

From matrix pivots to graphs in surfaces: exploring combinatorics through partial duals

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Abstract

To what extent is a graph determined by the trees in it? What changes if we ask this question not for graphs in the abstract, but graphs that are embedded on surfaces? By considering these questions we will see how a collection of seemingly disjoint topics in mathematics are brought together through the idea of a partial dual.

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1 Introduction

Consider two graphs \mathbb{G} and \mathbb{H} each of which is drawn on a plane so that its edges do not intersect (or consider two spherical polyhedra if you prefer). Then \mathbb{G} and \mathbb{H} are *geometric duals* if the vertices in one correspond to the faces in the other, and the edges between vertices in one correspond to the edges between faces in the other. (See Figure 2 for an example.)

Now consider two graphs G and H (not drawn on the plane this time). Each contains a set of *spanning trees*, these are the maximal acyclic subgraphs contained in them. Then G and H are *algebraic duals* if their sets of spanning trees correspond through complementation (i.e., the edge set of a spanning tree of one is the complement of the edge set of a spanning tree of the other).

It is a classical result of H. Whitney that a graph has an algebraic dual if and only if it can be drawn on the plane without its edges crossing, in which case the algebraic dual is exactly a geometric dual. This sets up a fundamental relationship between planarity, duality and spanning trees.

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But what happens if the graphs cannot be drawn on the plane in this way? It is this situation we examine here. We shall see that it is inexorably linked to graphs drawn on surfaces, duals and partial duals, matroids and delta-matroids, principal pivot transforms of matrices, and pivot-minors of simple graphs.

This exposition is aimed at a general mathematical reader. A familiarity with elementary graph theory and with orientable surfaces is assumed. We note that graphs here may have *multiple edges* (edges that have the same ends) and *loops* (an edge with both ends being the same vertex). For simplicity we shall only consider orientable surfaces, but (almost) everything here can be extended to non-orientable surfaces.

2 Graphs and their spanning trees

We start with a classical question with well-known answer. Recall that a graph is a *tree* if it is connected and contains no cycles. A *spanning tree* of a graph G is a subgraph that is a tree and that contains every vertex of G . (For example, the bold edges in the left and right images in Figure 2 define spanning trees.) Only connected graphs have spanning trees, and to simplify terminology here we shall generally restrict ourselves to connected graphs. This restriction does not result in any real loss of generality. This is since most of our results extend trivially and obviously to non-connected graphs by considering the *maximal spanning forests* of a graph, which are the subgraphs that restrict to a spanning tree in each connected component.

Our initial interest is in the question:

Is a graph determined by its spanning trees?

There are a few ways to interpret this question resulting in different answers. Here we are interested in what happens if the only information you have about any given spanning tree is the edges that are in it. But since loops will never appear in a spanning tree, we will also need to know if there are any loops. So our precise question is: *If you know the edge set of each spanning tree of a connected graph as well as any loops in the graph, do you then know the graph?* It is not hard to see that the answer is no. For example consider the two non-isomorphic trees on three edges. But this “no” is really a “more or less, yes”.

Consider the moves of *vertex identification*, *vertex cleaving* and *Whitney twisting* illustrated in Figure 1. Vertex identification is a move that identifies two vertices that lie in different connected components of a graph, and vertex cleaving is the inverse operation. For Whitney twisting, suppose u_1 and v_1 are vertices in a graph G_1 , and u_2 and v_2 are vertices in a graph G_2 . Construct a graph G by identifying u_1 and u_2 , and v_1 and v_2 . Construct also a graph G' by identifying u_1 and v_2 , and v_1 and u_2 . Then we say G and G' are related by Whitney twists. Two graphs are said to be *2-isomorphic* if one can be obtained from the other through isomorphism, vertex identification, vertex cleaving and Whitney twisting.

Whitney’s *2-Isomorphism Theorem* [59] provides an answer to our question. It states that if you know the edge set of each spanning tree of a graph as well as any loops in

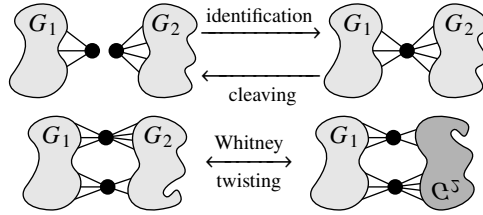


Figure 1: The moves for 2-isomorphism: vertex identification, vertex cleaving and Whitney twisting.

the graph, then you know the graph up to 2-isomorphism. Conversely, the collections of edge sets of spanning trees and loops in two 2-isomorphic graphs are equal. (We shall give a cleaner statement of Whitney's 2-Isomorphism Theorem below.)

Thus the spanning tree structure determines the graph up to some simple moves. In particular, it completely determines 3-connected graphs (ones in which there are three internally disjoint paths between each pair of vertices) up to isomorphism as the moves cannot be applied to such graphs. It turns out that many graph properties and results do not distinguish between 2-isomorphic graphs, and so can be understood in terms of spanning tree structure. In fact, considering the spanning tree structure of a graph, rather than the graph itself, turns out to be an extremely fruitful thing to do.

The spanning trees in a connected graph G have many nice standard properties. For example, every non-loop edge of G is in some spanning tree; all spanning trees have the same number of edges; and if G has n vertices, a subgraph is a spanning tree if and only if it has exactly $n - 1$ edges. Spanning trees also satisfy an *exchange property*: if T and T' are spanning trees and e is an edge in T but not T' , then there is always some edge f in T' but not T such that removing e from T then adding f results in another spanning tree. (A reader may spot that this exchange property also applies to the bases of a vector space.) These properties on the collection of spanning trees lead us to *matroids*.

Definition 2.1. Let E be a finite set, and \mathcal{B} be a non-empty collection of subsets of E . Then the pair $M := (E, \mathcal{B})$ is called a *matroid* if for distinct $A, B \in \mathcal{B}$ and for all $a \in A \setminus B$ there exists $b \in B \setminus A$ such that $(A \setminus a) \cup b \in \mathcal{B}$.

By the properties of trees mentioned above, if G is a connected graph with edge set E and \mathcal{B} is the set consisting of all edge sets of its spanning trees, then $C(G) := (E, \mathcal{B})$ is a matroid. It is called the *cycle matroid* of G .

Example 2.2. The graph on the left of Figure 2 has cycle matroid (E, \mathcal{B}) with $E = \{1, 2, 3, 4, 5, 6, 7\}$ and $\mathcal{B} = \{\{1, 2, 3, 5\}, \{1, 2, 4, 5\}, \{1, 3, 4, 5\}, \{2, 3, 4, 5\}, \{1, 3, 5, 7\}, \{1, 4, 5, 7\}, \{2, 3, 5, 7\}, \{2, 4, 5, 7\}\}$.

Our initial question of whether the spanning trees determine the graph then becomes a matroid theoretic one: *if you have a cycle matroid, can you determine the graph it came*

from? We can rephrase our previous answer (for the statement, matroid isomorphism is defined in the obvious way):

Theorem 2.3 (Whitney’s 2-Isomorphism Theorem). *Let G and H be connected graphs. Then $C(G)$ and $C(H)$ are isomorphic matroids if and only if G and H are 2-isomorphic.*

Whitney’s 2-Isomorphism Theorem nails down the connection between cycle matroids and graphs. Cycle matroids give rise to a class of matroids, but almost all matroids are *not* cycle matroids [43]. Nevertheless, cycle matroids are important in matroid theory and graph theory. On one hand, insights from matroid theory can lead to new results about graphs. On the other hand, graph theory can serve as an excellent guide for studying matroids. A good introduction to the mutually enriching relationship between graph theory and matroid theory can be found in [45].

Bibliographic remarks. The topics discussed in this section are classical. An excellent resource for this material is Chapter 5 of J. Oxley’s book [44]. Whitney’s 2-Isomorphism Theorem dates from the 1930’s and is due to H. Whitney, [59] (see also [52, 56]) and Theorem 2.3 is a modern formulation in terms of matroids.

Our motivational question was whether a graph is determined by its spanning trees or its cycle matroid. We restrict discussion here to characterising graphs that have the same cycle matroid, ignoring the algorithmic question about constructing the graphs from the cycle matroid. Discussion of the latter problem can be found in [55] (for what will follow, the equivalent problem for quasi-trees can be answered through the circle graph recognition methods of [32, 36, 48]).

H. Whitney introduced matroids in the 1930’s (see [60]) to capture ideas of dependence common to linear algebra and graph theory. There are many ways to define matroids and Definition 2.1 provides their definition in terms of “bases”. The cycle matroid $C(G)$ can also be defined through the cycles in a graph (using a “circuit definition” of a matroid), hence the name. Matroid theory is a major topic of study in combinatorics. Our encounter with matroids here is extremely brief and we refer the reader to the books [44, 57] for more on them.

A spectacular illustration of the mutually enriching relationship between graph theory and matroid theory can be found in J. Geelen, B. Gerards and G. Whittle’s recent and, at the time of writing, unpublished result that, for any finite field, the class of matroids that are representable over that field is well-quasi-ordered by the minor relation. Their results generalise N. Robertson and P. Seymour’s Graph Minors Project where it is shown that graphs are well-quasi-ordered by the minor relation [47]. In [35] Geelen, Gerards and Whittle wrote “it would be inconceivable to prove a structure theorem for matroids without the Graph Minors Structure Theorem as a guide”.

3 The appearance of topology

We want to make contact with topological graph theory, which is the study of graphs embedded in surfaces. We shall do this by considering duals. Suppose $M = (E, \mathcal{B})$ is

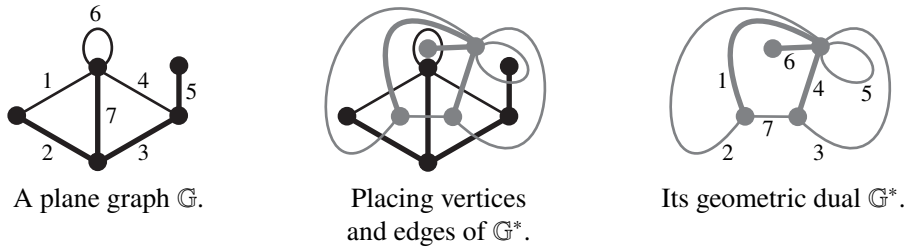


Figure 2: Forming the geometric dual \mathbb{G}^* of a plane graph \mathbb{G} .

a matroid. Define a collection of sets \mathcal{B}^* by taking the complement of each member of \mathcal{B} , so $\mathcal{B}^* := \{E \setminus B : B \in \mathcal{B}\}$. It is not hard to check that the pair (E, \mathcal{B}^*) also forms a matroid. This is called the *dual* of M and is denoted by M^* .

Example 3.1. The dual of the matroid in Example 2.2 has $\mathcal{B}^* = \{\{4, 6, 7\}, \{3, 6, 7\}, \{2, 6, 7\}, \{1, 6, 7\}, \{2, 4, 6\}, \{2, 3, 6\}, \{1, 4, 6\}, \{1, 3, 6\}\}$.

If G is a graph and $C(G)$ its cycle matroid, then the dual matroid $C(G)^*$ is always a matroid. However, it is not always the cycle matroid of a graph. If $C(G) = (E, \mathcal{B})$, the graph G is connected, and $C(G)^* = (E, \mathcal{B}^*)$, then \mathcal{B} consists of the edge sets of all the spanning trees of G . For $C(G)^*$ to be the cycle matroid of a graph we require the existence of some graph H on the edge set E such that the sets in \mathcal{B}^* define exactly the spanning trees of H . That is, we require H to have the property that T is a spanning tree of G if and only if $E \setminus T$ is a spanning tree of H . Such a graph H , if it exists, is called an *algebraic dual* (or *abstract dual* or *combinatorial dual*) of G . If it does exist, it may or may not be unique.

The existence of algebraic duals is tied to the topological properties of a graph. A connected *plane graph* consists of a connected graph drawn, or *embedded*, in the sphere (or, equivalently, the plane) in such a way that vertices are distinct points and edges only intersect at their ends. (So each vertex is a point on the sphere, each edge is simple curve between these points, and these curves do not intersect except when their ends share a vertex.) Plane graphs are *equivalent* if there is a homeomorphism of the sphere taking one graph drawing to the other (i.e., inducing a graph isomorphism). A plane graph divides the sphere into regions called *faces*. For example, with the page representing a portion of the sphere, the left-hand image of Figure 2 shows a plane graph with four faces. A connected graph is said to be *planar* if can be drawn in the sphere in the above way. (So a plane graph *is* drawn on the sphere, and a planar graph *can* be drawn on the sphere.) Inequivalent plane graphs can be drawings of the same planar graph. These definitions are extended to non-connected graphs by drawing each graph component in its own copy of the sphere.

Plane graphs have another type of dual. If \mathbb{G} is a plane graph then its *geometric dual*, denoted \mathbb{G}^* , is the plane graph obtained from \mathbb{G} by placing one vertex in each

of its faces, and embedding an edge of \mathbb{G}^* between two of these vertices whenever the faces of \mathbb{G} they lie in meet at an edge. Edges of \mathbb{G}^* are embedded so that they cross only the corresponding edge of \mathbb{G} . An example is given in Figure 2.

For a plane graph $\mathbb{G} = (V, E)$, Euler's Formula gives that $|V| - |E| + |F| = 2$, where $|F|$ is the number of faces. Thus if A is the edge set of a spanning tree in \mathbb{G} then $|V| - |A| = 1$ and so $|F| - |E \setminus A| = 1$ giving that $E \setminus A$ is the edge set of a spanning tree of \mathbb{G}^* . As $(\mathbb{G}^*)^* = \mathbb{G}$ it follows that geometric duals of plane graphs are algebraic duals, and so for plane graphs $C(\mathbb{G})^* = C(\mathbb{G}^*)$.

The converse is also true: if G and H are algebraic duals then the correspondence between their spanning tree structures guarantees there are plane graph \mathbb{G} and \mathbb{H} that are embeddings (i.e., drawings) of G and H that are geometric duals, $\mathbb{H} = \mathbb{G}^*$. Collecting all this together gives the following result of Whitney [58].

Theorem 3.2. *Let G be a connected graph with cycle matroid $C(G)$. Then the dual matroid $C(G)^*$ is the cycle matroid of a graph if and only if G is planar. Moreover, if G is planar then*

$$C(\mathbb{G})^* = C(\mathbb{G}^*),$$

where \mathbb{G} is any plane embedding of G , and \mathbb{G}^* its geometric dual.

In this theorem we see how the spanning tree (or cycle matroid) structure of a graph captures its topological properties. However, Theorem 3.2 illustrates that many of these properties are tied to planarity. What if you do not want to restrict yourself to plane or planar graphs? Let us examine what changes when you consider graphs on surfaces other than the plane.

As noted above, for expositional simplicity we shall only consider orientable surfaces. However (almost) everything here extends to non-orientable surfaces (with varying degrees of difficulty) and details of how to do this can be found in the references. We will often omit the word “orientable”, although we shall add it when it is crucial. We recall that the Classification of Surfaces states that every closed orientable surface is homeomorphic to a sphere with handles (or n -torus). Every orientable surface with boundary is homeomorphic to a sphere with handles with the interiors of some discs removed from it. In both cases the number of handles is its *genus*.

An *embedded graph* \mathbb{G} is a graph drawn on a closed surface Σ in such a way that edges only intersect at their ends, and the drawing divides Σ into regions that are homeomorphic to discs. (As in the plane case, each vertex is a point on the surface, each edge is simple curve between these points, and these curves do not intersect except when their ends share a vertex.) The regions of Σ determined by the graph drawing are called *faces* of \mathbb{G} . Thus a plane graph is a graph embedded in the sphere. We note that if \mathbb{G} has more than one component, then each component of the graph lies in its own surface. Figure 3a shows a graph embedded in a torus. It has two faces.

The *geometric dual* \mathbb{G}^* of an embedded graph \mathbb{G} is formed just as in the plane case by placing vertices in the faces and drawing edges between these vertices when the faces meet at an edge. Note that \mathbb{G} and \mathbb{G}^* are embedded in the same surface.

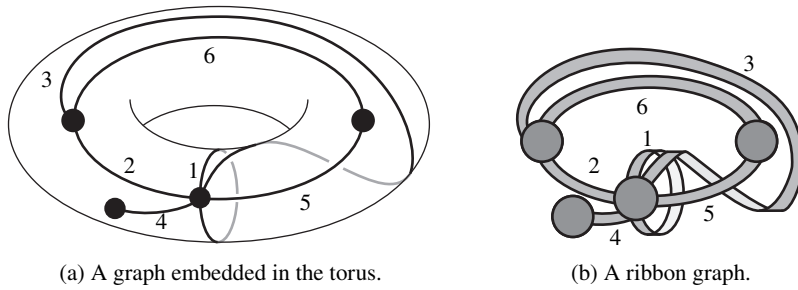


Figure 3: Realisations of the same embedded graph.

Suppose \mathbb{G} is a connected embedded graph and \mathbb{G}^* its geometric dual. Since the edge sets of \mathbb{G} and \mathbb{G}^* correspond, we may assume that a graph and its geometric dual have the same edge set E . The operation $*$: $A \mapsto E \setminus A$ sends edge sets of \mathbb{G} to edge sets of \mathbb{G}^* , or equivalently the set of spanning subgraphs of \mathbb{G} to the set of spanning subgraphs of \mathbb{G}^* .² (As an example, the bold edges in Figure 2 indicate a pair of spanning trees identified under this map.) Theorem 3.2 and the characterisation of planar graphs in terms of algebraic duals depend upon the fact that if \mathbb{G} (and so \mathbb{G}^*) is a plane graph, then $*$ sends spanning trees to spanning trees, and this happens if and only if \mathbb{G} is a plane graph.

If \mathbb{G} is embedded in an arbitrary closed surface Σ and A is the edge set of one of its spanning trees \mathbb{T} . Let $*(\mathbb{T})$ be the spanning subgraph of \mathbb{G}^* on the edge set $*(A)$. Then it is easy to see (e.g., by drawing a picture; as an example consider the bold edges in the middle image of Figure 2) that Σ can be written as the union of a neighbourhood of \mathbb{T} and a neighbourhood of $*(\mathbb{T})$. Since \mathbb{T} is a spanning tree its neighbourhood is a disc. Thus the neighbourhood of $*(\mathbb{T})$ consist of a once-punctured copy of Σ . In particular, it is a subgraph whose neighbourhood has exactly one boundary component. This is the property that is important to us.

A spanning subgraph of an embedded graph \mathbb{G} is said to be a *spanning quasi-tree* if its neighbourhood has exactly one boundary component. Notice that every spanning tree is a spanning quasi-tree, although in general an embedded graph will have many other spanning quasi-trees. The *genus* of a quasi-tree is the genus of its neighbourhood considered as a surface with boundary. (We shall reformulate these definitions in the next section.) If \mathbb{G} is in a surface Σ of genus n , then it will have spanning quasi-trees of genus $0, 1, 2, \dots, n$, and the spanning trees are just those of genus zero. The map

²At this point we are glossing over the issue of exactly how a subgraph of \mathbb{G} should be considered as an embedded graph. The difficulty is that restricting the drawing of \mathbb{G} to the edges and vertices in the subgraph may result in faces that are not discs, in which case the surface will need to be altered, by removing any redundant handles, to obtain an embedded graph. This issue will be resolved in the next section by switching to the language of ribbon graphs. For the present discussion it is safe, although not quite correct, to think of restricting the drawing of \mathbb{G} to the edges and vertices in the subgraph.

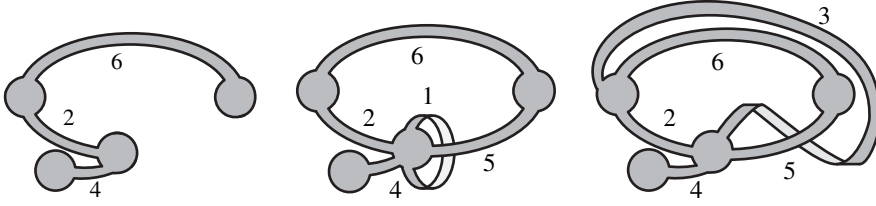


Figure 4: Neighbourhoods of the subgraphs on $\{2, 4, 6\}$, $\{1, 2, 4, 5, 6\}$, and $\{2, 3, 4, 5, 6\}$.

* then sends a tree to a quasi-tree of maximal genus n . More generally, * will send a spanning quasi-tree of genus g to a spanning quasi-tree of genus $n - g$.

Example 3.3. For the embedded graph shown in Figure 3a, each of the sets $\{2, 4, 6\}$, $\{1, 2, 4, 5, 6\}$, and $\{2, 3, 4, 5, 6\}$ induces a spanning quasi-tree. The neighbourhoods are shown in Figure 4. The set $\{2, 4, 6\}$ defines a spanning quasi-tree of genus zero, and the other two sets induce spanning quasi-trees of genus one.

We started with the question of whether the spanning trees in a graph determine the graph itself. Whitney's Theorem provided a complete answer to this question, and Theorem 3.2 tied together duality, spanning tree structure and planarity. If instead we want to work with non-plane embedded graphs, rather than looking at spanning trees, we should consider quasi-trees. Thus we are led to ask:

Is an embedded graph determined by its spanning quasi-trees?

Just as in the spanning trees case we formalise this by asking: *If you know the edge set of each spanning quasi-tree of ribbon graph, as well as any edges that appear in no quasi-trees, then do you then know the ribbon graph?*

Again the immediate answer is no. For example, if \mathbb{G} is a plane ribbon graph then its set of spanning quasi-trees is exactly its set of spanning trees, and we already know that these do not necessarily determine a plane embedding. In this plane case, however, Whitney's 2-Isomorphism Theorem will provide a way to characterise all plane ribbon graphs that have the same set of spanning quasi-trees. What if \mathbb{G} is non-plane? In this case Whitney's 2-Isomorphism Theorem does not help.

Bibliographic remarks. Dual matroids date back to H. Whitney's foundational work on matroids [60]. The construction of a geometric dual is classical and can be seen in J. Kepler's work on dual polyhedra (see p. 181 of his *Harmonices mundi* dating from 1619). Algebraic duals, as well as their connection with planarity and geometric duals, are due to H. Whitney [58]. Theorem 3.2 provides a modern statement of his results.

Embedded graphs are standard objects in graph theory. They have several alternative names and formulations including *combinatorial maps*, *rotation systems*, *ribbon*

graphs, graph encoded maps, and so on. Excellent introductions to embedded graphs and topological graph theory are J. Gross and T. Tucker's [37], and B. Mohar and C. Thomassen's [42].

4 Partial duals

Duality tied spanning tree structure to planarity. For non-plane embedded graphs and quasi-trees we consider a generalisation of geometric duality called *partial duality*. For our discussion of partial duals, it is convenient to describe embedded graphs as *ribbon graphs*.

A ribbon graph is a structure that arises by taking a regular neighbourhood of a graph embedded in a surface, but without throwing away the vertex–edge structure of the graph. See Figure 3. We can think of them informally as “graphs whose vertices consist of discs, and whose edges consist of ribbons”, as in Figure 3b. They can be defined formally as follows.

Definition 4.1. A ribbon graph $\mathbb{G} = (V, E)$ is a surface with boundary represented as the union of two sets of discs, a set V of *vertices*, and a set E of *edges* such that: (1) the vertices and edges intersect in disjoint line segments; (2) each such line segment lies on the boundary of precisely one vertex and precisely one edge; (3) every edge contains exactly two such line segments.

Ribbon graphs are equivalent to embedded graphs. Above we described how a ribbon graph can be obtained from an embedded graph. In the other direction, given a ribbon graph, the classification of surfaces with boundary ensures there is a unique way (up to homeomorphism) to embed it in a closed surface by ‘filling in the holes’. This gives an embedding of the ribbon graph in a closed surface from which it is clear how to obtain the embedded graph. Two ribbon graphs are *equivalent* if there is a homeomorphism from one to the other that sends vertices to vertices and edges to edges. Thus ribbon graphs are equivalent precisely when their corresponding embedded graphs are. Thus any result about ribbon graphs is a result about embedded graphs, and vice versa.

Graph theory terminology is extended to ribbon graphs in the obvious way. A *ribbon subgraph* \mathbb{H} of \mathbb{G} is a ribbon graph obtained from \mathbb{G} by removing some of its vertices and edges. It is *spanning* if it has the same vertices as \mathbb{G} . The spanning ribbon subgraph obtained from \mathbb{G} by deleting an edge e is denoted by $\mathbb{G} \setminus e$. Ribbon graphs have topological parameters in addition to their graph theoretic ones. Here we defined ribbon graphs to be *orientable* meaning that they are orientable when considered as a surface with boundary. (Recall for expository simplicity we restricted ourselves to orientable surfaces, and therefore to orientable ribbon graphs.) In general ribbon graphs may be *non-orientable* as well, and at times we will comment on this case. The *genus* of a ribbon graph is its genus as a surface. A connected ribbon graph is *plane* if it has genus 0 (i.e., if it corresponds to a graph on a sphere). We are often interested

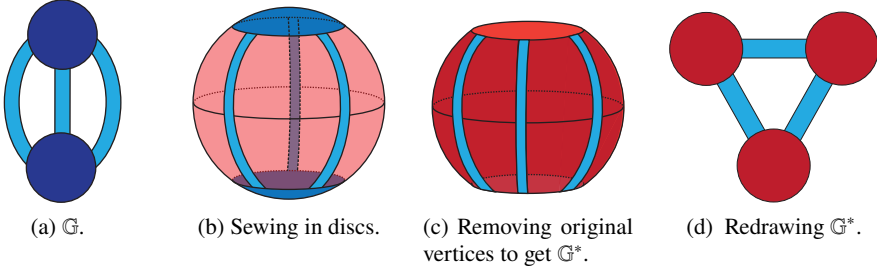


Figure 5: Forming the geometric dual of a ribbon graph.

in the *boundary components* of a ribbon graph, which are just the components of its boundary when it is considered as a surface with boundary. A ribbon graph that has exactly one vertex is called a *bouquet*. These form an important class of ribbon graphs.

Geometric duality has a very neat description in terms of ribbon graphs. If $\mathbb{G} = (V, E)$ is a ribbon graph then its *geometric dual* \mathbb{G}^* is the ribbon graph formed by taking one disc for each boundary component of \mathbb{G} (these will form the vertices of the dual); for each boundary component of \mathbb{G} (which is topologically a circle), identify it with the boundary of one of these discs (resulting in a surface without boundary); finally, in the resulting surface, delete the interiors of the vertex discs in V . This results in the ribbon graph \mathbb{G}^* . The discs that were added during the construction form the vertices of \mathbb{G}^* , and the edges of \mathbb{G} form the edges of \mathbb{G}^* but the parts of their boundary that are and are not attached to vertices are switched. This construction is illustrated in Figure 5.

It is not too hard to see our two constructions for geometric duals agree. The construction of \mathbb{G}^* in terms of embedded graphs is a global construction in the sense that it applies to the whole of \mathbb{G} at the same time. However, once you have switched to the language of ribbon graphs, the construction is easily adapted to give a local construction, where local here means that you can form the geometric dual \mathbb{G}^* at individual edges. Then, with this local construction in hand, we can form the dual at just some of edges while leaving the remaining edges alone. This observation leads to the surprising idea of *partial duals*.

Partial duals arise by modifying the description of geometric duality for ribbon graphs so that the dual is formed with respect to only a subset of edges. Let $\mathbb{G} = (V, E)$ be a ribbon graph and $A \subseteq E$. The *partial dual* of \mathbb{G} with respect to A , denoted \mathbb{G}^A , is the ribbon graph formed as follows. Consider the spanning ribbon subgraph (V, A) as a subset of \mathbb{G} . The boundary of (V, A) defines a set of closed curves on \mathbb{G} . For each of these closed curves, take a disc (which will form a vertex of \mathbb{G}^A) and identify the curve and the boundary of this disc. Finally, delete the interior of each vertex disc in V . The resulting ribbon graph is \mathbb{G}^A . This construction is illustrated in Figure 6.

The following properties of partial duals follow directly from the definition: $\mathbb{G}^* =$

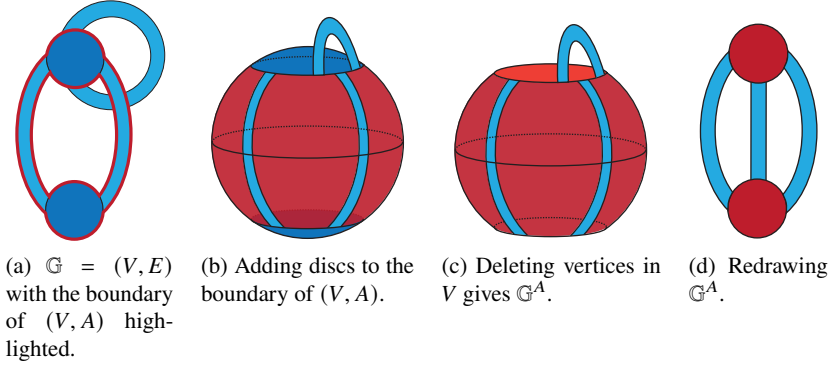


Figure 6: Forming a partial dual \mathbb{G}^A where A consist of the two non-loop edges of \mathbb{G} .

$\mathbb{G}^E(\mathbb{G})$; $\mathbb{G}^\emptyset = \mathbb{G}$; $(\mathbb{G}^A)^B = \mathbb{G}^{(A \cup B) \setminus (A \cap B)}$ and so partial duals can be formed one edge at a time; partial duality acts disjointly on the connected components of a ribbon graph; and \mathbb{G}^A is orientable if and only if \mathbb{G} is. Another useful fact for us is that if \mathbb{H} is a spanning ribbon subgraph of \mathbb{G} with exactly one boundary component (for example, if \mathbb{H} is a spanning tree) and A is the edge set of \mathbb{H} , then \mathbb{G}^A is a bouquet (i.e., has exactly one vertex). This is because the vertices of \mathbb{G}^A correspond to the boundary components of \mathbb{H} (just as the vertices of \mathbb{G}^* correspond to the boundary components of \mathbb{G}).

Bibliographic remarks. As with embedded graphs, ribbon graphs are standard objects in graph theory. They arise in several settings and under different names including *fat graphs*, *dessins d'Enfants*, and *reduced band decompositions*. However it should be remembered that they are just one of the many descriptions of embedded graphs. J. Ellis-Monaghan and I. Moffatt's book [31] offers an introduction to ribbon graphs and partial duals. Although we described partial duals in terms of ribbon graphs here, they can, of course, be described in other the models for embedded graphs. In particular their local nature is prominent when they are defined in the languages of arrow presentations [18], graph encoded maps [29], or permutation models [22].

Partial duality was introduced by S. Chmutov in [18] in order to reconcile the various results in [20, 22, 27] which constructed the Jones polynomial of a knot or link as an evaluation of the Bollobás–Riordan polynomial of a ribbon graph. The Bollobás–Riordan polynomial of [4, 5] is a graph polynomial that offers an analogue of the Tutte polynomial [53] for embedded graphs. The connections between ribbon graphs and knot theory extend Thistlethwaite's well-known connection [50] between the Tutte polynomial of a plane graph and the Jones polynomial of an alternating link; a connection that was integral to his proof of the Tait Conjectures. Chmutov used the term 'generalized duality' in his original paper. Its adopted name 'partial duality' was suggested to the author of the present article by D. Archdeacon and has been used in all subsequent papers. Partial duality has since entered topological graph theory as a topic of study in its own right and is proving to be a fundamental operation on embedded

graphs.

5 Ribbon graphs and their spanning quasi-trees

In the language of ribbon graphs, a *quasi-tree* is a ribbon graph that has exactly one boundary component. A ribbon subgraph \mathbb{H} is a *spanning quasi-tree* of \mathbb{G} if it is a quasi-tree that contains all of the vertices of \mathbb{G} . A ribbon graph of genus g has a spanning quasi-tree of genus $0, 1, \dots, g$, and its spanning trees are exactly its spanning quasi-trees of genus zero.

Recall from Section 2 that the set of spanning trees in a graph satisfies an exchange property: if T and T' are spanning trees and e is an edge in T but not T' , then there is always some edge f in T' but not T such that removing e from T then adding f results in another spanning tree. This exchange property does not hold for spanning quasi-trees in general.

However, spanning quasi-trees satisfy a more general *symmetric exchange property*. If \mathbb{H} and \mathbb{H}' are spanning quasi-trees and e is an edge that is in exactly one of \mathbb{H} or \mathbb{H}' , then there is always an edge f that is in exactly one of \mathbb{H}' or \mathbb{H} such that adding or removing each of e and f from \mathbb{H} results in a spanning quasi-tree. Proving that this symmetric exchange property holds does require a little work. A proof can be found in [23] or implicitly in [12], or see Figure 16 of [40] for a pictorial explanation. We shall return to this symmetric exchange property in the next section.

In Section 2 we used matroids to capture the spanning tree structure of a graph. A minor modification of the definition of a cycle matroid gives a way to similarly record the spanning quasi-trees in a ribbon graph.

Definition 5.1. Let $\mathbb{G} = (V, E)$ be a connected ribbon graph, and let

$$\mathcal{F} := \{F \subseteq E : F \text{ is the edge set of a spanning quasi-tree of } \mathbb{G}\}.$$

We call $D(\mathbb{G}) := (E, \mathcal{F})$ the *delta-matroid* of \mathbb{G} .

Example 5.2. Let \mathbb{G} be the ribbon graph of Figure 3b. Then $D(\mathbb{G}) = (E, \mathcal{F})$ where $E = \{1, 2, \dots, 6\}$ and $\mathcal{F} = \{\{2, 4, 5\}, \{2, 4, 6\}, \{3, 4, 5\}, \{3, 4, 6\}, \{4, 5, 6\}, \{1, 2, 3, 4, 5\}, \{1, 2, 3, 4, 6\}, \{1, 2, 4, 5, 6\}, \{2, 3, 4, 5, 6\}\}$.

Euler's Formula gives that if \mathbb{H} is an orientable quasi-tree with v vertices and e edges, then $(1 - v + e)/2$ gives the genus of \mathbb{H} (or half its genus if \mathbb{H} is non-orientable). As the spanning quasi-trees of \mathbb{G} have the same number of vertices, this relates the sizes of the sets in \mathcal{F} to the topology of the spanning quasi-trees. In particular, it follows that: every set in \mathcal{F} has the same parity (i.e., is of odd or even size) if and only if \mathbb{G} is orientable; the genus of \mathbb{G} is one half of the differences in sizes between the largest and smallest sets in \mathcal{F} ; and for \mathbb{G} connected, $D(\mathbb{G}) = C(\mathbb{G})$ if and only if \mathbb{G} is plane.

Rephrased in terms of ribbon graphs, the map $*$ from Section 3 sends a spanning ribbon subgraph (V, A) of $\mathbb{G} = (V, E)$ to the spanning ribbon subgraph $(V^*, E \setminus A)$ of

\mathbb{G}^* . Moreover, this map sends a spanning quasi-tree of genus g to a spanning quasi-tree of genus $n - g$ where n here is the genus of \mathbb{G} . Thus if $D(\mathbb{G}) = (E, \mathcal{F})$ and we define $\mathcal{F}^* := \{E \setminus F : F \in \mathcal{F}\}$, then for any ribbon graph \mathbb{G} we have that $D(\mathbb{G}^*) = (E, \mathcal{F}^*)$. The main insights for quasi-tree structure, however, come from partial duals rather than geometric duals.

Partial duality preserves the quasi-tree structure of a ribbon graph. Let $\mathbb{G} = (V, E)$ be a ribbon graph and $B \subseteq E$. We shall relate the quasi-trees of \mathbb{G} to those of its partial dual \mathbb{G}^B . For this recall that the *symmetric difference* $X \triangle Y$ of sets X and Y is $(X \cup Y) \setminus (X \cap Y)$. Then $A \subseteq E$ is the edge set of a quasi-tree of \mathbb{G} if and only if $A \triangle B$ is the edge set of a quasi-tree of \mathbb{G}^B . It is not hard to see why this is the case — essentially it follows from the observation that the boundary components of $\mathbb{G}^{\{e\}}$ and $\mathbb{G} \setminus e$ correspond. In terms of the delta-matroids, this means that if $D(\mathbb{G}) = (E, \mathcal{F})$ and we set $\mathcal{F}^B := \{B \triangle F : F \in \mathcal{F}\}$, then $D(\mathbb{G}^B) = (E, \mathcal{F}^B)$.

The significance of this result is that if we wish to study the spanning quasi-trees of \mathbb{G} , we may equivalently study the spanning quasi-trees of any of its partial duals \mathbb{G}^B . The partial duals of a ribbon graph can have quite different properties from each other and from the original ribbon graph. This means that we have some ability to choose the ribbon graphs to work with without losing any generality, something we did not have much scope to do when working with geometric duals alone. A specific instance of this principle, and one that we shall make much use of here, is that every ribbon graph has a partial dual that is a bouquet (i.e., a one-vertex ribbon graph). Thus we only ever need to consider the spanning quasi-tree structure of bouquets. But to make use of this, we need a better understanding of $D(\mathbb{G})$.

Bibliographic remarks. The definition and approach to the delta-matroids of ribbon graphs that we follow here is due to C. Chun, I. Moffatt, S. Noble, R. Rueckriemen [23, 24]. However, these delta-matroids are equivalent to A. Bouchet's delta-matroids of maps from [12]. There Bouchet associated a delta-matroid with the 4-regular medial graph of an embedded graph. The delta-matroid arises from its Eulerian circuits, and the Eulerian circuits correspond to the quasi-trees of the embedded graph. That $D(\mathbb{G})$ determines genus and orientability can be deduced from [12] through the correspondence ([23] gives the form stated here). The behaviour of $D(\mathbb{G})$ under partial duals is from [23].

6 Delta-matroids and quasi-tree structure

Recall from Section 3 that the dual of a matroid $M = (E, \mathcal{B})$ is $M^* = (E, \mathcal{B}^*)$ where $\mathcal{B}^* = \{E \setminus B : B \in \mathcal{B}\}$. We can write \mathcal{B}^* as $\{E \triangle B : B \in \mathcal{B}\}$, and, in light of the above, it becomes obvious that we can form a *partial dual* of a matroid by replacing E with the subset X of E . So we can define a partial dual of $M = (E, \mathcal{B})$ as $M^X := (X, \mathcal{B}^X)$, where, as above, $\mathcal{B}^X := \{X \triangle B : B \in \mathcal{B}\}$.

For example, if $M = (\{1, 2\}, \{\{1\}, \{2\}\})$ and $X = \{1\}$ then a partial dual is $M^X = (\{1, 2\}, \{\{\emptyset\}, \{1, 2\}\})$. The difficulty, as can be seen in this example, is that M^X

may no longer be a matroid. Instead its an example of a more general structure called a *delta-matroid*.

Definition 6.1. A *delta-matroid* D consists of a pair (E, \mathcal{F}) where E is a finite set and \mathcal{F} a non-empty collection of subsets of E . Furthermore, \mathcal{F} is required to satisfy the *Symmetric Exchange Axiom* which states that:

$$(\forall X, Y \in \mathcal{F}) (\forall u \in X \Delta Y) (\exists v \in X \Delta Y) (X \Delta \{u, v\} \in \mathcal{F}).$$

Since the collection of spanning quasi-trees of a ribbon graph \mathbb{G} satisfies the symmetric exchange property, it follows that $D(\mathbb{G})$, as introduced in Definition 5.1, is a delta-matroid (and so the name we gave $D(\mathbb{G})$ is an honest one). Not every delta-matroid arises in this way, as the following example shows. In fact, almost all delta-matroids do not come from ribbon graphs, although those that do play an important role.

Example 6.2. Let $E = \{1, 2, 3, 4\}$, $\mathcal{F} = \{\emptyset, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}$, and $\mathcal{F}' = \{\emptyset, \{1, 2\}, \{1, 4\}, \{2, 3\}, \{3, 4\}, \{1, 2, 3, 4\}\}$. Then (E, \mathcal{F}) and (E, \mathcal{F}') are both delta-matroids but neither is the delta-matroid of a ribbon graph. This can be verified by calculating the delta-matroids of the bouquets on four edges.

Matroids are also examples of delta-matroids: M is matroid if and only if it is a delta-matroid in which every member of \mathcal{F} has the same size. Most delta-matroids are not matroids though.

While the class of matroids is not closed under partial duals, the class of delta-matroids is. Let $D = (E, \mathcal{F})$ be a delta-matroid and $B \subseteq E$. The *partial dual* (or *twist*) D^B of D is defined as the pair (E, \mathcal{F}^B) where $\mathcal{F}^B := \{F \Delta B : F \in \mathcal{F}\}$. The *dual* D^* of D is D^E .

Example 6.3. If D is the delta-matroid from Example 5.2, then $D^{\{3,4\}} = (E, \mathcal{F}^{\{3,4\}})$ where $E = \{1, \dots, 6\}$ and $\mathcal{F}^{\{3,4\}} = \{\{2, 3, 5\}, \{2, 3, 6\}, \{5\}, \{6\}, \{3, 5, 6\}, \{1, 2, 5\}, \{1, 2, 6\}, \{1, 2, 3, 5, 6\}, \{2, 5, 6\}\}$.

Matroid duality captures the way that the spanning trees of a *plane* graph \mathbb{G} are transformed into the spanning trees of its geometric dual \mathbb{G}^* , giving the identity $C(\mathbb{G}^*) = C(\mathbb{G})^*$ for plane graphs. Delta-matroid duality captures that the spanning quasi-trees of *any* ribbon graph \mathbb{G} are transformed into the spanning quasi-trees of any partial dual \mathbb{G}^B . Indeed the following result follow from our previous discussion.

Theorem 6.4. Let \mathbb{G} be a connected ribbon graph. Then

1. $C(\mathbb{G}^*) = C(\mathbb{G})^*$ if and only if \mathbb{G} is a plane ribbon graph;
2. $D(\mathbb{G}^*) = D(\mathbb{G})^*$ for any ribbon graph \mathbb{G} ; and
3. $D(\mathbb{G}^B) = D(\mathbb{G})^B$ for any ribbon graph \mathbb{G} and any subset of its edges B .

Just as with ribbon graphs, we can use partial duality to transform a delta-matroid into one with desirable properties. A delta-matroid $D = (E, \mathcal{F})$ is said to be *normal* if $\emptyset \in \mathcal{F}$. Every delta-matroid has a normal partial dual: if $D = (E, \mathcal{F})$ and F is any element of \mathcal{F} , then D^F is normal. On the other hand, some properties are preserved by partial duals. For example, a delta-matroid $D = (E, \mathcal{F})$ is said to be *even* if every set in \mathcal{F} has the same parity (i.e., they are all of odd size or all of even size). If a delta-matroid is even then so is each of its partial duals.

By making use of the properties of spanning quasi-trees we observe that for a connected ribbon graph \mathbb{G} , the delta-matroid $D(\mathbb{G})$ is even if and only if \mathbb{G} is orientable, and that $D(\mathbb{G})$ is normal if and only if \mathbb{G} is a bouquet. As we are restricting to orientable ribbon graphs here, we shall focus on even delta-matroids.

Bibliographic remarks. Delta-matroids were introduced in the mid-1980s, independently, by A. Bouchet in [6]; R. Chandrasekaran and S. Kabadi, under the name of *pseudo-matroids*, in [17]; and A. Dress and T. Havel, under the name of *metroids*, in [28]. Delta-matroids are related to many different matroidal-objects, including: É. Tardos' *g*-matroids [49], J. Kung's Pfaffian structures [39], L. Qi's ditroids [46], A. Bouchet's symmetric matroids [6], L. Traldi's transition matroids [51], Bouchet's Isotropic systems [7], jump systems [14], and Bouchet's multimatroids [15]. This list is indicative, not exhaustive.

The discipline has adopted Bouchet's terminology and notation (most of the early development of the topic is due to Bouchet and his collaborators) and it is that we follow here except in the following instance. What we have called the "partial dual" and denoted D^B is usually called a "twist" and denoted $D * A$, but here we prefer to keep close to the ribbon graph terminology.

Bouchet, in [6], showed that the partial dual of a delta-matroid is indeed a delta-matroid. That $D(\mathbb{G}^*) = D(\mathbb{G})^*$ is implicit in [12] (it was translated into this form in [23]), and that $D(\mathbb{G}^B) = D(\mathbb{G})^B$ is from [23].

Additional background on delta-matroids can be found in the survey [40] or in the source papers.

7 Matrices and representability

We are interested in the spanning quasi-trees of a connected orientable ribbon graph \mathbb{G} . Since $D(\mathbb{G}^B) = D(\mathbb{G})^B$, partial duality preserves the spanning quasi-tree structure and so, without loss of generality, we may assume that \mathbb{G} is a bouquet. Then the ribbon subgraph of \mathbb{G} induced by any two of its edges forms either a genus one or a genus zero ribbon graph. We say that two edges of \mathbb{G} are *interlaced* if the ribbon subgraph \mathbb{G} they induce has genus one.

There is a method from algebraic topology (e.g., see Theorem 3 of [3] and its subsequent exercises) for determining via a matrix if an orientable bouquet is a quasi-tree. Let $\mathbb{G} = (V, E)$ be an orientable bouquet. Number the edges of \mathbb{G} by travelling around the boundary of the vertex from an arbitrary starting point in either direction

and assigning the numbers $1, 2, \dots, |E|$ in the order that you first encounter one of their ends. Now construct an $|E| \times |E|$ -matrix \mathbf{IM}_G^O by setting the (i, j) -entry to be $\text{sgn}(i - j)$ if the edges labelled i and j are interlaced, and 0 otherwise. (Here sgn is the signum function.) Then $\det(\mathbf{IM}_G^O) = 1$ if G is a quasi-tree and is 0 otherwise.

This construction can be simplified by working over the field of two elements, $\text{GF}(2)$. In this case, as we are forgetting the signs, we can construct an $|E| \times |E|$ -matrix \mathbf{IM}_G whose rows and columns are indexed by the edges of G by setting the (e, f) -entry to be 1 if edges e and f are interlaced, and to be 0 otherwise. Again $\det(\mathbf{IM}_G) = 1$ if G is a quasi-tree and is 0 otherwise, where here we compute the determinant over $\text{GF}(2)$.

The matrices \mathbf{IM}_G^O and \mathbf{IM}_G in fact determine the whole spanning quasi-tree structure of G (although not G itself). This is since we can test if a ribbon subgraph H of G is a quasi-tree by computing the determinant of the principal submatrix given by the edges of H (delete any rows and columns of \mathbf{IM}_G^O or \mathbf{IM}_G that correspond to edges not in H).

Thus the delta-matroid $D(G)$ can be recovered from the matrices \mathbf{IM}_G^O or \mathbf{IM}_G by computing determinants of their principal submatrices over \mathbb{R} or $\text{GF}(2)$ respectively. These matrices provide what is known as a *representation* of the delta-matroid $D(G)$.

Before continuing let us highlight one issue with this approach to studying spanning quasi-trees via matrices. As the matrices are only defined on bouquets, if we are interested in a ribbon graph G that has more than one vertex then we can obtain a matrix by choosing a one-vertex partial dual of G and computing a matrix from that. However, different choices of partial dual will result in different matrices, so we will need to understand how the matrices change under this choice.

A matrix \mathbf{A} is *symmetric* if $\mathbf{A}^t = \mathbf{A}$, is *skew-symmetric* if $\mathbf{A}^t = -\mathbf{A}$ and the diagonal entries are zero. (The condition on the diagonal is there for fields of characteristic 2.) Suppose that \mathbf{A} is a symmetric or skew-symmetric matrix over a field \mathbb{k} , and that a set E labels its rows and columns (in the same order). For $X \subseteq E$, let $\mathbf{A}[X]$ denote the principal submatrix of \mathbf{A} given by the rows and columns indexed by X . Define a collection \mathcal{F} of subsets of E by

$$X \in \mathcal{F} \iff \mathbf{A}[X] \text{ is non-singular,}$$

where $\mathbf{A}[\emptyset]$ is considered to be non-singular. Then the pair $D(\mathbf{A}) := (E, \mathcal{F})$ forms a delta-matroid. (This result is due to A. Bouchet [11].)

Since the principal submatrices of \mathbf{IM}_G^O or \mathbf{IM}_G are non-singular precisely when the corresponding edge sets of G define a quasi-tree, it follows that when G is an orientable bouquet

$$D(G) = D(\mathbf{IM}_G^O) = D(\mathbf{IM}_G),$$

where we work over \mathbb{R} for the middle expression and $\text{GF}(2)$ for the one on the right.

Since $\mathbf{A}[\emptyset]$ is non-singular, such a delta-matroid $D(\mathbf{A})$ is necessarily normal. We say a normal delta-matroid is *representable* if it can be obtained as the delta-matroid of a matrix. Every delta-matroid is a partial dual of a normal delta-matroid, so we can extend representability to non-normal delta-matroids by saying that a delta-matroid is *representable* if one of its partial duals is the delta-matroid of a matrix.

Definition 7.1. Let $D = (E, \mathcal{F})$ be a delta-matroid. We say that D is *representable* over \mathbb{k} , if there exists some $X \subseteq E$ and a symmetric or skew-symmetric matrix \mathbf{A} over a field \mathbb{k} such that

$$D^X = D(\mathbf{A}).$$

A delta-matroid is *binary* if it is representable over $\text{GF}(2)$, and is *regular* if it is representable over \mathbb{R} . Delta-matroids of orientable ribbon graphs are binary since

$$D(\mathbb{G})^X = D(\mathbb{G}^X) = D(\mathbf{IM}_{\mathbb{G}^X}^O),$$

where X is the edge set of any spanning quasi-tree of \mathbb{G} . Similarly, the matrix $\mathbf{IM}_{\mathbb{G}^X}^O$ shows that they are regular. (We note that orientability matters here as delta-matroids of non-orientable ribbon graphs are not regular, although they are binary.)

The definition of representability for delta-matroids requires a choice of a set X to make D^X normal. In general, there are many such sets to choose from, and therefore a delta-matroid D will have many representing matrices. How do the different representing matrices of a delta-matroid relate? That is, if $D(\mathbf{A}) = D(\mathbf{B})^X$ what can you say about the matrices \mathbf{A} and \mathbf{B} ?

The relevant matrix operation predates delta-matroids and can be found in work of A. Tucker [54] that appeared in 1960. Let \mathbf{A} be a square matrix over a field \mathbb{k} , whose rows and columns are labelled (in the same order) by a set E . Let $X \subseteq E$. Without loss of generality (reordering if necessary), suppose that X labels the first $|X|$ rows and columns of the matrix. Then \mathbf{A} has a block form

$$\begin{array}{cc} & \begin{array}{cc} X & E \setminus X \end{array} \\ \begin{array}{c} X \\ E \setminus X \end{array} & \left[\begin{array}{c|c} \alpha & \beta \\ \gamma & \delta \end{array} \right]. \end{array}$$

Suppose that $\mathbf{A}[X]$ is non-singular. Then the *principal pivot transform* of \mathbf{A} with respect to X is the matrix $\mathbf{A} * X$ with block form

$$\begin{array}{cc} & \begin{array}{cc} X & E \setminus X \end{array} \\ \begin{array}{c} X \\ E \setminus X \end{array} & \left[\begin{array}{c|c} \alpha^{-1} & \alpha^{-1}\beta \\ -\gamma\alpha^{-1} & \delta - \gamma\alpha^{-1}\beta \end{array} \right]. \end{array}$$

A. Bouchet, in [11], proved that principal pivot transformations correspond to partial duals of delta-matroids.

Theorem 7.2. Let \mathbf{A} be a symmetric or skew-symmetric matrix over a field \mathbb{k} , whose rows and columns are labelled (in the same order) by a set E . Let $X \subseteq E$ be such that $\mathbf{A}[X]$ is non-singular. Then $\mathbf{A} * X$ is a symmetric or skew-symmetric matrix (of the same type as \mathbf{A}), and

$$D(\mathbf{A} * X) = D(\mathbf{A})^X. \quad (7.1)$$

Thus if \mathbf{A} is a representing matrix for a delta-matroid D , then \mathbf{B} is also a representing matrix for D if and only if \mathbf{B} is a principal pivot transform of \mathbf{A} . Thus we have our answer to the problem in this section: all of the representing matrices for an orientable ribbon graph \mathbb{G} are principal pivot transformations of one another.

Bibliographic remarks. That $D(\mathbf{A})$ is a delta-matroid, that $D(\mathbf{A} * X) = D(\mathbf{A})^X$, and the definition of representability is due to A. Bouchet and from [11]. The representations for $D(\mathbb{G})$ can also be deduced from this reference (see also [9] for $\mathbf{IM}_{\mathbb{G}}^O$), although changes in language are needed (the interpretation in ribbon graph language is from [23]). However, a different route to showing that $D(\mathbb{G}) = D(\mathbf{IM}_{\mathbb{G}}^O) = D(\mathbf{IM}_{\mathbb{G}})$ was taken in this section. Here we deduced the result from a theorem on weight systems of Vassiliev invariants due to D. Bar-Natan and S. Garoufalidis [3]. This knot theory work seems to be entirely independent of Bouchet's work.

8 The reappearance of graphs

So far we have seen that the spanning quasi-tree structure of an orientable ribbon graph \mathbb{G} is described by its delta-matroid $D(\mathbb{G})$, and also by a binary representing matrix $\mathbf{IM}_{\mathbb{H}}$, where \mathbb{H} is any one-vertex partial dual of \mathbb{G} . The matrix $\mathbf{IM}_{\mathbb{H}}$ is a skew-symmetric 0-1 matrix. (Recall that skew-symmetric matrices here must have zeros on their diagonal.) Thus we can consider it as the adjacency matrix of a simple graph G . (A graph is *simple* if it does not have multiple edges or loops.) In this section we consider the properties of these simple graphs and what they tell us about ribbon graphs.

The *adjacency matrix*, \mathbf{AM}_G , of a simple graph G is the matrix over $\text{GF}(2)$ whose rows and columns correspond to the vertices of G ; and whose (u, v) -entry is 1 if there is an edge uv in G and is 0 otherwise.

Adjacency matrices are skew-symmetric, and every skew-symmetric matrix over $\text{GF}(2)$ is an adjacency matrix of some simple graph. This results in a 1-1 correspondence between skew-symmetric binary matrices and simple graphs. Every skew-symmetric binary matrix \mathbf{A} gives rise to a normal even binary delta-matroid $D(\mathbf{A})$. (The delta-matroid must be even since odd order skew-symmetric matrices are always singular.) On the other hand, a normal even binary delta-matroid D determines a unique skew-symmetric matrix \mathbf{A} such that $D = D(\mathbf{A})$. (Since if $D = (E, \mathcal{F})$ is binary then it must come from a binary matrix, and the sets of size two in \mathcal{F} determine which entries of this matrix are zero and which are one.) This means that there is a 1-1 correspondence between simple graphs and normal even binary delta-matroids.

However, we want work with all even binary delta-matroids not just normal ones. Obtaining a representing matrix for an arbitrary binary even delta-matroid D requires choosing a normal partial dual of it. Different choices will result in different matrices, however, from the results of Section 7, we know that these matrices will be related through principal pivot transforms. How are the simple graphs corresponding to these two matrices related? Once again we can find the relevant operation in the literature in a

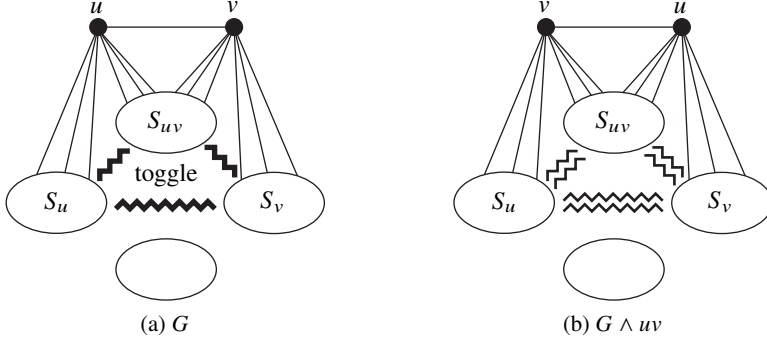


Figure 7: Pivoting (edges between the three sets, S_u , S_v , and $S_{u,v}$, are ‘toggled’, and the names of u and v are switched).

move introduced by A. Bouchet in [10, 16] and rediscovered by R. Arratia, B. Bollobás, and G. Sorkin in [1, 2].

Definition 8.1. Let G be a simple graph, and uv be an edge. Partition the vertices other than u and v into four classes: (1) vertices adjacent to u but not v , (2) vertices adjacent to v but not u , (3) vertices adjacent to both u and v , (4) vertices adjacent to neither u nor v . The *pivot* of the edge uv is the graph, $G \wedge uv$, constructed from G as follows. For any vertex pair x, y where x is in one of the classes (1)–(3), and y is in a different class (1)–(3), “toggle” the pair xy in the edge set (so if xy was an edge, make it a non-edge; and if xy was a non-edge, make it an edge). Finally, switch the names of the vertices u and v . See Figure 7.

Suppose G is a simple graph with adjacency matrix \mathbf{AM}_G , and uv is an edge of G . Then the principal submatrix $\mathbf{AM}_G[\{u, v\}]$ defined by the edge has zeros on the diagonal and ones elsewhere and is hence non-singular. This means we can form the principal pivot transform $\mathbf{AM}_G * \{u, v\}$ of \mathbf{AM}_G . This changes the matrix in a very nice way and its not too hard an exercise (remembering we are working over $\text{GF}(2)$) to track this change through to the corresponding simple graphs: the graphs will be pivots of one another. Passing to delta-matroids, for an edge uv of G we have that

$$D(\mathbf{AM}_G)^{\{u, v\}} = D(\mathbf{AM}_G * \{u, v\}) = D(\mathbf{AM}_{G \wedge uv}).$$

Thus we can identify even binary delta-matroids up to partial duals with simple graphs up to pivoting:

$$\left\{ \begin{array}{c} \text{even binary delta-matroids} \\ \text{up to partial duals} \end{array} \right\} \xleftrightarrow{1-1} \left\{ \begin{array}{c} \text{simple graphs} \\ \text{up to edge pivots} \end{array} \right\}.$$

As edge pivoting is of interest in graph theory in its own right, this identification opens up a new body of graph theory for studying delta-matroids, and vice versa.

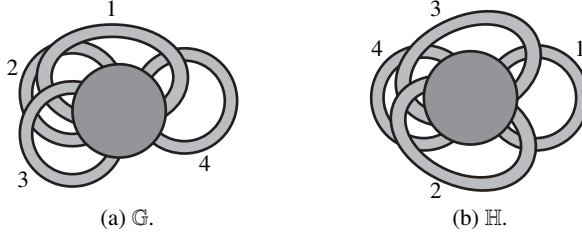


Figure 8: Two bouquets.

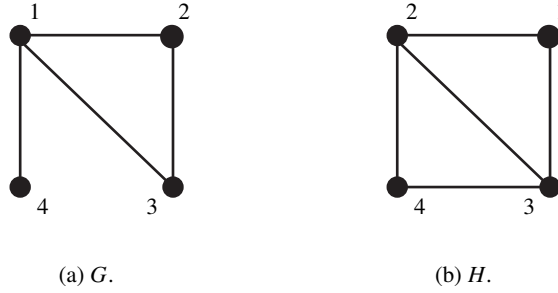


Figure 9: Two simple graphs.

However there is a catch when we want to use simple graphs and edge pivots to study ribbon graphs and their spanning quasi-trees. Although the delta-matroid $D(\mathbb{G})$ of an orientable ribbon graph is even and binary, not all even and binary delta-matroids arise from ribbon graphs. This means that the delta-matroids of ribbon graphs correspond with a proper subclass of simple graphs. We turn our attention to this class in the next section.

Example 8.2. As an illustration of the discussion from Section 6 onwards, consider the bouquets \mathbb{G} and \mathbb{H} of Figure 8. Both are on the edge set $E = \{1, 2, 3, 4\}$. Their binary representing matrices are:

$$\mathbf{IM}_{\mathbb{G}} = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{matrix}, \quad \text{and} \quad \mathbf{IM}_{\mathbb{H}} = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{matrix}.$$

Now let G and H be the simple graphs in Figure 9. It is readily checked that $\mathbf{AM}_G = \mathbf{IM}_{\mathbb{G}}$ and $\mathbf{AM}_H = \mathbf{IM}_{\mathbb{H}}$.

By direct computation from the bouquets and matrices we see that $D(\mathbb{G}) =$

$D(\mathbf{IM}_{\mathbb{G}}) = (E, \mathcal{F}_{\mathbb{G}})$ and $D(\mathbb{H}) = D(\mathbf{IM}_{\mathbb{H}}) = (E, \mathcal{F}_{\mathbb{H}})$ where $\mathcal{F}_{\mathbb{G}} = \{\emptyset, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 4\}, \{1, 2, 3, 4\}\}$ and $\mathcal{F}_{\mathbb{H}} = \{\emptyset, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}$.

The bouquets \mathbb{G} and \mathbb{H} are partial duals with $\mathbb{H} = \mathbb{G}^{\{1,2\}}$. In addition, the matrices $\mathbf{IM}_{\mathbb{G}}$ and $\mathbf{IM}_{\mathbb{H}}$ can be verified as principal pivot transforms with $\mathbf{IM}_{\mathbb{H}} = \mathbf{IM}_{\mathbb{G}} * \{1, 2\}$, and G and H are pivots with $H = G \wedge 12$. Thus we can see that

$$D(\mathbb{G}^{\{1,2\}}) = D(\mathbb{G})^{\{1,2\}} = D(\mathbf{IM}_{\mathbb{G}} * \{1, 2\}) = D(\mathbf{AM}_{G \wedge 12}),$$

and we can work with spanning quasi-trees in any of the settings.

Bibliographic remarks. Pivoting is a graph operation related to A. Kotzig's transformations on Eulerian circuits [38]. It was introduced by A. Bouchet in the context of isotropic systems [10] and multimatroids [16], and rediscovered by R. Arratia, B. Bollobás, and G. Sorkin when they introduced the interlace polynomial in [1, 2].

Further information on binary delta-matroids can be found in [13]. In particular this reference contains the result that a normal binary delta-matroid (D, \mathcal{F}) is completely determined by the members of \mathcal{F} of size at most two.

The identification of even binary delta-matroids considered up to partial duals with simple graphs considered up to edge pivots can be extended to all binary delta-matroids. They can be identified with looped simple graphs considered up to *elementary pivots* which are pivots on edges not adjacent to loops, and a *local complementation* move (toggle the edges and non-edges, and loops and non-loops in the neighbourhood of a looped vertex). This identification was first written down by J. Geelen in [33] (see also [34]) although he has said that the graph-theoretical point-of-view was used by both A. Bouchet and W. Cunningham in their discussions with him at the time of writing that paper.

9 Bringing it all together

A *chord diagram* consists of a circle in the plane and a number line segments, called *chords*, whose end-points lie on the circle. The end-points of chords should all be distinct. The *intersection graph* of a chord diagram is the graph $G = (V, E)$ where V is the set of chords, and where $uv \in E$ if and only if the chords u and v intersect. A graph is a *circle graph* if it is the intersection graph of a chord diagram. Figure 10 shows a circle graph and a corresponding chord diagram.

Now suppose that \mathbb{G} is an orientable bouquet. We may regard \mathbb{G} as a chord diagram with the vertex boundary forming the circle and chords defined by where the edges touch this circle. Let $I_{\mathbb{G}}$ denote the corresponding intersection graph. There is an edge ef of $I_{\mathbb{G}}$ whenever the edges e and f are interlaced in \mathbb{G} . In terms of the delta-matroid $D(\mathbb{G}) = (E, \mathcal{F})$ this means that there is an edge ef of $I_{\mathbb{G}}$ whenever $\{e, f\}$ is in \mathcal{F} . Thus, since $D(\mathbb{G})$ is binary, we can obtain a binary representing matrix \mathbf{A} for $D(\mathbb{G})$ by setting the (e, f) -entry to be 1 if ef is an edge in $I_{\mathbb{G}}$ and 0 otherwise, so \mathbf{A} is the adjacency matrix of $I_{\mathbb{G}}$. Thus the intersection graph $I_{\mathbb{G}}$ of \mathbb{G} is exactly the simple

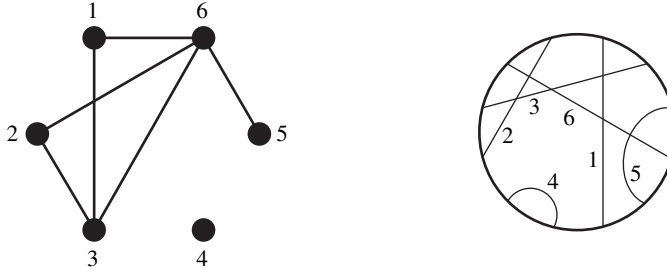


Figure 10: A circle graph and a corresponding chord diagram.

graph corresponding to the delta-matroid $D(\mathbb{G})$. (As an example, it can be checked that $G = I_{\mathbb{G}}$ and $H = I_{\mathbb{H}}$ in Example 8.2.)

We can then conclude that circle graphs are exactly the simple graphs that represent the delta-matroids of orientable ribbon graphs:

$$\left\{ \begin{array}{c} \text{Delta-matroids of orientable ribbon graphs} \\ \text{up to partial duals} \end{array} \right\} \xleftrightarrow{1-1} \left\{ \begin{array}{c} \text{circle graphs} \\ \text{up to edge pivots} \end{array} \right\}.$$

Circle graphs are well-studied in graph theory and their appearance in the present setting provides access to a large body of work that we can apply to ribbon graphs. Let us take advantage of this to characterise the delta-matroids that arise from ribbon graphs.

A *minor* of a graph is any graph that can be obtained from it by edge deletion (remove an edge), vertex deletion (remove a vertex and the edges it meets) and edge contraction (delete the edge then identify its ends). An excluded minor characterisation of a class of graphs is a result that states that a graph belongs to the class if and only if it has no minor in a given finite list. Possibly the best-known example of an excluded minor characterisation is Wagner's Theorem which states that a graph is planar if and only if it has no minor isomorphic to K_5 (the graph of five vertices and one edge between each pair of vertices) or $K_{3,3}$ (the graph with two sets of three vertices and an edge between all pairs of vertices in different sets). (The name Kuratowski's Theorem, which uses a different type of minor, is often associated with this result.) The spectacular Robertson–Seymour Theorem gives that every minor-closed class of graphs has an excluded minor characterisation [47].

Circle graphs, however, are not closed under the usual graph minor operations, and so it does not make sense to ask for an excluded minor characterisation of them with the usual type of graph minor. However, the set of circle graphs is closed under edge pivots and vertex deletions which leads to a different type of graph minor.

A *pivot-minor* of a graph is any graph that can be obtained from it by edge pivots and vertex deletions. Circle graphs have an excluded pivot-minor characterisation. J. Geelen and S. Oum [34] proved that a graph is a circle graph if and only if it has no pivot-minor isomorphic to any of the graphs shown in Figure 11.

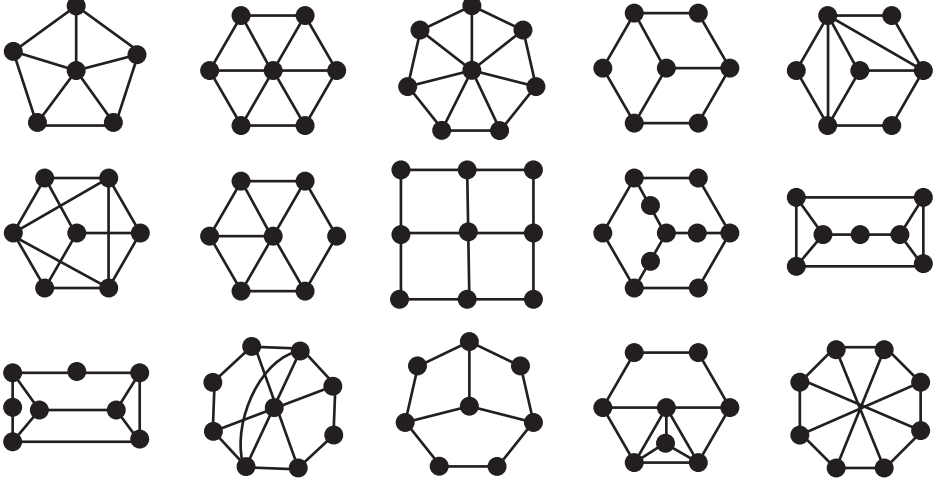


Figure 11: Excluded pivot-minors for circle graphs.

We can use the correspondence between delta-matroids and simple graphs to derive an excluded minor characterisation for the class of delta-matroids that arise from ribbon graphs. For this we need delta-matroid versions of the vertex minor operations. We know from Section 8 that the delta-matroid version of an edge pivot is a partial dual. Vertex deletion corresponds to the standard idea of deletion for delta-matroids.

Let $D = (E, \mathcal{F})$ be a delta-matroid, and $e \in E$. Then D delete e , denoted $D \setminus e$, is defined as $D \setminus e := (E \setminus e, \mathcal{F}')$, where $\mathcal{F}' = \{F : F \in \mathcal{F} \text{ and } e \notin F\}$ when e is not in every member of \mathcal{F} ; and $\mathcal{F}' = \{F \setminus e : F \in \mathcal{F} \text{ and } e \in F\}$ when e is in every member of \mathcal{F} . Although we do not use the fact here, it is worth noting that $D(\mathbb{G} \setminus e) = D(\mathbb{G}) \setminus e$. A delta-matroid D' is said to be a *minor* of a delta-matroid D if it can be obtained from D through the operations of deletion and partial duality.

By translating the excluded pivot-minor characterisation of circle graphs we obtain the following characterisation the even delta-matroids that arise from ribbon graphs.

Theorem 9.1. *Let D be an even delta-matroid. Then $D = D(\mathbb{G})$ for some ribbon graph \mathbb{G} if and only if \mathbb{G} has no minor isomorphic to $D(\mathbf{AM}_G)$ where G is one of the graphs shown in Figure 11, or to one of the delta-matroids given in Example 6.2.*

The excluded minors from Example 6.2 are included to ensure that an even delta-matroid is binary and hence comes from a simple graph.

Finally we come to the question from which our journey into delta-matroids began: *Do the spanning quasi-trees of an embedded graph determine it?* In terms of delta-matroids we are asking:

If $D(\mathbb{G}) = D(\mathbb{H})$ then how are the ribbon graphs \mathbb{G} and \mathbb{H} related?

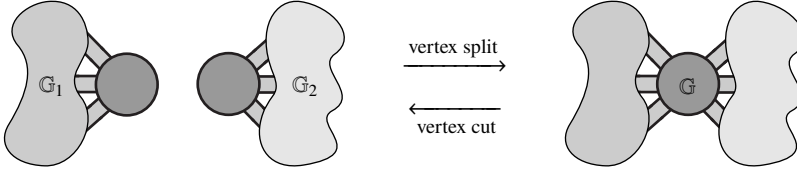


Figure 12: Vertex joins and vertex splits.

So we are looking for a version of Whitney’s Theorem that applies to ribbon graphs and their delta-matroids.

Again we can make use of the circle graph literature. There has been extensive work on recovering chord diagrams from circle graphs, and on determining which chord diagrams correspond to the same circle graph. Appearing implicitly in [8, 25, 32], and explicitly in [19], is an operation on chord diagrams called *mutation* that relates all chord diagrams that have the same intersection graph. This operation cuts out a certain substructure in a chord diagram, rotates it then glues it back in (we omit a definition of the move as we do not use its details here). The result uses Cunningham’s theory of graph decompositions from [26] to decompose an intersection graph into ‘prime’ graphs that have unique intersection graphs. Mutation then corresponds to the choices that are made when reassembling a corresponding chord diagram from these prime graphs.

In the present setting, if two ribbon graphs \mathbb{G} and \mathbb{H} have equal delta-matroids, then there must be some set of edges X such that the partial duals \mathbb{G}^X and \mathbb{H}^X are both bouquets with the same delta-matroid. The delta-matroids $D(\mathbb{G}^X)$ and $D(\mathbb{H}^X)$ therefore correspond to the same simple graph. As this simple graph can be considered as the intersection graphs of \mathbb{G}^X and \mathbb{H}^X , it follows that \mathbb{G}^X and \mathbb{H}^X must be related by mutation (technically, a version of mutation for bouquets). Then by analysing how mutation changes under partial duality, we can pull back the operations to the original ribbon graphs \mathbb{G} and \mathbb{H} . This approach results in a characterisation of ribbon graphs that have the same delta-matroid. We describe the relevant moves then the characterisation. The first move is the analogue of the vertex identification and vertex cleaving that are used in Whitney’s Theorem and illustrated in Figure 1.

Suppose that \mathbb{G}_1 and \mathbb{G}_2 are ribbon graphs. For $i = 1, 2$, suppose that α_i is an arc that lies on the boundary of \mathbb{G}_i and entirely on a vertex boundary. If a ribbon graph \mathbb{G} can be obtained from \mathbb{G}_1 and \mathbb{G}_2 by identifying the arc α_1 with α_2 (where the identification merges the vertices), then we say that \mathbb{G} is obtained from \mathbb{G}_1 and \mathbb{G}_2 by a *vertex join*, and that \mathbb{G}_1 and \mathbb{G}_2 are obtained from \mathbb{G} by a *vertex split*. The operations are illustrated in Figure 12 and are standard operations in ribbon graph theory. It is important to observe that the definition of a vertex join does not allow for any “interlacing” of the edges of \mathbb{G}_1 and \mathbb{G}_2 .

The next operation we need is called *mutation*. It is illustrated in Figure 13. The figure shows a local change in a ribbon graph (so the ribbon graphs are identical outside

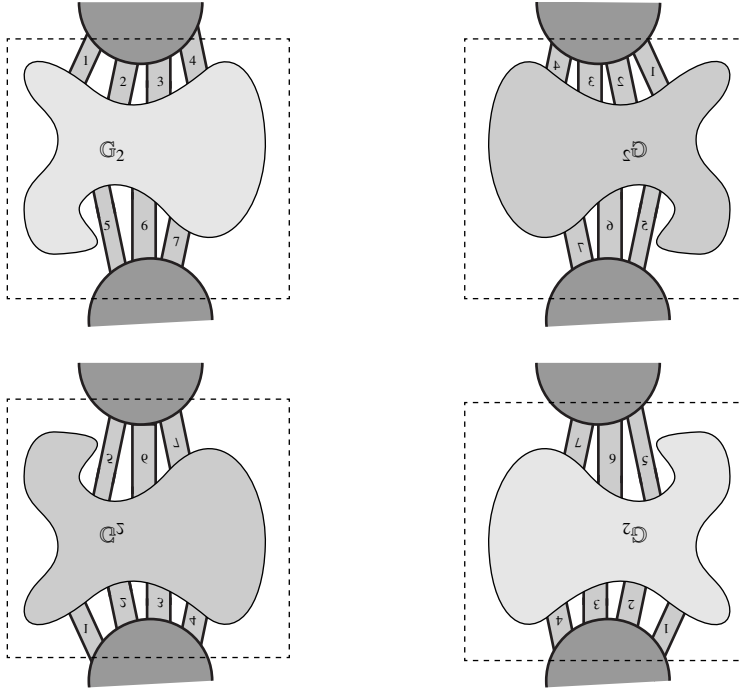


Figure 13: Mutation for ribbon graphs.

of the region shown) and the two parts of vertices that are shown in it may come from the same vertex. To define the move, let \mathbb{G}_1 and \mathbb{G}_2 be ribbon graphs. For $i = 1, 2$, let α_i and β_i be two disjoint directed arcs that lie on the boundary of \mathbb{G}_i and lie entirely on boundaries of (one or two) vertices. Furthermore suppose that \mathbb{G} is a ribbon graph that is obtained by identifying the arcs α_1 with α_2 , and β_1 with β_2 , where both identifications are consistent with the direction of the arcs. (The identification merges the vertices.) Suppose further that \mathbb{H} is a ribbon graph obtained by either: (1) identifying α_1 with α_2 , and β_1 with β_2 , where the identifications are inconsistent with the direction of the arcs; (2) identifying α_1 with β_2 , and β_1 with α_2 , where the identifications are consistent with the direction of the arcs; (3) identifying α_1 with β_2 , and β_1 with α_2 , where the identifications are inconsistent with the direction of the arcs. Then we say that \mathbb{G} and \mathbb{H} are related by *mutation*.

With these definitions in hand, we can complete our tour with an answer (due to I. Moffatt and J. Oh [41]) to our original question as to what extend the spanning quasi-trees determine the ribbon graph.

Theorem 9.2. *Let \mathbb{G} and \mathbb{H} be a connected orientable ribbon graphs, and let $D(\mathbb{G})$ and $D(\mathbb{H})$ be their delta-matroids. Then $D(\mathbb{G}) = D(\mathbb{H})$ if and only if \mathbb{G} can be obtained from \mathbb{H} by ribbon graph isomorphism, vertex joins, vertex splits, or mutation.*

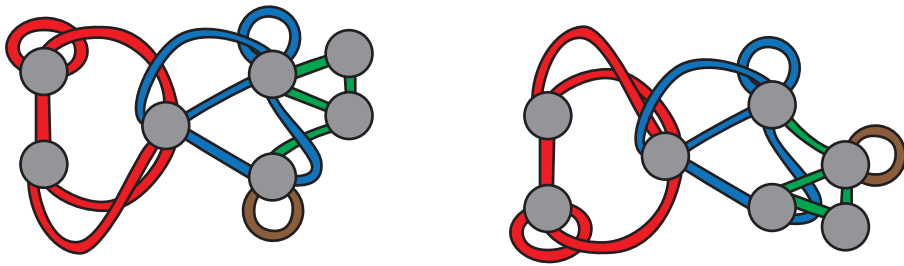


Figure 14: Two ribbon graphs with the same delta-matroid.

As an example of Theorem 9.2, the two non-equivalent ribbon graphs in Figure 14 can be obtained from each other by isomorphism, vertex joins, vertex splits, and mutation. Therefore their delta-matroids are isomorphic.

Bibliographic remarks. The excluded minor characterisation for the delta-matroids of orientable ribbon graphs stated in Theorem 9.1 is implicit in J. Geelen and S. Oum’s paper [34]. There it was stated for even Eulerian delta-matroids which, from [23], are equivalent to the delta-matroids of ribbon graphs. The ribbon graph formulation given here is from [23]. The characterisation extends to non-orientable ribbon graphs. Again this was given in for Eulerian delta-matroids in [34] and translated to the ribbon graph setting in [23]. There are 171 excluded minors in this case.

The excluded minor characterisation of binary delta-matroids alluded to after the statement of Theorem 9.1 is due to A. Bouchet and A. Duchamp [13]. There are five excluded minors for binary delta-matroids, and the two appearing in Example 6.2 are the even ones.

Theorem 9.2 is due to I. Moffatt and J. Oh, and from [41]. It is given there more generally for non-orientable and non-connected ribbon graphs. Extending to the non-connected case is straightforward, but additional work is required for the non-orientable case.

10 Now we can get started...

We set out with the classical question of whether the spanning trees in a graph determine the graph itself. This led to a topological version it, if the spanning quasi-trees in a ribbon graph determine it. In answering this question we were guided by the idea of partial duality which appeared in different forms and settings. This took us to ribbon graphs, matroids and delta-matroids, matrices, as well as simple and circle graphs. Moreover, we saw that delta-matroids provided the central unifying framework for all of these ideas. It is this common framework that we should really take away from our journey.

As mentioned earlier, there is a well-known and successful symbiotic relationship between graph theory and matroid theory, with each area informing the other. As

reported in [45], W. Tutte famously observed that, “If a theorem about graphs can be expressed in terms of edges and circuits alone it probably exemplifies a more general theorem about matroids.” An analogous correspondence between embedded graphs and delta-matroids was proposed in [23, 24]. This view of delta-matroid is proving to be successful. It has led implicitly and explicitly to advances in, especially, the topics of graph polynomials, and the structural theory of both delta-matroids and ribbon graphs. But we really are only at the beginning of this journey. Many fundamental questions remain unanswered and directions remain unexplored, but our knowledge is rapidly advancing.

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