The efficient index hypothesis and its implications in the BSM model

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Abstract

This note studies the behavior of an index I_t which is assumed to be a tradable security, to satisfy the BSM model $dI_t/I_t = \mu dt + \sigma dW_t$, and to be *efficient* in the following sense: we do not expect a prespecified trading strategy whose value is almost surely always nonnegative to outperform the index greatly. The efficiency of the index imposes severe restrictions on its growth rate; in particular, for a long investment horizon we should have $\mu \approx r + \sigma^2$, where r is the interest rate. This provides another partial solution to the equity premium puzzle. All our mathematical results are extremely simple.

1 Introduction

The efficient index hypothesis (EIH) is a version of the random walk hypothesis and the efficient market hypothesis. It is a statement about a specific index, such as S&P 500, and says that we do not expect a prespecified trading strategy to beat the index by a factor of $1/\delta$ or more, for a given threshold δ (such as $\delta = 0.1$). The trading strategy is assumed to be *prudent*, in the sense of its value being nonnegative a.s. at all times. By saying that it beats the index by a factor of $1/\delta$ or more we mean that its initial value is $\mathcal{K}_0 > 0$ and its final value \mathcal{K}_T satisfies $\mathcal{K}_T/I_T \geq (1/\delta)(\mathcal{K}_0/I_0)$. (By the value of a trading strategy we always mean the undiscounted dollar value of its current portfolio.) We will see that the EIH has several interesting implications, such as $\mu \approx r + \sigma^2$ for the growth coefficient μ of the index.

We use the EIH in the interpretation of our results, but their mathematical statements do not involve this hypothesis. For example, in Section 2 we prove that there is a prudent trading strategy that, almost surely, beats the index by a factor of at least 10 unless

$$\frac{I_T}{\mathrm{e}^{rT}} \in \left(\mathrm{e}^{\sigma^2 T/2 - 1.64\sigma\sqrt{T}}, \mathrm{e}^{\sigma^2 T/2 + 1.64\sigma\sqrt{T}}\right) \tag{1.1}$$

(see Proposition 2.1). If we believe in the EIH (for $\delta = 0.1$), we should believe in (1.1). But even if we do not believe in the EIH, the proposition gives us a way of beating the index when (1.1) is violated. As used in this note, the EIH is a weaker assumption than it appears to be. There might be sophisticated prudent trading strategies that do beat the index (by a large factor), but we are not interested in such strategies. It is sufficient that the primitive strategies considered in this note be not expected to beat the index.

Our EIH is obviously related, and has a similar motivation, to the standard efficient markets hypothesis [2]. There are, however, important differences. For example, the EIH does not assume that the security prices are "correct" in any sense, or that investors' expectations are rational (individually or *en masse*). The EIH controls for risk only by insisting that our trading strategies be prudent. Admittedly, this is a weak requirement, and so the threshold value of δ should be a small number; in our examples, we use $\delta = 0.1$. (If a trader is worried about losing all money, nothing prevents her from investing only part of her capital in prudent strategies that can lose everything.)

Remark. In [9, 11], the EIH was referred to as the "efficient market hypothesis", whereas the standard hypothesis of market efficiency as the "efficient markets hypothesis", with "markets" in plural. However, nowadays the standard hypothesis is more often called the "efficient market hypothesis" than the "efficient markets hypothesis", and so it is safer to use a different term for our hypothesis. The results of this note agree with the results of [11] (see, e.g., (1) of [11] as applied to $s_n := r, \forall n$), which were obtained using very different methods.

We start the main part of the note with results about the growth rate of the index under the EIH (Section 2). The main insight here is that the index outperforms the bond approximately by a factor of $e^{\sigma^2 T/2}$ (cf. (1.1)). In the following section, Section 3, we show that, under the EIH, $\mu \approx r + \sigma^2$. Section 4 applies this result to the equity premium puzzle; the equity premium of σ^2 is closer to the observed levels of the equity premium than the predictions of standard theories. Section 5 discusses our findings from the point of view of game-theoretic probability (see, e.g., [9]).

2 Growth rate of the index

The time interval in this note is [0, T], T > 0; in the interpretation of our results the horizon T will be assumed to be a large number. The value of the index at time t is denoted I_t . We assume that it satisfies the BSM (Black–Scholes– Merton) model

$$\frac{\mathrm{d}I_t}{I_t} = \mu \mathrm{d}t + \sigma \mathrm{d}W_t \tag{2.1}$$

and that $I_0 = 1$. The interest rate r is assumed constant. We will sometimes interpret e^{rt} as the price at time t of a zero-coupon bond whose initial price is 1.

The risk-neutral version of (2.1) is

$$\frac{\mathrm{d}I_t}{I_t} = r\mathrm{d}t + \sigma\mathrm{d}W_t. \tag{2.2}$$

The explicit strong solution to this SDE is $I_t = e^{(r-\sigma^2/2)t+\sigma W_t}$.

Let $E \subseteq \mathbb{R}$ be a Borel set. If $F : \mathbb{R} \to \mathbb{R}$, we let F(E) stand for the set $\{F(x) \mid x \in E\}$; this convention defines the meaning of expressions such as $\frac{\ln E}{2} - 1$. The BSM price at time 0 of the European contingent claim whose payoff at time T is

$$F(I_T) := \begin{cases} I_T & \text{if } I_T \in E\\ 0 & \text{otherwise} \end{cases}$$
(2.3)

can be computed as the discounted expected value

$$e^{-rT} \mathbb{E} \left(e^{(r-\sigma^2/2)T + \sigma\sqrt{T}\xi} \mathbf{1} \left\{ e^{(r-\sigma^2/2)T + \sigma\sqrt{T}\xi} \in E \right\} \right)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\frac{\ln E}{\sigma\sqrt{T}} - \frac{r}{\sigma}\sqrt{T} + \frac{\sigma}{2}\sqrt{T}} e^{-\sigma^2 T/2 + \sigma\sqrt{T}x - x^2/2} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\frac{\ln E}{\sigma\sqrt{T}} - \frac{r}{\sigma}\sqrt{T} + \frac{\sigma}{2}\sqrt{T}} e^{-(x-\sigma\sqrt{T})^2/2} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\frac{\ln E}{\sigma\sqrt{T}} - \frac{r}{\sigma}\sqrt{T} - \frac{\sigma}{2}\sqrt{T}} e^{-y^2/2} dy$$

$$= N_{0,1} \left(\frac{\ln E}{\sigma\sqrt{T}} - \frac{r}{\sigma}\sqrt{T} - \frac{\sigma}{2}\sqrt{T} \right), \qquad (2.4)$$

where $\xi \sim N_{0,1}$, $N_{0,1}$ is the standard Gaussian distribution on \mathbb{R} , and $\mathbf{1}\{\ldots\}$ is defined to be 1 if the condition in the curly braces is satisfied and 0 otherwise.

Since the BSM price can be hedged perfectly (see, e.g., [3], Theorem 5.8.12), there is a prudent trading strategy Σ with initial value (2.4) and final value (2.3) a.s. We can see that Σ beats the market by the reciprocal to (2.4) if *E* happens.

Two special cases

Let $\delta \in (0,1)$ and $E := (-\infty, A] \cup [B, \infty)$, where A and B are chosen such that

$$\frac{\ln A}{\sigma\sqrt{T}} - \frac{r}{\sigma}\sqrt{T} - \frac{\sigma}{2}\sqrt{T} = -z_{\delta/2}, \quad \frac{\ln B}{\sigma\sqrt{T}} - \frac{r}{\sigma}\sqrt{T} - \frac{\sigma}{2}\sqrt{T} = z_{\delta/2}, \quad (2.5)$$

where z_p is the upper *p*-quantile of the standard Gaussian distribution, i.e., is defined by the requirement that $N_{0,1}([z_p, \infty)) = p$. Then $N_{0,1}(E) = \delta$. Equations (2.5) give

$$A = \mathrm{e}^{rT} \, \mathrm{e}^{\sigma^2 T/2 - z_{\delta/2} \sigma \sqrt{T}}, \quad B = \mathrm{e}^{rT} \, \mathrm{e}^{\sigma^2 T/2 + z_{\delta/2} \sigma \sqrt{T}}.$$

We can state the result of our calculations as follows.

Proposition 2.1. Let $\delta > 0$. There is a prudent trading strategy (depending on σ, r, T, δ) that, almost surely, beats the index by a factor of $1/\delta$ unless

$$\frac{I_T}{\mathrm{e}^{rT}} \in \left(\mathrm{e}^{\sigma^2 T/2 - z_{\delta/2}\sigma\sqrt{T}}, \mathrm{e}^{\sigma^2 T/2 + z_{\delta/2}\sigma\sqrt{T}}\right).$$
(2.6)

Equation (2.6) says that for large T the efficient index can be expected to outperform the bond $e^{\sigma^2 T/2}$ -fold. The case $\delta \geq 1$ in Proposition 2.1 is trivial, but we do not exclude it to simplify the statement of the proposition.

If we are only interested in a lower or upper bound on I_T , we should instead consider the set $E := (-\infty, A]$ or $E := [B, \infty)$, respectively. We will obtain (2.5) with δ in place of $\delta/2$ and, therefore, will obtain the following proposition.

Proposition 2.2. Let $\delta > 0$. There is a prudent trading strategy that, almost surely, beats the index by a factor of $1/\delta$ unless

$$\frac{I_T}{\mathrm{e}^{rT}} > \mathrm{e}^{\sigma^2 T/2 - z_\delta \sigma \sqrt{T}} \,.$$

There is another prudent trading strategy that, almost surely, beats the index by a factor of $1/\delta$ unless

$$\frac{I_T}{z^{rT}} < e^{\sigma^2 T/2 + z_\delta \sigma \sqrt{T}} \,. \tag{2.7}$$

It is clear that Propositions 2.1 and 2.2 are tight in the sense that the factor $1/\delta$ cannot be improved.

3 Implications for μ

The following corollary of Proposition 2.1 shows that the EIH and the BSM model (2.1) imply $\mu \approx r + \sigma^2$.

Proposition 3.1. For each $\delta > 0$ there exists a prudent trading strategy $\Sigma = \Sigma(\sigma, r, T, \delta)$ that satisfies the following condition. For each $\epsilon > 0$, either

$$\left|r + \sigma^2 - \mu\right| < \frac{(z_{\delta/2} + z_{\epsilon})\sigma}{\sqrt{T}} \tag{3.1}$$

or Σ beats the index by a factor of at least $1/\delta$ with probability at least $1-\epsilon$.

Intuitively, $\mu \approx r + \sigma^2$ unless we can beat the index or a rare event happens (assuming that δ and ϵ are small and T is large).

Proof of Proposition 3.1. Without loss of generality, assume $\delta, \epsilon \in (0, 1)$. As Σ we take a prudent trading strategy that beats the index by a factor of $1/\delta$ unless (2.6) holds. Therefore, we are only required to prove that the event that (2.6) holds but (3.1) does not has probability at most ϵ . We can rewrite (2.6) as

$$\left|\ln I_T - rT - \frac{\sigma^2}{2}T\right| < z_{\delta/2}\sigma\sqrt{T}.$$
(3.2)

Remembering that (2.1) has explicit solution $I_t = e^{(\mu - \sigma^2/2)t + \sigma W_t}$, we can rewrite (3.2) as

$$\left|\sigma\sqrt{T}\xi - (r+\sigma^2-\mu)T\right| < z_{\delta/2}\sigma\sqrt{T},$$

where $\xi \sim N_{0,1}$, i.e., as

$$\left| \xi - \frac{(r + \sigma^2 - \mu)\sqrt{T}}{\sigma} \right| < z_{\delta/2}.$$
(3.3)

If (3.1) is violated, we have either $r + \sigma^2 - \mu < -(z_{\delta/2} + z_{\epsilon})\sigma/\sqrt{T}$ or $r + \sigma^2 - \mu > (z_{\delta/2} + z_{\epsilon})\sigma/\sqrt{T}$. The two cases are analogous, and we consider only the first. In this case, (3.3) implies $\xi < -z_{\epsilon}$, the probability of which is ϵ .

Proposition 3.1 shows that the arbitrariness of μ in the BSM model (2.1) for the index is to a large degree illusory if we accept the EIH.

The strategy Σ of Proposition 3.1 depends only on σ , r, T, and δ . If we allow, additionally, dependence on μ and ϵ , we can use Proposition 2.2 instead of Proposition 2.1 and strengthen (3.1) by replacing $\delta/2$ with δ .

Proposition 3.2. Let $\delta > 0$ and $\epsilon > 0$. Unless

$$\left|r + \sigma^2 - \mu\right| < \frac{(z_{\delta} + z_{\epsilon})\sigma}{\sqrt{T}},\tag{3.4}$$

there exists a prudent trading strategy $\Sigma = \Sigma(\mu, \sigma, r, T, \delta, \epsilon)$ that beats the index by a factor of at least $1/\delta$ with probability at least $1 - \epsilon$.

Proof. Suppose (3.4) is violated. Since the cases $r + \sigma^2 - \mu < -(z_{\delta} + z_{\epsilon})\sigma/\sqrt{T}$ and $r + \sigma^2 - \mu > (z_{\delta} + z_{\epsilon})\sigma/\sqrt{T}$ are analogous, we will assume

$$r + \sigma^2 - \mu < -\frac{(z_\delta + z_\epsilon)\sigma}{\sqrt{T}}.$$
(3.5)

(Our trading strategy depends on which of the two cases holds, and so depends on μ and ϵ .) As Σ we take a prudent trading strategy that beats the index by a factor of $1/\delta$ unless (2.7) holds. We are required to prove that the probability of (2.7) is at most ϵ . We can rewrite (2.7) as

$$\ln I_T - rT - \frac{\sigma^2}{2}T < z_\delta \sigma \sqrt{T},$$

i.e.,

$$\xi - \frac{(r + \sigma^2 - \mu)\sqrt{T}}{\sigma} < z_{\delta},$$

where $\xi \sim N_{0,1}$. The last inequality and (3.5) imply $\xi < -z_{\epsilon}$, whose probability is ϵ .

4 Equity premium puzzle

The equity premium is the excess of stock returns over bond returns, and it appears to be higher in the real world than suggested by standard economic

theories. This has been dubbed the equity premium puzzle [7]. There is no consensus as to the explanation, or even to the existence, of the equity premium puzzle; for recent reviews see, e.g., [5, 8]. In this section we will see that our results can be interpreted as providing a partial solution to the puzzle.

According to Proposition 3.1, under the EIH we can expect $\mu \approx r + \sigma^2$. This gives the equity premium σ^2 . The annual volatility of S&P 500 is approximately 20% (see, e.g., [6], p. 3, or [5], p. 8), which translates into an expected 4% equity premium. The standard theory predicts an equity premium of at most 1% ([5], p. 11).

The empirical study by Mehra and Prescott reported in [6], Table 2, estimates the equity premium over the period 1889–2005 as 6.36%. Taking into account the later years 2006–2010 reduces it, but not much, to 6.05%. (The recent news about bonds outperforming stocks over the past 30 years [4] were about 30-year Treasury bonds, whereas Mehra and Prescott use short-term Treasury bills for this period.) Our figure of 4% is below 6.05%, but the difference is much less significant than for the standard theory. If the years 1802–1888 are also taken into account (as done by Siegel [10], updated until 2004 by Mehra and Prescott [6], Table 2, and until 2010 by myself), the equity premium goes down to 5.17%.

Equation (2.6) allows us to estimate the accuracy of our estimate σ^2 of the equity premium. Namely, we have, almost surely,

$$\frac{1}{T} \int_0^T \frac{\mathrm{d}I_t}{I_t} - r - \sigma^2 = \frac{\ln I_T + \sigma^2 T/2 - rT - \sigma^2 T}{T} \in \left(-\frac{z_{\delta/2}\sigma}{\sqrt{T}}, \frac{z_{\delta/2}\sigma}{\sqrt{T}}\right) \quad (4.1)$$

unless a prespecified prudent trading strategy beats the index by a factor of $1/\delta$. Plugging $\delta := 0.1$ (to obtain a reasonable accuracy), $\sigma := 0.2$, and T := 2010 - 1888, we evaluate $z_{\delta/2}\sigma/\sqrt{T}$ in (4.1) to 2.98% for the period 1889–2010, and changing T to 2010-1801, we evaluate it to 2.28% for the period 1802–2010. For both periods, the observed equity premium falls well within the prediction interval.

5 Three kinds of probabilities for the index

In this section we will take a broader view of the simple results of the previous sections. We started from the "physical" probability measure (2.1), used the risk-neutral probability measure (2.2), and saw the importance of the "EIH measure"

$$\frac{\mathrm{d}I_t}{I_t} = (r + \sigma^2)\mathrm{d}t + \sigma\mathrm{d}W_t.$$
(5.1)

We will see that the last two are essentially special cases of game-theoretic probability, as defined in [9]. If E is a Borel subset of the Banach space $\Omega := C([0,T])$ of all continuous functions on [0,T], we define its upper probability with bond as numéraire by

$$\overline{\mathbb{P}}_{\mathbf{b}}(E) := \inf \left\{ \mathcal{K}_0 \ \middle| \ \frac{\mathcal{K}_T}{\mathbf{e}^{rT}} \ge \mathbf{1}_E \ \text{a.s.} \right\},\$$

where $\mathbf{1}_E$ is the indicator function of E, \mathcal{K} ranges over the value processes of prudent trading strategies, and "a.s." means with probability one under the physical measure (2.1) (equivalently, under (2.2) or under (5.1)). In other words, $\overline{\mathbb{P}}_{\mathrm{b}}(E)$ is the infimum of $\delta > 0$ such that a prudent trading strategy can beat the bond by a factor of $1/\delta$ or more on the event E (except for its subset of zero probability). We define the *upper probability* of E with index as numéraire by

$$\overline{\mathbb{P}}_{I}(E) := \inf \left\{ \mathcal{K}_{0} \mid \frac{\mathcal{K}_{T}}{I_{T}} \geq \mathbf{1}_{E} \text{ a.s.} \right\}.$$

In other words, $\overline{\mathbb{P}}_{I}(E)$ is the infimum of $\delta > 0$ such that a prudent trading strategy can beat the index by a factor of $1/\delta$ or more on the event E.

For each Borel E, $\overline{\mathbb{P}}_{b}(E)$ is the risk-neutral measure of E and $\overline{\mathbb{P}}_{I}(E)$ is its EIH measure. It is standard in game-theoretic probability to define the corresponding *lower probabilities*

$$\underline{\mathbb{P}}_{\mathbf{b}}(E) := 1 - \overline{\mathbb{P}}_{\mathbf{b}}(E^c) \text{ and } \underline{\mathbb{P}}_I(E) := 1 - \overline{\mathbb{P}}_I(E^c),$$

where $E^c := \Omega \setminus E$. Since our market is complete, upper and lower probabilities always coincide. A major difference of the definitions of $\overline{\mathbb{P}}_b$ and $\overline{\mathbb{P}}_I$ from the usual definitions of upper probabilities in game-theoretic probability is the presence of "a.s."; in game-theoretic probability "a.s." is absent as there is no probability measure to begin with.

The processes (2.2) and (5.1) are in some sense reciprocal. By Itô's formula, if I_t satisfies (2.2), then $I_t^* := e^{2rt}/I_t$ will satisfy (5.1) with I^* in place of Iand -W in place of W, and vice versa. (The definition of I_t^* makes the bond's growth rate e^{rt} the geometric mean of I_t and I_t^* .) In particular, the growth rate of typical trajectories of (2.2) is approximately $e^{(r-\sigma^2/2)t}$, and the growth rate of typical trajectories of (5.1) is approximately $e^{(r+\sigma^2/2)t}$.

Acknowledgments

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