Holding a group together: non-game-theory vs. game-theory

Running Title: non-game-theory vs. game-theory

Michael Richter[†]

Department of Economics, Royal Holloway, University of London

Ariel Rubinstein[‡]

School of Economics, Tel Aviv University
and Department of Economics, New York University

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ABSTRACT: Each member of a group chooses a position and has preferences regarding his chosen position. The group's harmony depends on the profile of chosen positions meeting a specific condition. We analyse a solution concept (Richter and Rubinstein, 2020) based on a permissible set of individual positions, which plays a role analogous to that of prices in competitive equilibrium. Given the permissible set, members choose their most preferred position. The set is tightened if the chosen positions are inharmonious and relaxed if the restrictions are unnecessary. This new equilibrium concept yields more attractive outcomes than does Nash equilibrium in the corresponding game.

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[†]E-mail: Michael.Richter@rhul.ac.uk
[‡]E-mail: rariel@tauex.tau.ac.il

1. Introduction

Each member of a group has to choose a position on the real line and has a single-peaked preference relation over his position. Each member cares only about the position he chooses and does not attempt to influence, what we call the group's *overall position*. If the members' choice profile does not satisfy a harmony condition related to the overall position, then a crisis erupts. We analyse three such conditions. In the first, a profile is harmonious if there is a position which a majority (or super majority) of members agree on. In the other two, harmony requires that no member's chosen position is too far from the overall position, which is either the median or the average of the chosen positions.

A leading scenario we have in mind is a debate within a political party. Each of the party's members has his own ideal policy. Each expresses a position and the profile of expressed positions determines the party line. The unity of the party is dependent on the members' positions not deviating too much from that party line. If there is wide variation among the members' ideals (as is the case in many large political parties in Western countries) and all of the party's members stick to their ideals, then a crisis tears the party apart. In order to survive, the party needs some mechanism which directs the party's members to express positions that preserve unity.

We employ the non-conventional *Y-equilibrium* concept developed in Richter and Rubinstein (2020). A Y-equilibrium specifies the convex set of permissible positions and each member's chosen position. In a Y-equilibrium:

(i) *the rationality condition*: each member's choice is his most preferred position from among the permissible ones;
(ii) *the harmony condition*: the profile is harmonious;
(iii) *the maximality condition:* there is no larger set of permissible positions from which a profile satisfying (i) & (ii) can be assigned.

Thus, a Y-equilibrium is required to be resistent to two forces. The first modifies the permissible set when the profile of chosen positions leads to a crisis. The second loosens restrictions on the permissible set when they are unnecessarily tight, that is, when members' optimal choices after the loosening would still avoid a crisis.

We regard the permissible set as an expression of *social norms* which, like competitive prices, apply uniformly to all members of the society. This uniformity might be viewed as the outcome of a societal desire for fairness but primarily it is a simplicity condition. Norms must be simple and easy to understand and apply. An important aspect of simplicity is that the norms do not depend on personal characteristics.

The Y-equilibrium notion offers a decentralized way to obtain harmony in a society, without introducing an extraneous medium. No authority dictates the permissible and forbidden sets, just as there is no authority that sets prices in the market.

We created the concept of Y-equilibrium as an abstract generalization of the concept of competitive equilibrium to non-market social situations. We require that in equilibrium the permissible set yields a profile of optimal choices that preserves a harmony condition. The permissible set plays a role analogous to that of the price system. The harmony condition is analogous to the market requirement that total demand not exceed total supply. The maximality condition is akin to the condition that total demand equals total supply.

In each of the three parts of the paper, we require that in equilibrium the profile of positions satisfy some version of group cohesion related to an overall position. In Richter and Rubinstein (2020), we analyzed a pairwise cohesion requirement that the maximum distance between any two positions must not exceed a certain bound.

The paper can be viewed as an analysis of a Political Economics model in which an individual does not try to influence the overall outcome. Rather he cares about the position he expresses and some sort of unity is required for the existence of the group.

However, the main target of this paper is methodological. We suggest the reader to consider using a non-strategic approach before instinctively grabbing a conventional game-theoretic analysis off the shelf. For each of the three harmony conditions, we study the corresponding game in which each member selects a position, where his first priority is to avoid a crisis and his second priority is the position he takes. In a Nash equilibrium, either the group collapses and no single member can save it by changing his position, or the group survives and no member can improve his position without causing a crisis.

As mentioned above, our approach is closer to competitive equilibrium theory than to game theory. Although game theory is rich and beautiful, we do not find it as appealing for the class of situations analyzed here. In particular, in the game-theoretic approach, each member faces a dual mandate: maximizing his individual position and avoiding a crisis. The latter requires that each member possesses information about all of the other members' choices. In our approach, an individual's task is much simpler: each member simply maximizes his position from among the set of permissible positions. We will see that the three corresponding games have a vast multiplicity of Nash equilibria, most of which even lack the natural monotonicity property (the more rightish individuals have a weakly more rightish equilibrium position). Of course, it is up to the reader to judge between the approaches; we just plea for researchers in Economics not to automatically apply a particular solution concept.

2. The model

Each member of a group $N = \{1, \ldots, n\}$ chooses a position along the line $X = \mathbb{R}$. The group has an odd number of members. Each member i has continuous and strict convex preferences over his position with a finite peak at $peak^i$. For simplicity, we assume that all peaks are distinct. Without loss of generality, we assume that the members are ordered by their peaks from left to right. Denote the left-most peak by L, the median by M and the right-most by R. A *profile* is a vector in X^N and for ease of notation, we denote $x = (x^i)_{i\in N}$, $y = (y^i)_{i\in N}$ and $z = (z^i)_{i\in N}$. The group's *overall position* is $O(x)$ where O is a function of the positions chosen by the members. In each of the three versions of the model, we use a different aggregation scheme: the mode, the median and the average.

Unlike in many familiar models, a member in our model does not care about the group's overall position. While in Hotelling (1929) a member cares only about the group's position and in Downs (1957) he also cares about his chosen position, here we go a step further by positing that a member cares only about his chosen position.

The final component of the model is a set $F \subseteq X^N$ of harmonious choice profiles that avoid a crisis. Each of the three cases that we consider is characterized by a different function O and set F. In all three, the function O and the set F are anonymous.

2.1 The equilibrium concept

A Y-equilibrium consists of two ingredients: The first is a convex permissible set, consisting of the alternatives that all members choose from. It is a reflection of the social norms that dictate the limits on the positions that are acceptable. The second is a profile of choices from the permissible set – one choice for each member.

The following definition formalizes the three conditions (rationality, harmony and maximality) described in the introduction:

Definition 1 *A **Y-equilibrium** is a pair $\langle Y,(y^i)\rangle$ where Y is a a convex set and (y^i) is a profile of choices satisfying:*
(i) for all i, y^i is a $\succsim^i$-maximal position in Y;
(ii) $(y^i)\in F$; and
(iii) for no convex set $Z \supset Y$ is there a profile $(z^i) \in F$ such that z^i is a $\succsim^i$-maximal alternative in Z for all i.

There are two motivations for our requirement that the permissible set is *convex*. First, there is a natural asymmetry between the permissible and the forbidden. For example, it is conceivable that a social norm would consider acceptable a social visit between 1-3 hours but it would be strange if the norm was the reverse (where the only socially acceptable visits are either short visits of less than 1 hour or long visits of more than 3, but medium visits of 2 hours are a social faux pas). "Forbidden" is associated with extreme conditions and "permissible" with moderate conditions. Second, in our one-dimensional setting, convexity implies that permitted behavior is an interval. An interval is defined by a lower and upper bound which is the simplest way to describe a subset of the line. Simplicity is a merit, and perhaps necessary, for a public norm to be understood and internalized by a large group of members.

Note that in Richter and Rubinstein (2020), we allow the grand set X to be finite or otherwise non-Euclidean. Therefore, we considered two variants of the solution concept: without a convexity requirement (called Y-equilibrium) and with a convexity requirement (called convex Y-equilibrium). In the current paper, the set of alternatives is convex and therefore we only consider the second notion, which we simply refer to as Y-equilibrium.

Given that the set Y is required to be convex and that all preference relations are single-peaked, each member's optimal choice is unique. Therefore, a Y-equilibrium can be specified simply by the permissible set. Thus, we often refer to a Y-equilibrium $\langle Y, (y^i)\rangle$ by its permissible set Y only.

The permissible set is *uniform* for all members. Although one can imagine situations in which the norms are different for different members, fair normative principles tend to be uniform. Moreover, uniformity has the merit of simplicity.

It is worthwhile comparing the Y-equilibrium concept to that of standard competitive equilibrium for an exchange economy:

(a) The two notions have similar structures. The permissible set in our solution is analogous to a system of prices in the competitive equilibrium model, in that both determine the choice sets.

(b) The permissible set is uniform, as is the competitive price system. In the standard competitive market model, all members face the same (linear) set of "net trades". (c) The competitive market model, unlike ours, has initial endowments, which leads to nonuniform final choice sets (even though the net trade sets are uniform, subject to feasibility).

(d) There is an analogy between a hidden assumption in the concept of competitive equilibrium without endowments and our maximality requirement. Consider an environment in which each agent has some money and there is a finite supply of goods available. One can think about a weak equilibrium concept, which is a price vector such that, for each good, the agents' total demand does not exceed the total supply. In contrast, the standard competitive equilibrium notion is a price vector in which the sum of all agents' demands *equals* the total bundle. Thus, one can think about competitive equilibrium as a weak equilibrium with the additional requirement of price minimality or equivalently the maximality of agents' opportunity sets.

We view Y-equilibrium as a decentralized concept. In competitive equilibrium, prices adjust when there is excess supply or demand in order to achieve feasibility of the agents' optimal choices (market clearing). Likewise, we imagine that the permissible set adjusts by analogous equilibrating forces. If the profile of optimal choices induces members to choose an inharmonious profile, then there is a pressure that tightens the permissible set. On the other hand, if the limits can be loosened and the members' updated optimal choices are feasible then there is a pressure to relax the limits.

2.2 The corresponding game

The following is the *strategic game* that corresponds to our model. The players are the members of the group. The set of actions for each player is the set of positions X. Each member j has a preference relation $\succsim_*^j$ on the set of choice profiles, such that $x=(x^i)\succsim_*^j y=(y^i)$ if either:

(i) $x\in F$ and $y\notin F$, or

(ii) both $x,y\in F$ or both $x,y\notin F$ and $x^j\succsim^j y^j$.

In other words, every member lexicographically first prefers harmony and then his own position.

The structure of the members' preferences is a more extreme version of what is referred to in the literature as conformism (see Jones (1984)). In that literature, each member of society faces a tradeoff between choosing an alternative that is close to his ideal and his wish to conform to the group's average behavior. These two considerations also appear in our model but the smooth tradeoff is replaced by a lexicographic priority for conformity.

The standard Nash equilibrium is applied to this game. A *crisis equilibrium* is a Nash equilibrium profile outside of F. If it exists, then it must be the profile in which each member chooses his peak since otherwise a member could deviate to his peak – which he prefers – regardless of whether that results in harmony.

Note that every Pareto-efficient profile (z^i) in F is trivially a Nash equilibrium. If (z^i) is not a Nash equilibrium, then there must be some member who strictly prefers a different position and harmony is not disturbed if he moves to that new position. Since all other members are indifferent to that move, it is a Pareto improvement, contradicting the Pareto efficiency of (z^i). However, as demonstrated later, a Nash equilibrium does not need to be Pareto-efficient.

The notion of a non-crisis Nash equilibrium in this game is identical to that of social equilibrium in Debreu (1952). The relation between Y-equilibrium and Debreu's social equilibrium was discussed in detail in Richter and Rubinstein (2020). The results presented here confirm what we have argued before, namely that the two concepts of Y-equilibrium and Debreu's social equilibrium may lead to very different outcomes.

3. A voting model

In this section, the harmony constraint is that at least τ of the n members propose the same position where $(n+1)/2 \leq \tau \leq n$. That is, τ is a threshold between simple majority and full unanimity. This threshold could be interpreted as the minimal number of members who are needed for a resolution to be made and a resolution is necessary for the group to be in harmony. A profile satisfies the harmony condition if there is a position chosen by at least τ members, and the unique majority-chosen position is taken to be the overall position. Recall that the members are ordered so that $peak^1 < peak^2 < \ldots < peak^n$.

3.1 Y-equilibrium

Proposition 1 characterizes all Y-equilibria for every τ. The set of Y-equilibrium overall positions is $[peak^{n+1-\tau}, peak^{\tau}]$. In particular, for the case of simple majority, the only Y-equilibrium overall position is M. It follows that *higher voting thresholds, rather than promoting compromise, support additional extreme equilibrium outcomes.*

Regarding permissible sets, for every threshold above a simple majority, each supported overall outcome is part of a unique Y-equilibrium, whereas for the simple majority case, there are two Y-equilibria that support the unique overall position M.

Proposition 1 *(i) For the case of simple majority, there are exactly two Y-equilibria. Each has overall position M.*
(ii) For any other threshold, the set of Y-equilibrium overall positions is $[peak^{n+1-\tau}, peak^{\tau}]$. Each overall position is supported by a unique Y-equilibrium.

Proof. (i) If a permissible set contains points both to the left and the right of M, then no alternative attracts majority support: one member chooses M, a minority chooses to the right of M and a different minority chooses to the left of M. Therefore any Y-equilibrium must be a subset of $[M, \infty)$ or $(-\infty, M]$. A majority supports M from $[M, \infty)$, and therefore $[M, \infty)$ and $(-\infty, M]$ are the only Y-equilibria.
(ii) We say that a convex set is a *d-set* if it contains at least two points in $[peak^{n+1-\tau}, peak^{\tau}]$. Observe that from any d-set, there is no alternative which is chosen by at least τ members. This is because if $peak^{n+1-\tau} \leq s < t \leq peak^{\tau}$ are in a permissible set, then less

than τ members prefer any alternative above s to s and less than τ members prefer any alternative below t to t.

The set $[peak^\tau, \infty)$ is a Y-equilibrium with the overall outcome $peak^\tau$: members $1, \ldots, \tau$ all choose $peak^\tau$ while all others choose their peaks. Any larger set $[l, \infty)$ is a d-set. No other Y-equilibrium Y could have the overall position $peak^\tau$ since Y cannot be a subset of $[peak^\tau, \infty)$ and thus would be a d-set. Analogous arguments apply for $(-\infty, peak^{n-\tau+1}]$.

For any position $peak^{n+1-\tau} < t < peak^\tau$, the set $\{t\}$ is a Y-equilibrium (any larger set is a d-set) and is also the unique Y-equilibrium with overall position t.

There is no Y-equilibrium Y with an overall position larger than $peak^\tau$ since then either Y is a subset of $[peak^\tau, \infty)$ or it is a d-set. An analogous argument rules out Y-equilibria with overall positions to the left of $peak^{n-\tau+1}$. □

Notice that any position outside $[L, R]$ is not an Y-equilibrium overall position for any threshold. In the simple majority case, we have a "median voter theorem": M is the only Y-equilibrium overall position. While the unique overall position is M, which is often taken to be an ideal compromise outcome, the permissible set is not in the spirit of compromise since it allows choosing positions only to one side of M.

3.2 The voting game

A crisis equilibrium typically exists (except the case $n = 3$ and $\tau = 2$). The Nash equilibrium notion allows for any overall position, even those outside of $[L, R]$. All non-crisis Nash equilibria have a specific structure: τ members choose an overall position and the rest choose their peaks.

Proposition 2 *The set of non-crisis Nash equilibria of the voting game consists of all profiles where a bare majority τ choose a position t, while all other members choose their peaks.*

Proof. Clearly, these are Nash equilibria. To see that there are no others, consider a Nash equilibrium in which at least τ members choose a common position t. Every member who is not at t is not instrumental in maintaining harmony, and so must be at his peak. If strictly more than τ members choose t, then none of them is instrumental for harmony, and at least one of them is not at his peak and would deviate. □

3.3 Comparing the solution concepts

The difference between the two solution concepts is starkest for the simple majority case. While the Y-equilibrium concept yields M as the only overall position, all positions, including those outside the range $[L, R]$, are Nash equilibrium overall positions.

The Nash equilibria of this game require extreme coordination between the members because exactly τ members must support the overall position while all others choose their peak. In a Y-equilibrium, the permissible set handles the coordination, just as competitive prices coordinate supply and demand.

4. The near-median model

In this section, the overall position $O(x)$ is the median of the members' choices, and feasibility requires that all chosen positions are within d from the median. Formally, $F = \{x \mid d(x^j, O(x)) \leq d \text{ for all } j\}$. We focus on the richest case, $L + d < M < R - d$. We avoid boring details about the case in which the inequalities don't hold, since it adds nothing to the discussion. Without loss of generality, we set $d = 1$.

4.1 Y-equilibrium

Proposition 3 characterizes the Y-equilibria. There is a multiplicity of Y-equilibria, of which $[M-1, M+1]$ is special. It is the only one with overall position M and the only one with a Pareto-efficient choice profile. It is also the only one that involves choices both to the right and to the left of the median choice; in all other Y-equilibria, a majority of members choose one of the endpoints of the permissible set.

Proposition 3
(i) The Y-equilibria of the near-median model are the following sets:

(a) $[M-1, M+1]$ *(with overall position M);*

(b) $[t-1, t]$ *with* $L+1 < t < M$*, or* $[t, t+1]$ *with* $M < t < R-1$ *(with overall position t);*

(c) $(-\infty, L+1]$ *or* $[R-1, \infty)$ *(with overall position* $L+1$ *or* $R-1$*).*

(ii) The only Pareto-efficient Y-equilibrium is (a).

Proof. *(i)* We first verify that the above are indeed Y-equilibria.

(a) $[M-1, M+1]$. Obviously the median of the members' choices is M and it remains M from any larger permissible set. From $[M-1, M+1]$ member 1 (with peak L) chooses $M-1$, but would rather go further left while member n (with peak R) chooses $M+1$ and would rather go further right. Thus, from any larger permissible set, at least one member must be further than 1 from the median. Therefore, there is no larger permissible set with a feasible profile of optimal choices.

(b-c) $[t-1, t]$ with $L+1 < t < M$. A majority of members choose t and member 1 chooses $t-1$. Any right extension of the set would move the median to the right while any left extension would move the choice of member 1 to the left. Thus, the profile of optimal choices from any larger permissible set violates the harmony condition. A similar argument holds for the other three cases.

We now show that there are no other Y-equilibria:

First, we show that in any other Y-equilibrium $\langle Y, y\rangle$ the overall position is in $[L+1, R-1]$. Suppose that $O(y) > R-1$. Observe that Y necessarily contains an alternative z which is to the left of $R-1$ because it is not a subset of the Y-equilibrium $[R-1, \infty)$. Since $z, O(y) \in Y$ and Y is convex it follows that $R-1 \in Y$. The majority of members who have peaks to the left of $R-1$ will choose a position to the left of $R-1$, a contradiction to $O(y)$ being to the right of $R-1$. Similarly, $L+1 \leq O(y)$.

Second, for each $L+1 \leq t \leq R-1$ we have already described a Y-equilibrium with overall position t. Thus, it is sufficient to show that there are no two Y-equilibria, $\langle Y, y\rangle$ and $\langle Z, z\rangle$, with $O(y) = O(z) = t$. If $M < t$, then the left endpoint of Y and Z is t. Therefore, the sets Y, Z are nested, contradicting the smaller one being a Y-equilibrium. Similarly, it cannot be that $t < M$. If $t = M$, then $Y, Z \subseteq [M-1, M+1]$, which is a Y-equilibrium, and thus $Y = Z = [M-1, M+1]$.

(ii) The Y-equilibrium $[M-1, M+1]$ is Pareto-efficient. To see this, notice that all interior members are at their peak and cannot be improved. Thus, any Pareto-improvement must move members at the right endpoint further to the right, or members at the left endpoint further to the left, or both. Such a Pareto improvement does not change the median position. Since both endpoints are 1 away from M, then any new position is farther than 1 away from M, violating the harmony requirement.

The Y-equilibria in (b) and (c) are Pareto-inefficient. Consider, for example, any Y-equilibrium $[t-1, t]$ with overall position $t < M$. A majority of members are at t and would like to move to the right. Thus, a Pareto-improvement can be achieved by moving a single member (or any number of members less than a majority) closer to his peak. □

4.2 *The near-median game*

We will now study the set of non-crisis Nash equilibria and show that the set of Nash equilibrium overall positions is also $[L+1, R-1]$. When preferences are sufficiently diverse, for example if $L+2 < M < R-2$, then the game has a crisis equilibrium where (as in any crisis equilibrium) all members choose their peaks and the overall position is M.

Proposition 4 *(i) The only Y-equilibrium of the near-median model in which the profile is a Nash equilibrium of the near-median game is* $[M-1, M+1]$.
(ii) The set of Nash equilibria overall positions of the near-median game is $[L+1, R-1]$.
(iii) In every Nash equilibrium with an overall position $t < M$*, there are two members* i, j *with* $peak^i, peak^j \geq M$ *such that* i *chooses* t*, and* j *chooses strictly to the right of* t.

Proof. *(i)* The position profile chosen from the Y-equilibrium $[M-1, M+1]$ is a Nash equilibrium of the game. A player who chooses an internal point is at his first-best. A player who chooses an endpoint, say $M+1$, does not prefer to move to the left since his peak is to the right and does not prefer to move to the right since he will then be further away from M, thus causing a crisis.

On the other hand, consider the Y-equilibria $(-\infty, L+1]$ or $[t-1, t]$ with $L+1 < t < M$. A majority of members choose the right endpoint of the permissible set, which is also the overall position, and prefer to move to the right. Each of them has a profitable deviation, a move of size less than 1 towards his ideal position, which does not change the median position and preserves harmony.

(ii) We now construct a Nash equilibrium with overall position $t \in [L+1, M)$ (the case $t \in [L+1, M)$ is analogous). Assign to each i with $peak^i \leq t$ his most preferred position in $[t-1, t]$. The number m of such members satisfies $1 \leq m < (n+1)/2$. Assign t to any $(n+1)/2 - m$ members with peaks greater than t. Assign to every other i his most preferred position in $[t, t+1]$. This assignment is harmonious and its median is t. No member at his peak wishes to move. Members at $t-1$ only wish to move leftward, while members at $t+1$ (if there are any) only wish to move rightward; but any such move would violate harmony. Each of the rightist members at t wishes to move to the right, but each is pivotal for harmony.

For example, for $n = 3$ and peaks at 0, 5 and 10, the Nash equilibrium constructed for $t = 4½$ is $(3½, 5, 4½)$. Note that this Nash equilibrium is not monotonic in members' peaks.

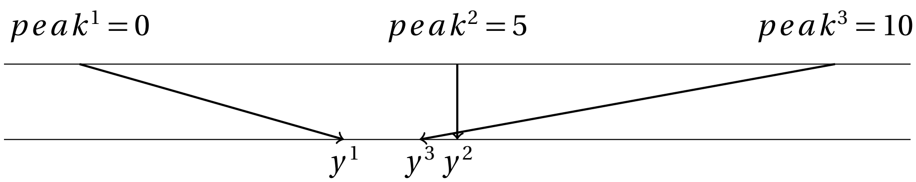

Figure 1: Example Nash equilibrium with median 4½.

There is no Nash equilibrium with an overall position $t < L+1 < M$. If there were, then no member chooses the position $t-1 < L$, since any member who chooses that point would prefer to move right and can do so without disturbing feasibility. As a majority of members choose a position in $(-\infty, t]$ and a majority of members have peaks in $[M, R]$, there is a member who wishes to move to the right and can do so without violating harmony (even if he moves the median to the right).

(iii) No member j with $peak^j < t$ chooses a position $y^j > t$ since he would benefit by deviating to t. Likewise, no member j with $peak^j > t$ chooses a position $y^j < t$. Thus, since there is a majority of members with peak weakly above M and there is a majority of members with choice weakly below t, there must be at least one member i with $peak^i \geq M$ who chooses t. Member i would move to the right unless he would shift the median to the right. Thus, the number of members who choose positions in $[t-1, t]$ is exactly $(n+1)/2$ and it includes member 1. Thus, there is some member j

whose peak is weakly above M and whose choice is strictly above t. $\square$

4.3 Comparing the solution concepts

The set of non-crisis Nash equilibria is huge and one can verify that it is equal to the set of all Pareto-efficient profiles. The Y-equilibrium approach and the Nash equilibrium approach yield the same set of overall positions but each overall position is supported by many Nash equilibria and only by a unique permissible set and a unique profile.

The Nash equilibria differ from the Y-equilibrium notion on multiple fronts: First, in every Nash equilibrium (except the one which is also a Y-equilibrium), some members are trapped. That is, when the overall position is below the median, the trapped member is “rightist” and wishes to move further to their peaks, as other rightist members are allowed to, but they cannot because it would cause some other "extreme left" members to be too far from the new median, thus causing a crisis.

Second, as a consequence of this trapping and as demonstrated in Proposition 4, the Nash equilibria typically are non-monotonic in the sense that member i – who is more rightish than member j – chooses a position to the left of the position chosen by j. In contrast, a Y-equilibrium is based on choice from the same set and a member’s choice is always monotonic in his peak.

5. The near-average model

In this section, the overall position $O(x)$ of the profile x is the average of the choices, denoted by $avg(x)$. The harmony condition requires that all chosen positions are "near the average". Formally, $F = \{x \mid d(x^j, avg(x)) \leq 1 \text{ for all } j\}$. We focus on the more interesting case where $R - L > 2$, in which the ideal positions are diverse.

5.1 Y-equilibrium

There is a continuum of Y-equilibria. The left limit of a Y-equilibrium permissible set is either $-\infty$ or a point in $[L, L']$ where $L' < R$. The right limit is either ∞ or a point in $[R', R]$ where $L < R'$. No two Y-equilibria have the same overall position.

Proposition 5 *In the near-average model:*
(a) There are $L < L'$ and $R' < R$ and a continuous and strictly increasing function $r : [L, L'] \to [R', R]$ with $r(L) = R'$ and $r(L') = R$ such that the Y-equilibria are: $(-\infty, R']$, $[L', \infty)$ and for each $l \in (L, L')$ the set $[l, r(l)]$.
(b) There are no two Y-equilibria with the same overall position.
(c) Every Y-equilibrium profile is Pareto-efficient but there may be other Pareto-efficient profiles.

Proof. *(a)* For any $s \leq t$, let $x(s,t)$ be the profile where $x^i(s,t)$ is member i's most preferred location in $[s,t]$ and $\Phi(s,t) = \max_i d(x^i(s,t),\ avg(x(s,t)))$, which is the maximal distance between a member's position and the average. The function $\Phi(s,t)$ is continuous and $\Phi(s,s) = 0$ for all s. The function is also strictly decreasing in s when $s \in [L, R]$ and constant when $s \notin [L, R]$ and is strictly increasing in t when $t \in [L, R]$ and constant when $t \notin [L, R]$.

Since $R - L > 2$, it follows that $\Phi(L, R) > 1$. Thus, there is a unique $R' < R$ with $\Phi(L, R') = 1$ and a unique L' such that $\Phi(L', R) = 1$. For every $l \in (L, L')$, we have $\Phi(l, R) > 1$, and so there is a unique point $r(l)$ such that $\Phi(l, r(l)) = 1$.

It follows that $(-\infty, R'], [L', \infty)$ are Y-equilibria and there is no other unbounded Y-equilibrium. Also, for every $l \in (L, L')$ the set $[l, r(l)]$ is a Y-equilibrium. There is no other Y-equilibrium $[l, r]$ where $-\infty < l < L$ since $\Phi(l, R') = 1$ and therefore, if $R' < r$ then $\Phi(l, r) > 1$ and if $r \leq R'$ then $[l, r]$ is strictly included in the Y-equilibrium

$Y = (-\infty, R']$. There is no Y-equilibrium $[l, r]$ with $l > L'$ since such a set is a subset of the Y-equilibrium $[L', \infty)$.

(b) Notice that $avg(x(l, r(l)))$ is an increasing function of $l \in [L, L']$ since the intervals move to the right, the members' choices weakly move to the right and the members at L and R strictly move to the right. Thus, no two Y-equilibria have the same average choice.

(c) Consider a bounded Y-equilibrium of the type $[l, r]$ with the overall position t. Since $L < l$ and $r < R$, some members choose l while others choose r. By the maximality condition, $t = l+1$ or $t = r-1$.

WLOG, assume that $t = l+1$. Members with peaks to the left of l choose l but wish to move further left. Members with peaks between l and r choose their peaks. Members with peaks to the right of r choose r but wish to move further right. Suppose there is a Pareto-improvement with average t'. Members with peaks in $[l, r]$ remain at their peak. If members move only to the right, then $t' > t$, the members who choose l do not move and $d(l, t') > 1$, thus violating harmony. Otherwise, some members move to the left. Let λ be the largest move leftward. Since not all members move to the left, the average decreases by less than λ and therefore $d(l-\lambda, t') > 1$, thus violating feasibility.

A similar argument demonstrates the Pareto efficiency of the two unbounded Y-equilibria.

Finally, consider a three member group where $peak^1 = -2$, $peak^2 = -1$ and $peak^3 = 1$. The profile $(0, -1, +1)$ is Pareto-efficient since members 2 and 3 are at their peaks and any move by member 1 violates the harmony condition. However, the profile is not a Y-equilibrium outcome since member 1 prefers to move to -1 which must be in the permissible set . $\square$

For illustration, suppose that there are three members with peaks at 1, 4 and 7, respectively. The Y-equilibria are depicted in Figure 2. Notice that $R' = 2.5$ and $L' = 5.5$.

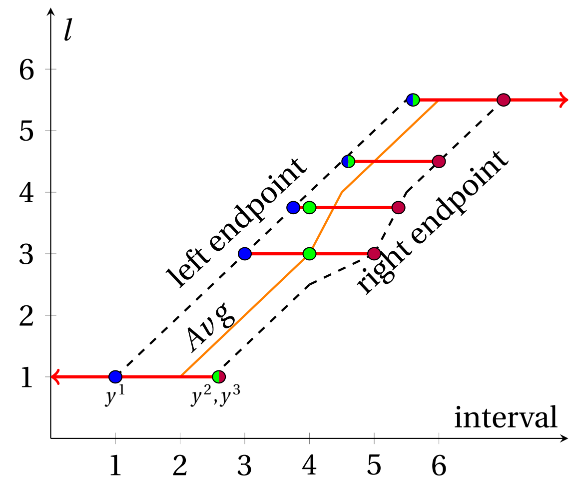

Figure 2: The two infinite Y-equilibrium permissible sets, $(-\infty, 2.5]$ and $[5.5, \infty)$, are depicted by the red arrows. The bounded permissible sets have left endpoints l ranging from 1 to 5.5. Each bounded Y-equilibrium is a horizontal interval between the dashed lines. Three such equilibria are depicted by the red line segments. The average of each permissible set is depicted by the orange curve. Choices of members 1, 2 and 3 are respectively depicted by blue, green and burgundy dots.

5.2 The near-average game

Every Y-equilibrium profile is Pareto-efficient and thus is also a Nash equilibrium of the near-average game. However, when there are at least five members, there are many more Nash equilibria, and in fact any position can be a Nash equilibrium overall position.

Proposition 6 *When $n \geq 5$, all positions, even those outside of $[L, R]$, are Nash equilibrium overall positions of the near-average game.*

Proof. Consider a profile of choices in which one member is at t and all others are split equally between $t-1$ and $t+1$. This is a Nash equilibrium of the near-average game regardless of the members' preferences, since any individual move to the left (right) will bring the average to below (above) t and makes it of distance larger 1 for at least one member who stays at $t+1$ ($t-1$). □

5.3 Comparing the solution concepts

Every possible position, including those outside $[L, R]$ is an overall Nash equilibrium position. Many of the Nash equilibria are Pareto-dominated. For example, any Nash equilibrium with overall position to the right of R is dominated by another Nash equilibrium. In contrast, the set of Y-equilibrium overall positions is restricted to $[L, R]$. Each overall position in this range is supported by a unique Y-equilibrium and the Y-equilibria profiles are Pareto efficient.

A Nash equilibrium like the one described in *(ii)* depends on a high degree of coordination between the players. A player's optimization requires knowledge of all other players' choices. Tragic coordination might leave players stuck at positions far from their peaks.

These Nash equilibria are especially tragic because from these profiles, no agent has a feasible deviation, and thus, they are Nash equilibria regardless of preferences. This means that all previously identified flaws can simultaneously apply to these Nash equilibria: in addition to Pareto inefficiency, they may be non-monotonic, they may feature envy where the middle agent wishes to move to one of the endpoints, and they may be unstable (a pair of agents might like to swap positions, but cannot individualistically move).

6. Final comments

This paper is a part of a larger project exploring "price-like" though "non-price" institutions that can bring harmony in conflicting social environments (see Piccione and Rubinstein (2007), Richter and Rubinstein (2015, 2020) and Rubinstein and Wolinksy (2021)).

Three models were presented in which each member of a group chooses a position along a line and stability depended on the combination of positions satisfying a specific constraint: "having enough support for one of the positions", "not being too far from the median" or "not being too far from the average". We characterized the Y-equilibrium concept and compared it to Nash equilibrium, and arrived at four observations:

(a) In all three models, there is a much larger multiplicity of Nash equilibria than Y-equilibrium choice profiles.

(b) Many of the Nash equilibria require an extraordinary degree of coordination between the members in order for them to jointly satisfy the harmony constraint. The Y-equilibrium solution concept only requires members to know the social restrictions, just as in the marketplace individuals need only know prices but not other members' actions.

(c) In two of the models, all positions – even outside the range of the members' peaks – are Nash equilibria overall positions. This is not the case for Y-equilibrium.

(d) In all three models there are non-monotonic Nash equilibria. A Y-equilibrium is always monotonic. Furthermore, most of the Nash equilibria have an envious flavor: some members are trapped with the responsibility of upholding feasibility while others are freer in their choices. The trapped members would like to move towards the freer members, but they cannot because doing so would destroy harmony either for themselves (as in the voting model) or for some other members (as in the other two models).

Needless to say, we are not arguing that the common game-theoretical approach is "wrong" or "valueless". But we do urge the reader to put a question mark before automatically applying Nash-equilibrium-like concepts and to consider alternative solution concepts in the spirit of the one presented here.

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