APPENDIX III

Chapter Three model

We begin with the annual model:

$$\begin{pmatrix} N_1' \\ N_2' \end{pmatrix} = \begin{bmatrix} (1-g_1)(1-r)+(1-d_1(1-c))S_{1,t}g_1 & (1-\mu)d_2(1-c)S_{2,t}g_2 \\ (1-\mu)d_2(1-c)S_{1,t}g_1 & (1-g_2)(1-r)+(1-d_2(1-c))S_{2,t}g_2 \end{bmatrix} \cdot \begin{pmatrix} N_1 \\ N_2 \end{pmatrix}.$$

By scaling $d_i' = d_i(1-c)$ and then dropping primes we get:

$$\begin{pmatrix} N_1' \\ N_2' \end{pmatrix} = \begin{bmatrix} (1-g_1)(1-r)+(1-d_1)S_{1,t}g_1 & (1-\mu)d_2S_{2,t}g_2 \\ (1-\mu)d_1S_{1,t}g_1 & (1-g_2)(1-r)+(1-d_2)S_{2,t}g_2 \end{bmatrix} \cdot \begin{pmatrix} N_1 \\ N_2 \end{pmatrix}.$$

We will now consider plants and seeds in the two patches. We collect these variables in the vector $\mathbf{x} = (N_1, P_1, N_2, P_2)^T$.

Starting at the beginning of the season with seeds, after overwintering and germination we have in the next generation:

$$\mathbf{M} \cdot \cdot \mathbf{x}$$

with

$$\mathbf{M} = \begin{bmatrix} 1-g_1 & 0 & 0 & 0 \\ g_1 & 0 & 0 & 0 \\ 0 & 0 & 1-g_2 & 0 \\ 0 & 0 & g_2 & 0 \end{bmatrix}.$$

After setting seed and redistribution of the seeds the number of plants and seeds is given by multiplication with the matrix

$$\mathbf{S}_t = \begin{bmatrix} (1-r) & (1-d_1)S_{1,t} & 0 & (1-\mu)d_2S_{2,t} \\ 0 & 0 & 0 & 0 \\ 0 & (1-\mu)d_1S_{1,t} & (1-r) & (1-d_2)S_{2,t} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The combined matrix $\mathbf{S}_t \cdot \mathbf{M}$ has the form

$$\mathbf{S}_t \cdot \mathbf{M} = \begin{bmatrix} (1-g_1)(1-r)+(1-d_1)g_1S_{1,t} & 0 & (1-\mu)d_2g_2S_{2,t} & 0 \\ 0 & 0 & 0 & 0 \\ (1-\mu)d_1g_1S_{1,t}(1-\mu) & 0 & (1-g_2)(1-r)+(1-d_2)g_2S_{2,t} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The number of seeds thus changes over time as:

$$\begin{aligned} N_1' &= ((1-g_1)(1-r)+(1-d_1)g_1S_{1,t})\,N_1 + (1-\mu)d_2g_2S_{t,2}N_2 \\ N_2' &= (1-\mu)d_1g_1S_{t,1}N_1 + ((1-g_2)(1-r)+(1-d_2)g_2S_{2,t})\,N_2 \end{aligned}$$

and we have recovered the original model with multiplication matrix

$$\begin{bmatrix} (1-g_1)(1-r)+(1-d_1)g_1S_{1,t} & (1-\mu)d_2g_2S_{2,t} \\ (1-\mu)d_1g_1S_{1,t}(1-\mu) & (1-g_2)(1-r)+(1-d_2)g_2S_{2,t} \end{bmatrix}.$$

Perennials. For perennial plants we need to modify the matrices $\mathbf{M}$ and $\mathbf{S}_t$. For $\mathbf{M}$:

$$\mathbf{M} = \begin{bmatrix} 1-g_1 & 0 & 0 & 0 \\ g_1 & \pi_1^w & 0 & 0 \\ 0 & 0 & 1-g_2 & 0 \\ 0 & 0 & g_2 & \pi_2^w \end{bmatrix}.$$

The matrix $\mathbf{S}_t$, as plants can now survive from one season to the next (for simplicity assuming that all plants survive the growing season and that mortality takes place over the winter), becomes:

$$\mathbf{S}_t = \begin{bmatrix} (1-r) & (1-d_1)S_{1,t} & 0 & (1-\mu)d_2S_{2,t} \\ 0 & \pi_1^s & 0 & 0 \\ 0 & (1-\mu)d_1S_{1,t} & (1-r) & (1-d_2)S_{2,t} \\ 0 & 0 & 0 & \pi_2^s \end{bmatrix}.$$

The combined matrix $\mathbf{S}_t \cdot \mathbf{M}$ now has the form

$$\begin{bmatrix} (1-g_1)(1-r)+(1-d_1)g_1S_{1,t} & (1-d_1)\pi_1^wS_{1,t} & (1-\mu)d_2g_2S_{2,t} & (1-\mu)d_2\pi_2^wS_{2,t} \\ g_1\pi_1^s & \pi_1^s\pi_1^w & 0 & 0 \\ (1-\mu)d_1g_1S_{1,t} & (1-\mu)d_1\pi_1^wS_{1,t} & (1-g_2)(1-r)+(1-d_2)g_2S_{2,t} & (1-d_2)\pi_2^wS_{2,t} \\ 0 & 0 & g_2\pi_2^s & \pi_2^s\pi_2^w \end{bmatrix}.$$

Note that by scaling $P_i' = \pi_i^w P_i$ and redefining $\pi_i = \pi_i^s\pi_i^w$ two parameters can be scaled out of the model. The matrix then reads:

$$\begin{bmatrix} (1-g_1)(1-r)+(1-d_1)g_1S_{1,t} & (1-d_1)S_{1,t} & (1-\mu)d_2g_2S_{2,t} & (1-\mu)d_2S_{2,t} \\ g_1\pi_1 & \pi_1 & 0 & 0 \\ (1-\mu)d_1g_1S_{1,t} & (1-\mu)d_1S_{1,t} & (1-g_2)(1-r)+(1-d_2)g_2S_{2,t} & (1-d_2)S_{2,t} \\ 0 & 0 & g_2\pi_2 & \pi_2 \end{bmatrix}.$$