+a_1\tau\right).$$

Recall that $a_0-1+a_1\tau=a_0-a_0\tau^2$.

$$\begin{aligned}
\mathcal{U}(J_\infty) &= (i+1)\tau+\tau^2-\left[a_1\tau^3+\sum_{k=1}^{\infty}(a_0-1)\tau^{2k+2}+a_1\tau^{2k+3}\right]\\
&= (i+1)\tau+\tau^2-\left[a_1\tau^3+\sum_{k=1}^{\infty}\tau^{2k+2}\left((a_0-1)+a_1\tau\right)\right]\\
&= (i+1)\tau+\tau^2-\left[a_1\tau^3+\sum_{k=1}^{\infty}\tau^{2k+2}\left(a_0-a_0\tau^2\right)\right]\\
&= (i+1)\tau+\tau^2-\left[a_1\tau^3+a_0\tau^4\right]\\
&= (i+1)\tau+\tau^2-\left[\tau^2\right]\\
&= (i+1)\tau^2.
\end{aligned}$$

Therefore $\mathcal{U}(J_r)>(i+1)\tau^2$ for all $r\in\mathbb{N}$. Therefore $(i+1)\tau\in J_r$ for all $r\in\mathbb{N}$.

Thus, if increasing the type of the root-caret of the (a_1,a_0)-tree $\mathcal{T}$ to be the same type as the root-caret as $\mathcal{T}'$ requires adding carets, then $\mathcal{T}\not\sim\mathcal{T}'$. □

Corollary 4.3.12. Let the (a_1,a_0)-trees $\mathcal{T}$ and $\mathcal{T}'$ are leaf equivalent with root-carets of type C_i and C_j respectively with $i<j$. Then it is possible to increase the type of C_i to C_j using the Increase type algorithm, without adding any new (a_1,a_0)-carets.

Proof. If $j=i+1$, then we have shown in Lemma 4.3.11 that it is possible to increase type of the root-caret of $\mathcal{T}$ to be of type C_{i+1}.

If $j=i+2$, then the breakpoint $(i+1)\tau$ is still a breakpoint in $\mathcal{T}'$, and therefore if we are unable to increase the type of the root-caret of $\mathcal{T}$ to C_{i+1}, then the (a_1,a_0)-tree $\mathcal{T}$ must resemble the (a_1,a_0)-tree $\mathcal{T}^*$, shown in figure 4.2. In Lemma 4.3.11, we showed that $\mathcal{T}^*$ cannot contain the breakpoint $(i+1)\tau$, and so this is a contradiction, so it is in fact possible to use the increase type of the root-caret of $\mathcal{T}$ without adding any new (a_1,a_0)-carets. Therefore there exists an (a_1,a_0)-tree $\mathcal{T}_1$ where $\mathcal{T}\sim\mathcal{T}_1\sim\mathcal{T}'$, and the root-caret of $\mathcal{T}_1$ is of type C_{i+1}. Since $j=(i+1)+1$ then we can increase the type of the root-caret of $\mathcal{T}_1$ to be the same as the type of the root-caret of $\mathcal{T}'$. Therefore we have increased the type of the root-caret of $\mathcal{T}$ to be the same as the type of the root-caret of $\mathcal{T}'$.

Now suppose that we can increase the type of C_i to be of type C_j for all $j \leq i + N$, for some $N \in \mathbb{N}$, and consider the case where $j = N+1$. Then since $j = i+N+1 > i$, then $(i+1)\tau$ is certainly a breakpoint in the (a_1, a_0)-tree $\mathcal{T}'$. Therefore by Lemma 4.3.11, if we are unable to increase the type of the root-caret of $\mathcal{T}$ to be of type C_{i+1} then $(i+1)\tau$ is not a breakpoint in $\mathcal{T}$. Since $\mathcal{T} \sim \mathcal{T}'$, this is a contradiction, so we are able to increase the type of the root-caret of $\mathcal{T}$ to be of type C_1. We call this new (a_1, a_0)-tree $\mathcal{T}_1$, and note that $\mathcal{T} \sim \mathcal{T}_1 \sim \mathcal{T}'$, and the type of the root-caret of $\mathcal{T}_1$ is C_{i+1}.

In particular $\mathcal{T}_1 \sim \mathcal{T}'$ and the root-carets are of type C_{i+1} and C_j respectively, where $j = (i+1)+N$. Therefore it is possible to increase the type of the root-caret of $\mathcal{T}_1$ to be of type C_j without adding any new (a_1, a_0)-carets. Therefore we have increased the type of the root-caret of $\mathcal{T}$ to be of type C_j.

By induction, we have reached our result. □

Lemma 4.3.13. If $\mathcal{T}_1$ and $\mathcal{T}_2$ are leaf-equivalent connected (a_1, a_0)-trees, then it is possible to graft from $\mathcal{T}$ to $\mathcal{T}'$ by using the Increase type and Decrease type algorithms.

Proof. We prove this by induction on the depth of the (a_1, a_0)-trees $\mathcal{T}$ and $\mathcal{T}'$. The result is trivial for $\mathcal{T}$ and $\mathcal{T}'$ of depth 0 and 1. Suppose $\mathcal{T}$ is a connected (a_1, a_0)-tree of depth 2.

Suppose that for if $\mathcal{T}$ and $\mathcal{T}'$ are any (a_1, a_0)-trees of depth $d \leq N$ for some $N \in \mathbb{N}$ with $\mathcal{T} \sim \mathcal{T}'$, then it is possible to graft from $\mathcal{T}$ to $\mathcal{T}_2$ using the Increase type and Decrease type algorithms.

Now let $\mathcal{T}$ and $\mathcal{T}'$ be connected (a_1, a_0)-trees of depth $N+1$ and let $\mathcal{T} \sim \mathcal{T}\prime$. Let R and R' be the root-carets of $\mathcal{T}_1$ and $\mathcal{T}'$ respectively. If $type(R) = type(R')$, then we consider the sub-trees $\mathcal{T}_{R(j)}$ and $\mathcal{T}'_{R'(j)}$ for $1 \leq j \leq K$. These are both connected (a_1, a_0)-trees of depth N or $N-1$ and must be leaf-equivalent, and as such we by our inductive hypothesis it is possible to graft from one to the other using the Increase type and Decrease type algorithms. Therefore if $type(R) = type(R')$ then we can graft from $\mathcal{T}$ to $\mathcal{T}'$ by using the Increase type and Decrease type algorithms.

If $type(R) \neq type(R')$, then without loss of generality suppose that $type(R) = C_i < type(R') = C_j$. Then by Corollary 4.3.12 we can increase the type of R to be of type C_j, without adding any new (a_1, a_0)-carets. Call this new (a_1, a_0)-tree $\bar{\mathcal{T}}$. Then $\bar{\mathcal{T}}$ and $\mathcal{T}'$ are leaf-equivalent connected (a_1, a_0)-trees of depth $N+1$ with root-carets of the same type, and as shown earlier, this allows us to graft the sub-trees $\bar{\mathcal{T}}_{R(j)}$ to be $\mathcal{T}'_{R(j)}$ for each $1 \leq j \leq K$.

Therefore, by induction we can graft any connected (a_1, a_0)-tree $\mathcal{T}$ to a leaf-equivalent connected (a_1, a_0)-tree $\mathcal{T}'$ using only the Increase type and Decrease type algorithms. □

We have already seen that the positive connected (a_1, a_0)-generators form a generating set for F_β,

and have found two types of relations:

$$R_1 : \alpha_i\gamma_j = \gamma_{j+K-1}\alpha_i \qquad \text{for } i < j, \alpha, \gamma \in \{[C_0], \ldots, [C_{a_0}]\},$$

$$R_3 : [C_r]_{j+i+a_0}[C_i]_j = [C_{r+a_0}]j + i[C_{i+1}]_j \qquad \text{for } 0 \leq r \leq a_1 - a_0.$$

If there is any other type of relation, which cannot be derived from R_1 and R_3, on the positive (a_1, a_0)-generators, then for some $i_1, \ldots, i_{a_0}$

$$[C_{i_1}]_{j_1} \cdots [C_{i_t}]_{j_t} = [C'_{i_1}]_{j'_1} \cdots [C'_{i_t}]_{j'_t}.$$

This equates to there being two equivalent positive (a_1, a_0)-tree pairs $(\mathcal{T}_1, Sp(\mathcal{T}_1))$ and $(\mathcal{T}_2, Sp(\mathcal{T}_2))$ in which $\mathcal{T}_1 \sim \mathcal{T}_2$. The (a_1, a_0)-trees $\mathcal{T}_1$ and $\mathcal{T}_2$ are leaf-equivalent connected (a_1, a_0)-trees and so by Lemma 4.3.13, we can graft from $\mathcal{T}_1$ to $\mathcal{T}_2$ using the Increase type and Decrease type algorithms. These algorithms describe the relation R_3, and so there cannot be any other relations on the positive connected (a_1, a_0)-generators.

Thus we have proved the following.

Proposition 4.3.14. A presentation for F_β

$$F_\beta = \left\langle [C_0]_{j_1}, \ldots, [C_{a_1}]_{j_{a_1}} \text{ for } j_i \geq 0 \,\middle|\, R_1, R_3 \right\rangle$$

where the relations R_1 and R_3 are:

$$R_1 : \alpha_i\gamma_j = \gamma_{j+K-1}\alpha_i \qquad \text{for } i < j, \alpha, \gamma \in \{[C_0], \ldots, [C_{a_0}]\},$$

$$R_3 : [C_r]_{j+i+a_0}[C_i]_j = [C_{r+a_0}]j + i[C_{i+1}]_j \qquad \text{for } 0 \leq r \leq a_1 - a_0.$$

4.3.15 x, z-caret relations

Definition 4.3.16.

The connected (a_1, a_0)-carets of type C_0 and C_{a_0} are called (a_1, a_0)**-carets of type** x, z.

The connected (a_1, a_0)-generators of type x, z are $\{x_j = [C_0]_j, z_j = [C_{a_0}]_j\}_{j \geq 0}$.

Lemma 4.3.17. Let $[C_i]_j$ be the connected (a_1, a_0)-generator for some $0 \leq i \leq a_0$. Then

$$z_j z_{j+1} \cdots z_{j+i-1} x_{j+i+a_0} x_{j+i+a_0+1} \cdots x_{j+2a_0} [C_i]_j = x_{j+a_0} x_{j+a_0+1} \cdots x_{j+2a_0-1}.$$

Proof. Much like in Lemma 4.3.7, for some positive element $g = (\mathcal{T}, Sp(\mathcal{T}))$ the (a_1, a_0)-tree constructed by taking

$$z_j z_{j+1} \cdots z_{j+i-1} x_{j+i+a_0} x_{j+i+a_0+1} \cdots x_{j+2a_0} [C_i]_j \circ g$$

hangs an (a_1, a_0)-tree $\mathcal{T}_X$ from the $(j+1)^{th}$ leaf of $\mathcal{T}$. Call this leaf X. The (a_1, a_0)-tree $\mathcal{T}$ is the same as the tree constructed in Lemma 4.2.12, and so has a leaf-equivalent (a_1, a_0)-tree which contains only x-type carets. Hence

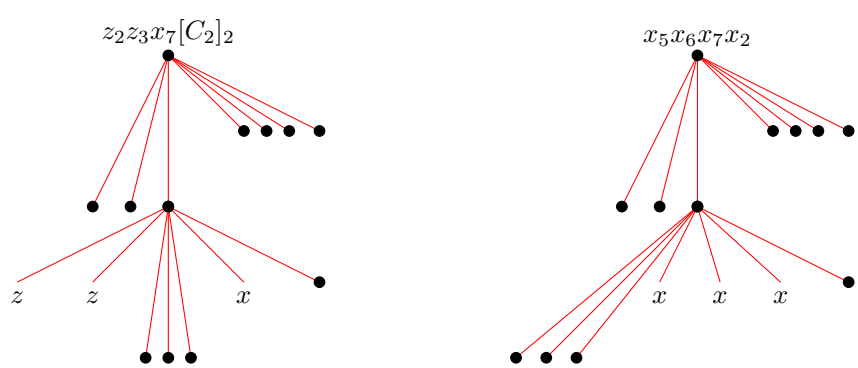
$$z_j z_{j+1} \cdots z_{j+i-1} x_{j+i+a_0} x_{j+i+a_0+1} \cdots x_{j+2a_0} [C_i]_j = x_{j+a_0} x_{j+a_0+1} \cdots x_{j+2a_0-1}.$$

□

Example 31. Consider the connected $(4, 3)$-generator $[C_2]_2$. Lemma 4.3.17 tells us that

$$z_2 z_3 x_7 [C_2]_2 = x_5 x_6 x_7 x_2.$$

We will show the $(4, 3)$-trees pairs representing these maps is shown below, not including the $(4, 3)$-spines which would be of size 5.



The leaf sequence of both of these $(4,3)$-trees is

$$(2,2,4,4,4,5,5,5,4,4,4,4,5,5,5,4,4,4,4,5,5,5,4,4,4,4,3,1,1,1,1).$$

Therefore

$$[C_2]_2 = [3,4,5]_2 = x_7^{-1} z_3^{-1} z_2^{-1} x_5 x_6 x_7 x_2.$$

Remark 40. For any connected (a_1, a_0)-generator $[C_i]_j$, there exists maps P, Q such that P, Q are products of (a_1, a_0)-generators of type x, z, and $Q \circ [C_i]_j = P$. Therefore

$$[C_i]_j = Q^{-1} P,$$

This remark leads us directly to the following proposition.

Proposition 4.3.18. The set $\{x_0, x_1, \ldots, z_0, z_1 \ldots\}$ is a generating set for F_β.

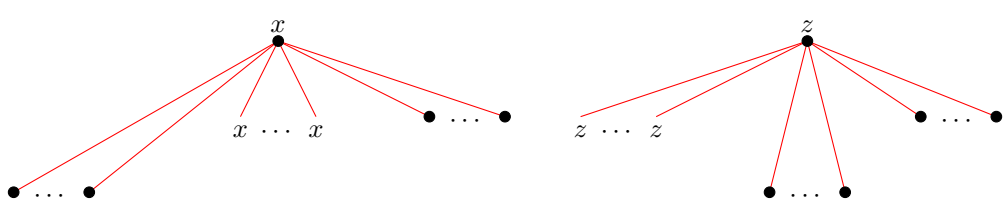
We will consider two kinds relations on our x, z-type (a_1, a_0)-generators.

We have already seen the first kind,

$$R_1\colon\ \alpha_i \gamma_j = \gamma_{j+K-1} \alpha_i \text{ if } i < j, \text{ for all } \alpha, \gamma \in \{x, z\}.$$

The second kind of relation on the (a_1, a_0)-generators of x, z-type, comes from our basic moves and is a variation of Lemma 4.3.17.

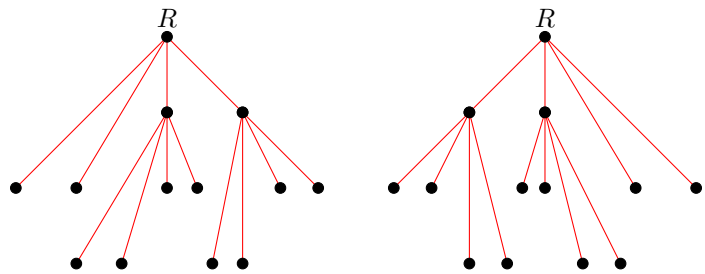
$$R_2\colon\ x_{i+a_0} x_{i+a_0+1} \cdots x_{i+2a_0-1} x_i = z_i z_{i+1} \cdots z_{i+a_0-1} z_i, \text{ for } i \geq 0.$$



Example 32. Let $\mathcal{T}$ be a right aligned $(2,2)$-tree, and let R be the ith leaf of $\mathcal{T}$. If $g = (\mathcal{T}, SP(\mathcal{T}))$, a positive element of F_β, we will see that

$$x_{i+2} \circ x_{i+3} \circ x_i \circ g = z_i \circ z_{i+1} \circ z_i \circ g.$$

Below are the sub-trees hanging from R after having post-composed by $x_{i+2} \circ x_{i+3} \circ x_i$ and $z_i \circ z_{i+1} \circ z_i$ respectively.



These (a_1, a_0)-trees are leaf-equivalent, and so the composition of (a_1, a_0)-generators satisfy

$$x_{i+2} \circ x_{i+3} \circ x_i = z_i \circ z_{i+1} \circ z_i.$$

Lemma 4.3.19. For all $g \in F_\beta$, there exists P, Q such that $g = Q^{-1}P$ and P and Q are of the form

$$P = \alpha_{j_1}^{(1)} \cdots \alpha_{j_r}^{(r)}$$

for $\alpha^{(i)} \in \{x, z\}$.

Proof. In Remark 40, we noted that any $g \in F_\beta$ can be expressed as the composition of (a_1, a_0)-generators of type x, z and their inverses. We want to be able to move all of the inverse (a_1, a_0)-generators of type x, z to the left of this list. We will use the relations R_1 and R_2. We have

$$R_1\colon\ \alpha_i \gamma_j = \gamma_{j+K-1} \alpha_i \text{ for } i < j \text{ and } \alpha, \gamma \in \{x, z\}.$$

From R_1, we can find the following expressions for $i < j$ and $\alpha, \gamma \in \{x, z\}$:

$$\gamma_j \alpha_i^{-1} = \alpha_i^{-1} \gamma_{j+K-1} \quad \text{and} \quad \alpha_i \gamma_j^{-1} = \gamma_{j+k-1}^{-1} \alpha_i.$$

Therefore, if the pair of (a_1, a_0)-generators $\alpha_i \gamma_j^{-1}$ for $i \neq j$, appears in the expression for g in terms of x, z-type generators, then we can move the inverses to the left.

If $i = j$, then we need to consider the relation R_2:

$$R_2\colon\ x_{j+a_0} x_{j+a_0+1} \cdots x_{j+a_0} x_j = z_j z_{j+1} \cdots z_{j+a_0-1} z_j.$$

From R_2, we can find the following expressions for $x_j z_j^{-1}$ and $z_j x_j^{-1}$:

$$x_j z_j^{-1} = x_{j+2a_0}^{-1} \cdots x_{j+a_0+1}^{-1} x_{j+a_0}^{-1} z_j z_{j+1} \cdots z_{j+a_0-1}$$
$$z_j x_j^{-1} = z_{j+a_0-1}^{-1} \cdots z_{j+1}^{-1} z_j^{-1} x_{j+a_0} x_{j+a_0+1} \cdots x_{j+2a_0}.$$

We have now covered a method to swap any expression $\alpha_i\gamma_j^{-1}$ for an expression in which all negative (a_1, a_0)-generators are to the left of the positive generators. This means that we can find P, Q generated by positive (a_1, a_0)-generators of type x, z, such that $g = Q^{-1}P$.

□

For all β the root of a quadratic Pisot subdivision polynomial $f_\beta = X^2 - a_1X - a_0$, we can find a presentation for F_β.

Theorem 4.3.20. For β the positive zero of the Pisot subdivision polynomial $f_\beta = X^2 - a_1X - a_0$,

$$F_\beta = \langle x_0, x_1, x_2, \ldots, z_0, z_1, z_2, \ldots \,|\, R_1, R_2 \rangle$$

with the relations:

$$\begin{aligned} R_1: \quad x_ix_j &= x_{j+K-1}x_i \ \forall\ i < j \\ x_iz_j &= z_{j+K-1}x_i \ \forall\ i < j \\ z_ix_j &= x_{j+K-1}z_i \ \forall\ i < j \\ z_iz_j &= z_{j+K-1}z_i \ \forall\ i < j \\ R_2: \quad x_{i+a_0}x_{i+a_0+1}\cdots x_{i+2a_0-1}x_i &= z_iz_{i+1}\cdots z_{i+a_0-1}z_i \ \forall\ i \geq 0. \end{aligned}$$

Proof. We have already shown that the (a_1, a_0)-generators of type x, z form a generating set for F_β.

We consider the relations on the generating set of connected (a_1, a_0)-generators. We will show that these relations reduce to the relations R_1 and R_2 when we substitute each $[C_i]_j$ for an expression in terms of (a_1, a_0)-generators of type x, z.

The relation R_1 is the same, and is representative of the ability to hang non-intersecting sub-trees in any order. We have already noted that the relation R_2 can be derived by repeating the relation R_3, and by choosing $r = 0$ every time:

$$R_3\colon\ [C_r]_{j+i+a_0}[C_i]_j = [C_{r+a_0}]_{j+i}[C_{i+1}]_j.$$

We can rearrange this to find an expression for $[C_{i+1}]_j$ in terms of $[C_i]_j, [C_r]_{j+i+a_0}, [C_{r+a_0}]_{j+i}$, for some $0 \leq r \leq a_1 - a_0$. This gives us

$$\begin{aligned} [C_r]_{j+i+a_0}[C_i]_j &= [C_{r+a_0}]_{j+i}[C_{i+1}]_j \\ [C_{r+a_0}]_{j+i}^{-1}[C_r]_{j+i+a_0}[C_i]_j &= [C_{i+1}]_j. \end{aligned}$$

In particular we can choose $r = 0$, and replace $[C_0], [C_{a_0}]$ with x, z:

$$\begin{aligned}
[C_{i+1}]_j &= z_{j+i}^{-1} x_{j+i+a_0} [C_i]_j \\
&= z_{j+i}^{-1} x_{j+i+a_0} z_{j+i-1}^{-1} x_{j+i+a_0-1} [C_{i-1}]_j \\
&= z_{j+i}^{-1} x_{j+i+a_0} z_{j+i-1}^{-1} x_{j+i+a_0-1} z_{j+i-2}^{-1} x_{j+i+a_0-2} [C_{i-2}]_j.
\end{aligned}$$

Thus for all $[C_i]_j$, we can find an expression in terms of (a_1, a_0)-generators of type x, z.

$$\begin{aligned}
[C_i]_j &= z_{j+i-1}^{-1} x_{j+i+a_0-1} [C_{i-1}]_j \\
&= z_{j+i-1}^{-1} x_{j+i+a_0-1} z_{j+i-2}^{-1} x_{j+i+a_0-2} [C_{i-2}]_j \\
&\vdots \\
&= z_{j+i-1}^{-1} x_{j+i+a_0-1} z_{j+i-2}^{-1} x_{j+i+a_0-2} \cdots z_{j+1}^{-1} x_{j+a_0+1} [C_{i-(i-1)}]_j \\
&= z_{j+i-1}^{-1} x_{j+i+a_0-1} z_{j+i-2}^{-1} x_{j+i+a_0-2} \cdots z_{j+1}^{-1} x_{j+a_0+1} z_j^{-1} x_{j+a_0} [C_{i-i}]_j \\
&= z_{j+i-1}^{-1} x_{j+i+a_0-1} z_{j+i-2}^{-1} x_{j+i+a_0-2} \cdots z_{j+1}^{-1} x_{j+a_0+1} z_j^{-1} x_{j+a_0} x_j.
\end{aligned}$$

We consider the relation R_3, and start by replacing $[C_{i+1}]_j$. Then

$$\begin{aligned}
[C_r]_{j+i+a_0} [C_i]_j &= [C_{r+a_0}]_{j+i} [C_{i+1}]_j \\
&= [C_{r+a_0}]_{j+i} z_{j+i}^{-1} x_{j+i+a_0} [C_i]_j \\
[C_r]_{j+i+a_0} &= [C_{r+a_0}]_{j+i} z_{j+i}^{-1} x_{j+i+a_0}.
\end{aligned}$$

Now we can replace $[C_r]_{j+i+a_0}$ with (a_1, a_0)-generators of type x, z. Then

$$\begin{aligned}
[C_r]_{j+i+a_0} &= z_{j+i+a_0+r-1}^{-1} x_{j+i+a_0+r+a_0-1} [C_{r-1}]_{j+i+a_0} \\
&\vdots \\
&= z_{j+i+a_0+r-1}^{-1} x_{j+i+a_0+r+a_0-1} \cdots z_{j+i+a_0+(r-s)}^{-1} x_{j+i+a_0+a_0+(r-s)} [C_{r-s}]_{j+i+a_0} \\
&\vdots \\
&= z_{j+i+a_0+r-1}^{-1} x_{j+i+a_0+r+a_0-1} \cdots z_{j+i+a_0}^{-1} x_{j+i+a_0+a_0} [C_0]_{j+i+a_0} \\
&= z_{j+i+a_0+r-1}^{-1} x_{j+i+a_0+r+a_0-1} \cdots z_{j+i+a_0}^{-1} x_{j+i+2a_0} x_{j+i+a_0}.
\end{aligned}$$

We can also replace $[C_{r+a_0}]_{j+i}$:

$$\begin{aligned}
[C_{r+a_0}]_{j+i} &= z^{-1}_{j+i+r+a_0-1} x_{j+i+r+a_0+a_0-1} [C_{r+a_0-1}]_{j+i} \\
&\vdots \\
&= z^{-1}_{j+i+r+a_0-1} x_{j+i+r+a_0+a_0-1} \cdots z^{-1}_{j+i+(r+a_0-s)} x_{j+i+a_0+(r+a_0-s)} [C_{r+a_0-s}]_{i+j} \\
&\vdots \\
&= z^{-1}_{j+i+r+a_0-1} x_{j+i+r+a_0+a_0-1} \cdots z^{-1}_{j+i+1} x_{j+i+a_0+1} [C_1]_{i+j} \\
&= z^{-1}_{j+i+r+a_0-1} x_{j+i+r+a_0+a_0-1} \cdots z^{-1}_{j+i} x_{j+i+a_0} x_{i+j}.
\end{aligned}$$

Note here that the first (a_1, a_0)-generators of type x, z in the expressions for $[C_r]_{j+i+a_0}$ and $[C_{r+a_0}]_{j+i}$ are identical, as $j + i + a_0 + r + a_0 - 1 = j + i + r + a_0 + a_0 - 1$. In fact

$$\begin{aligned}
[C_r]_{j+i+a_0} &= [C_{r+a_0}]_{j+i} z^{-1}_{j+i} x_{j+i+a_0} \\
x_{j+i+a_0} &= z^{-1}_{j+i+a_0-1} x_{j+i+a_0+a_0-1} \cdots z^{-1}_{j+i} x_{i+j+a_0} x_{j+i} z^{-1}_{j+i} x_{j+i+a_0}.
\end{aligned}$$

Pre-composing with $x^{-1}_{j+1+a_0} z_{i+j}$ gives us

$$z_{j+i} = z^{-1}_{j+i+a_0-1} x_{j+i+a_0+a_0-1} \cdots z^{-1}_{j+i} x_{i+j+a_0} x_{j+i}.$$

If we let $t = j + i$, this becomes.

$$z_t = z^{-1}_{t+a_0-1} x_{t+a_0+a_0-1} \cdots z^{-1}_{t+1} x_{t+a_0+1} z_t^{-1} x_{t+a_0} x_t. \tag{4.1}$$

In Lemma 4.3.19, we saw that $x_j z_i^{-1} = z_i^{-1} x_{j+(K-1)}$. If we consider the right hand side of equation (4.1), we notice that for each $x_j z_i^{-1}$, $i < j$. In fact, every x-type generator has higher index than every negative z-type generator. Therefore we can move all our x-type generators to the right of the z-type generators.

$$\begin{aligned}
z_t &= z^{-1}_{t+a_0-1} \cdots z^{-1}_{t+1} z_t^{-1} x_{t+a_0+a_0-1+(K-1)(a_0-1)} \cdots x_{t+a_0+1+(K-1)} x_{t+a_0} x_t \\
z_t z_{t+1} \cdots z_{t+a_0-1} z_t &= x_{t+a_0+a_0-1+(K-1)(a_0-1)} \cdots x_{t+a_0+1+(K-1)} x_{t+a_0} x_t.
\end{aligned}$$

From the relation R_1, we see that $x_{j+(K-1)}x_i = x_ix_j$ for all $i < j$. In the equation above, the first two terms $x_{t+2a_0-1+(K-1)(a_0-1)}x_{t+2a_0-2+(K-1)(a_0-2)}$ satisfy the conditions of the first relation:

$$x_{t+2a_0-1+(K-1)(a_0-1)}x_{t+2a_0-2+(K-1)(a_0-2)} = x_{t+2a_0-2+(K-1)(a_0-2)}x_{t+2a_0-1+(K-1)(a_0-2)}.$$

In fact, using R_1, we can *move move* $x_{t+2a_0-1+(K-1)(a_0-1)}$ past the next $a_0 - 1$ x-type generators:

$$z_tz_{t+1}\cdots z_{t+a_0-1}z_t = x_{t+a_0+a_0-2+(K-1)(a_0-2)}\cdots x_{t+a_0+1+(K-1)}x_{t+a_0}x_{t+2a_0-1}x_t.$$

Similarly we can now *move* $x_{t+2a_0-2+(K-1)(a_0-2)}$ past the next $a_0 - 2$ x-type generators. We can repeat this process until there are no x-type generators have an added $(K-1)$ in their subscripts.

$$\begin{aligned}
z_tz_{t+1}\cdots z_{t+a_0-1}z_t &= x_{t+a_0+a_0-1+(K-1)(a_0-1)}\cdots x_{t+a_0+1+(K-1)}x_{t+a_0}x_t \\
&= x_{t+a_0+a_0-2+(K-1)(a_0-2)}\cdots x_{t+a_0+1+(K-1)}x_{t+a_0}x_{t+2a_0-1}x_t \\
&\quad\vdots \\
&= x_{t+a_0+1+(K-1)}x_{t+a_0}x_{t+a_0+2}\cdots x_{t+2a_0-1}x_t \\
&= x_{t+a_0}x_{t+a_0+1}x_{t+a_0+2}x_{t+a_0+3}\cdots x_{t+2a_0-1}x_t.
\end{aligned}$$

This is exactly the relation R_2. Therefore, the relations R_1 and R_3 in connected (a_1, a_0)-generators collapse down to the two relations R_1 and R_2 in just (a_1, a_0)-generators of type x, z. So a presentation for F_β is

$$F_\beta = \langle x_0, x_1, x_2, \ldots, z_0, z_1, z_2, \ldots \,|\, R_1, R_2\rangle$$

with the relations

$$\begin{aligned}
R_1: \quad x_ix_j &= x_{j+K-1}x_i \ \forall\ i < j \\
x_iz_j &= z_{j+K-1}x_i \ \forall\ i < j \\
z_ix_j &= x_{j+K-1}z_i \ \forall\ i < j \\
z_iz_j &= z_{j+K-1}z_i \ \forall\ i < j \\
R_2: \quad x_{i+a_0}x_{i+a_0+1}\cdots x_{i+2a_0-1}x_i &= z_iz_{i+1}\cdots z_{i+a_0-1}z_i \ \forall\ i \geq 0.
\end{aligned}$$

□

4.4 Abelianizations

4.4.1 Orbits in F_β

In [5], the case $a_0 = 1$, Brown found a presentation for F_β, and this presentation has been used to find F_β^{ab}. The abelianisation of all of the cases contained 2-torsion, and a free abelian group of rank $K = a_1 + a_0$.

In fact, given an irreducible subdivision polynomial $f_\beta = X^n - a_{n-1}X^{n-1} - \cdots - a_1X - a_0$ and corresponding positive real zero β, β not necessarily Pisot, Nucinkis has given a proof that the abelianisation of F_β has an embedded free group of rank K, where $K = a_{n-1} + a_{n-2} + \cdots + a_1 + a_0$. This result is in fact a variant on a result by Bieri and Strebel, and so whilst the work below was completed in private communications with Nucinkis, the credit for the result goes to Bieri and Strebel. This result can be found in section 5 of their work [10].

Result from Bieri and Strebel

Let β be the positive real zero of an irreducible subdivision polynomial

$$f_\beta = X^n - a_{n-1}X^{n-1} - a_{n-2}X^{n-2} - \cdots - a_1X - a_0$$

and let $\tau = \dfrac{1}{\beta}$ Recall that $\mathbb{Z}[\tau] = \mathbb{Z}[\beta][\dfrac{1}{\beta}]$, as $\beta \in \mathbb{Z}[\tau]$.

Lemma 4.4.2.
There is a well defined surjective ring homomorphism

$$\pi : \mathbb{Z}[\beta] \to \mathbb{Z}/(K-1)\mathbb{Z}$$

where $K = a_{n-1} + \cdots + a_1 + a_0$.

Proof. There is a well defined surjective ring-homomorphism (evaluation at $X = 1$)

$$\begin{array}{rccc} p: & \mathbb{Z}[X] & \twoheadrightarrow & \mathbb{Z} \\ & \sum_{i=0}^{m} b_i X^i & \longmapsto & \sum_{i=0}^{m} b_i. \end{array}$$

Hence $p(f_\beta(X)) = -(K-1)$, and the projection onto $\mathbb{Z}/(K-1)\mathbb{Z}$ now extends to

$$\pi : \mathbb{Z}[X]/(f_\beta(X)) \twoheadrightarrow \mathbb{Z}/(K-1)\mathbb{Z}.$$

The claim follows from the fact that

$$\mathbb{Z}[X]/(f_\beta(X)) \cong \mathbb{Z}[\beta].$$

□

Recall the corollary to Theorem 2.2.19: Every element $x \in \mathbb{Z}[\tau]$ has an of the form

$$x = \frac{b_0 + b_1\beta + \cdots + b_{n-1}\beta^{n-1}}{\beta^m}$$

where $b_i, m \in \mathbb{Z}_{\geq 0}$.

Note that $\beta^0, \beta^1, \ldots, \beta^{n-1} \geq 1$, which leads to the following remark.

Remark 41. Let $x \in \mathbb{Z}[\tau] \cap (0,1)$, there is an expression for x in the form

$$x = \frac{b_0 + b_1\beta + \cdots + b_{n-1}\beta^{n-1}}{\beta^m}$$

where $m, b_i \in \mathbb{Z}_{\geq 0}$.

In fact, for any given x there will not necessarily be an expression of this form with $m = 0$. Once a value for m for which there is an expression for x is found, say $m = \mu$, then there will also be an expression in this form where we take $m = \mu + t$ for any $t \in \mathbb{Z}_{\geq 0}$. This means there must always be a minimal choice of $m \in \mathbb{Z}_{\geq 0}$ for such an expression for each $x \in \mathbb{Z}[\tau] \cap (0,1)$.

Proposition 4.4.3.

There is a well-defined surjective ring-homomorphism

$$\begin{array}{rccc} \pi : & \mathbb{Z}[\tau] & \twoheadrightarrow & \mathbb{Z}/(K-1)\mathbb{Z} \\ & \dfrac{b_0 + b_1\beta + \cdots + b_{n-1}\beta^{n-1}}{\beta^m} & \longmapsto & b_0 + b_1 + \cdots + b_{n-1}. \end{array}$$

Proof. This follows from Remark 41, and we see that the well-definedness relies on β being mapped to $\bar{1}$ by the homomorphism of Lemma 4.4.2. □

Orbits of G_β

By Proposition 4.4.3, the breakpoints of the elements in G_β fall into $K-1$ classes. Now we will show that this implies that there are $K-1$ orbits in $\mathbb{Z}[\tau] \cap (0,1)$ under the action of G_β.

Lemma 4.4.4.
Let $g \in G_\beta$, and $x \in \mathbb{Z}[\tau] \cap (0,1)$, a breakpoint of g such that $\pi(x) \equiv_{(K-1)} i$. Then $\pi(f(x)) \equiv_{(K-1)} i$.

Proof. Let $l \in \mathbb{Z}$. Suppose $0 \leq l \leq n-1$. Then clearly $\pi(\beta^l) = 1$. Note that for $l < 0$, $\beta^l = \tau^{|l|} = \dfrac{1}{\beta^{|l|}}$, and so $\pi(\beta^l) = 1$. Now if $l \geq n$, $\beta^l = a_{n-1}\beta^{l-1} + a_{n-2}\beta^{l-2} + \cdots + a_1\beta^{l-(n-1)} + a_0\beta^{l-n}$. Therefore

$$\begin{aligned}\pi(\beta^n) &= \pi\left(a_{n-1}\beta^{n-1} + a_{n-2}\beta^{n-2} + \cdots + a_1\beta + a_0\right)\\ &= a_{n-1} + a_{n-2} + \cdots + a_1 + a_0\\ &= K \equiv_{(K-1)} 1.\end{aligned}$$

As π is a ring homomorphism, $\pi(\beta^{n+1}) = \pi(\beta) \times \pi(\beta^n) = 1 \times K = K \equiv_{(K-1)} 1$. We can repeat this for $\pi(\beta^{n+2})$, and realise that for all $l \in \mathbb{Z}$

$$\pi(\tau^l) = \pi(\beta^{-l}) \equiv_{(K-1)} 1.$$

Now suppose that $x = x_1$ is the first breakpoint of g. Then there is there is $l_1 \in \mathbb{Z}$ such that $g(x_1) = \beta^{l_1}x_1 = y_1$. Hence

$$\pi(y_1) = \pi(g(x_1)) = \pi(\beta^{l_1}x_1) = \pi(\beta^{l_1})\pi(x_1) = \pi(x_1).$$

Now suppose that $x = x_t$, the tth breakpoint in g, and that $\pi(y_{t-1}) = \pi(g(x_{t-1})) \equiv_{(K-1)} \pi(x_{t-1})$, where (x_{t-1}, y_{t-1}) is the $(t-1)^{th}$ breakpoint of g. Then the line segment from the $(t-1)^{th}$ breakpoint to the tth breakpoint is found by taking

$$y_t - y_{t-1} = \tau^{l_t}(x_t - x_{t-1})$$

for some $l_t \in \mathbb{Z}$. Therefore

$$\pi(y_t - y_{t-1}) = \pi\left(\beta^{l_t}(x_t - x_{t-1})\right)$$
$$\pi(y_t) - \pi(y_{t-1}) = \pi(\beta^{l_t})(\pi(x_t) - \pi(x_{t-1}))$$
$$\pi(y_t) = \pi(x_t) - \pi(x_{t-1}) + \pi(y_{t-1}).$$

Since $\pi(y_{t-1}) = \pi(g(x_{t-1})) \equiv_{(K-1)} \pi(x_{t-1})$,

$$\pi(y_t) = \pi(g(x_t)) \equiv_{(K-1)} \pi(x_t).$$

By induction, if $x \in \mathbb{Z}[\tau] \cap (0,1)$ is a breakpoint in $g \in G_\beta$ then $\pi(g(x)) \equiv_{(K-1)} \pi(x)$. □

Therefore for any two breakpoints $x, y \in \mathbb{Z}[\tau] \cap (0,1)$ such that x and y lie in the same G_β-orbit, $\pi(x) \equiv_{(K-1)} \pi(y)$.

Lemma 4.4.5.

Any two elements $x, y \in \mathbb{Z}[\tau] \cap (0,1)$ such that $\pi(x) \equiv_{(K-1)} \pi(y)$ lie in the same G_β-orbit.

Proof. Remark 41, tells us that we can find expressions for x and y in the form

$$x = \frac{b_0 + b_1\beta + \cdots + b_{n-1}\beta^{n-1}}{\beta_1^m} \text{ and } y = \frac{c_0 + c_1\beta + \cdots + c_{n-1}\beta^{n-1}}{\beta_2^m}$$

where $b_i, c_i \in \mathbb{Z}_{\geq 0}$, and $m_1, m_2 > 0$. Furthermore, by assumption

$$\pi(x) = b_0 + b_1 + \cdots + b_{n-1} \equiv_{(K-1)} c_0 + c_1 + \cdots + c_{n-1} = \pi(y).$$

Since $b_i, c_i \geq 0$ for each i, we can subdivide the intervals $(0,x)$ and $(0,y)$ into the sub-intervals which are powers of β. The subdivision of $(0,x)$ will contain $\pi(x) = \sum_{i=0}^{n-1} b_i$ sub-intervals, b_0 of length β^{-m_1}, b_1 of length β^{1-m_1}, …, b_i of length β^{i-m_1}, …, and b_{n-1} of length β^{n-1-m_1}. Similarly the interval $(0,y)$ can be subdivided into $\pi(y) = \sum_{i=0}^{n-1} c_i$ sub-intervals.

Without loss of generality, we can assume that $\pi(x) = \displaystyle\sum_{i=0}^{n-1} b_i \leq \sum_{i=0}^{n-1} c_i = \pi(y)$. I.e.,

$$\pi(x) = \sum_{i=0}^{n-1} b_i + l(K-1) = \sum_{i=0}^{n-1} c_i = \pi(y)$$

for some $l \in \mathbb{Z}_{\geq 0}$. If $l = 0$, then the subdivisions of $(0, x)$ and $(0, y)$ contain the same number of sub-intervals. If $l > 0$, then consider the first sub-interval in the subdivision of $(0, x)$. Let this interval be I_1 and have length β^{i-m_1} for some $0 \leq i \leq n-1$. Then we can subdivide the interval $I_1 = [0, \beta^{i-m_1}]$ into K smaller sub-intervals using

$$\beta^{i-m_1} = a_{n-1}\beta^{i-m_1-1} + \cdots + a_1\beta^{i-m_1-(n-1)} + a_0\beta^{i-m_1-n}.$$

By replacing I_1 in the subdivision of $(0, x)$ with this subdivision, we have subdivided $(0, x)$ into $\pi(x) + K - 1 = \sum_{i=0}^{n-1} b_i + (K-1)$ sub-intervals, each of length which is a power of β. We can repeat this process l times, until we have subdivided the interval $(0, x)$ into $\pi(x) + l(K-1) = \sum_{i=0}^{n-1} b_i + l(k-1) = \sum_{i=0}^{n-1} c_i = \pi(y)$ sub-intervals each with length a power of β.

As $\pi(x) = \pi(y)$, then

$$\pi(1-x) = \pi(1) - \pi(x) = \pi(1) - \pi(y) = \pi(1-y)$$

and so we can similarly subdivide the intervals $(x, 1)$ and $(1-y)$ into the same number of sub-intervals. We have then found two β-subdivisions S_1 and S_2 of $[0, 1]$ such that the $\pi(y)^{th}$ of S_1 is x and of S_2 is y. We can therefore construct an element $g = (S_1, S_2) \in G_\beta$ such that $y = g(x)$, and (x, y) is a breakpoint of g. Therefore, x and y lie in the same G_β-orbit. □

There are now $K - 1$ possible orbits for the elements $x \in \mathbb{Z}[\tau] \cap (0, 1)$. By including the points 0 and 1, which are fixed points under action by elements of $\mathcal{G}_\beta$, the combination of Lemma 4.4.4 and Lemma 4.4.5 proves the following theorem.

Theorem 4.4.6. There are $K + 1$ orbits of elements in $\mathbb{Z}[\tau] \cap [0, 1]$ under the action of F_β.

We now consider the abelianisation of G_β.

Theorem 4.4.7.
The abelianisation of G_β contains a free abelian sub-group of rank K.

Proof. We begin by showing that there is a homomorphism ϕ from G_β to the free abelian group of rank $N + 2$. Let $g \in G_\beta$.

For each breakpoint $b \in [0, 1]$ of g, we denote by l_b^g the gradient of the left slope, and by r_b^g the gradient of the right slope of g at b.

The breakpoints of g fall into $K-2$ distinct orbits $\mathcal{O}_i$, for $i \in \{1, \dots, K-1\}$. For each orbit $\mathcal{O}_i$, we define a number s_i^g as follows

$$s_i^g = \sum_{j=1}^{t_i} \left(-log(l_{b_j}^g) + log(r_{b_j}^g) \right).$$

Here k_i is the number of breakpoints of g that lie in $\mathcal{O}_i$, $b_j \in [0,1]$ is one of those breakpoints in g, and the log is of base β. We then define

$$\begin{aligned} \phi : G_\beta \ &\to \qquad\qquad \mathbb{Z}^{K+1} \\ g &\longmapsto \left(log(r_0^g), s_1^g, \dots, s_{K-1}^g, log(l_1^g) \right). \end{aligned}$$

This does indeed satisfy the properties of a group homomorphism, as slopes of linear functions are multiplicative, and since all gradients of these slopes are powers of β the logarithms are additive.

Note that if we were to refine the list of breakpoints, and include points in which the gradient of g does not change, then $\phi(g)$ will remain the same, as for any "non-proper" breakpoint b, $l_b^g = r_b^g$.

We also observe that

$$log(r_0^g) + s_1^g + \dots + s_{K-1}^g + log(l_1^g) = 0.$$

Therefore we can induce the following surjective homomorphism:

$$\begin{aligned} \phi : G_\beta \ &\twoheadrightarrow \qquad\qquad \mathbb{Z}^{K} \\ g &\longmapsto \left(log(r_0^g) - log(l_1^g), s_1^g, \dots, s_{K-1}^g \right) \end{aligned}$$

as required. □

This result is in line with previous results on F_n for $n \in \mathbb{N}$, and for $\mathbb{F}_\beta$ where β is the golden mean. In general if the group F_β has a description of elements in tree-pair diagrams then K is equal to the number of legs in a caret.

4.4.8 The group $F_{\beta_n}^{ab}$

We will look at the case with β_n, positive real zero of the subdivision polynomial

$$f_{\beta_n} = X^2 - (n+1)X - n$$

for some $n \in \mathbb{N}$. We will aim to find the group $F^{ab}_{\beta_n}$. Note, that here $K = 2n+1$.

From Theorem 4.3.20 we have the presentation for the group F_β:

$$F_\beta = \langle x_0, x_1, x_2, \ldots, z_0, z_1, z_2, \ldots \big| R_1, R_2 \rangle \text{ with the relations}$$

$$\begin{aligned} R_1 : \quad x_i x_j &= x_{j+2n} x_i \ \forall\ i < j \\ x_i z_j &= z_{j+2n} x_i \ \forall\ i < j \\ z_i x_j &= x_{j+2n} z_i \ \forall\ i < j \\ z_i z_j &= z_{j+2n} z_i \ \forall\ i < j \\ R_2 : \quad x_{i+n} x_{i+n+1} \cdots x_{i+2n-1} x_i &= z_i z_{i+1} \cdots z_{i+n-1} z_i \ \forall\ i \in \mathbb{N}. \end{aligned}$$

A presentation for $F^{ab}_{\beta_n}$ is gained by adding a third relation

$$R_3 : g_i h_j = h_j g_i \ \forall\ i, j \text{ and for } h, g \in \{x, z\}.$$

The addition of of the relation R_3, allows us to find a smaller generating set for $F^{ab}_{\beta_n}$.

Lemma 4.4.9. $F^{ab}_{\beta_n}$ is generated by the set

$$\{x_0, x_1, \ldots, x_{2n}, z_0, z_1, \ldots, z_{2n}\}.$$

Proof. The set $\{x_0, x_1, x_2, \ldots, z_0, z_1, z_2, \ldots\}$ is clearly a generating set. We consider what happens when we use a combination of the relations R_1 and R_3.

$$\begin{aligned} x_i x_j &= x_{j+2n} x_i = x_i x_{j+2n} \\ x_j &= x_{j+2n} \text{ for } j \geq 1 \\ x_i z_j &= z_{j+2n} x_i = x_i x_{j+2n} \\ z_j &= z_{j+2n} \text{ for } j \geq 1. \end{aligned}$$

Thus we only need $\{x_0, x_1, \ldots, x_{2n}, z_0, z_1, \ldots, z_{2n}\}$ to generate $F^{ab}_{\beta_n}$. □

We can then reduce this generating set using the relations R_2 and R_3.

Lemma 4.4.10. For even $n \in \mathbb{N}$, $F^{ab}_{\beta_n}$ is generated by the set

$$\{x_0, z_0, z_1, \ldots, z_{2n}\}$$

Proof. Consider the family of relations R_2. For each value $i \in \mathbb{N}_0$, we denote the relation by $R_2(i)$.

$$
\begin{aligned}
i = 0: &\quad x_n x_{n+1} \cdots x_{2n-1} x_0 = z_0 z_1 \cdots z_{n-1} z_0 \\
i = 1: &\quad x_{n+1} x_{n+2} \cdots x_{2n} x_1 = z_1 z_2 \cdots z_n z_1 \\
i = 2: &\quad x_{n+2} x_{n+3} \cdots x_{2n+1} x_2 = z_2 z_3 \cdots z_{n+1} z_2 \\
\vdots & \qquad\qquad \vdots \\
i = n: &\quad x_{2n} x_{2n+1} \cdots x_{3n-1} x_n = z_n z_{n+1} \cdots z_{2n-1} z_n \\
i = n+1: &\quad x_{2n+1} x_{2n+2} \cdots x_{3n} x_{n+1} = z_{n+1} z_{n+2} \cdots z_{2n} z_{n+1} \\
i = n+2: &\quad x_{n+2} x_{2n+3} \cdots x_{3n+1} x_{n+2} = z_{n+2} z_{n+3} \cdots z_{2n+1} z_{n+2} \\
\vdots & \qquad\qquad \vdots \\
i = 2n-2: &\quad x_{3n-2} x_{3n-1} \cdots x_{4n-3} x_{2n-2} = z_{2n-2} z_{2n-1} \cdots z_{3n-3} z_{2n-2} \\
i = 2n-1: &\quad x_{3n-1} x_{3n} \cdots x_{4n-2} x_{2n-1} = z_{2n-1} z_{2n} \cdots z_{3n-2} z_{2n-1} \\
i = 2n: &\quad x_{3n} x_{3n+1} \cdots x_{4n-1} x_{2n} = z_{2n} z_{2n+1} \cdots z_{3n-1} z_{2n} \\
\vdots & \qquad\qquad \vdots
\end{aligned}
$$

In lemma 4.4.9 we saw that $x_j = x_{j+2n}$, and $z_j = z_{j+2n}$ for all $j \geq 1$. This means that we can reduce the number of generators in the $2n+1$ relations above. Since the generators x_0, z_0 do not appear in the list aside from when $i = 0$, we can confirm that if $i = j + 2n$ for some $j \geq 1$, then

$$R_2(i) = R_2(i - 2n) = R_2(j).$$

We can therefore ignore all relations $R_2(i)$ for $i \geq 2n+1$, as they are equivalent to some relation already in this list.

We can also remove any generator α_j in a given relation in which $j \geq 2n+1$, as shown in the

proof of Lemma 4.4.9. This process repeats until we are left with only the $2n+1$ equations below.

$$
\begin{aligned}
i = 0: &\quad x_n x_{n+1} \cdots x_{2n-1} x_0 = z_0 z_1 \cdots z_{n-1} z_0 \\
i = 1: &\quad x_{n+1} x_{n+2} \cdots x_{2n} x_1 = z_1 z_2 \cdots z_n z_1 \\
i = 2: &\quad x_{n+2} x_{n+3} \cdots x_1 x_2 = z_2 z_3 \cdots z_{n+1} z_2 \\
\vdots & \qquad\qquad \vdots \\
i = n: &\quad x_{2n} x_1 \cdots x_{n-1} x_n = z_n z_{n+1} \cdots z_{2n-1} z_n \\
i = n+1: &\quad x_1 x_2 \cdots x_n x_{n+1} = z_{n+1} z_{n+2} \cdots z_{2n} z_{n+1} \\
i = n+2: &\quad x_2 x_3 \cdots x_{n+1} x_{n+2} = z_{n+2} z_{n+3} \cdots z_1 z_{n+2} \\
\vdots & \qquad\qquad \vdots \\
i = 2n-2: &\quad x_{n-2} x_{n-1} \cdots x_{2n-3} x_{2n-2} = z_{2n-2} z_{2n-1} \cdots z_{n-3} z_{2n-2} \\
i = 2n-1: &\quad x_{n-1} x_n \cdots x_{2n-2} x_{2n-1} = z_{2n-1} z_{2n} \cdots z_{n-2} z_{2n-1} \\
i = 2n: &\quad x_n x_{n+1} \cdots x_{2n-1} x_{2n} = z_{2n} z_1 \cdots z_{n-1} z_{2n}
\end{aligned}
$$

We will be assuming that all generators have been reduced to be of type $x_1, \ldots, x_{2n}, z_1, \ldots, z_{2n}$ whenever possible.

We will now convert this to an easier to read form, namely an additive form. This is clearly possible through the map $x_i x_j \longrightarrow x_i + x_j$.

$$
\begin{aligned}
i = 0: &\quad x_0 + x_n + x_{n+1} + \cdots + x_{2n-1} = 2z_0 + z_1 + \cdots + z_{n-1} \\
i = 1: &\quad x_1 + x_{n+1} + x_{n+2} + \cdots + x_{2n} = 2z_1 + z_2 + \cdots + z_n \\
i = 2: &\quad x_2 + x_{n+2} + x_{n+3} + \cdots + x_1 = 2z_2 + z_3 + \cdots + z_{n+1} \\
\vdots & \qquad\qquad \vdots \\
i = n: &\quad x_n + x_{2n} + x_1 + \cdots + x_{n-1} = 2z_n + z_{n+1} + \cdots + z_{2n-1} \\
i = n+1: &\quad x_{n+1} + x_1 + x_2 + \cdots + x_n = 2z_{n+1} + z_{n+2} + \cdots + z_{2n} \\
i = n+2: &\quad x_{n+2} + x_2 + x_3 + \cdots + x_{n+1} = 2z_{n+2} + z_{n+3} + \cdots + z_1 \\
\vdots & \qquad\qquad \vdots \\
i = 2n-2: &\quad x_{2n-2} + x_{n-2} + x_{n-1} + \cdots + x_{2n-3} = 2z_{2n-2} + z_{2n-1} + \cdots + z_{n-3} \\
i = 2m-1: &\quad x_{2n-1} + x_{n-1} + x_n + \cdots + x_{2n-2} = 2z_{2n-1} + z_{2n} + \cdots + z_{n-2} \\
i = 2n: &\quad x_{2n} + x_n + x_{n+1} + \cdots + x_{2n-1} = 2z_{2n} + z_1 + \cdots + z_{n-1}
\end{aligned}
$$

Our goal is to eliminate $x_1, \ldots, x_{2n}$. To do this we will consider $R_2(1) - R_2(2)$, $R_2(2) - R_2(3)$,..., $R_2(2n-1) - R_2(2n)$, and $R_2(2n) - R_2(1)$.

$$
\begin{aligned}
R_2(1) - R_2(2): &\quad x_{n+1} - x_2 = 2z_1 - z_2 - z_{n+1} \\
R_2(2) - R_2(3): &\quad x_{n+2} - x_3 = 2z_2 - z_3 - z_{n+2} \\
\vdots & \qquad\qquad \vdots \\
R_2(n-1) - R_2(n): &\quad x_{2n-1} - x_n = 2z_{n-1} - z_n - z_{2n-1} \\
R_2(n) - R_2(n+1): &\quad x_{2n} - x_{n+1} = 2z_n - z_{n+1} - z_{2n} \\
R_2(n+1) - R_2(n+2): &\quad x_1 - x_{n+2} = 2z_{n+1} - z_{n+2} - z_1 \\
\vdots & \qquad\qquad \vdots \\
R_2(2n-1) - R_2(2n): &\quad x_{n-1} - x_{2n} = 2z_{2n-1} - z_{2n} - z_{n-1} \\
R_2(2n) - R_2(1): &\quad x_n - x_1 = 2z_{2n} - z_1 - z_n
\end{aligned}
$$

We have not yet eliminated any generator. Now consider $R_2(2n) - R_2(0)$

$$R_2(2n) - R_2(0): \qquad x_{2n} - x_0 = 2z_{2n} - z_0.$$

This rearranges to give

$$x_{2n} = x_0 - 2z_0 + 2z_{2n}.$$

We have now eliminated the generator x_{2n} as it can be found by composing other generators.

Below, the previous $R_2(i) - R_2(i+1)$ have been relabelled. Looking at equation $1, \ldots, 2n$, we see that each of the generators $x_1, \ldots, x_{2n}$ occur exactly twice. This means that x_{2n} can be substituted in to two of these equations.

$$\begin{aligned}
1: &\qquad x_{n+1} - x_2 = 2z_1 - z_2 - z_{n+1} \\
2: &\qquad x_{n+2} - x_3 = 2z_2 - z_3 - z_{n+2} \\
\vdots &\qquad \vdots \\
n-1: &\qquad x_{2n-1} - x_n = 2z_{n-1} - z_n - z_{2n-1} \\
n: &\qquad x_{2n} - x_{n+1} = 2z_n - z_{n+1} - z_{2n} \\
n+1: &\qquad x_1 - x_{n+2} = 2z_{n+1} - z_{n+2} - z_1 \\
\vdots &\qquad \vdots \\
2n-1: &\qquad x_{n-1} - x_{2n} = 2z_{2n-1} - z_{2n} - z_{n-1} \\
2n: &\qquad x_n - x_1 = 2z_{2n} - z_1 - z_n
\end{aligned}$$

We can substitute x_{2n} into either equation n or equation $2n-1$.

Case 1: First we will choose to substitute x_{2n} into equation $2n-1$. In case 1, we will call our generating set $G_1 = \{x_0, x_1, \ldots, x_{2n-1}, z_0, z_1, \ldots, z_{2n}\}$. We have

$$\begin{aligned}
x_{n-1} - x_{2n} &= 2z_{2n-1} - z_{2n} - z_{n-1} && \text{so} \\
x_{n-1} &= x_{2n} + 2z_{2n-1} - z_{2n} - z_{n-1} && \text{so} \\
&= x_0 - 2z_0 + 2z_{2n} + 2z_{2n-1} - z_{2n} - z_{n-1} && \text{so} \\
&= x_0 - 2z_0 - z_{n-1} + 2z_{2n-1} + z_{2n}.
\end{aligned}$$

We have now written x_{n-1} in terms of other generators of $F^{ab}_{\beta_n}$ and so we can eliminate it from our generating set G_1. We will note that each of the equations $1, \ldots, 2n$ follows a similar form

$$i: \qquad x_{i+n} - x_{i+1} = 2z_i - z_{i+1} - zi + n.$$

This means that the other occurrence of x_{n-1} is in equation $n-2$

$$\begin{aligned}
x_{2n-2} - x_{n-1} &= 2z_{n-2} - z_{n-1} - z_{2n-2} && \text{so} \\
x_{2n-2} &= x_{n-1} + 2z_{n-2} - z_{n-1} - z_{2n-2} && \text{so} \\
&= x_0 - 2z_0 - z_{n-1} + 2z_{2n-1} + z_{2n} + 2z_{n-2} + 2z_2 - z_{2n-1} - z_{n-2} && \text{so} \\
&= x_0 - 2z_0 + 2z_{n-2} - 2z_{n-1} - Z_{2n-2} + 2Z_{2n-1} + z_{2n}.
\end{aligned}$$

So we have now eliminated $x_{2n}, x_{n-1}, x_{2n-2}$ from our list of generators. If we continue this process, we would look at equation $2n-3$. Here we would be able to eliminate the first x generator which would be $x_{2n-3+n} \equiv x_{n-3}$. Through this process, once we have eliminated the generator x_i, we consider the equation $i-1$, and can then eliminate $x_{i+(n-1)}$.

Case 2: Alternatively, we could start by substituting the generator x_{2n} into equation n. Our generating set will be called $G_2 = \{x_0, x_1, \ldots, x_{2n-1}, z_0, z_1, \ldots, z_{2n}\}$.

$$\begin{aligned}
x_{2n} - x_{n+1} &= 2z_n - z_{n+1} - z_{2n} && \text{so} \\
x_{n+1} &= x_{2n} - 2z_n + z_{n+1} + z_{2n} && \text{so} \\
&= x_0 - 2z_0 + 2z_{2n} - 2z_n + z_{n+1} + z_{2n} && \text{so} \\
&= x_0 - 2z_0 - 2z_n + z_{n+1} + 3z_{2n}.
\end{aligned}$$

We can thus eliminate x_{n+1} from the generating set G_2. We can find then substitute x_{n+1} into equation 1.

$$\begin{aligned}
x_{n+1} - x_2 &= 2z_1 - z_2 - z_{n+1} && \text{so} \\
x_2 &= x_{n+1} - 2z_1 + z_2 + z_{n+1} && \text{so} \\
&= x_0 - 2z_0 - 2z_n + z_{n+1} + 3z_{2n} - 2z_1 + z_2 + z_{n+1} && \text{so} \\
&= x_0 - 2z_0 - 2Z_1 + z_2 - 2Z_n + 2z_{n+1} + 3Z - 2n.
\end{aligned}$$

So we have now eliminated x_{2n}, x_{n+1}, x_2 from our list of generators. If we continue this process, we would look at equation $n+3$. Here we would be able to eliminate the second x-type generator which will be $x_{n+3+1} \equiv x_{n+4}$. Through this process, once we have eliminated the generator x_i, we consider the equation $i+n$, and can similarly eliminate $x_{i+(n+1)}$.

Claim: If n is even, we can express the variable x_i in terms of $x_0, z_0, z_1, \ldots, z_{2n}$ for $i \in \{1, \ldots, 2n\}$.

If $gcd(n-1, 2n) = 1$ or $gcd(n+1, 2n) = 1$, then either of these processes will reach all of the variables $x_1, x_2, \ldots, x_{2n}$. Note that $gcd(n-1, 2n) = 1$ if and only if n is even. Similarly $gcd(n+1, 2n) = 1$ if and only if n is even.

In either generating set G_1 or G_2, we will be able to eliminate all of the generators $x_1, \ldots, x_{2n}$, as long as n is even. Thus we can find a generating set for $F^{ab}_{\beta_n}$, namely

$$G = \{x_0, z_0, z_1, \ldots, z_{2n}\}.$$

□

Now that we have a reduced generating set for the abelianization of $F^{ab}_{\beta_n}$, we can look at the properties of the generators.

Theorem 4.4.11. If $n \in \mathbb{N}$ is even,

$$F^{ab}_{\beta_n} \cong \mathbb{Z}^{2n+1} \oplus \mathbb{Z}/(n+1)\mathbb{Z}.$$

Proof. From Theorem 4.4.6, we know that if G_β is a Bieri-Strebel group where β is the root of the subdivision polynomial $X^n - a_{n-1}X^{n-1} - a_{n-2}X^{n-2} - \cdots - a_1X - a_0$, then the G^{ab}_β has at least K free generators, where $K = a_{n-1} + a_{n-2} + \cdots + a_1 + a_0$.

In the case of our β_n, $f_{\beta_n} = X^2 - (n+1)X - n$, so $K = 2n+1$. We know that our Thompson like group $\mathbb{F}_{\beta_n}$ is a Bieri-Strebel group, so we also know that any generating set for $F^{ab}_{\beta_n}$ must contain at least $2n+1$ free generators. Since our generating set for $F^{ab}_{\beta_n}$ from Lemma 4.4.10 is of size $2n+2$, there must be $2n+1$ free generators and so each must be isomorphic to a generator of $\mathbb{Z}$.

We will substitute the new expressions for $x_n, x_{n+1}, \ldots, x_{2n-1}$ into the relation from Lemma 4.4.10, $R_2(0)$, which we will relabel as equation 0.

$$0: \qquad x_0 + x_n + x_{n+1} + \cdots + x_{2n+1} = 2z_0 + z_1 + \cdots + z_{n-1}.$$

In Lemma 4.4.10 we deduced two possible substitutions for the generator x_{2n}, each of which led to the creation of a different generating set. These were labelled G_1 and G_2. The order in which the generators were eliminated from G_i are as follows

$$G_1: \qquad x_{n-1}, x_{2n-2}, x_{n-3}, x_{2n-4}, x_{n-5}, \ldots, x_n.$$

$$G_2: \qquad x_{n+1}, x_2, x_{n+3}, x_4, x_{n+5}, \ldots, x_n.$$

So if $1 \leq i \leq n$ is even then x_i was eliminated first by G_2, and if $0 \leq i \leq n$ is odd then x_i is first eliminated by G_1. Conversely, if $n+1 \leq j \leq 2n$ is even, then x_j was first eliminated by G_1, and if $n+1 \leq j \leq 2n-1$ is odd then x_i is first eliminated by G_2. It should be recognised that since n is even, then all x_i will be eliminated in both G_1 and G_2, and the expression for x_i in terms of $x_0, z_0, z_1, \ldots, z_{2n}$ will be the same in both G_1 and G_2.

So we will consider the expressions for the eliminated generators first eliminated from G_1:

$$\begin{aligned}
x_{2n} &= x_0 - 2z_0 + 2z_{2n} \\
x_{n-1} &= x_0 - 2z_0 - z_{n-1} + 2z_{2n-1} + z_{2n} \\
x_{2n-2} &= x_0 - 2z_0 + 2z_{n-2} - 2z_{n-1} - z_{2n-2} + 2z_{2n-1} + z_{2n} \\
x_{n-3} &= x_0 - 2z_0 - z_{n-3} + 2z_{n-2} - 2z_{n-1} + 2z_{2n-3} - 2z_{2n-2} + 2z_{2n-1} + z_{2n} \\
&\vdots \qquad\qquad\qquad\qquad \vdots \\
x_i &= x_{i-(n-1)} \quad + 2z_{i-n} - z_{i-(n-1)} - z_i \\
&\vdots \qquad\qquad\qquad\qquad \vdots
\end{aligned}$$

We want to find the expressions for x_i, where $n \leq i \leq 2n$ and i even. This allows us to skip every

other generator in the above list, and only consider the evenly labelled generators.

$$
\begin{aligned}
x_{2n} &= x_0 - 2z_0 + 2z_{2n} \\
x_{2n-2} &= x_0 - 2z_0 + 2z_{n-2} - 2z_{n-1} - z_{2n-2} + 2z_{2n-1} + z_{2n} \\
x_{2n-4} &= x_0 - 2z_0 + 2z_{n-4} - 2z_{n-3} + 2z_{n-2} - 2z_{n-1} - z_{2n-4} + 2z_{2n-3} - 2z_{2n-2} + 2z_{2n-1} + z_{2n} \\
&\vdots \qquad\qquad\qquad\qquad \vdots \\
x_i &= x_{i-(n-1)} \quad + 2z_{i-n} - z_{i-(n-1)} - z_i \\
&= x_{i-2} \quad + 2z_{i+1} - z_{i+2} - z_{i-(n-1)} \quad + 2z_{i-n} - z_{i-(n-1)} - z_i \\
&= x_{i-2} \quad + 2z_{i-n} - 2z_{i-(n-1)} - z_i + 2z_{i+1} - z_{i+2}.
\end{aligned}
$$

This gives us an expression for the generators x_i for $n+2 \leq i \leq 2n-2$ and i even. We will consider $i = n$ as a special case later. For $k \in \{1, \ldots, \frac{n}{2} - 1\}$,

$$
x_{2n-2k} = x_0 - 2z_0 + \sum_{j=1}^{2k} \left((-1)^j 2z_{n-j}\right) - z_{2n-2k} + \sum_{j=1}^{2k-1} \left((-1)^{j+1} 2z_{2n-j}\right) + z_{2n}.
$$

We now consider the expressions for generators first eliminated from G_2

$$
\begin{aligned}
x_{2n} &= x_0 - 2z_0 + 2z_{2n} \\
x_{n+1} &= x_0 - 2z_0 - 2z_n + z_{n+1} + 3z_{2n} \\
x_2 &= x_0 - 2z_0 - 2z_1 + z_2 - 2Z_n + 2z_{n+1} + 3Z - 2n \\
x_{n+3} &= x_0 - 2z_0 - 2z_1 + 2z_2 - 2z_n + 2z_{n+1} - 2z_{n+2} + z_{n+3} + 3z_{2n} \\
&\vdots \qquad\qquad\qquad\qquad \vdots \\
x_i &= x_{i-(n+1)} \quad - 2z_{i-1} + z_i + z_{i-(n+1)} \\
&\vdots \qquad\qquad\qquad\qquad \vdots
\end{aligned}
$$

We want to find the expressions for x_i, $n \leq i \leq 2n$, i odd, in terms of $x_0, z_0, \ldots, z_{2n}$. We only need

to consider the odd labelled generators in the list above.

$$
\begin{aligned}
x_{n+1} &= x_0 - 2z_0 - 2z_n + z_{n+1} + 3z_{2n} \\
x_{n+3} &= x_0 - 2z_0 - 2z_1 + 2z_2 - 2z_n + 2z_{n+1} - 2z_{n+2} + z_{n+3} + 3z_{2n} \\
x_{n+5} &= x_0 - 2z_0 - 2z_1 + 2z_2 - 2z_3 + 2z_4 - 2z_n + 2z_{n+1} - 2z_{n+2} + 2z_{n+3} + 2z_{n+4} + z_{n+5} + 3z_{2n} \\
&\vdots \qquad\qquad\qquad\qquad \vdots \\
x_i &= x_{i-(n+1)} \quad - 2z_{i-1} + z_i + z_{i-(n+1)} \\
&= x_{i-2} \quad - 2z_{i-(n+2)} + z_{i-(n+1)} + z_{i-2} \quad - 2z_{i-1} + z_i + z_{i-(n+1)} \\
&= x_{i-2} \quad - 2z_{i-(n+2)} + 2z_{i-(n+1)} + z_{i-2} - 2z_{i-1} + z_i.
\end{aligned}
$$

This gives us an expression for the generators x_i for $n+1 \leq i \leq 2n-1$ and i odd. For $k \in \{1, \ldots, \frac{n}{2}\}$,

$$
x_{n+(2k-1)} = x_0 - 2z_0 + \sum_{j=1}^{2(k-1)} \left((-1)^j 2z_j\right) + \sum_{j=1}^{2k-1} \left((-1)^j 2z_{n+j-1}\right) + 3z_{2n}.
$$

An expression for the generator x_n in terms of $x_0, z_0, \ldots, z_{2n}$ can be found from either G_1 or G_2.

The even generator eliminated immediately before x_n from G_1 is x_1. We will substitute x_1 into equation $2n$:

$$
\begin{aligned}
x_n - x_1 &= 2z_{2n} - z_1 - z_n \\
x_n &= x_1 + 2z_{2n} - z_1 - z_n.
\end{aligned}
$$

We can find an expression x_1 by rearranging equation $n+1$, and using an expression for $x_{n+2} = x_{2n-n-2}$.

$$
\begin{aligned}
x_1 &= x_{n+2} + 2z_{n+1} - z_1 - z_n \\
x_1 &= x_0 - 2z_0 + \sum_{j=1}^{n-2} \left((-1)^j 2z_{n-j}\right) - z_{2n-2k} + \sum_{j=1}^{n-3} \left((-1)^{j+1} 2z_{2n-j}\right) + z_{2n} \\
&\quad + 2z_{n+1} - z_{n+1} - z_{2n}.
\end{aligned}
$$

Thus we get

$$
\begin{aligned}
x_n &= x_1 + 2z_{2n} - z_1 - z_n \\
x_n &= x_0 - 2z_0 + \sum_{j=1}^{n-2} \left((-1)^j 2z_{n-j}\right) - z_{2n-2k} + \sum_{j=1}^{n-3} \left((-1)^{j+1} 2z_{2n-j}\right) + z_{2n} \\
&\quad + \ 2z_{n+1} - z_{n+1} - z_{2n} \qquad + \ 2z_{2n} - z_1 - z_n \\
x_n &= x_0 - 2z_0 + \sum_{j=1}^{n-1} \left((-1)^j 2z_{n-j}\right) - z_n + \sum_{j=1}^{n-1} \left((-1)^{j+1} 2z_{2n-j}\right) + 3z_{2n}.
\end{aligned}
$$

The generator eliminated immediately before x_n from G_2 is x_{2n-1}. We can substitute x_{2n-1} into equation $n-1$ to obtain

$$
\begin{aligned}
x_{2n-1} - x_n &= 2z_{n-1} - z_n - z_{2n-1} \\
x_n &= x_{2n-1} - 2z_{n-1} + z_n + z_{2n-1} \\
x_n &= x_0 - 2z_0 + \sum_{j=1}^{n-2} \left((-1)^j 2z_j\right) + \sum_{j=1}^{n-1} \left((-1)^j 2z_{n+j-1}\right) + z_{2n-1} + 3z_{2n} \\
&\quad - \ 2z_{n-1} + z_n + z_{2n-1} \\
x_n &= x_0 - 2z_0 + \sum_{j=1}^{n-1} \left((-1)^j 2z_j\right) + z_n + \sum_{j=1}^{n} \left((-1)^j 2z_{n+j-1}\right) + 3z_{2n}.
\end{aligned}
$$

These are equivalent expressions for x_n as in both cases

$$
x_n = x_0 - 2z_0 - 2z_1 + 2z_2 - \cdots - 2z_{n-1} - z_n + 2z_{n+1} - 2z_{n-2} + \cdots + 2z_{2n-1} + 3z_{2n}.
$$

Consider equation 0.

$$
0\text{:}\ x_0 + x_n + x_{n+1} + \cdots + x_{2n-1} = 2z_0 + z_1 + \cdots + z_{n-1}.
$$

We want to substitute our expressions for $x_n, x_{n+1}, \ldots, x_{2n-1}$ in terms of $x_0, z_0, z_1, \ldots, z_{2n}$ into equation 0. For the left hand side of this equation, we create a table of coefficients, table 4.1, for the generators $x_0, z_0, z_1, \ldots, z_{2n}$. We also make note of the occurrences in the expressions for which of the eliminated generators they appear in.

Generator	Appears in term for	Coefficient in LHS of eq.0
x_0	$x_0, x_n, \ldots, x_{2n-1}$	$n+1$
z_0	$x_n, \ldots, x_{2n-1}$	$-2n$
z_1	$x_n, x_{n+3}, x_{n+5}, \ldots, x_{2n-1}$	$(-2)\times 1 + (-2)\times(\frac{n}{2}-1) = -n$
z_2	$x_n, x_{n+2}, x_{n+3}, x_{n+5}, \ldots, x_{2n-1}$	$(2)\times 2 + (2)\times(\frac{n}{2}-1) = n+2$
z_1	$x_n, x_{n+2}, x_{n+5}, \ldots, x_{2n-1}$	$(-2)\times 2 + (-2)\times(\frac{n}{2}-2) = -n$
z_2	$x_n, x_{n+2}, x_{n+4}, x_{n+5}, \ldots, x_{2n-1}$	$(2)\times 3 + (2)\times(\frac{n}{2}-2) = n+2$
$\vdots$	$\vdots$	$\vdots$
z_{n-1}	$x_n, x_{n+2}, \ldots, x_{2n-2}$	$(-2)\times(\frac{n}{2}) + (-2) = -n$
z_n	$x_n, x_{n+1}, x_{n+3}, \ldots, x_{2n-1}$	$(-1)\times 1 + (-2)\times(\frac{n}{2}) = -(n+1)$
z_{n+1}	$x_n, x_{n+2}, x_{n+3}, \ldots, x_{2n-1}$	$(2)\times 2 + (1)\times 1 + (2)\times(\frac{n}{2}-2) = n+1$
z_{n+2}	$x_n, x_{n+2}, x_{n+3}, \ldots, x_{2n-1}$	$(-1)\times 1 + (2)\times 2 + (2)\times(\frac{n}{2}-2) = -(n+1)$
$\vdots$	$\vdots$	$\vdots$
z_{2n-1}	$x_n, x_{n+2}, \ldots, x_{2n-2}, x_{2n-1}$	$(2)\times\frac{n}{2} + (1)\times 1 = n+1$
z_{2n}	$x_n, \ldots, x_{2n-2}, x_{n+1}, \ldots, x_{2n-1}$	$(3)\times 1 + (1)\times\frac{n}{2} - 1 + (3)\times\frac{n}{2} = 2(n+1)$

Table 4.1: Coefficients and occurrences of generators in LHS of equation 0

We use this table that all of these substitutions reduce equation 0 to

$$\begin{aligned} 2z_0 + z_1 + \cdots + z_{n-1} = (n+1)x_0 - 2nz_0 - nz_1 + (n+2)z_2 - \cdots \\ \cdots - nz_{n-1} - (n+1)z_n + (n+1)z_{n+1} - \cdots \\ \cdots - (n+1)z_{2n-2} + (n+1)z_{2n-1} + 2(n+1)z_{2n}. \end{aligned}$$

We can rearrange the equation to give

$$\begin{aligned} 0 = (n+1)x_0 - 2(n+1)z_0 - (n+1)z_1 + (n+1)z_2 - \cdots \\ \cdots - (n+1)z_{n-1} - (n+1)z_n + (n+1)z_{n+1} - \cdots \\ \cdots - (n+1)z_{2n-2} + (n+1)z_{2n-1} + 2(n+1)z_{2n}. \end{aligned}$$

There is now a common factor of $n+1$ in every coefficient in this expression. This implies that there is a generator whose order divides $n+1$. It is not yet clear that

We will take a step back and recall that Lemma 4.4.9 showed us that $S = \{x_1, \ldots, x_{2n}, x_0, z_0, \ldots, z_{2n}\}$ is a finite generating set for $F^{ab}_{\beta_n}$, and is a set of size $4n+2$. We define $\phi : \mathbb{Z}^{4n+2} \to F^{ab}_{\beta_n}$, an onto homomorphism where

$$\phi(i_1, i_2, \ldots, i_{2n}, i_{2n+1}, i_{2n+2}, \ldots, i_{4n+2}) = i_1x_1 + \cdots + i_{2n}x_{2n} + i_{2n+1}x_0 + i_{2n+2}z_0 + \cdots + i_{4n+2}z_{2n}.$$

Consider the kernel of ϕ, $Ker\phi$. From Lemma 4.4.10, we that for some $P_j \in \mathbb{Z}[X_1, \ldots, X_{2n}]$, there are $2n$ linear sums of generators which equate to the additive identity.

$$\begin{aligned}
0 =& x_1 - x_0 + 2z_0 + P_1(z_1, \ldots, z_{2n}) \\
0 =& x_2 - x_0 + 2z_0 + P_2(z_1, \ldots, z_{2n}) \\
&\vdots \qquad\qquad\qquad \vdots \\
0 =& x_{2n-1} - x_0 + 2z_0 + P_{2n-1}(z_1, \ldots, z_{2n}) \\
0 =& x_{2n} - x_0 + 2z_0 + P_{2n}(z_1, \ldots, z_{2n}).
\end{aligned}$$

We also have the following linear sum,

$$\begin{aligned}
0 = (n+1)x_0 - 2(n+1)z_0 - (n+1)z_1 + (n+1)z_2 - \cdots \\
\cdots - (n+1)z_{n-1} - (n+1)z_n + (n+1)z_{n+1} - \cdots \\
\cdots - (n+1)z_{2n-2} + (n+1)z_{2n-1} + 2(n+1)z_{2n}.
\end{aligned}$$

These sums form the basis for $Ker\phi$. Let $A \in M_{4n+2}(\mathbb{Z})$ be the $(2n+2) \times (4n+2)$ integer matrix representing $Ker\phi$, with respect to the ordering given in basis S. Then the first $2n+2$ columns of A resemble the matrix shown below.

$$A = \begin{pmatrix}
1 & 0 & 0 & \ldots & 0 & 0 & -1 & 2 & \ldots \\
0 & 1 & 0 & \ldots & 0 & 0 & -1 & 2 & \ldots \\
0 & 0 & 1 & \ldots & 0 & 0 & -1 & 2 & \ldots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \\
0 & 0 & 0 & \ldots & 1 & 0 & -1 & 2 & \ldots \\
0 & 0 & 0 & \ldots & 0 & 1 & -1 & 2 & \ldots \\
0 & 0 & 0 & \ldots & 0 & 0 & (n+1) & -2(n+1) & \ldots \\
0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots
\end{pmatrix}$$

Note that for each non-zero row, the first non-zero entry will divide all other non-zero entries within the row. Therefore, by performing column operations this matrix can be reduced to the following diagonal matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & n+1 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

This matrix can also be written as

$$Diag(\underbrace{1, \dots, 1}_{2n}, n+1, \underbrace{0, \dots, 0}_{2n+1})$$

This matrix has been reduced to the Smith normal form, and thus we can use a variant of the classification of finitely generated modules over PIDs [24]. We will use this to show that

$$Ker\phi \cong \underbrace{\mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{2n} \oplus (n+1)\mathbb{Z}.$$

We also know that $\phi : \mathbb{Z}^{4n+2} \to F^{ab}_{\beta_n}$ is a surjective homomorphism, so we are able to use the first isomorphism theorem.

$$F^{ab}_{\beta_n} \cong \mathbb{Z}^{4n+2} \big/ Ker\phi \cong \mathbb{Z}^{4n+2-2n-1} \oplus \mathbb{Z}\big/(n+1)\mathbb{Z} \cong \mathbb{Z}^{2n+1} \oplus \mathbb{Z}\big/(n+1)\mathbb{Z}$$

□

We have shown that there exists Thompson-like Bieri-Strebel groups with arbitrarily large torsion in their abelianisations. We will offer up the following two conjectures.

Conjecture 4.4.12. Let $n \in \mathbb{N}$. Then

$$F^{ab}_{\beta_n} \cong \mathbb{Z}^{2n+1} \oplus \mathbb{Z}\big/(n+1)\mathbb{Z}.$$

An example of this is the $(2,1)$ group which has been shown to contain 2-torsion in the abelianisation.

Conjecture 4.4.13. Let β be the unique positive real zero of the irreducible Pisot polynomial $f_\beta = X^2 - a_1 X - a_0$. Then

$$F_\beta^{ab} \cong \mathbb{Z}^{a_1+a_0} \oplus \mathbb{Z}/(a_0+1)\mathbb{Z}.$$

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