# Representation Theory of the Symmetric Group

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## Declaration of Authorship

I, Jasdeep Kochhar, hereby declare that this thesis and the work presented in it is either entirely my own, or completed in collaboration with others as indicated in the text. Where I have consulted the work of others, this is always clearly stated.

Signed:

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## Dedication

For my late mother Bhagwant. Whilst she never saw me start this PhD, I know that she would be filled with pride now that I have finished it.

## Notation

We write $\mathbf{N}$ for the set of natural numbers and $\mathbf{N}_0$ for $\mathbf{N} \cup \{0\}$. We write $\mathbf{Z}, \mathbf{Q}$, and $\mathbf{C}$ for the set of integers, rationals and complex numbers, respectively. If $p$ is a prime number, then we write $\mathbf{Q}_p$ for the field of $p$-adic numbers and $\mathbf{Z}_p$ for the ring of $p$-adic integers. Also $\mathbf{F}_p$ denotes the finite field with $p$ elements.

Given a set $X$, we write $\mathrm{Sym}(X)$ for the symmetric group on $X$. Given $n \in \mathbf{N}$, we write $S_n$ for the symmetric group $\mathrm{Sym}(\{1, 2, \ldots, n\})$.

Given a field $F$ and a finite group $G$, we work with left $FG$-modules throughout. Over an appropriate field, $\mathrm{Irr}(G)$ denotes the set of ordinary irreducible characters of $G$, and $\mathrm{Lin}(G)$ denotes the group of degree-one characters of $G$.

We write $U \otimes V$ for the inner tensor product of $FG$-modules $U$ and $V$. Given a finite group $H$, if $U$ is an $FG$-module and $V$ is an $FH$-module, then $U \boxtimes V$ denotes the $F[G \times H]$-module given by the outer tensor product of $U$ and $V$.

The induction and restriction of modules over finite dimensional group algebras are denoted by $\uparrow$ and $\downarrow$, respectively.

Given a subgroup $H \leq G$, we write $C_G(H)$ and $N_G(H)$ for the centraliser and normaliser subgroups of $H$ in $G$, respectively.

## Abstract

In this thesis we consider problems in the representation theory of $S_n$ and the representation theory of the imprimitive wreath product $G \wr S_n$, for a finite group $G$.

In §1 we give the background from the representation theory of $S_n$ required throughout this thesis. We also collect the required background on the representation theory of $G \wr S_n$, where $G$ is a finite group, noting that, in most of this thesis, we specialise this background to the case when $G = C_2$.

In §2 we provide a new proof of the Murnaghan–Nakayama rule. We do this by computing the trace of the matrix representing the action of an $n$-cycle on the standard basis of a skew Specht module indexed by a border strip partition. This work in this chapter is joint with Mark Wildon.

In §3 we consider the odd-degree irreducible characters of $G \wr S_{2^n}$ for particular groups $G$. We consider the restrictions of these irreducible characters to the normaliser of a Sylow 2-subgroup for each of these groups, and give bijective proofs of the McKay conjecture for the groups considered. We also consider the low degree constituents of the restriction of an odd-degree irreducible $S_{2^n}$-character to its Sylow 2-subgroup.

In §4 we consider the modular representation theory of the symmetric group. We express the $FS_n$-permutation module $M^{(\lambda_1,\lambda_2)}$ as a sum of its indecomposable summands, where $F$ is a field of characteristic 3. We do this using the endomorphism algebra of this permutation module via the Schur algebra.

From §5 onwards we consider the representation theory of wreath products. In §5 we determine certain decomposition numbers of $C_2 \wr S_n$. We do this using Brauer reciprocity by determining projective summands of a module whose ordinary character forms an involution model of $C_2 \wr S_n$.

In §6 we consider $C_2 \wr S_n$ as the symmetry group of the $n$-hypercube, and we determine the homology of the chain complex induced by the boundary map of the $n$-hypercube. We do this both in fields of characteristic 0 and in fields of strictly positive characteristic.

In §7 we consider two generalisations of the Foulkes characters. The Foulkes characters are the subject of Foulkes' conjecture, which remains a fundamental open problem in the representation theory of symmetric groups and their wreath products.

## Contents

CHAPTER 1

# Introduction and background

Let $G$ be a finite group, and let $F$ be a field. Define the *group algebra* $FG$ to be the associative $F$-algebra with basis

$$\{v_g : g \in G\},$$

and basis multiplication given by $v_g v_h := v_{gh}$, which we extend linearly to $FG$. For ease of notation, we write $g$ for the basis vector $v_g$.

Given an $F$-vector space $V$, we define a *representation of* $G$ to be a homomorphism

$$\rho : FG \to \mathrm{End}_F(V),$$

where $\mathrm{End}_F(V)$ denotes the vector space of all $F$-linear transformations from $V$ to itself. Throughout this thesis we write our maps on the left. We say that $V$ is a *left $FG$-module* in this case, and we see that there is an action of $FG$ on $V$ given by the linear extension of

$$gv = \rho(g)v,$$

where $g \in G$ and $v \in V$. In this thesis we only consider $FG$-modules $V$ that are finite dimensional.

We say that $V$ is *irreducible* if it contains no proper non-zero subspace $U$ such that $U$ is also an $FG$-module. The following theorem, known as Maschke's Theorem, demonstrates the importance of irreducible modules in the representation theory of finite groups.

Theorem (Maschke's Theorem). *Let $F$ be a field of characteristic $p$. Then every $FG$-module can be written as the direct sum of irreducible $FG$-modules if and only if $p \nmid |G|$.*

Maschke's Theorem shows that if $p \nmid |G|$, then irreducible modules are the building blocks of all $FG$-modules. The representation theory of $G$ in this case is referred to as the *ordinary representation theory* of $G$.

We refer to the representation theory in the case when $p \mid |G|$ as the *modular representation theory* of $G$. Whilst it is no longer true in this setting that an arbitrary $FG$-module can be written as a direct sum of irreducible modules, it can still always be written as a direct sum of indecomposable modules, which we now define. We say that an $FG$-module $M$ is *indecomposable* if whenever there exists an equality of $FG$-modules $M = U \oplus V$, then either $U = 0$, or $V = 0$. However, as we will see in §4 of this thesis, it is a difficult problem in general to write an $FG$-module as a sum of indecomposable submodules.

Fix $n \in \mathbf{N}$. A central subject of study in this thesis is the representation theory of the symmetric group $S_n$. If $F$ is a field of characteristic $p$ such that $p \nmid n!$, then the irreducible $FS_n$-modules are completely understood. Moreover, we can get a considerable way in understanding the ordinary representation theory of $S_n$ using ideas from combinatorics (see for instance §2). Nevertheless there are still many open problems in the ordinary case, an example of which we will see in §3. In the modular case, the situation is much less understood. For instance when $p \mid n!$, we are able to construct the irreducible $FS_n$-modules, however determining simple properties, such as their dimensions, remain unknown in general. In this thesis we therefore concern ourselves with both problems in the ordinary and modular representation theories of $S_n$.

Also of significant interest in this thesis is the representation theory of the imprimitive wreath product $G \wr S_n$, for certain finite groups $G$. In particular we will see that the representation theory of $S_n$ is closely related to the representation theory of the imprimitive wreath product $C_2 \wr S_n$. However the representation theory of $C_2 \wr S_n$ is significant in its own right, and there remain problems that cannot be approached solely using the representation theory of $S_n$ (see for example Theorem 5.1.1).

We now provide a survey of the chapters and main results of this thesis. This thesis can be thought of as being made up of two parts. The first part consists of §2, §3 and §4, in which we consider problems in the representation theory of $S_n$. The second part is made up of §5, §6, and §7, in which we consider the representation theory of $G \wr S_n$ for certain finite groups $G$. In particular we consider problems in the representation theory of $C_2 \wr S_n$ in §5 and §6. In §7 we consider problems motivated by the ordinary representation theory of $S_m \wr S_n$, where $m \in \mathbf{N}$.

In the remainder of this chapter we collect the background that we use throughout. We start by giving the relevant background on skew partitions, Young diagrams, and the ordinary representation theory of the symmetric group in §1.1. In order to state the main theorems from §2 to §7, we require the following notation from §1.1. Given a skew partition $\lambda/\mu$, we write $\chi^{\lambda/\mu}$ for the ordinary $S_n$-character afforded by $\lambda/\mu$. If $\mu = \varnothing$, then we write $\chi^{\lambda}$ for $\chi^{\lambda/\varnothing}$.

In §1.2 we introduce the imprimitive wreath product $G \wr S_n$, where $G$ is a finite group. In particular we define $G \wr S_n$, and we give a complete description of the irreducible $\mathbf{C}G \wr S_n$-modules. On the way to determining the irreducible $\mathbf{C}G \wr S_n$-modules, we also give a description of $G \wr S_n$-conjugacy classes.

In §1.3 we give the background results that we require on the modular representation theory of finite groups. We then specialise this background in §1.3.6 to give the required results on the modular representation theory of $S_n$. Furthermore, in §1.4 we give the required background on the representation theory of $C_2 \wr S_n$, thus specialising the results in §1.2 and §1.3 to this case.

In §2 we provide a new proof of the Murnaghan–Nakayama rule, which is a combinatorial rule for calculating character values of $S_n$. In order to state the rule, we require the following elementary definitions. Given partitions $\lambda$ and $\mu$, if $\mu$ is a subpartition (see §1.1.1) of $\lambda$, then write $\mu \subset \lambda$. In this case, write $\mathrm{ht}(\lambda/\mu)$ for one less than the number of non-empty rows of the skew diagram $[\lambda/\mu]$. If $\lambda/\mu$ has size $n$, then we write $|\lambda/\mu| = n$. We also require the definition of a border strip, which can be found in §1.1.1.

Theorem 2.1.1 (Murnaghan–Nakayama rule). *Let $m, n \in \mathbf{N}$, and let $\lambda$ be a partition of $m+n$. Let $\rho \in S_{m+n}$ be an $n$-cycle and let $\pi$ be a permutation of the remaining $m$ numbers. Then*

$$\chi^\lambda(\pi\rho) = \sum (-1)^{\mathrm{ht}(\lambda/\mu)} \chi^\mu(\pi),$$

*where the sum is over all $\mu \subset \lambda$ such that $|\mu| = m$ and $\lambda/\mu$ is a border strip.*

The proof of the rule that we give requires only the basic definitions of polytabloids and Garnir relations, and the relatively elementary Young and Pieri rules. The work in §2 is joint work with Mark Wildon, and is based on the paper [**42**], which is to appear in Annals of Combinatorics.

In §3 we consider a problem in the ordinary representation theory of $S_n$ surrounding local-global conjectures. An aim of these conjectures is to understand the representation theory of a finite group by considering the representation theory of a smaller group. The conjecture that motivates §3 is the McKay Conjecture, which we now describe. Let $G$ be a finite group, with Sylow 2-subgroup $P$. Also let $\mathrm{Irr}_{2'}(G)$ denote the set of irreducible odd-degree characters of $G$. Then the McKay Conjecture states that $|\mathrm{Irr}_{2'}(G)| = |\mathrm{Irr}_{2'}(N_G(P))|$. Although a proof of the conjecture is known, finding a canonical bijection between the relevant sets is of increasing interest. Giannelli accomplishes this for $S_n$ in [**22**], and our contribution to this problem is the following theorem.

Theorem 3.0.1. *Let $G$ be one of the following groups:*

- $S_{2^a}$, *where $a \in \mathbf{N}$*
- $C_2^a$, *where $a \in \mathbf{N}$*
- *any finite abelian $p$-group, where $p$ is an odd prime,*

*and let $P$ be a Sylow 2-subgroup of $G \wr S_{2^n}$. Given $\chi \in \mathrm{Irr}_{2'}(G \wr S_{2^n})$, the restricted character $\chi\big\downarrow_{N_{G \wr S_{2^n}}(P)}$ has a unique degree-one constituent, denoted*

$\Phi(\chi)$. *Moreover, the map* $\chi \mapsto \Phi(\chi)$ *is a bijection between* $\mathrm{Irr}_{2'}(G \wr S_{2^n})$ *and* $\mathrm{Irr}(N_{G\wr S_{2^n}}(P))$.

In §4 we turn to the modular representation theory of $S_n$. Define $M^{(\lambda_1,\lambda_2)}$ to be the $FS_n$-permutation module corresponding to the action of $S_n$ on the cosets of the Young subgroup $S_{\{1,2,\dots,\lambda_1\}} \times S_{\{\lambda_1+1,\dots,n\}}$. A notoriously difficult open problem is to express $M^{(\lambda_1,\lambda_2)}$ as a direct sum of indecomposable $FS_n$-modules. Previously this problem has only been solved over fields of characteristic 2, and in §4 we give a complete solution over fields of characteristic 3. We do this by determining a complete set of central primitive idempotents in the endomorphism algebra $S_F(\lambda) := \mathrm{End}_{FS_n}(M^{(\lambda_1,\lambda_2)})$. The work in §4 is based on the paper [**40**].

Over a field of characteristic 2 the primitive idempotents of $S_F(\lambda)$ are constructed as follows: to each $(m, g) \in \mathbf{N}_0^2$ assign an element $\widetilde{e}_{m,g} \in S_F(\lambda)$. Then the set of $\widetilde{e}_{m,g}$ such that $g \leq \lambda_2$ and the binomial coefficient

$$B(m, g) := \binom{m + 2g}{g}$$

is non-zero modulo 2 is a complete set of primitive idempotents in $S_F(\lambda)$. When $F$ has characteristic 3, our construction uses the same idea, and we assign elements $e_{m,g} \in S_F(\lambda)$ to the $(m, g) \in \mathbf{N}_0^2$ such that $B(m, g)$ is non-zero modulo 3. For the complete definition of the elements $e_{m,g}$, we refer the reader to §4.1.1. Our first main result in §4 is the following theorem.

THEOREM 4.1.3. *Given* $n \in \mathbf{N}$, *let* $\lambda = (\lambda_1, \lambda_2) \vdash n$ *and* $m = \lambda_1 - \lambda_2$. *The set of elements* $e_{m,g}$, *with* $B(m, g)$ *non-zero modulo* 3 *and* $g \leq \lambda_2$, *give a complete set of primitive orthogonal idempotents for* $S_F(\lambda)$.

Our second main result in §4 determines the Young module summand $Y^\mu$ of $M^\lambda$ that the idempotent $e_{m,g}$ corresponds to. For a definition of the Young module $Y^\mu$, see §4.1.

THEOREM 4.1.4 *Let* $\lambda = (\lambda_1, \lambda_2)$ *and* $\mu = (\mu_1, \mu_2)$ *be partitions of* $n$ *such that* $Y^\mu$ *is a direct summand of* $M^\lambda$. *Define*

$$m = \lambda_1 - \lambda_2 \text{ and } g = \lambda_2 - \mu_2.$$

*Then* $e_{m,g}$ *is the primitive idempotent in* $S_F(\lambda)$ *such that* $e_{m,g}M^\lambda \cong Y^\mu$.

For §5 onwards we consider problems arising in the representation theory of $G \wr S_n$. Before we continue, we remark that the work in §5 is based on the paper [**41**], which is to appear in Algebras and Representation Theory. In §5 we consider decomposition numbers of $C_2 \wr S_n$, the definition of which we give in §1.4.4. Determining decomposition numbers of $C_2 \wr S_n$ remains an open problem. Although this problem can be reduced to the representation theory of $S_n$, our approach relies on first characterising the vertices (see §1.3) of the indecomposable summands of the twisted Baddeley module $M_{(2a,b,c)}$, which we define in §5.1.

In order to state our result on vertices, the following preliminaries are required. Briefly write $S_{2n}$ for the symmetric group

$$\mathrm{Sym}(\{1,2,\ldots,n,\overline{1},\ldots,\overline{n}\}).$$

We view $C_2 \wr S_n$ as the subgroup of $S_{2n}$ generated by the set

$$\{(1\ \overline{1}),(1\ 2)(\overline{1}\ \overline{2}),(1\ 2\ldots n)(\overline{1}\ \overline{2}\ldots\overline{n})\}.$$

Given $a \in \mathbf{N}$, define $V_a$ to be equal to the subgroup

$$\langle(1\ \overline{1})(a+1\ \overline{a+1}),(2\ \overline{2})(a+2\ \overline{a+2}),\ldots,(a\ \overline{a})(2a\ \overline{2a})\rangle \rtimes \xi(S_2 \wr S_a),$$

where $\xi$ is as defined in §1.4.1. Also define $V_\lambda$ to be the subgroup of $V_a$ equal to

$$\langle(1\ \overline{1})(a+1\ \overline{a+1}),(2\ \overline{2})(a+2\ \overline{a+2}),\ldots,(a\ \overline{a})(2a\ \overline{2a})\rangle \rtimes \xi(S_2 \wr S_\lambda),$$

where $\lambda$ is a partition of $a$, and $S_\lambda$ is the corresponding Young subgroup of $S_a$ (defined in §1.1.1).

Given a prime $p$ and $r \in \mathbf{N}$ such that $rp \le n$, define

$$T'_r := \{(\lambda,t,u) : \lambda \in \Lambda(2,s), 2s+t+u=r \text{ and } sp \le a, tp \le b, up \le c\},$$

where $\Lambda(2,s)$ denotes the set of all compositions of $s$ in at most 2 parts.

THEOREM 5.1.1. *Let $(a,b,c) \in \mathbf{N}_0^3$ be such that $2a+b+c=n$, and let $U$ be a non-projective indecomposable summand of $M_{(2a,b,c)}$. Then $U$ has a vertex equal to a Sylow $p$-subgroup of*

$$V_{p\lambda} \times C_2 \wr S_{tp} \times C_2 \wr S_{up},$$

*for some $r \in \mathbf{N}$, where $rp \le n$, and $(\lambda,t,u) \in T'_r$.*

In order to state our second main theorem in §5, the following preliminaries are required. Given a $p$-core partition $\gamma$ (see §1.3.6) and given $b \in \mathbf{N}_0$, let $w_b(\gamma)$ be the minimum number of border strips of size $p$ such that when added to $\gamma$, we obtain a partition with exactly $b$ odd parts. Let $\mathcal{E}_b(\gamma)$ be the set of all partitions of $|\gamma| + w_b(\gamma)p$ obtained in this way.

We also require the definition of the dominance order on partitions, which we give in §1.1.

THEOREM 5.1.2 *Let $\gamma$ and $\delta$ be $p$-core partitions, and let $b, c \in \mathbf{N}_0$. If $b \ge p$ (resp. $c \ge p$), suppose that $w_{b-p}(\gamma) \ne w_b(\gamma) - 1$ (resp. $w_{c-p}(\delta) \ne w_c(\delta) - 1$). Then there exists a set partition of $\mathcal{E}_b(\gamma) \times \mathcal{E}_c(\delta)$, say $\Lambda_1,\ldots,\Lambda_t$, such that each $\Lambda_i$ has a unique pair $(\nu_i,\widetilde{\nu_i})$ with $\nu_i$ and $\widetilde{\nu_i}$ both maximal in the dominance orders on $\mathcal{E}_b(\gamma)$ and $\mathcal{E}_c(\delta)$, respectively. Moreover, $\nu_i$ and $\widetilde{\nu_i}$ are $p$-regular for each $i$, and the decomposition number $d_{\lambda\nu_i,\mu\widetilde{\nu_i}}$ equals one if $(\lambda,\mu) \in \Lambda_i$, and equals zero otherwise.*

In §6 we consider $C_2 \wr S_n$ as the symmetry group of the $n$-hypercube $I^n$, where $I$ denotes the closed unit interval $[0, 1]$. In particular we equip the $n$-hypercube with an orientation, and we define $U_i$ to be the $F$-span of the set of oriented $i$-hypercubes lying on the oriented $n$-hypercube. We define the *boundary* map $\delta_i : U_i \to U_{i-1}$, and we show that $\delta_i\delta_{i+1} = 0$ (when composed from right to left) for all $0 \leq i < n$. Our main result in §6 is the following theorem.

THEOREM 6.0.6. *The chain complex*

$$U_n \xrightarrow{\delta_n} U_{n-1} \xrightarrow{\delta_{n-1}} \cdots \xrightarrow{\delta_2} U_1 \xrightarrow{\delta_1} U_0 \xrightarrow{\delta_0} \mathbf{Q} \tag{6.1}$$

*is exact in all places.*

The map $\delta_i$ is a natural generalisation of the boundary map of an oriented $i$-simplex lying on an $n$-simplex. The symmetry group of an oriented $n$-simplex is $S_n$, and so Theorem 6.0.6 is a generalisation of the representation theory of $S_n$ to that of $C_2 \wr S_n$. In fact our proof of Theorem 6.0.6 uses an analogous result for the simplex, and therefore demonstrates the links between the representation theories of these groups.

When $F$ has characteristic 2, we further generalise the boundary maps to multistep maps $\psi_i^{(t)}$, which we define in §6.2. The map $\psi_i^{(t)}$ has domain equal to the $F$-span of the $i$-dimensional hypercubes, and range equal to the $F$-span of the $(i-t)$-dimensional hypercubes for $t \geq 2$. The multistep maps satisfy the relation $\psi_i^{(t)}\psi_{i+t}^{(t)} = 0$, and so we consider the corresponding chain complex. In particular we demonstrate several differences between these modules and the analogous modules for $FS_n$.

In §7 we consider the ordinary representation theory of the imprimitive wreath product $S_m \wr S_n$, where $m \in \mathbf{N}$. In particular we define the Foulkes characters for wreath products of symmetric groups, which are the subject of the long standing Foulkes' Conjecture (stated in §7). The main result in [**19**] is a recursive formula for the Foulkes characters, which is used to prove the conjecture in certain cases. Our result is the extension of this recursive formula to a generalisation of the Foulkes characters, which we now define.

Given partitions $\vartheta$ and $\nu$ of $m$ and $n$, respectively, define the *plethysm*

$$\varphi_\vartheta^\nu = \left(\widetilde{\chi^\vartheta}^{\times n} \operatorname{Inf}_{S_n}^{S_m \wr S_n} \chi^\nu\right) \big\uparrow_{S_m \wr S_n}^{S_{mn}},$$

where the notation in this display is defined in §1.2.

THEOREM 7.1.2 *Let $m, n \in \mathbf{N}$. Let $\vartheta = (a, 1^b)$ for some $a + b = m$, and let $\nu \vdash n$. If $\lambda \vdash mn$, then*

$$\left\langle \varphi_\vartheta^\nu, \chi^\lambda \right\rangle = \frac{1}{n} \sum_{j=1}^{n} \sum_{\mu \subset \lambda} \varepsilon_j(\lambda/\mu) |\mathcal{B}_{a,b}^{\lambda/\mu}| \sum (-1)^{\mathrm{ht}(\nu/\rho)} \left\langle \varphi_\vartheta^\rho, \chi^\mu \right\rangle,$$

*where the third sum runs over all $\rho \subset \nu$ such that $\nu/\rho$ is a border strip.*

In §7.2 we generalise the Foulkes characters in a different way. The Foulkes character $\varphi^{(n)}_{(2)}$ is the unique $S_{2n}$-character satisfying the following two conditions:

(U1) the constituents of $\chi\downarrow_{S_{2n-1}}$ are the $\chi^\mu$ such that $\mu$ has exactly one odd part, each appearing with multiplicity one,

(U2) $\chi^{(2n)}$ is a constituent of $\chi$, appearing with multiplicity one.

This remarkable fact can be used to give a complete decomposition of the Foulkes character $\varphi^{(n)}_{(2)}$ as a direct sum of its irreducible constituents. Our main result in §7.2 is the following theorem.

THEOREM 7.2.1. *There is a unique $S_{2n}$-character $\chi$ such that*

(U1) *the constituents of $\chi\downarrow_{S_{2n-1}}$ are the $\chi^\mu$ such that $\mu$ has exactly one odd part, each appearing with multiplicity one,*

(U2′) *$\chi^{(2n)}$ is not a constituent of $\chi$.*

Similar to the case of $\varphi^{(n)}_{(2)}$, the proof of this result determines the complete decomposition of $\chi$ (in the statement of the theorem) as a sum of its irreducible constituents.

## 1.1. The representation theory of the symmetric group

Our exposition in this section follows that of the paper [**42**]. Given $n \in \mathbf{N}$, we define a *composition* of $n$ to be a sequence

$$\lambda = (\lambda_1, \ldots, \lambda_t)$$

such that $\lambda_i \in \mathbf{N}$ for all $i$ and $\sum_{i=1}^t \lambda_i = n$. In this case we write $|\lambda| = n$. If the parts $\lambda_i$ of $\lambda$ are non-increasing, then we say that $\lambda$ is a *partition* of $n$, and we write $\lambda \vdash n$. We denote by $\ell(\lambda)$ the number of parts of $\lambda$. In some places we adopt the usual index abbreviation for partitions, for instance we write $(5^2, 3^3, 1)$ for $(5, 5, 3, 3, 3, 1)$. Given $r \in \mathbf{N}$ such that $r \leq n$, we write $\Lambda(r, n)$ for the set of compositions of $n$ with at most $r$ parts.

We define a *multi-partition* to be a sequence of partitions $(\lambda^1, \ldots, \lambda^t)$ such that $\sum_{i=1}^t |\lambda^i| = n$. In this case we say that the multi-partition has length $t$. We write $\mathcal{P}^t(n)$ for the set of multi-partitions of $n$ of length $t$.

We define a partial order, known as the dominance order, on the set of compositions of $n$, as follows. We write $\mu \trianglerighteq \lambda$ if and only if $\ell(\mu) \leq \ell(\lambda)$ and $\sum_{i=1}^k \mu_i \geq \sum_{i=1}^k \lambda_i$ whenever $1 \leq k \leq \ell(\mu)$. In the case that $\mu \trianglerighteq \lambda$, we say that $\mu$ *dominates* $\lambda$.

The combinatorics of partitions is of fundamental importance in the representation theory of the symmetric group, both in the ordinary and modular cases. A notable example is Theorem 1.1.2 in this section, which shows that the irreducible $\mathbf{Q}S_n$-modules are labelled by the set of partitions of $n$. In order to construct the Specht modules $S^\lambda$ in the statement of Theorem 1.1.2, we take the unusual approach of constructing the more general skew Specht modules $S^{\lambda/\mu}$. This is because the skew Specht modules are required in §2, where we prove the Murnaghan–Nakayama rule. The usual definition of the Specht modules follows by taking $\mu = \varnothing$. For details on the representation theory of the symmetric group, we refer to [**33**] and [**35**].

**1.1.1. Skew Specht modules.** Given partitions $\mu$ and $\lambda$ of $m$ and $m+n$ respectively, we say that $\mu$ is a *subpartition* of $\lambda$, and write $\mu \subseteq \lambda$, if $\ell(\mu) \leq \ell(\lambda)$ and $\mu_i \leq \lambda_i$ for $1 \leq i \leq \ell(\mu)$. We define the *skew diagram* (or *Young diagram*) $[\lambda/\mu]$ to be the set of *boxes*

$$\{(i,j) : 1 \leq i \leq t \text{ and } \mu_i < j \leq \lambda_i\},$$

and call $\lambda/\mu$ a *skew partition.* We define *row* $k$ (resp. *column* $k$) of $\lambda/\mu$ to be the subset of $[\lambda/\mu]$ of boxes whose first (resp. second) coordinate equals $k$. Let $\mathrm{ht}(\lambda/\mu)$ be one less than the number of non-empty rows of $[\lambda/\mu]$.

In various places in this thesis, we consider skew diagrams that are border strips. By definition a *border strip* is a skew partition whose skew diagram is connected and which contains no four boxes forming the Young diagram $[(2,2)]$.

Fix $m, n \in \mathbf{N}$. Let $\lambda$ be a partition of $m+n$ and let $\mu$ be a subpartition of $\lambda$ of size $m$. We define a $\lambda/\mu$*-tableau* $t$ to be a bijective function $t : [\lambda/\mu] \to \{1, 2, \ldots, n\}$, and call $t$ a *skew tableau* of *shape* $\lambda/\mu$. We call $t(i,j)$ the *entry of* $t$ in position $(i,j)$. Thus a $\lambda/\mu$-tableau can be visualized (see Example 1.1.1) as a filling of the boxes $[\lambda/\mu]$ with distinct entries from $\{1, \ldots, n\}$. We draw skew diagrams with the largest part at the top of the page: thus the *top row* is row 1, and so on.

There is a natural action of $S_n$ on the set of $\lambda/\mu$-tableaux defined by $(\sigma t)(i,j) = \sigma\big(t(i,j)\big)$ for $\sigma \in S_n$. Given a $\lambda/\mu$-tableau $t$, let $R(t)$ (resp. $C(t)$) be the subgroup of $S_n$ consisting of all permutations that setwise fix the entries in each row (resp. column) of $t$. We define an equivalence relation $\backsim$ on the set of $\lambda/\mu$-tableaux by $t \backsim u$ if and only if there exists $\pi \in R(t)$ such that $u = \pi t$. The $\lambda/\mu$*-tabloid* $\{t\}$ is the equivalence class of $t$. A short calculation shows that there is a well-defined action of $S_n$ on the set of $\lambda/\mu$-tabloids given by $\sigma\{t\} = \{\sigma t\}$.

We say that a $\lambda/\mu$-tableau is *row standard* if the entries in the rows are increasing when read from left to right, and *column standard* if the entries in the columns are increasing when read from top to bottom. A tableau $t$ that is both row standard and column standard is a *standard* tableau. Define $\widetilde{t}$ to be the unique column standard $\lambda/\mu$-tableau whose columns agree setwise with $t$. We call $\widetilde{t}$ the *column straightening* of $t$. We define the *row straightening* $\overline{t}$ of $t$ in the analogous way.

EXAMPLE 1.1.1. Consider the following $(5,4,2,1)/(2,1)$-tableaux:

$$v := \begin{array}{|c|c|c|c|c|} \cline{3-5} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 7 & 1 & 4 \\ \cline{2-5} \multicolumn{1}{c|}{} & 2 & 9 & 6 & \multicolumn{1}{c}{} \\ \cline{1-4} 5 & 3 & \multicolumn{3}{c}{} \\ \cline{1-2} 8 & \multicolumn{4}{c}{} \\ \cline{1-1} \end{array} \quad \text{and } w := \begin{array}{|c|c|c|c|c|} \cline{3-5} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 1 & 4 & 7 \\ \cline{2-5} \multicolumn{1}{c|}{} & 2 & 6 & 9 & \multicolumn{1}{c}{} \\ \cline{1-4} 3 & 5 & \multicolumn{3}{c}{} \\ \cline{1-2} 8 & \multicolumn{4}{c}{} \\ \cline{1-1} \end{array}.$$

By definition $w$ is a standard tableau. Also observe that $\{v\} = \{w\}$, and so $w = \overline{v}$.

Let $M^{\lambda/\mu}$ be the $\mathbf{Z}S_n$-permutation module spanned by the $\lambda/\mu$-tabloids. Observe that as $M^{\lambda/\mu}$ is a permutation module, it is isomorphic to $\mathbf{Z}\big\uparrow_H^{S_n}$, for some subgroup $H$ of $S_n$. Indeed, given a composition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_t)$ of $n$, define the *Young subgroup* $S_\lambda \le S_n$ to be equal to

$$S_{\{1,2,\dots,\lambda_1\}} \times S_{\{\lambda_1+1,\lambda_1+2,\dots,\lambda_1+\lambda_2\}} \times \cdots \times S_{\{\lambda_1+\cdots+\lambda_{t-1}+1,\dots,n\}}.$$

If $\nu$ is the composition of $n$ recording the lengths of the non-empty rows of $\lambda/\mu$, then $M^{\lambda/\mu}$ is the permutation module corresponding to the action of $S_n$ on the cosets of $S_\nu$. In particular $M^{\lambda/\mu}$ can be defined over any ring. If we need to specify the ring $R$, then we write $M_R^{\lambda/\mu}$ for $R\uparrow_{S_\nu}^{S_n}$. Taking $\mu = \varnothing$, this is the usual *Young permutation module* $M^\lambda$ corresponding to the partition $\lambda$.

We define the $\lambda/\mu$-*polytabloid* $e(t) \in M_R^{\lambda/\mu}$ by

$$e(t) = \sum_{\sigma \in C(t)} \operatorname{sgn}(\sigma)\sigma\{t\}.$$

If $t$ is a standard tableau then we say that $e(t)$ is a *standard polytabloid.* The *skew Specht module* $S_R^{\lambda/\mu}$ is then the $RS_n$-module spanned by all $\lambda/\mu$-polytabloids. Taking $\mu = \varnothing$ this is the Specht module $S_R^\lambda$, defined over any ring $R$. If the ring $R$ is clear, then we omit the subscript $R$ in $S_R^{\lambda/\mu}$ (resp. $S_R^\lambda$).

Given a skew partition $\lambda/\mu$, we write $\chi^{\lambda/\mu}$ for the character of the $\mathbf{Q}S_n$-module $S_{\mathbf{Q}}^{\lambda/\mu}$. Again taking $\mu = \varnothing$ we write $\chi^\lambda$ for the character of the Specht module $S_{\mathbf{Q}}^\lambda$.

THEOREM 1.1.2. *Let $n \in \mathbf{N}$. Then the set*

$$\{S_{\mathbf{Q}}^\lambda : \lambda \textit{ is a partition of } n\}$$

*is a complete set of pairwise non-isomorphic irreducible $\mathbf{Q}S_n$-modules. Furthermore*

$$\operatorname{Irr}(S_n) = \{\chi^\lambda : \lambda \textit{ is a partition of } n\}.$$

Let $F$ be a field. Theorem 4.9 in [**33**] states that if $S_F^\lambda$ is irreducible, then $S_E^\lambda$ is irreducible for any extension field $E$ of $F$. It follows from Theorem 1.1.2 that $S_F^\lambda$ is irreducible, where $F$ is any field of characteristic zero. Moreover, the irreducible characters $\chi^\lambda$ are integer valued, and so we can write $\chi^\lambda$ (resp. $\chi^{\lambda/\mu}$) for the character of $S_F^\lambda$ (resp. $S_F^{\lambda/\mu}$) in this case.

Theorem 1.1.2 implies that there exists a Specht module that labels the trivial $\mathbf{Q}S_n$-module. Indeed this is the Specht module labelled by the one part partition $(n)$.

For all $n > 1$, there exists exactly one other one-dimensional Specht module. This is the Specht module corresponding to the partition $(1^n)$, on which every element of the symmetric group $S_n$ acts by its sign. We therefore refer to $S^{(1^n)}$ as the *sign module*, and we write $\mathrm{sgn}_n$ (or sgn when the index $n$ is clear) for this module.

It is a basic character theoretic fact that the product of a degree-one character with an irreducible character is again an irreducible character. Therefore given a partition $\nu$ of $n$, we have $\chi^\nu \times \chi^{(1^n)} = \chi^\lambda$ for some partition $\lambda$ of $n$. It is proved in [**33**, (6.6)] that $\lambda$ is the unique partition such that the Young diagram $[\lambda]$ is the transpose of the Young diagram $[\nu]$. As is usual we write $\nu'$ for this partition, and we refer to $\nu'$ as the *conjugate partition* of $\nu$. Observe that multiplying by $\chi^{(1^n)}$ is an involution, and therefore so is conjugating a partition.

**1.1.2. Garnir relations and the Standard Basis Theorem.** In this section we consider various relations in the skew Specht modules. If $\sigma \in S_n$ then an easy calculation shows that

$$\sigma e(t) = e(\sigma t). \tag{1.1}$$

Hence $S^{\lambda/\mu}$ is cyclic, generated by any $\lambda/\mu$-polytabloid. Moreover if $\tau \in C(t)$, then

$$\tau e(t) = \mathrm{sgn}(\tau) e(t). \tag{1.2}$$

Therefore $S^{\lambda/\mu}$ is spanned by the $\lambda/\mu$-polytabloids $e(t)$ for $t$ a column standard $\lambda/\mu$-tableau. Recall that we define $\widetilde{t}$ to be the unique column standard $\lambda/\mu$-tableau whose columns agree setwise with $t$. Let $\varepsilon_t \in \{+1, -1\}$ be defined by $e(\widetilde{t}) = \varepsilon_t e(t)$.

Suppose that $(i,j)$ and $(i,j+1)$ are boxes in $[\lambda/\mu]$. Given a $\lambda/\mu$-tableau $t$, let $X = \{t(i,j), t(i+1,j), \ldots\}$ be the set of entries in column $j$ of $t$ weakly below box $(i,j)$, and let $Y = \{\ldots, t(i-1,j+1), t(i,j+1)\}$ be the set of entries in column $j+1$ of $t$ weakly above box $(i,j+1)$. Let $C_{X,Y}$ be the set of all products of transpositions $(x_1,y_1)\ldots(x_k,y_k)$ for $x_1 < \ldots < x_k$ and $y_1 < \ldots < y_k$ where $\{x_1,\ldots,x_k\} \subseteq X$ and $\{y_1,\ldots,y_k\} \subseteq Y$ are non-empty $k$-sets. We define the *Garnir element for $X$ and $Y$* by

$$G_{X,Y} = 1 + \sum_{\sigma \in C_{X,Y}} \mathrm{sgn}(\sigma)\sigma \in \mathbf{Z}S_{X \cup Y}. \tag{1.3}$$

Restated, replacing ideals in the group ring $\mathbf{Z}S_n$ with polytabloids, (3.8) in [**20**] implies that

$$G_{X,Y} e(t) = 0. \tag{1.4}$$

Similarly restated, Theorem 3.9 in [**20**] is as follows.

THEOREM 1.1.3 (Standard Basis Theorem).

(i) *Any $\lambda/\mu$-polytabloid can be expressed as a* $\mathbf{Z}$*-linear combination of standard $\lambda/\mu$-polytabloids by applications of column relations* (1.2) *and Garnir relations* (1.4).

(ii) *The $\mathbf{Z}S_n$-module $S^{\lambda/\mu}$ has the set of standard $\lambda/\mu$-polytabloids as a $\mathbf{Z}$-basis.*

We remark that the proofs of Theorem 7.2 and 8.4 in [**33**], for the case when $\mu = \varnothing$, but defined using polytabloids, generalise easily to prove (1.4) and Theorem 1.1.3 exactly as stated above. We also remark that the Garnir relations and therefore the Standard Basis theorem hold over any field. We give a small example of Garnir relations in Example 1.1.9.

**1.1.3. Restricted Specht modules.** Fix throughout this section $m$, $n \in \mathbf{N}$ and a partition $\lambda$ of $m+n$. Recall that the Young subgroup $S_{(m,n)}$ is defined to be $S_{\{1,2,\ldots,m\}} \times S_{\{m+1,m+2,\ldots,m+n\}}$. We shall prove the following theorem, which determines the restriction of a Specht module to $S_{(m,n)}$. We will use this result in §2 and §3.

THEOREM 1.1.4. *The module $S^\lambda \downarrow_{S_{(m,n)}}$ has a filtration by $\mathbf{Z}S_{(m,n)}$-modules whose successive quotients are isomorphic to $S^\mu \boxtimes S^{\lambda/\mu}$, where each subpartition $\mu$ of $\lambda$ of size $m$ occurs exactly once.*

Theorem 1.1.4 is the main result in [**36**]. The proof in [**36**] constructs skew Specht modules as ideals in the group algebra of $S_n$ over a field. Our proof using polytabloids instead generalizes James' proof of the modular branching rule for Specht modules [**33**, Ch. 9]. In this way we obtain a stronger isomorphism for integral modules that replaces the lexicographic order used in [**33**] and [**36**] with the dominance order. The following preliminaries are required.

Suppose that $\lambda$ has first part $c$. Given a $\lambda$-tableau $t$ we define the *$m$-shape* of $t$ to be the composition $(\gamma_1, \ldots, \gamma_c)$ such that $\gamma_j$ equals the number of entries in column $j$ of $t$ that are at most $m$. For each composition $\gamma$ such that $\ell(\gamma) \le c$ we define

$$V^{\trianglerighteq\gamma} = \langle e(t) : t \text{ a column standard } \lambda\text{-tableau of } m\text{-shape } \delta \text{ where } \delta \trianglerighteq \gamma \rangle_{\mathbf{Z}}.$$

Note that the definition of the $m$-shape agrees with the notation $b(y)$ in the proof of [**36**, Theorem 3.1]. We require the following total ordering on the set of column standard $\lambda$-tableaux, defined implicitly in [**33**, page 30].

DEFINITION. Let $u$ and $t$ be column standard $\lambda$-tableaux. We write $u > t$ if and only if the greatest entry appearing in a different column in $u$ to $t$ appears further right in $u$ than $t$.

For instance, the $\geq$ order on the set of column standard $(2,2)$-tableaux is

$$\begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline\end{array} > \begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline\end{array} > \begin{array}{|c|c|}\hline 2&1\\\hline 3&4\\\hline\end{array} > \begin{array}{|c|c|}\hline 1&2\\\hline 4&3\\\hline\end{array} > \begin{array}{|c|c|}\hline 2&1\\\hline 4&3\\\hline\end{array} > \begin{array}{|c|c|}\hline 3&1\\\hline 4&2\\\hline\end{array}.$$

Note that here, as in general, the greatest tableau under $>$ is standard. Several times below we use that if $x > y$ and $x$ is to the left of $y$ in the column standard tableau $u$ then $\widetilde{(x,y)u} > u$.

PROPOSITION 1.1.5. *Let $u$ be a column standard $\lambda$-tableau of $m$-shape $\gamma$. Then $e(u)$ is equal to a $\mathbf{Z}$-linear combination of standard $\lambda$-polytabloids $e(t)$ where each $t$ has $m$-shape $\mu'$ for some partition $\mu$ such that $\mu' \trianglerighteq \gamma$.*

PROOF. If $u$ is standard then $\gamma$ is a partition, and there is nothing to prove. If $u$ is not standard then there exists $(i,j) \in [\lambda]$ such that $u(i,j) > u(i,j+1)$. Let $X$ and $Y$ be as defined in (1.3). By (1.4) we have

$$0 = e(u) + \sum_{\sigma \in C_{X,Y}} \varepsilon_{\sigma u} \operatorname{sgn}(\sigma) e(\widetilde{\sigma u})$$

where $\widetilde{\sigma u}$ and $\varepsilon_{\sigma u} \in \{+1,-1\}$ are as defined at the start of §1.1.2. Let $\sigma \in C_{X,Y}$. Since the minimum of $X$ exceeds the maximum of $Y$, we have $x > y$ for each transposition $(x,y)$ in $\sigma$. Hence $\widetilde{\sigma u} > u$. Write $\delta$ for the $m$-shape of $\widetilde{\sigma u}$. If there are exactly $k$ transpositions $(x,y)$ such that $x > m \geq y$ then $\delta_j = \gamma_j + k$, $\delta_{j+1} = \gamma_{j+1} - k$ and $\delta_{j'} = \gamma_j$ for $j' \neq j, j+1$. Hence $\delta \trianglerighteq \gamma$. The lemma now follows by induction on the $\geq$ and $\trianglerighteq$ orders. $\square$

COROLLARY 1.1.6. *Let $\mu$ be a subpartition of $\lambda$ of size $m$. Then $V^{\trianglerighteq\mu'}$ is a $\mathbf{Z}S_{(m,n)}$-submodule of $S^\lambda$ with $\mathbf{Z}$-basis given by the standard $\lambda$-tableaux of $m$-shape $\nu'$ such that $\nu' \trianglerighteq \mu'$.*

PROOF. Since the standard $\lambda$-polytabloids are linearly independent by Theorem 1.1.3(ii), it follows immediately from Proposition 1.1.5 that $V^{\trianglerighteq\mu'}$ has a $\mathbf{Z}$-basis as claimed. If $\pi \in S_{(m,n)}$ and $s$ is a standard $\lambda$-tableau of $m$-shape $\nu'$ then $\pi s$ also has $m$-shape $\nu'$, as does $\widetilde{\pi s}$. By (1.2) and Proposition 1.1.5, $e(\pi s) = \pm e(\widetilde{\pi s}) \in V^{\trianglerighteq\nu'} \subseteq V^{\trianglerighteq\mu'}$. Hence $V^{\trianglerighteq\mu'}$ is a $\mathbf{Z}S_{(m,n)}$-module. $\square$

Given a $\mu$-tableau $u$ with (as usual) entries $\{1,\ldots,m\}$ and a $\lambda/\mu$-tableau $v$ with entries $\{m+1,\ldots,m+n\}$, let $u \cup v$ denote the $\lambda$-tableau defined by

$$(u \cup v)(i,j) = \begin{cases} u(i,j) & \text{if } (i,j) \in [\mu] \\ v(i,j) & \text{if } (i,j) \in [\lambda/\mu]. \end{cases}$$

Clearly every $\lambda$-tableau of $m$-shape $\mu'$ is of this form. We shall show that the action of $S_{(m,n)}$ on standard $\lambda$-polytabloids is compatible with this factorization. We require the following lemma and proposition, which are illustrated in Example 1.1.9.

In the following lemma, the tableau $u$ and $v$ are as described on the previous page.

LEMMA 1.1.7. *Let $\mu$ be a subpartition of $\lambda$ of size $m$. Let $u$ be a column standard $\mu$-tableau and let $v$ be a $\lambda/\mu$-tableau. Let $(i,j) \in [\mu]$ be a box such that*

$$m \geq u(i,j) > u(i,j+1).$$

*Let $r = \mu'_j$ so $(r,j)$ is the lowest box in column $j$ of $u$, and define*

$$\begin{aligned} X &= \{u(i,j), u(i+1,j), \ldots, u(r,j), v(r+1,j), \ldots\}, \\ Y &= \{\ldots, u(i-1,j+1), u(i,j+1)\}, \\ X^\star &= \{u(i,j), u(i+1,j), \ldots, u(r,j)\}. \end{aligned}$$

*Let $C_{X^\star,Y} = \{\sigma \in C_{X,Y} : \sigma x = x \text{ for all } x \in X \backslash X^\star\}$. Then*

$$0 = e(u \cup v) + \sum_{\sigma^\star \in C_{X^\star,Y}} \operatorname{sgn}(\sigma^\star)\sigma^\star e(u \cup v) + \sum_{\sigma \in C_{X,Y} \backslash C_{X^\star,Y}} \operatorname{sgn}(\sigma)\sigma e(u \cup v)$$

*where*

(i) *for each $\sigma^\star$, we have $\sigma^\star e(u \cup v) = e(\sigma^\star u \cup v)$ and $\widetilde{\sigma^\star u} > u$;*
(ii) *for each $\sigma$, $\sigma e(u \cup v)$ is a $\mathbf{Z}$-linear combination of polytabloids $e(s)$ for standard tableaux $s$ of $m$-shape $\nu'$ where $\nu' \rhd \mu'$.*

PROOF. Since

$$G_{X,Y} = 1 + \sum_{\sigma^\star \in C_{X^\star,Y}} \operatorname{sgn}(\sigma^\star)\sigma^\star + \sum_{\sigma \in C_{X,Y} \backslash C_{X^\star,Y}} \operatorname{sgn}(\sigma)\sigma,$$

the displayed equation follows from (1.4). Since $C_{X^\star,Y} \subseteq S_{\{1,\ldots,m\}}$, (i) follows from the observation after the definition of the $\geq$ order. Take $\sigma \in C_{X,Y} \backslash C_{X^\star,Y}$ and let $w = \sigma(u \cup v)$. Since $\sigma$ involves a transposition $(x,y)$ with $x > m \geq y$, the statistic $k$ in the proof of Proposition 1.1.5 is non-zero. Hence the $m$-shape of $e(\widetilde{w})$ is $\delta$ for some composition $\delta$ with $\delta \rhd \mu'$. The statement of Proposition 1.1.5 now implies that $e(\widetilde{w})$ is a $\mathbf{Z}$-linear combination of standard polytabloids $e(s)$ for $s$ of $m$-shape $\nu'$ where $\nu' \unrhd \delta$. Hence $\nu' \rhd \mu'$, as required for (ii). $\square$

PROPOSITION 1.1.8. *Let $\mu$ be a subpartition of $\lambda$ of size $m$. Let $u$ be a column standard $\mu$-tableau and let $t$ be a standard $\lambda/\mu$-tableau. If $e(u) = \sum_S \alpha_S e(S)$ where the sum is over all standard $\mu$-tableaux $S$ and $\alpha_S \in \mathbf{Z}$ for each $S$ then*

$$e(u \cup t) \in \sum_S \alpha_s e(S \cup t) + \sum_{\nu' \rhd \mu'} V^{\unrhd \nu'}.$$

PROOF. If $u$ is standard the result is obvious. If not, there exists a box $(i,j) \in [\mu]$ such that $m \geq u(i,j) > u(i+1,j)$. Let $X^\star$ and $Y$ be as in Lemma 1.1.7. By Lemma 1.1.7(ii) we have

$$e(u \cup t) \in -\sum_{\sigma^\star \in C_{X^\star,Y}} \operatorname{sgn}(\sigma^\star)\sigma^\star e(u \cup t) + \sum_{\nu' \rhd \mu'} V^{\unrhd \nu'}.$$

Using Lemma 1.1.7(i), the result now follows by induction on the $\geq$ order. $\square$

We also need the analogous lemma in which $u(i,j) > u(i,j+1) > m$, $Y^\star = \{u(r,j+1),\ldots,u(i,j+1)\}$ where now $r = \mu'_{j+1}+1$, and the relevant sets of coset representatives are $C_{X,Y^\star}$ and $C_{X,Y}\backslash C_{X,Y^\star}$. It implies the analogous proposition in which $e(t \cup v)$ is written as a sum of standard polytabloids, where $t$ is a standard $\mu$-tableau and $v$ is a column standard $\lambda/\mu$-tableau. The proofs are entirely analogous.

EXAMPLE 1.1.9. Let $u$, $t$ and $u \cup t$ be the skew tableaux shown below.

$$u = \begin{array}{|c|c|}\hline 1&2\\\hline 4&3\\\hline\end{array}, \quad t = \begin{array}{ccc}\cline{3-3} &&\multicolumn{1}{|c|}{5}\\\cline{3-3} &&\multicolumn{1}{|c|}{7}\\\hline \multicolumn{1}{|c|}{6}&\multicolumn{1}{c|}{8}&\\\cline{1-2}\end{array}, \quad u \cup t = \begin{array}{|c|c|c|}\hline 1&2&5\\\hline 4&3&7\\\hline 6&8\\\cline{1-2}\end{array}.$$

As $4 = (u \cup t)(2,1) > (u \cup t)(2,2) = 3$, we define $X = \{4,6\}$ and $Y = \{2,3\}$. The relation $G_{X,Y}e(u \cup t) = 0$ gives

$$\begin{aligned} e(u \cup t) = &- e\left(\begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4&7\\\hline 6&8\\\cline{1-2}\end{array}\right) + e\left(\begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4&7\\\hline 6&8\\\cline{1-2}\end{array}\right) \\ &+ e\left(\begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&6&7\\\hline 4&8\\\cline{1-2}\end{array}\right) - e\left(\begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&6&7\\\hline 4&8\\\cline{1-2}\end{array}\right) - e\left(\begin{array}{|c|c|c|}\hline 1&4&5\\\hline 2&6&7\\\hline 3&8\\\cline{1-2}\end{array}\right). \end{aligned}$$

In the notation of Lemma 1.1.7, we have $X^\star = \{4\}$. The standard polytabloids in the top and bottom lines come from the permutations in $C_{X^\star,Y}$ and $C_{X,Y}\backslash C_{X^\star,Y}$, respectively. Furthermore, the 4-shape of each polytabloid in the top line is $(2,2)$ and in the bottom line is $(3,1)$. Therefore

$$e(u \cup t) \in -e\left(\begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4&7\\\hline 6&8\\\cline{1-2}\end{array}\right) + e\left(\begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4&7\\\hline 6&8\\\cline{1-2}\end{array}\right) + V^{\unrhd(3,1)},$$

as expected from Proposition 1.1.8.

PROOF OF THEOREM 1.1.4. We start by proving that there exists a $\mathbf{Z}S_{(m,n)}$-module isomorphism

$$\frac{V^{\unrhd \mu'}}{\sum_{\nu' \rhd \mu'} V^{\unrhd \nu'}} \overset{\phi}{\cong} S^\mu \boxtimes S^{\lambda/\mu}.$$

By Corollary 1.1.6, the module on the left-hand side has a $\mathbf{Z}$-basis given by the set of standard $\lambda$-tableaux of $m$-shape $\mu'$. Therefore the linear extension $\phi$ of the map $\phi e(s \cup t) = e(s) \otimes e(t)$, where $s \cup t$ is a standard $\lambda$-tableau of $m$-shape $\mu'$, is a well-defined $\mathbf{Z}$-linear morphism. Since the tensors $e(s) \otimes e(t)$

for $s$ a standard $\mu$-tableau and $t$ a standard $\lambda/\mu$-tableau form a basis for $S^\mu \boxtimes S^{\lambda/\mu}$, $\phi$ is a $\mathbf{Z}$-linear isomorphism.

To show that $\phi$ is a $\mathbf{Z}S_{(m,n)}$-module homomorphism, it suffices to consider the actions of $S_{\{1,\ldots,m\}}$ and $S_{\{m+1,\ldots,m+n\}}$ separately. Let $\pi \in S_{\{1,\ldots,m\}}$ and let $s \cup t$ be a standard $\lambda$-tableau. Observe that $\pi\widetilde{(s \cup t)} = \widetilde{\pi s} \cup t$ and $\varepsilon_{\pi(s\cup t)} = \varepsilon_{\pi s}$. Suppose that $e(\widetilde{\pi s}) = \sum_S \alpha_S e(S)$ where the sum is over all standard $\mu$-tableaux $S$. On the one hand

$$\pi\big(e(s) \otimes e(t)\big) = -\varepsilon_{\pi s} \sum_S \alpha_S e(S) \otimes e(t).$$

On the other hand, by Proposition 1.1.8 we have

$$\pi e(s \cup t) \in -\varepsilon_{\pi s} \sum_S \alpha_S e(S \cup t) + \sum_{\nu' \rhd \mu'} V^{\unrhd \nu'}.$$

The argument is entirely analogous for the action of $S_{\{m+1,\ldots,m+n\}}$.

We now write $\geq$ for the lexicographic order of compositions. We define $V^{\geq \mu'}$ in a similar way to $V^{\unrhd \mu'}$, replacing the condition $\delta \unrhd \mu'$ with $\delta \geq \mu'$. Since $\nu' \unrhd \mu'$ implies that $\nu' \geq \mu'$, replacing every instance of $\unrhd$ with $\geq$ in Proposition 1.1.5 and Corollary 1.1.6 implies that $V^{\geq \mu'}$ is also a $\mathbf{Z}S_{(m,n)}$-module. Moreover, $V^{\geq \mu'}$ has a $\mathbf{Z}$-basis given by the standard $\lambda$-tableaux of $m$-shape $\nu'$ such that $\nu' \geq \mu'$, and so there is an isomorphism

$$\frac{V^{\geq \mu'}}{\sum_{\nu' > \mu'} V^{\geq \mu'}} \cong \frac{V^{\unrhd \mu'}}{\sum_{\nu' \rhd \mu'} V^{\unrhd \nu'}} \cong S^\mu \boxtimes S^{\lambda/\mu}.$$

Therefore the modules $V^{\geq \mu'}$, where $\mu$ ranges over all subpartitions of $\lambda$ of size $m$, give the required filtration. □

Corollary 1.1.10. *Let $\rho \in S_{m+n}$ be an $n$-cycle and let $\pi$ be a permutation of the remaining $m$ numbers. Then*

$$\chi^\lambda(\pi\rho) = \sum_\mu \chi^\mu(\pi)\chi^{\lambda/\mu}(\rho)$$

*where the sum is over all subpartitions $\mu$ of $\lambda$ of size $m$.*

Proof. By taking a suitable conjugate of $\pi\rho$ we may assume that $\pi \in S_{\{1,\ldots,m\}}$ and $\rho \in S_{\{m+1,\ldots,m+n\}}$. Taking characters in Theorem 1.1.4 gives

$$\chi^\lambda\big\downarrow_{S_{(m,n)}} = \sum_\mu \chi^\mu \times \chi^{\lambda/\mu} \tag{1.5}$$

where the sum is over all subpartitions $\mu$ of $\lambda$ of size $m$. Now evaluate both sides at $\pi\rho$. □

The following useful lemma follows from Corollary 1.1.10.

Lemma 1.1.11. *Let $\lambda$ be a partition of $m+n$ and let $\mu$ be a subpartition of $\lambda$ of size $m$. If $\psi$ is a character of $S_n$ then*

$$\langle \chi^{\lambda/\mu}, \psi \rangle_{S_n} = \big\langle \chi^\lambda, \chi^\mu \times \psi\big\uparrow_{S_m \times S_n}^{S_{m+n}} \big\rangle_{S_{m+n}}.$$

PROOF. By Frobenius reciprocity and (1.5),

$$\begin{aligned}\langle \chi^\lambda, \chi^\mu \times \psi\big\uparrow_{S_m\times S_n}^{S_{m+n}}\rangle &= \langle \chi^\lambda\big\downarrow_{S_m\times S_n}^{S_{m+n}}, \chi^\mu \times \psi\rangle \\ &= \langle \sum_\nu \chi^\nu \times \chi^{\lambda/\nu}, \chi^\mu \times \psi\rangle\end{aligned}$$

where the sum runs over all partitions $\nu$ of $m$ such that $\nu \subset \lambda$. The only non-zero summand is $\langle \chi^\mu \times \chi^{\lambda/\mu}, \chi^\mu \times \psi\rangle = \langle \chi^{\lambda/\mu}, \psi\rangle$. $\square$

**1.1.4. Pieri's rule and Young's rule.** In this section we provide module theoretic proofs of the well-known Pieri and Young rules. These follow as a consequence of Theorem 1.1.4.

We require the following definition. A skew partition $\lambda/\mu$ is a *vertical* (resp. *horizontal*) *strip* if $[\lambda/\mu]$ has at most one box in each row (resp. column).

THEOREM 1.1.12 (Young's rule). *Let $\lambda$ be a partition of $m+n$. If $\mu$ is a subpartition of $\lambda$ of size $m$ then*

$$\langle \chi^\lambda\big\downarrow_{S_m\times S_n}, \chi^\mu \times 1_{S_n}\rangle = \begin{cases} 1 & \text{if } \lambda/\mu \text{ is a horizontal strip} \\ 0 & \text{otherwise.}\end{cases}$$

PROOF. By Maschke's Theorem and (1.5) it suffices to prove that the multiplicity of $1_{S_n}$ as a direct summand of $S_{\mathbf{C}}^{\lambda/\mu}$ is 1 if $\lambda/\mu$ has no boxes in the same column and otherwise 0. For this we use the corresponding idempotent $E = \frac{1}{n!}\sum_{\tau\in S_n} \tau \in \mathbf{C}S_n$.

Suppose that $\lambda/\mu$ contains boxes $(i,j)$, $(i+1,j)$ in the same column. If $t$ is a $\lambda/\mu$-tableau then $(1+(x,y))e(t) = 0$ where $x = t(i,j)$ and $y = t(i+1,j)$. Since $E = \frac{1}{n!}\big(1+(x,y)\big)\sum_\pi \pi$, where the sum is over a set of right coset representatives for the cosets of $\langle (x,y)\rangle$ in $S_n$, it follows that $ES^{\lambda/\mu} = 0$ as required.

Suppose that $\lambda/\mu$ has no two boxes in the same column. Let $t$ be a $\lambda/\mu$-tableau. By assumption the column stabiliser $C(t)$ is trivial, and so the tabloid $\{t\}$ equals the polytabloid $e(t)$. It follows that $M_{\mathbf{C}}^{\lambda/\mu} = S_{\mathbf{C}}^{\lambda/\mu}$, and so $S_{\mathbf{C}}^{\lambda/\mu}$ is a transitive permutation module. Therefore

$$\langle S_{\mathbf{C}}^{\lambda/\mu}, 1_{S_n}\rangle = 1,$$

as required. $\square$

EXAMPLE 1.1.13. The unique submodule of $S_{\mathbf{C}}^{(3,1)/(1)}$ affording the character $1_{S_3}$ is spanned by $\{t\} + \{(2,3)t\} + \{(1\ 2\ 3)t\}$, where

$$t = \begin{array}{ccc} & \boxed{1} & \boxed{2} \\ \boxed{3} & & \end{array}, \quad (2\ 3)t = \begin{array}{ccc} & \boxed{1} & \boxed{3} \\ \boxed{2} & & \end{array}, \quad (1\ 2\ 3)t = \begin{array}{ccc} & \boxed{2} & \boxed{3} \\ \boxed{1} & & \end{array}.$$

Using Lemma 1.1.11 we immediately obtain the more usual statement of Young's rule that if $\nu$ is a partition of $n$ then $(\chi^\nu \times 1_{S_\ell})\big\uparrow_{S_n\times S_\ell}^{S_{n+\ell}} = \sum_\kappa \chi^\kappa$ where the sum is over all partitions $\kappa$ of $n+\ell$ such that $\kappa/\nu$ is a horizontal strip.

Multiplying by the sign character then gives Pieri's rule: $(\chi^\nu \times \chi^{(1^\ell)})\uparrow_{S_n\times S_\ell}^{S_{n+\ell}} = \sum_\kappa \chi^\kappa$ where the sum is over all partitions $\kappa$ of $n+\ell$ such that $\kappa/\nu$ is a vertical strip.

REMARK 1.1.14. A similarly explicit proof of Pieri's rule can be given, using a similar argument to the proof of Theorem 1.1.12. To reduce to vertical strips, observe that if $t$ is a standard $\lambda/\mu$-tableau with boxes $(i,j)$ and $(i,j+1)$ then $\big(1-(x,y)\big)\{t\} = 0$ where $x = t(i,j)$ and $y = t(i,j+1)$.

## 1.2. Wreath products and their representations

In this section we describe the representation theory of $FG \wr S_n$, where $G$ is a finite group and $F$ is an algebraically closed field. Define the imprimitive wreath product $G \wr S_n$ to be the semidirect product $G^n \rtimes S_n$, where the action of $S_n$ on $G^n$ is given by place permutation. Explicitly the multiplication in $G \wr S_n$ is given by:

$$(g_1, \dots, g_n; \sigma)(h_1, \dots, h_n; \tau) = (g_1 h_{\sigma^{-1}(1)}, \dots, g_n h_{\sigma^{-1}(n)}; \sigma\tau),$$

where $g_i, h_i \in G$ for all $1 \le i \le n$ and $\sigma, \tau \in S_n$. We have that $B_n := G^n \times \{1\}$ is a subgroup of $G \wr S_n$. We refer to $B_n$ as the *base group*, and we remark that $B_n$ is a normal subgroup in $G \wr S_n$. The quotient of $G \wr S_n$ by $B_n$ is isomorphic to $S_n$. As is usual with semidirect products, the quotient group $S_n$ can be realised as a subgroup of $G \wr S_n$. Indeed we have that the subgroup

$$T_n := \langle (1_G, \dots, 1_G; \sigma) : \sigma \in S_n \rangle$$

of $G \wr S_n$ is isomorphic to $S_n$, and we refer to $T_n$ as the *top group*.

DEFINITION. Let $G$ be a finite group, and let $K \le S_n$. Define the subgroup $G \wr K$ of $G \wr S_n$ as follows:

$$G \wr K = G^n \rtimes K.$$

Given $\nu := (\nu_1, \dots, \nu_t)$ a composition of $n$, there is an obvious isomorphism

$$G \wr S_\nu \cong \prod_{i=1}^{t} G \wr S_{\nu_i}.$$

### 1.2.1. Conjugacy in the wreath product.

In this section we describe the conjugacy classes of $G \wr S_n$. We follow §4.2 in [**35**].

DEFINITION. Given $g := (g_1, \dots, g_n; \sigma) \in G \wr S_n$, let $\nu := (a_1\ a_2 \dots a_k)$ be a cycle in $\sigma$, where $a_1 = \min_{1 \le i \le k}\{a_i\}$. Define the *cycle product* $g_\nu \in G$ as follows:

$$g_\nu = g_{a_k} \cdots g_{a_2} g_{a_1}.$$

We refer to $g_\nu$ as the *cycle product of $g$ corresponding to $\nu$*, and we say that $g_\nu$ has *length $k$*.

Definition. Let $C^1, \dots, C^t$ denote the conjugacy classes of $G$. Given $(g_1, \dots, g_n; \sigma) \in G \wr S_n$, denote by $a_{ik}((g_1, \dots, g_n; \sigma))$ the number of cycle products of $(g_1, \dots, g_n; \sigma)$ that have length $k$ and lie in the conjugacy class $C^i$ of $G$. Furthermore, define the *cycle product matrix* of $(g_1, \dots, g_n; \sigma)$ as follows:

$$a((g_1, \dots, g_n; \sigma)) = (a_{ik}((g_1, \dots, g_n; \sigma))).$$

The following theorem describes the conjugacy classes of $G \wr S_n$.

Theorem 1.2.1. [**35**, Theorem 4.2.8] *Two elements in $G \wr S_n$ are conjugate if and only if they have the same cycle product matrix.*

Remark 1.2.2. It follows from Theorem 1.2.1 that the $G \wr S_n$-conjugacy classes are in bijection with the elements of $\mathcal{P}^t(n)$. Indeed given $g := (g_1, \dots, g_n; \sigma) \in G \wr S_n$, let $\sigma_{i_1}, \dots, \sigma_{i_\ell}$ be all the disjoint cycles of $\sigma$ such that the cycle product $g_{\sigma_{i_j}}$ is in $C^i$. Let $\lambda^i$ denote the cycle type of the permutation $\sigma_{i_1} \dots \sigma_{i_\ell}$ for all $i$. Then the multi-partition $(\lambda^1, \dots, \lambda^t) \in \mathcal{P}^t(n)$ labels the conjugacy class of $G \wr S_n$ containing $g$.

We also prove the following useful lemma, which considers conjugating subgroups of the top group. We prove the result for semidirect products in general.

Lemma 1.2.3. *Let $G$ and $H$ be finite groups, and let $K \leq H$. Then*

$$N_{G \rtimes H}(K) = C_G(K) \rtimes N_H(K),$$
$$C_{G \rtimes H}(K) = C_G(K) \rtimes C_H(K).$$

Proof. We prove that $N_{G \rtimes H}(K) = C_G(K) \rtimes N_H(K)$, as the proof for $C_{G \rtimes H}(K)$ is entirely similar. Given $g \in G$ and $h \in H$, we write ${}^h g$ for the image of $g$ under the action of $h$.

It is clear that $N_{G \rtimes H}(K)$ contains $C_G(K) \rtimes N_H(K)$. Fix elements $(g; h) \in N_{G \rtimes H}(K)$ and $k \in K$. Define $\tilde{h} = hkh^{-1}$, and so

$$(g; h)(1; k)({}^{h^{-1}} g^{-1}; h^{-1}) = (g({}^{\tilde{h}} g^{-1}); \tilde{h}). \tag{1.6}$$

By assumption, $(g({}^{\tilde{h}} g^{-1}); \tilde{h}) \in K$, and so $h \in N_H(K)$.

We now define the group homomorphism $\vartheta : N_{G \rtimes H}(K) \to N_H(K)$ by $(g; h) \mapsto h$. Therefore $\ker(\vartheta) = N_G(K)$, and applying (1.6) with $h = 1$ gives

$$(g; 1)(1; k)(g^{-1}; 1) = (g({}^k g^{-1}); k).$$

If $g({}^k g^{-1}) = 1$, then $g \in C_G(K)$. Therefore $\ker(\vartheta) = C_G(K)$. As $\vartheta$ is clearly surjective, the first isomorphism theorem gives that

$$|N_{G \rtimes H}(K)| = |C_G(K)||N_H(K)|. \qquad \square$$

**1.2.2. The irreducible representations of $G \wr S_n$.** We remind the reader that $F$ is an algebraically closed field. We follow §4.3 in [**35**].

DEFINITION. Let $V$ be an $FG$-module. Then define the $FG \wr S_n$-module $\widetilde{V}^{\otimes n}$ to be the vector space $V^{\otimes n}$, on which $G \wr S_n$ acts as follows:

$$(g_1, \dots, g_n; \sigma)(v_1 \otimes \dots \otimes v_n) = g_1 v_{\sigma^{-1}(1)} \otimes \dots \otimes g_n v_{\sigma^{-1}(n)}.$$

We then extend this action linearly to $FG \wr S_n$. Moreover, if $\vartheta$ is the character of $V$, we write $\widetilde{\vartheta}^{\times n}$ for the character of $\widetilde{V}^{\otimes n}$.

By restricting the action of $\widetilde{V}^{\otimes n}$ to the base group, we see that $\widetilde{V}^{\otimes n}$ is an irreducible $FG \wr S_n$-module if and only if $V$ is an irreducible $FG$-module. We can also further extend this module to an irreducible module of $G \wr S_n$. In order to do this we require the following definition using the language of representations.

DEFINITION. Let $\rho$ be an $FS_n$-representation. Define $\operatorname{Inf}_{S_n}^{G \wr S_n} \rho$ to be the $FG \wr S_n$-representation such that

$$(\operatorname{Inf}_{S_n}^{G \wr S_n} \rho)(g_1, \dots, g_n; \sigma) = \rho(\sigma).$$

If $W$ is a module corresponding to $\rho$, then we write $\operatorname{Inf}_{S_n}^{G \wr S_n} W$ for the module corresponding to $\operatorname{Inf}_{S_n}^{G \wr S_n} \rho$. We refer to $\operatorname{Inf}_{S_n}^{G \wr S_n} W$ as the *inflation* of $W$ from $S_n$ to $G \wr S_n$.

It is an elementary fact that $\operatorname{Inf}_{S_n}^{G \wr S_n} W$ is an irreducible $FG \wr S_n$-module if and only if $W$ is an irreducible $FS_n$-module. We now complete the extension procedure mentioned above by considering the inner tensor product of modules $\widetilde{V}^{\otimes n} \otimes \operatorname{Inf}_{S_n}^{G \wr S_n} W$. Generally the inner tensor product of irreducible modules is not irreducible, however the following lemma shows that this module is.

LEMMA 1.2.4. *Let $\vartheta \in \operatorname{Irr}(G)$, and let $\lambda$ be a partition of $n$. Then*

$$\widetilde{\vartheta}^{\times n} \operatorname{Inf}_{S_n}^{G \wr S_n} \chi^\lambda \in \operatorname{Irr}(G \wr S_n).$$

PROOF. By Frobenius reciprocity

$$\langle \vartheta^{\times n} {\uparrow}_{G^n}^{G \wr S_n}, \widetilde{\vartheta}^{\times n} \operatorname{Inf}_{S_n}^{G \wr S_n} \chi^\lambda \rangle = \chi^\lambda(1).$$

It follows from Theorem 1.1.2 and by counting dimensions that

$$\vartheta^{\times n} {\uparrow}_{G^n}^{G \wr S_n} = \sum_{\lambda \vdash n} \chi^\lambda(1) \widetilde{\vartheta}^{\times n} \operatorname{Inf}_{S_n}^{G \wr S_n} \chi^\lambda. \tag{1.7}$$

Using (1.7), Frobenius reciprocity and Theorem 1.1.2 once more shows that

$$\langle \vartheta^{\times n} {\uparrow}_{G^n}^{G \wr S_n}, \vartheta^{\times n} {\uparrow}_{G^n}^{G \wr S_n} \rangle = \sum_{\lambda \vdash n} \chi^\lambda(1)^2 = n!.$$

This implies that

$$n! = \langle \sum_{\lambda \vdash n} \chi^\lambda(1) \widetilde{\vartheta}^{\times n} \operatorname{Inf}_{S_n}^{G \wr S_n} \chi^\lambda, \sum_{\mu \vdash n} \chi^\mu(1) \widetilde{\vartheta}^{\times n} \operatorname{Inf}_{S_n}^{G \wr S_n} \chi^\mu \rangle,$$

and so

$$\langle \widetilde{\vartheta}^{\times n} \operatorname{Inf}_{S_n}^{G \wr S_n} \chi^\lambda, \widetilde{\vartheta}^{\times n} \operatorname{Inf}_{S_n}^{G \wr S_n} \chi^\mu \rangle = \begin{cases} 1 & \text{if } \lambda = \mu \\ 0 & \text{otherwise.} \end{cases} \qquad \square$$

We have determined certain irreducible $FG \wr S_n$-modules, however this list is by no means complete. Nevertheless, using the modules that we have introduced, we can completely describe the irreducible $FG \wr S_n$-modules. A fundamental result in doing this will be Proposition 1.2.5 below, which plays a central part in the Clifford theory of group representations. We require the following preliminaries.

Recall that $\mathcal{P}^t(n)$ denotes the set of multi-partitions of $n$ with length equal to $t$.

Definition. Let $G$ be a finite group, and let

$$\{M_1, M_2, \ldots, M_t\}$$

be a complete set of representatives of the isomorphism classes of irreducible $FG$-modules. Given $(\lambda_1, \ldots, \lambda_t) \in \mathcal{P}^t(n)$, define

$$M_{\lambda^1, \lambda^2, \ldots, \lambda^t} = \left( \boxtimes_{i=1}^t \widetilde{M_i}^{\otimes n_i} \operatorname{Inf}_{S_{n_i}}^{G \wr S_{n_i}} S^{\lambda^i} \right) \big\uparrow_{G \wr S_{(n_1, \ldots, n_t)}}^{G \wr S_n},$$

where $\lambda^i$ is a partition of $n_i$ for each $i \in \{1, \ldots, t\}$.

We now give the background from Clifford theory required to show that these modules are irreducible. Given $K \trianglelefteq G$, let $M$ be an $FK$-module. We define the *inertial group* of $M$ to be the subgroup of $G$ consisting of all $g \in G$ such that ${}^g M \cong M$.

Proposition 1.2.5. [**3**, Proposition 3.13.2] *Let $T$ be the inertial group of $M$. Suppose that $M$ is indecomposable, and that*

$$M \big\uparrow_K^T = M_1 \oplus \cdots \oplus M_r,$$

*where each $M_i$ is an indecomposable $FT$-module. Then $M_i \big\uparrow_T^G$ is indecomposable, and $M_i \big\uparrow_T^G \cong M_j \big\uparrow_T^G$ if and only if $M_i \cong M_j$.*

The following theorem completely describes the irreducible $FG \wr S_n$-modules.

Theorem 1.2.6. *The set*

$$\{M_{\lambda^1, \ldots, \lambda^t} : (\lambda^1, \ldots, \lambda^t) \in \mathcal{P}^t(n)\}$$

*is a complete set of pairwise non-isomorphic irreducible $FG \wr S_n$-modules.*

PROOF. It follows from Lemma 1.2.4 and Proposition 1.2.5 that the modules in this set are irreducible and pairwise non-isomorphic. The result now follows from Remark 1.2.2. □

## 1.3. Modular representation theory

In §4 and §5 we study the modular representation theories of $FS_n$ and $FC_2 \wr S_n$, respectively. In this section we give the background on the modular representation theory of finite groups that we use. We start by stating the following result, which we refer to the Krull–Schmidt Theorem throughout.

THEOREM 1.3.1. [**1**, §4, Theorem 3] *Let $F$ be a field, and let $M$ be an $FG$-module such that*

$$M = U_1 \oplus \cdots \oplus U_r$$
$$M = V_1 \oplus \cdots \oplus V_s,$$

*are two decompositions of $M$ into the direct sum of indecomposable modules. Then $r = s$, and, after a suitable renumbering, $U_i \cong V_i$ for all $i$.*

**1.3.1. Induced modules and relative projectivity.** In this section we state several results from [**1**] that will be used throughout. A detailed account on induced modules can be found in [**1**, §8] and on relative projectivity in [**1**, §9]. The results in this section highlight that induced modules are both useful tools in the representation theory of finite groups and are interesting objects of study in their own right.

A notable case of induced modules is when the subgroup we induce from is the trivial subgroup. Then the induced module $F\uparrow_1^G$ is isomorphic to the module $FG$, where the action is given by the linear extension of the multiplication of $G$ on itself. An $FG$-module that is isomorphic to a direct sum of $r$ copies of $FG$ for some $r \in \mathbf{N}$ is known as a *free module of rank $r$*.

We state the following lemma, which gives a very useful characterisation of induced modules.

LEMMA 1.3.2. *Given $H \leq G$, let $X$ be an $FH$-module, and let $U$ be an $FG$-module. Suppose that $U$ is generated by $X$. Then $U \cong X\big\uparrow_H^G$ if and only if $\dim_F U = [G : H] \dim_F X$.*

We now define the vertex of an indecomposable module, which will be the central object of study in §5.

DEFINITION. Let $U$ be an indecomposable $FG$-module, and let $H \leq G$. We say that $U$ has *vertex $H$* if $H$ is minimal, with respect to inclusion, such that $U$ is a summand of $V\big\uparrow_H^G$, for some indecomposable $FH$-module $V$.

We note that in this definition, we are implicitly assuming that vertices exist. It is not obvious that this should be true, however Theorem 1.3.3 in this section shows that it is.

Before we state this result, we introduce the following special class of modules, known as projective modules. Suppose that the indecomposable $FG$-module $M$ has vertex equal to the trivial subgroup. It follows from the definition of a vertex that $M$ is a summand of $F\uparrow_1^G$. We have seen that $F\uparrow_1^G$ is isomorphic to the free module $FG$, and so $M$ is a summand of a free module. Generally we say that the $FG$-module $V$ is *projective* if $V$ is a summand of a free module of rank $r$ for some $r \in \mathbf{N}$. As we will see in §1.3.4, projective modules are useful objects in relating the modular and ordinary representation theories of finite groups.

Even more generally, if the indecomposable $FG$-module $U$ is a summand of $V\uparrow_H^G$, for some indecomposable $FH$-module $V$, then we say that $U$ is *relatively $H$-projective*. Therefore we can restate the definition of the vertex of $U$ as the minimal subgroup $H$ of $G$ such that $U$ is relatively $H$-projective. In this context we see that vertices of indecomposable modules are of interest as they provide a measure of 'how far' a module is from being projective.

THEOREM 1.3.3. *Let $U$ be an indecomposable $FG$-module, and let $F$ be a field of positive characteristic $p$. Then there exists a $p$-subgroup $P$ of $G$ minimal such that $U$ is a summand of $S\uparrow_P^G$, for some indecomposable $FP$-module $S$. Moreover, $P$ is unique up to conjugacy in $G$, and $S$ is unique up to conjugacy in $N_G(P)$.*

We refer to the $FP$-module $S$ in the statement of Theorem 1.3.3 as the source of $U$. In the case that $S$ is the trivial $FP$-module, we say that $U$ is a *trivial source module*. We have the following useful result on trivial source modules.

LEMMA 1.3.4. *Let $U$ be an indecomposable $FG$-module. Then $U$ is a trivial source module if and only if $U$ is a summand of $F\uparrow_H^G$, for some subgroup $H$ of $G$.*

It follows from this lemma that trivial source modules are the precisely summands of permutation modules.

In general, it is difficult to determine the vertex of an indecomposable module. In the case of trivial source modules, there is an algorithmic description for determining their vertices. We give details of this method in §1.3.3 below.

We end this section by stating the following lemma, which gives an example of when we can immediately determine the vertex of a module.

LEMMA 1.3.5. *Let $F$ be a field of characteristic $p > 0$. Given a $p$-group $P$, let $Q \leq P$. Then $F\Big\uparrow_Q^P$ is an indecomposable $FP$-module with a vertex equal to $P$.*

**1.3.2. Blocks of the group algebra.** Let $G$ be a finite group, and let $F$ be any field. Unless stated otherwise, the definitions and results in this section are as given in [**1**, §13].

The group algebra $FG$ can be written uniquely as a sum of minimal two-sided ideals

$$FG = B_1 \oplus \cdots \oplus B_t,$$

and each such $B_i$ is referred to as a *block* of $FG$. We say that an $FG$-module $M$ *lies in the block* $B_i$ if there exists a unique $1 \leq i \leq t$ such that $B_i M = M$, and $B_j M = 0$ for all $j \neq i$. We have the following result.

LEMMA 1.3.6. *Let $\{B_1, \ldots, B_t\}$ be the blocks of $FG$. If $M$ is an $FG$-module, then $M$ has a unique decomposition*

$$M = M_1 \oplus \cdots \oplus M_t$$

*such that $M_i$ lies in the block $B_i$.*

It follows immediately from Lemma 1.3.6 that every indecomposable $FG$-module lies in some block $B_i$.

An alternative way to define blocks is by considering $FG$ as an $F[G \times G]$-module, by extending the action $(g_1, g_2)g := g_1 g g_2^{-1}$ linearly. The blocks of $FG$ are therefore precisely the indecomposable summands of $FG$ under this action.

It seems difficult to give a description of the blocks of $FG$ in general. In the case of the symmetric group, there is a beautiful combinatorial description of the blocks of $FS_n$. This result is known as Nakayama's conjecture, which we state in §1.3.6. When $F$ is a field of characteristic $p \neq 2$, the blocks of the group algebra $FC_2 \wr S_n$ have a description that closely resembles that of $FS_n$. We state and prove the characterisation of the blocks of $FC_2 \wr S_n$ in §1.4.4.

It is sometimes useful to relate the blocks of $FG$ to the blocks of $FH$, where $H \leq G$. We do this using the following definition.

DEFINITION. Given $H \leq G$, let $B$ a block of $G$, and let $b$ be a block of $H$. We say that the block $B$ *corresponds to* $b$ if $b$ is a summand of $B{\downarrow_{H \times H}}$, and $B$ is the unique block of $FG$ with this property. In this case, we write $b^G = B$.

**1.3.3. The Brauer morphism.** Throughout this section let $G$ be a finite group, and let $F$ be a field of characteristic $p > 0$. Unless stated otherwise, the definitions and results in this section are as in [**8**].

Let $H \leq G$, and let $M$ be an $FG$-module. Define $M^H$ to be the set of vectors in $M$ fixed by $H$. Given $L \leq H \leq G$, we define the map

$$\begin{aligned} \mathrm{Tr}_L^H : M^L &\rightarrow M^H \\ x &\mapsto \sum gx, \end{aligned}$$

where the sum runs over a transversal of the cosets of $L$ in $H$.

When $P$ is a $p$-subgroup of $G$, we define

$$M(P) = M^P / \sum_{Q<P} \mathrm{Tr}_Q^P M^Q.$$

It is easy to prove that $M^P$ is an $FN_G(P)$-module, on which $P$ acts trivially. The same is true for $\sum_{Q<P} \mathrm{Tr}_Q^P M^Q$, and therefore also for $M(P)$. The quotient map $M^P \mapsto M(P)$, is known as the *Brauer morphism*, and this map is an $FN_G(P)$-module homomorphism.

The module $M$ is a *p-permutation module* if for all $p$-subgroups of $G$, there exists an $F$-basis of $M$ that is permuted by $P$. If $\mathcal{B}$ is such a basis, then we say that $\mathcal{B}$ is a *p-permutation basis* of $M$ with respect to $P$.

LEMMA 1.3.7. *The module $M$ is a p-permutation module if and only if there exists a subgroup $H$ of $G$ such that $M$ is a summand of $F{\uparrow}_H^G$.*

It follows that $p$-permutation modules are the familiar trivial source modules. We state the following proposition, which allows us to identify $p$-permutation modules using existing $p$-permutation modules.

PROPOSITION 1.3.8.

(1) *Suppose that $M$ and $N$ are two p-permutation $FG$-modules. Then the modules $M \oplus N$ and $M \otimes N$ are both p-permutation modules*
(2) *Given $H \leq G$, if $M$ (resp. $N$) is a p-permutation $FG$-module (resp. $FH$-module), then $M \downarrow_H^G$ (resp. $N \uparrow_H^G$) is a p-permutation $FH$-module (resp. $FG$-module).*
(3) *Any summand of a p-permutation module is a p-permutation module.*

We now assume that $M$ is a $p$-permutation module, and that $P$ is a $p$-subgroup of $G$. The following lemmas show how the Brauer morphism can be used to determine the vertices of an indecomposable $p$-permutation module.

LEMMA 1.3.9. *Let $M$ be an indecomposable p-permutation $FG$-module. Then $M$ has a vertex equal to $P$ if and only if $P$ is a maximal p-subgroup of $G$ such that $M(P) \neq 0$.*

LEMMA 1.3.10. *Let $\mathcal{B}$ be a p-permutation basis of $M$ with respect to $P$, and let $\mathcal{B}^P$ be the set of points in $\mathcal{B}$ that are fixed by $P$. Then $\mathcal{B}^P$ is a basis of $M(P)$.*

It follows that $P$ is a vertex of $M$ if there exists a vector in a $p$-permutation basis of $M$ (with respect to $P$) that has non-zero $P$-fixed points, and $P$ is maximal with this property.

LEMMA 1.3.11. [**24**, Lemma 4.7] *Let $R \trianglelefteq P \leq G$, and let $K = N_G(R)$. Then $M(R)$ is a $p$-permutation $FK$-module. Moreover, $M(P) \cong M(R)(P)$, where the isomorphism is of $FN_K(P)$-modules.*

We state the following lemma from [**65**], which relates the block of $FG$ to the Brauer morphism.

LEMMA 1.3.12. [**65**, Lemma 7.4] *Suppose that $M$ lies in the block $B$ of $G$. If $M(P)$ has a summand lying in the block $b$ of $N_G(P)$, then $b^G = B$.*

**1.3.4. Relating ordinary and modular representation theory.** In this section we consider how we can use results in ordinary representation theory to determine information on the modular representation theory of finite groups. Unless stated otherwise, the background that we give is a special case of the results in [**63**, §9.4].

Let $\mathbf{F}_p$ denote the finite field with $p$ elements, and let $\mathbf{Z}_p$ denote the ring of $p$-adic integers. With $\mathbf{Q}_p$ defined to be the field of $p$-adic numbers, we have that the triple $(\mathbf{Q}_p, \mathbf{Z}_p, \mathbf{F}_p)$ is an example of a *$p$-modular system.* Note that the definitions in the next paragraph can be generalised to an arbitrary $p$-modular system $(\mathcal{O}, R, F)$.

Given a $\mathbf{Z}_pG$-module $U$, we define the *reduction modulo $p$* of $U$ to be the $\mathbf{F}_pG$-module equal to

$$U_{\mathbf{F}_p} := \mathbf{F}_p \otimes_{\mathbf{Z}_p} U.$$

We remark that this notation is consistent with the notations $M^{\lambda/\mu}_{\mathbf{F}_p}$ and $S^{\lambda/\mu}_{\mathbf{F}_p}$ given in §1.1.1. If $V$ is an $\mathbf{F}_pG$-module such that $V = U_{\mathbf{F}_p}$ for some $\mathbf{Z}_pG$-module $U$, then we say that $V$ can be *lifted* to $U$. Note that it is not always possible for an $\mathbf{F}_pG$-module to be lifted to a $\mathbf{Z}_pG$-module. Moreover, if a module can be lifted then the lift may not be unique.

In §5 we consider the lifts of certain trivial source modules. The following theorem from [**3**] shows that trivial source modules can always be lifted; moreover, the lift of a trivial source module is unique up to isomorphism. We refer to this result as Scott's Lifting Theorem.

THEOREM 1.3.13. [**3**, Corollary 3.11.4] *Every trivial source $\mathbf{F}_pG$-module lifts to a trivial source $\mathbf{Z}_pG$-module, unique up to isomorphism.*

It is an immediate consequence of Scott's Lifting Theorem that projective $\mathbf{F}_pG$-modules can be lifted uniquely to $\mathbf{Z}_pG$-modules. We can use the lift of a projective module to understand the relationship between $\mathbf{Z}_pG$-modules and $\mathbf{F}_pG$-modules.

DEFINITION. Given a finite group $G$, we say that a field $F$ is a *splitting field* for $G$ if every irreducible $FG$-module $S$ is such that $E \otimes_F S$ is also an irreducible module for every field extension $E$ of $F$.

EXAMPLE 1.3.14. By the remark immediately after Theorem 1.1.2, the rational field $\mathbf{Q}$ is a splitting field for $S_n$ for all $n \in \mathbf{N}$. As stated in the proof of [**33**, Theorem 11.5], it follows from [**33**, Theorem 11.1] that in fact *every* field is a splitting field for $S_n$.

DEFINITION. Suppose that $\mathbf{Q}_p$ and $\mathbf{F}_p$ are splitting fields for $G$. Let $S$ be an irreducible $\mathbf{Q}_pG$-module, and let $D$ be an irreducible $\mathbf{F}_pG$-module. The *decomposition number* $d_{SD}$ is equal to the number of composition factors of $S_{\mathbf{F}_p}$ isomorphic to $D$.

Note that it follows from the Jordan–Hölder theorem that decomposition numbers are well-defined. Determining the decomposition numbers of $S_n$ is a fundamental open problem. Theorem 1.3.16 below, which we refer to as Brauer reciprocity, is a tool that enables us to determine decomposition numbers of finite groups using projective modules. In order to state this result, we need the following theorem from [**1**].

THEOREM 1.3.15. [**1**, §5, Theorem 3] *Let $G$ be a finite group, and let $F$ be a field. There is a one-to-one correspondence between isomorphism classes of projective indecomposable modules and isomorphism classes of irreducible $FG$-modules given by associating the indecomposable projective $FG$-module $P$ to the irreducible module $P/\operatorname{rad} P$.*

THEOREM 1.3.16. [**63**, Proposition 9.5.1] *Suppose that $\mathbf{Q}_p$ and $\mathbf{F}_p$ are splitting fields for $G$. Write $P_D$ for the projective indecomposable $\mathbf{F}_pG$-module corresponding to the irreducible $\mathbf{F}_pG$-module $D$. Let $\hat{P}_D$ denote the $\mathbf{Q}_pG$-module that is the lift of $P_D$. Then*

$$\hat{P}_D \cong \bigoplus_S d_{SD} S,$$

*where the sum runs over all irreducible $\mathbf{Q}_pG$-modules $S$.*

**1.3.5. Binomial coefficients modulo $p$.** Throughout this section fix a prime $p$, and fix $a, b \in \mathbf{N}_0$ such that $b \leq a$. Let

$$a = a_0p^0 + a_1p^1 + \cdots + a_tp^t$$
$$b = b_0p^0 + b_1p^1 + \cdots + b_tp^t$$

be the $p$-adic expansions of $a$ and $b$, respectively. In §3 and §4 we require the value of the binomial coefficient $\binom{a}{b}$ modulo $p$. We can compute this using the $p$-adic expansions of $a$ and $b$ via the following elementary lemma.

LEMMA 1.3.17 (Lucas' Theorem). *There is a congruence of binomial coefficients*

$$\binom{a}{b} \equiv \prod_{u=0}^{t} \binom{a_u}{b_u} \bmod p.$$

We refer to the product $\prod_{u=0}^{t}\binom{a_u}{b_u}$ in the statement of Lucas' Theorem as the *p-adic expansion* of $\binom{a}{b}$. Observe that this $p$-adic expansion is well-defined since the $p$-adic expansions of $a$ and $b$ are unique.

Lemma 1.3.18 uses Lucas' Theorem to give a necessary and sufficient condition for $\binom{a}{b}$ to be non-zero modulo $p$. We require the following notation.

Consider the following representation of the $p$-ary addition of $a-b$ and $b$:

$$\begin{array}{r|cccccc} a-b & (a-b)_0 & (a-b)_1 & \dots & (a-b)_u & \dots & (a-b)_t \\ b & b_0 & b_1 & \dots & b_u & \dots & b_t \\ \hline a & a_0 & a_1 & \dots & a_u & \dots & a_t \end{array},$$

where $(a-b) = (a-b)_0p^0 + (a-b)_1p^1 + \dots + (a-b)_tp^t$ is the $p$-adic expansion of $a-b$. Given $0 \leq u < t$, define $c_u \in \{0, 1, 2, \dots, p-1\}$ to be such that

$$(a-b)_u + b_u + c_{u-1} = a_u + pc_u,$$

so that $c_u$ is the carry *leaving* column $u$ in this addition. We say that the $p$-ary addition of $a-b$ and $b$ is *carry free* if $c_u = 0$ for all $0 \leq u < t$.

LEMMA 1.3.18. *The binomial coefficient $\binom{a}{b}$ is non-zero modulo $p$ if and only if the $p$-ary addition of $a-b$ and $b$ is carry free.*

PROOF. By definition of the carries $c_u$, the $p$-ary addition of $a-b$ and $b$ is carry free if and only if $(a-b)_u + b_u = a_u$ for all $0 \leq u \leq t$. This occurs if and only if $b_u \leq a_u$ for all $0 \leq u \leq t$, since $0 \leq (a-b)_u < p$ for all such $u$. The result now follows by applying Lucas' Theorem. □

COROLLARY 1.3.19. *Fix $a \in \mathbf{N}_0$ such that $0 \leq a \leq 2^n$. The binomial coefficient $\binom{2^n}{a}$ is odd if and only if either $a = 0$, or $a = 2^n$.*

PROOF. The binomial coefficient is odd if and only if it is non-zero modulo 2. Now apply Lemma 1.3.18. □

**1.3.6. The modular representation theory of $S_n$.** In this section we assume that $F$ is a field of positive characteristic $p$.

Let $\nu$ be a partition of $n$. Define $\langle -, - \rangle$ to be the unique bilinear form on $M^\nu$ such that

$$\langle \{t_1\}, \{t_2\} \rangle = \begin{cases} 1 & \text{if } \{t_1\} = \{t_2\} \\ 0 & \text{otherwise,} \end{cases}$$

where $t_1$ and $t_2$ are $\nu$-tableaux.

We then have the following result, known as James' Submodule theorem.

THEOREM 1.3.20. [**33**, Theorem 4.8] *If $U$ is a submodule of $M^\nu$, then either $S^\nu \subset U$ or $U \subset (S^\nu)^\perp$.*

We now describe the irreducible $FS_n$-modules. We require the following definition.

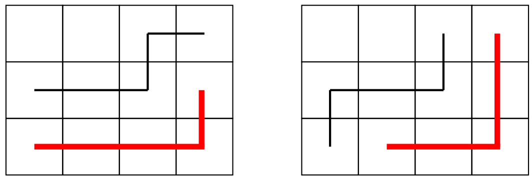


FIGURE 1.1. The two ways in which we can remove border strips of size 5 from the Young diagram of $(4,4,4)$. In both cases we first remove the thick red strip followed by the black strip.

DEFINITION. We say that a partition $\nu$ is *p-regular* if there does not exist $i \in \mathbf{N}$ such that

$$\nu_{i+1} = \nu_{i+2} = \cdots = \nu_{i+p}.$$

For example the partition $(5,1,1)$ of 7 is $p$-regular if and only if $p \geq 3$.

THEOREM 1.3.21. [**33**, Theorems 4.9, 11.5] *Let $\nu$ be a $p$-regular partition of $n$. Then the submodule $S^\nu \cap (S^\nu)^\perp$ is the unique maximal submodule of $S^\nu$, and so $D^\nu := S^\nu / S^\nu \cap (S^\nu)^\perp$ is an irreducible $FS_n$-module. Moreover, the set*

$$\{D^\nu : \nu \textit{ is a } p\textit{-regular partition of } n\}$$

*is a complete set of pairwise non-isomorphic irreducible $FS_n$-modules.*

Given partitions $\lambda$ and $\nu$ of $n$ such that $\nu$ is a $p$-regular partition, we specialise the definition in §1.3.4 and write $d_{\lambda\nu}$ for the decomposition number of $S_n$ equal to the number of composition factors of $S^\lambda_{\mathbf{F}_p}$ that are isomorphic to $D^\nu$.

We now turn to the blocks of $FS_n$, which are given by Nakayama's conjecture. In order to state this result, we require the following definition.

DEFINITION. Given a partition $\lambda$ of $n$, the *p-core* of $\lambda$ is the partition whose Young diagram is obtained by repeatedly removing border strips of size $p$ from $[\lambda]$.

We remark that implicit in the definition of a $p$-core is that it is unique, and so it is independent from the order in which the border strips are removed from $[\lambda]$. This is proved in [**35**, Theorem 2.7.16] using the abacus notation for partitions (see [**35**, §2.7]).

EXAMPLE 1.3.22. We determine the 5-core of $(4,4,4)$. As shown in Figure 1.1 above we can remove two 5-strips (highlighted in red), namely $(4,4,4)/(4,3)$ and $(4,4,4)/(3,3,1)$ from $[(4,4,4)]$. In the first case removing the strip $(4,3)/(2)$ from $[(4,3)]$ yields the 5-core $(2)$. Similarly, in the second case removing the strip $(3,3,1)/(2)$ from $[(3,3,1)]$ also yields the 5-core $(2)$, as claimed.

THEOREM 1.3.23 (Nakayama's conjecture). *The blocks of $FS_n$ are labelled by pairs $(\gamma, v)$ such that $\gamma$ is a $p$-core partition, and $|\gamma| + vp = n$. Moreover, the $FS_n$-module $S^\lambda$ lies in the block labelled by the $p$-core of $\lambda$.*

We remark that Nakayama's conjecture was first proved by Brauer and Robinson in [**5**] and [**58**]. In [**56**] O'Donovan gives an accessible proof of the result using the Brauer morphism (see §1.3.3 above) for $p \in \{2, 3\}$.

## 1.4. The hyperoctahedral group $C_2 \wr S_n$

In this section we specialise the background given in §1.2 and §1.3 to the group $C_2 \wr S_n$. We start by giving the presentation of $C_2 \wr S_n$ that we use in §5 and §6, which we have briefly encountered in §1. Write $S_{2n}$ for the symmetric group $\mathrm{Sym}(\{1, 2, \ldots, n, \overline{1}, \ldots, \overline{n}\})$, and define $C_2 \wr S_n$ to be the subgroup of $S_{2n}$ generated by the set

$$\{(1\ \overline{1}), (1\ 2)(\overline{1}\ \overline{2}), (1\ 2 \ldots n)(\overline{1}\ \overline{2} \ldots \overline{n})\}.$$

In this case the notation in §1.2 becomes

$$\begin{aligned} B_n &= \langle (1\ \overline{1}), (2\ \overline{2}), \ldots, (n\ \overline{n}) \rangle \\ T_n &= \langle (1\ 2)(\overline{1}\ \overline{2}), (1\ 2 \ldots n)(\overline{1}\ \overline{2} \ldots \overline{n}) \rangle. \end{aligned}$$

### 1.4.1. Subgroups of $C_2 \wr S_n$.

Given $\sigma \in \mathrm{Sym}(\{1, 2, \ldots, n\})$, define the permutation

$$\overline{\sigma} \in \mathrm{Sym}(\{\overline{1}, \ldots, \overline{n}\})$$

to be such that $\overline{\sigma}(\overline{x}) = \overline{\sigma(x)}$. Also write $\xi(H)$ to be the subgroup of $T_n$ consisting precisely of the permutations $\sigma\overline{\sigma}$ such that $\sigma \in H$, where $H \leq \mathrm{Sym}(\{1, 2, \ldots, n\})$.

Given $h \in C_2 \wr S_n$, we write $\widehat{h}$ for the image of $h$ under the natural surjection $C_2 \wr S_n \twoheadrightarrow S_n$. Then for $Q \leq C_2 \wr S_n$, define

$$\widehat{Q} = \{\widehat{h} : h \in Q\}.$$

Also given $X \subset \{1, 2, \ldots, n\}$, we write $C_2 \wr S_X$ for the subgroup of $C_2 \wr S_n$ generated by the set

$$\{(x\ \overline{x}) : x \in X\} \cup \{(x\ y)(\overline{x}\ \overline{y}) : x, y \in X, x \neq y\}.$$

We now consider $p$-subgroups of $C_2 \wr S_n$, where $p$ is an odd-prime. In this case the cardinality of a Sylow $p$-subgroup of $C_2 \wr S_n$ equals the cardinality of a Sylow $p$-subgroup of $S_n$. It follows that $C_2 \wr S_n$ has a Sylow $p$-subgroup contained in $T_n$, and so any $p$-subgroup of $C_2 \wr S_n$ has a conjugate in $T_n$.

**1.4.2. Conjugacy in the hyperoctahedral group.** We have described the conjugacy classes of $G \wr S_n$ for a finite group $G$ in Theorem 1.2.1. In Lemma 1.4.2 we see that the conjugacy classes of $C_2 \wr S_n$ afford a simpler description than in the general case. Nevertheless, we use Theorem 1.2.1 to prove the result in this special case.

Given $i \in \{1, 2, \ldots, n\}$, we define $\overline{\overline{i}} = i$. Given $g \in C_2 \wr S_n$, we say that $g$ is a *positive $r$-cycle* if

$$g = (a_1, a_2, \ldots, a_r)(\overline{a_1}, \overline{a_2}, \ldots, \overline{a_r}),$$

and that $g$ is a *negative $r$-cycle* if

$$g = (a_1, a_2, \ldots, a_r, \overline{a_1}, \overline{a_2}, \ldots, \overline{a_r}),$$

where $a_1, \ldots, a_r \in \{1, \overline{1}, \ldots, n, \overline{n}\}$.

Example 1.4.1. Let $n = 1$. The identity permutation $(1)(\overline{1})$ is a positive 1-cycle, and the permutation $(1\ \overline{1})$ is a negative 1-cycle.

We now have the following lemma, which describes the conjugacy classes of $C_2 \wr S_n$.

Lemma 1.4.2. *Every element of $C_2 \wr S_n$ can be expressed uniquely, up to the order of the factors, as a product of disjoint positive and negative cycles. Moreover, two elements in $C_2 \wr S_n$ are conjugate if and only if they have the same cycle type.*

Proof. Fix $g \in C_2 \wr S_n$, and write $g = bt$, for some unique $b \in B_n$ and $t \in T_n$. Observe that here $b$ and $t$ are unique by the definition of $C_2 \wr S_n$ as the semidirect product $B_n \rtimes T_n$. Let

$$\nu := (a_1 \ldots a_k)(\overline{a_1} \ldots \overline{a_k})$$

be a cycle in $t$. Define $J(\nu)$ to be the set of $j \in \{a_1, \ldots, a_k\}$ such that $(a_j\ \overline{a_j})$ is a factor of $b$, and let

$$c = \prod_{j \in J(\nu)} (a_j\ \overline{a_j}).$$

For the first statement, since $c\nu$ is an element of the symmetric group on the set

$$\{a_1, a_2, \ldots, a_k, \overline{a_1}, \overline{a_2}, \ldots, \overline{a_k}\},$$

it has a unique factorisation into disjoint cycles. Moreover, it follows that this factorisation is into either positive or a negative cycles in $C_2 \wr S_n$ since $\sigma(\overline{i}) = \overline{(\sigma(i))}$ for all $1 \leq i \leq n$.

For the second statement of the lemma, define $m(k)$ to be the cardinality of $J(\nu)$. Also define

$$c' = \prod_{j=1}^{m(k)} (a_j\ \overline{a_j}).$$

Since $c$ and $c'$ are conjugate in $C_2 \wr S_n$, we determine the cycle type of $c'\nu$. We now distinguish two cases, determined by the parity of $m(k)$.

*Case* (1). Suppose that $m(k)$ is odd. Then

$$\begin{aligned} c'\nu &= (a_1\ \overline{a_1})(a_2\ \overline{a_2})\dots(a_{m(k)}\ \overline{a_{m(k)}})(a_1\ a_2\dots a_k)(\overline{a_1}\ \overline{a_2}\dots\overline{a_k}) \\ &= (a_1\ \overline{a_2}\ a_3\dots a_{m(k)}\ a_{m(k)+1}\dots a_k\ \overline{a_1}\ a_2\ \overline{a_3}\dots\overline{a_{m(k)}}\ \overline{a_{m(k)+1}}\dots\overline{a_k}), \end{aligned}$$

which is a negative $k$-cycle.

*Case* (2). Suppose that $m(k)$ is even. Then

$$\begin{aligned} c'\nu &= (a_1\ \overline{a_1})(a_2\ \overline{a_2})\dots(a_{m(k)}\ \overline{a_{m(k)}})(a_1\ a_2\dots a_k)(\overline{a_1}\ \overline{a_2}\dots\overline{a_k}) \\ &= (a_1\ \overline{a_2}\ a_3\dots\overline{a_{m(k)}}\ \overline{a_{m(k)+1}}\dots\overline{a_k})(\overline{a_1}\ a_2\ \overline{a_3}\dots a_{m(k)}\ a_{m(k)+1}\dots a_k), \end{aligned}$$

which is a positive $k$-cycle.

It follows that the cycle product matrix of $g$ determines its cycle type. Furthermore, since the expression of each positive (resp. negative) cycle as an element $c\nu$, for some $c \in B_n$ and $\nu \in T_n$, is unique, we have that the cycle type of $c\nu$ determines its cycle product. $\square$

Given $g \in C_2 \wr S_n$, the number of positive (resp. negative) $r$-cycles of $g \in C_2 \wr S_n$ is denoted by $p_r$ (resp. $n_r$), and we say that $g$ has *cycle type* $((p_r),(n_r))_{1\le r\le n}$. We then have the following lemma.

LEMMA 1.4.3. *Let $g \in C_2 \wr S_n$ have cycle type $((p_r),(n_r))_{1\le r\le n}$. Then the $C_2 \wr S_n$-conjugacy class containing $g$ has order equal to*

$$\frac{2^n n!}{\prod_{r=1}^{n}(2r)^{p_r+n_r}(p_r!)(n_r!)}.$$

PROOF. Let $g \in C_2 \wr S_n$ have cycle type $((p_r),(n_r))_{1\le r\le n}$. We count the number of possible ways to arrange the letters

$$1, 2, \dots, n, \overline{1}, \overline{2}, \dots, \overline{n}$$

in $g$ and obtain a distinct permutation in $C_2 \wr S_n$. Since $g \in C_2 \wr S_n$, we have $g(\overline{i}) = \overline{g(i)}$ for all $1 \le i \le n$, and so once we have chosen the position of $i$ in the unique expression of $g$ as a product of disjoint cycles, then the position of $\overline{i}$ in $g$ is determined. The analogous statement holds if we first choose the position of $\overline{i}$ in $g$. For each $1 \le i \le n$, we therefore have two choices, namely $i$ or $\overline{i}$, for the element we can place in $g$. There are then $n!$ ways of arranging these $n$ chosen elements in $g$.

Cyclic shifts of letters within any cycle of $g$ leave $g$ invariant. Every negative $r$-cycle has length $2r$, and so we must divide $2^n n!$ by $2r$ for each negative $r$-cycle to take into account these cyclic shifts. Similarly, each cycle within a positive $r$-cycle has length $r$, and so we also divide by $r$ to account for cyclic shifts within each positive $r$-cycle. Moreover, we can transpose the two cycles within a positive $r$-cycle and still leave $g$ invariant, and so we must further divide by 2 for each positive $r$-cycle.

Finally, for each $1 \leq r \leq n$, reordering the $p_r$ positive $r$-cycles or the $n_r$ negative $r$-cycles leaves $g$ invariant, and so, for each $r$, we must divide by the $(p_r)!$ ways of ordering the positive $r$-cycles and the $(n_r)!$ ways of ordering the negative $r$-cycles.

We therefore have that there are

$$\frac{2^n n!}{\prod_{r=1}^{n} (2r)^{p_r+n_r}(p_r!)(n_r!)}$$

possible choices for $g$, and so the lemma is proved. □

**1.4.3. Hyperoctahedral Specht modules.** Throughout this section let $F$ be a field of characteristic $p \neq 2$. It follows that there are exactly two isomorphism classes of irreducible $FC_2$-modules. We write $N$ for the non-trivial irreducible $FC_2$-module.

Given $x \in \{1, 2, \ldots, n\}$, we define $[x, \overline{x}]$ to be the image of $(x, \overline{x})$ in the quotient of the $FC_2 \wr S_n$-permutation module $F[\{1, \ldots, n, \overline{1}, \ldots, \overline{n}\}]$ by the submodule generated by the set

$$\{(x, \overline{x}) + (\overline{x}, x) : 1 \leq x \leq n\}.$$

Therefore the $F$-span of $[x, \overline{x}]$ is isomorphic to $N$ as an $F[\mathrm{Sym}(\{x, \overline{x}\})]$-module.

Given $(\lambda, \mu) \in \mathcal{P}^2(n)$, let $t$ be the disjoint union of a $\lambda$-tableau and a $\mu$-tableau, such that

(1) the $\lambda$-tableau has entries $\{x, \overline{x}\}$, and the $\mu$-tableau has entries $[y, \overline{y}]$
(2) the set $\{x, \overline{x}\}$ is an entry of the $\lambda$-tableau if and only if $[x, \overline{x}]$ is not an entry of the $\mu$-tableau, for all $1 \leq x \leq n$.

In this case we say that $t$ is a $(\lambda, \mu)$-tableau. We write $t^+$ for the $\lambda$-tableau, and $t^-$ for the $\mu$-tableau.

Example 1.4.4. The following is a $((3), (3, 1))$-tableau.

$$\begin{array}{|c|c|c|c|c|c|} \cline{4-6} \multicolumn{3}{c|}{} & [2,\overline{2}] & [3,\overline{3}] & [6,\overline{6}] \\ \cline{4-6} \multicolumn{3}{c|}{} & [7,\overline{7}] & \multicolumn{2}{c}{} \\ \cline{1-4} \{1,\overline{1}\} & \{4,\overline{4}\} & \{5,\overline{5}\} & \multicolumn{3}{c}{} \\ \cline{1-3} \end{array}$$

Given a $(\lambda, \mu)$-tableau $t$, let $R(t)$ (resp. $C(t)$) be the subgroup of $T_n$ consisting of all permutations that setwise fix the entries in each row (resp. column) of $t$. We define an equivalence relation $\backsim$ on the set of $(\lambda, \mu)$-tableaux by $t \backsim u$ if and only if there exists $\pi \in R(t)$ such that $u = \pi t$. The $(\lambda, \mu)$-*tabloid* $\{t\}$ is the equivalence class of $t$. We define the $(\lambda, \mu)$-*polytabloid* $e(t)$ by

$$e(t) = \sum_{\sigma \in C(t)} \mathrm{sgn}(\sigma) \sigma \{t\}.$$

Define the hyperoctahedral Specht module $S_F^{(\lambda,\mu)}$ to be the $FC_2 \wr S_n$-module spanned by the set of all $(\lambda,\mu)$-polytabloids. When the field $F$ is clear, we omit the subscript $F$ in $S_F^{(\lambda,\mu)}$.

In order to describe a basis of $S^{(\lambda,\mu)}$, we order the sets $\{x,\overline{x}\}$ by setting $\{x,\overline{x}\} \leq \{y,\overline{y}\}$ if and only if $x \leq y$. We also define an ordering on the set of $[x,\overline{x}]$ in the same way. We say that $t$ is *standard* if both $t^+$ and $t^-$ are standard tableaux with respect to the orders just defined. It follows from the Standard Basis theorem that over any field $S^{(\lambda,\mu)}$ has a basis given by the set of polytabloids $e(t)$ such that $t$ is a standard $(\lambda,\mu)$-tableau.

It follows from [**35**, Corollary 4.4.11] that the ordinary characters of $C_2 \wr S_n$ are integer valued. This allows us to define $\chi^{(\lambda,\mu)}$ to be the ordinary character of the hyperoctahedral Specht module $S_F^{(\lambda,\mu)}$, where $F$ is any field of characteristic zero. By Theorem 1.2.6, we have the following theorem.

Theorem 1.4.5. *Let $F$ be a field of characteristic zero. The set*

$$\{S^{(\lambda,\mu)} : (\lambda,\mu) \in \mathcal{P}^2(n)\}$$

*is a complete set of pairwise non-isomorphic irreducible $FC_2 \wr S_n$-modules. Moreover,*

$$\operatorname{Irr}(C_2 \wr S_n) = \{\chi^{(\lambda,\mu)} : (\lambda,\mu) \in \mathcal{P}^2(n)\}.$$

**1.4.4. Modular representation theory of $C_2 \wr S_n$.** Assume now that $F$ is a field of characteristic $p > 0$ such that $p \neq 2$. The main result in this section is a complete description of the blocks of $FC_2 \wr S_n$, which we give in Proposition 1.4.8. In order to prove Proposition 1.4.8, we prove the stronger Theorem 1.4.7 below. We prove Theorem 1.4.7 as it is also used in this section to describe the irreducible $FC_2 \wr S_n$-modules, and in §5.4 to determine the blocks of $N_{C_2 \wr S_n}(R_r)$, where $R_r$ is as defined in §5.3.

We now give the required preliminaries for Theorem 1.4.7. Assume that $G = C_2^a \rtimes H$, where $a \in \mathbf{N}$ and $H$ is a finite group. Recall that $\operatorname{Lin}(C_2^a)$ denotes the set of linear characters of $C_2^a$. There is an action of $G$ on $\operatorname{Lin}(C_2^a)$ given by conjugation, and we have the following easy lemma.

Lemma 1.4.6. *The $G$-conjugacy classes of $\operatorname{Lin}(C_2^a)$ are labelled by pairs $(a_1,a_2) \in \mathbf{N}_0^2$ such that $a_1 + a_2 = a$.*

Given $0 \leq i \leq a$, write $\operatorname{Lin}_i(C_2^a)$ for the conjugacy class of $\operatorname{Lin}(C_2^a)$ labelled by $(i, a-i)$. Fix $\chi_i \in \operatorname{Lin}_i(C_2^a)$ and define $G_i = C_2^a \rtimes H_i$, where $H_i$ is the stabiliser of $\chi_i$ in $H$. Given an $FG$-module $V$ and $\chi \in \operatorname{Lin}(C_2^a)$, let

$$V^\chi = \{v \in V : gv = \chi(g)v \text{ for all } g \in C_2^a\}.$$

For $g \in G$, we have that $gV^\chi = V^{g\chi}$, and so $V^{\chi_i}$ is an $FG_i$-module. Furthermore, $V(i) := \bigoplus_{\chi \in \operatorname{Lin}_i(C_2^a)} V^\chi$ is an $FG$-module. Then

$$V = \bigoplus_{i=0}^{n} V(i), \tag{1.8}$$

as a direct sum of $FG$-modules. We say that $V$ *belongs to* $i$ if $V = V(i)$ for some $i$. Clearly every indecomposable $FG$-module belongs to $i$ for some $i$.

Let $\vartheta \in \mathrm{Hom}_{FG}(U, V)$. By considering the action of $C_2^a$, we see that $\vartheta(U^\chi) \subseteq V^\chi$. Therefore $\mathrm{Hom}_{FG}(U, V) = 0$ if $U$ belongs to $i$ and $V$ belongs to $j$ for $i \neq j$. It follows that the $FG$-modules belonging to $i$ generate a subcategory of the module category $\mathbf{mod}(G)$. We write $\mathbf{mod}_i(G)$ for this subcategory.

THEOREM 1.4.7. *The rings $FG$ and $\bigoplus_{i=0}^a FH_i$ are Morita equivalent.*

PROOF. Let $M$ be an $FH_i$-module, and write $K_i$ for the one-dimensional $FG_i$-module on which $C_2^a$ acts according to $\chi_i$ and $H_i$ acts trivially. Define the functor $\mathcal{F}_i : \mathbf{mod}(H_i) \to \mathbf{mod}_i(G)$ by

$$M \mapsto (K_i \otimes \mathrm{Inf}_{H_i}^{G_i} M)\big\uparrow_{G_i}^{G}.$$

It is sufficient to prove that $\mathcal{F}_i$ is an equivalence of categories, which we do by showing that it is essentially surjective, full, and faithful.

To prove that $\mathcal{F}_i$ is essentially surjective, it is sufficient to consider the case when $U$ is an indecomposable $FG$-module. Therefore $U$ belongs to $i$, and so by definition

$$U = \bigoplus_{\chi \in \mathrm{Lin}_i(C_2^a)} U^\chi \cong U^{\chi_i}\big\uparrow_{G_i}^{G},$$

where the isomorphism follows from Lemma 1.3.2. By definition, $U^{\chi_i}$ is such that $C_2^a$ acts according to $\chi_i$. Therefore $U^{\chi_i}$ is isomorphic to the tensor product of $K_i$ and a module on which $C_2^a$ acts trivially. This is equivalent to writing $U^{\chi_i} \cong K_i \otimes \mathrm{Inf}_{H_i}^{G_i} U'$, where $U'$ is an $FH_i$-module. This proves that $\mathcal{F}_i$ is essentially surjective.

Suppose that $0 \neq \vartheta \in \mathrm{Hom}_{FG}(U, V)$, where $V$ also belongs to $i$. Write $\varphi$ for $\vartheta$ restricted to $U^{\chi_i}$, which we view as an $FG_i$-module homomorphism. We have that $U$ is generated by $U^{\chi_i}$, and so $\varphi(U^{\chi_i}) \neq 0$. Moreover, let $u \in U$ be such that $gu = u'$ for some $g \in G/G_i$ and $u' \in U^{\chi_i}$. By the remark preceding this proof, we have $\varphi(U^{\chi_i}) \subseteq V^{\chi_i}$. Furthermore, by the discussion in the previous paragraph, we have that $U^{\chi_i} \cong U'$ as an $FH_i$-module. Writing $\varphi'$ for $\varphi$ viewed as an $FH_i$-module homomorphism, we have

$$\vartheta(u) = \vartheta(gu') = g\vartheta(u') = g\varphi(u') = g\varphi'(u').$$

It follows from part (4) of [**1**, §8, Lemma 6] that $\vartheta = \mathcal{F}_i(\varphi')$, and so $\mathcal{F}_i$ is full. Moreover, $\varphi'$ is determined by the restriction of $\vartheta$ to $U^{\chi_i}$, and so $\mathcal{F}_i$ is faithful. □

PROPOSITION 1.4.8. *The rings $FC_2 \wr S_n$ and $\bigoplus_{i=0}^n FS_{(i,n-i)}$ are Morita equivalent. Moreover, the blocks of $FC_2 \wr S_n$ are labelled by pairs*

$$((\gamma, v), (\delta, w)),$$

*where $\gamma$ and $\delta$ are $p$-core partitions such that $|\gamma|+vp+|\delta|+wp = n$. The hyperoctahedral Specht module $S^{(\lambda,\mu)}$ lies in the block labelled by $((\gamma, v),(\delta, w))$ if and only if $\lambda$ is a partition of $|\gamma| + vp$ with $p$-core $\gamma$, and $\mu$ is a partition of $|\delta| + wp$ with $p$-core $\delta$.*

PROOF. Given $i \in \{0, 1, \dots, n\}$, let $\chi_i \in \mathrm{Lin}(C_2^n)$ be such that

$$\chi_i((1\ \bar{1})) = \cdots = \chi_i((i\ \bar{i})) = 1$$
$$\chi_i((i+1\ \overline{i+1})) = \cdots = \chi_i((n\ \bar{n})) = -1.$$

In this case $H_i = S_{(i,n-i)}$. The first statement of the result is now immediate using Theorem 1.4.7. The remaining statements then follow from the definition of $S^{(\lambda,\mu)}$ and Nakayama's conjecture for the symmetric group. $\square$

We write $B((\gamma, v),(\delta, w))$ for the block of $FC_2 \wr S_n$ labelled by the pair $((\gamma, v),(\delta, w))$.

Given $(\nu, \widetilde{\nu}) \in \mathcal{P}^2(n)$ such that $\nu$ and $\widetilde{\nu}$ are $p$-regular, we define

$$D^{(\nu,\widetilde{\nu})} = (\mathrm{Inf}_{S_{|\nu|}}^{C_2 \wr S_{|\nu|}} D^{\nu} \boxtimes N^{\otimes|\widetilde{\nu}|} \otimes \mathrm{Inf}_{S_{|\widetilde{\nu}|}}^{C_2 \wr S_{|\widetilde{\nu}|}} D^{\widetilde{\nu}})\big\uparrow_{C_2 \wr S_{(|\nu|,|\widetilde{\nu}|)}}^{C_2 \wr S_n},$$

where $D^{\nu}$ is defined in the statement of Theorem 1.3.21.

The following proposition follows immediately from Theorem 1.3.21 and Proposition 1.4.8.

PROPOSITION 1.4.9. *Let $n \in \mathbf{N}$. The set*

$$\{D^{(\nu,\widetilde{\nu})} : (\nu, \widetilde{\nu}) \in \mathcal{P}^2(n) \text{ and } \nu, \widetilde{\nu} \text{ are } p\text{-regular}\},$$

*is a complete set of pairwise non-isomorphic irreducible $FC_2 \wr S_n$-modules.*

It follows from Corollary 4.4.9 of Theorem 4.4.8 in [**35**] that every field is a splitting field for $C_2 \wr S_n$. We can therefore specialise the definition in §1.3.4 and write $d_{\lambda\nu,\mu\widetilde{\nu}}$ for the decomposition number of $C_2 \wr S_n$ equal to the number of composition factors of $S^{(\lambda,\mu)}$ that are isomorphic to $D^{(\nu,\widetilde{\nu})}$.

CHAPTER 2

# A combinatorial proof of the Murnaghan–Nakayama rule

Throughout this chapter we fix $n \in \mathbf{N}$. Recall from Theorem 1.1.2 that the ordinary characters of $S_n$ are labelled by partitions of $n$. Moreover, we have seen that the partitions of $n$ give an explicit construction of the irreducible $\mathbf{Q}S_n$-modules using the combinatorics of Young diagrams via Specht modules.

The well-known Murnaghan–Nakayama rule (see Theorem 2.1.1) further utilises the combinatorics of Young diagrams by providing a formula for calculating the character values of $\chi^\lambda$ using *only* the Young diagram of $\lambda$. Moreover, this formula is recursive, and so it provides a computationally efficient algorithm for calculating single character values. Indeed, it is noted in the documentation of the computer algebra system MAGMA that the rule is used in this way, except when computing the value of a character on the identity element (see [**4**, §92.3.1]).

In this chapter we give a new combinatorial proof of the Murnaghan–Nakayama rule. As Stanley notes in [**61**, page 401], the rule was first proved by Littlewood and Richardson in [**44**, §11]. Their proof derives it, essentially as stated in Theorem 2.1.1 below, as a corollary of the Frobenius formula for the characters of symmetric groups. For a statement of the Frobenius formula see [**61**, (7.77)] or [**21**, (4.10)]. Murnaghan [**54**, page 462, (13)] and Nakayama [**55**, page 183] gave independent derivations of the rule, still using the Frobenius formula. James gave a different proof in [**33**, Ch. 11] using the relatively deep Littlewood–Richardson rule. More recently, elegant involutive proofs have been given by Mendes and Remmel [**52**, Theorem 6.3] using Pieri's rule and Young's rule and by Loehr [**45**, §11] using his labelled abacus representation of antisymmetric functions. Our proof identifies the unique standard polytabloid (see §1.1) that makes a non-zero contribution to the trace of the matrix representing the action of an $n$-cycle on the standard basis of a skew Specht module.

## 2.1. The Murnaghan–Nakayama rule

We remind the reader that a *border strip* is a skew partition whose skew diagram is connected and which contains no four boxes forming the Young diagram $[(2, 2)]$.

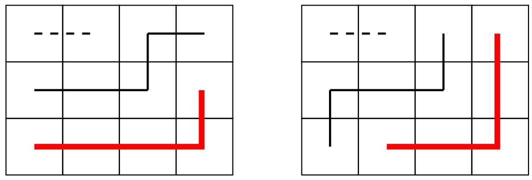


FIGURE 2.1. The border strips of size 5 (black and thick red) and 2 (dashed) removed to compute the character value $\chi^{(4,4,4)}\big((1\ 2)(3\ 4\ 5\ 6\ 7)(8\ 9\ 10\ 11\ 12)\big)$.

THEOREM 2.1.1 (Murnaghan–Nakayama rule). *Let $m, n \in \mathbf{N}$, and let $\lambda$ be a partition of $m+n$. Let $\rho \in S_{m+n}$ be an $n$-cycle and let $\pi$ be a permutation of the remaining $m$ numbers. Then*

$$\chi^\lambda(\pi\rho) = \sum (-1)^{\mathrm{ht}(\lambda/\mu)} \chi^\mu(\pi),$$

*where the sum is over all $\mu \subset \lambda$ such that $|\mu| = m$ and $\lambda/\mu$ is a border strip.*

We provide an example of the Murnaghan–Nakayama rule, showing how it can be applied recursively to calculate single character values.

EXAMPLE 2.1.2. Let $\sigma = (1\ 2)(3\ 4\ 5\ 6\ 7)(8\ 9\ 10\ 11\ 12) \in S_{12}$. We evaluate $\chi^{(4,4,4)}(\sigma)$. Taking $\rho = (8\ 9\ 10\ 11\ 12)$, we begin by removing border strips of size 5 from $(4,4,4)$. As shown in Figure 2.1 there are two such strips (highlighted in red), namely $(4,4,4)/(4,3)$ and $(4,4,4)/(3,3,1)$, of heights 1 and 2, respectively. Therefore by the Murnaghan–Nakayama rule

$$\chi^{(4,4,4)}(\sigma) = (-\chi^{(4,3)} + \chi^{(3,3,1)})\big((1\ 2)(3\ 4\ 5\ 6\ 7)\big).$$

Two further applications of the Murnaghan–Nakayama rule to each summand now show that $\chi^{(4,4,4)}(\sigma) = (\chi^\varnothing + \chi^\varnothing)(\mathrm{id}) = 1 + 1 = 2$, where id denotes the identity permutation in $S_0$.

**Outline.** Recall that Corollary 1.1.10 of Theorem 1.1.4 states that

$$\chi^\lambda(\pi\rho) = \sum_\mu \chi^\mu(\pi)\chi^{\lambda/\mu}(\rho),$$

where $\chi^{\lambda/\mu}$ is as defined in §1.1.1. By this corollary, it suffices to show that if $\rho$ is an $n$-cycle then

$$\chi^{\lambda/\mu}(\rho) = \begin{cases} (-1)^{\mathrm{ht}(\lambda/\mu)} & \text{if } \lambda/\mu \text{ is a border strip of size } n \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

We do this by explicitly computing the trace of the matrix representing the $n$-cycle $\rho$ in the standard basis (see Theorem 1.1.3) of $S^{\lambda/\mu}$.

In the critical case where $\lambda/\mu$ is a border strip, we show that there is a unique basis element giving a non-zero contribution to the trace. This gives a new and essentially bijective proof of the Murnaghan–Nakayama rule. To this end we prove Lemma 2.2.2 in §2.2, which gives a necessary condition for a standard polytabloid to appear with a non-zero coefficient when a given $\lambda/\mu$-polytabloid is written as a linear combination of standard polytabloids. This generalises Proposition 4.1 in [**65**] to skew tableaux. In §2.3 we give the proof of (2.1) when $\lambda/\mu$ is a border strip. We then deal with the remaining case in §2.4 by a short argument using Pieri's rule and Young's rule (see Theorem 1.1.12).

## 2.2. A dominance lemma for skew tableaux

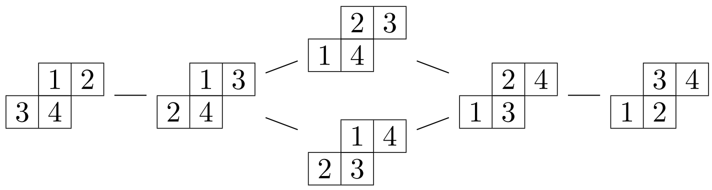
The dominance order for tabloids is defined in [**33**, Definition 3.11], or, in a way more convenient for us, in [**59**, Definition 2.5.4]. We extend it to compare row standard skew tableaux of shape a fixed skew partition.

Definition. Let $t$ be a row standard $\lambda/\mu$-tableau where $|\lambda/\mu| = n$. Given $1 \le y \le n$, we define $\mathrm{sh}_{\le y}(t)$ to be the composition $\beta$ such that

$$\beta_i = \bigl|\{x : x \in \text{row } i \text{ of } t,\ x \le y\}\bigr|$$

for $1 \le i \le \ell(\lambda)$. If $s$ is another row standard $\lambda/\mu$-tableau, then we say that $s$ *dominates* $t$, and write $s \trianglerighteq t$, if $\mathrm{sh}_{\le y}(s) \trianglerighteq \mathrm{sh}_{\le y}(t)$ for all $y \in \{1, \ldots, n\}$, where on the right-hand side $\trianglerighteq$ denotes the dominance order of compositions defined in §1.1.3.

Example 2.2.1. The $\trianglerighteq$ order on the row standard $(3,2)/(1)$-tableaux is shown below.



Recall that given a $\lambda/\mu$-tableau $t$, its row straightening $\bar{t}$ is the unique row standard $\lambda/\mu$-tableau whose rows agree setwise with $t$. We extend the dominance order to $\lambda/\mu$-tabloids by setting $\{s\} \trianglerighteq \{t\}$ if and only if $\bar{s} \trianglerighteq \bar{t}$.

Lemma 2.2.2 (Dominance Lemma). *If $t$ is a column standard $\lambda/\mu$-tableau then $\bar{t}$ is standard and*

$$e(t) = e(\bar{t}) + w,$$

*where $w$ is a $\mathbf{Z}$-linear combination of standard polytabloids $e(s)$ such that $\bar{t} \triangleright s$.*

*Preliminaries for the proof of the Dominance Lemma.* We first show that $\bar{t}$ is standard. Suppose, for a contradiction, that there exist boxes $(i,j)$ and $(i+1,j) \in [\lambda/\mu]$ such that $\bar{t}(i,j) > \bar{t}(i+1,j)$. Define

$$R = \{\bar{t}(i,k) : j \le k \le \lambda_i\}$$
$$S = \{\bar{t}(i+1,k) : \mu_{i+1} < k \le j\}.$$

Since

$$\bar{t}(i+1,\mu_{i+1}+1) < \ldots < \bar{t}(i+1,j) < \bar{t}(i,j) < \ldots < \bar{t}(i,\lambda_i)$$

we have $x > y$ for each $x \in R$ and $y \in S$. But since $|R|+|S| = \lambda_i - \mu_{i+1}+1$, the pigeonhole principle implies that there exist $x \in R$ and $y \in S$ lying in the same column of the column standard skew tableau $t$, a contradiction.

The following two lemmas generalise Lemmas 3.15 and 8.3 in [**33**] to skew tableaux.

LEMMA 2.2.3. *Let $t$ be a $\lambda/\mu$-tableau. Let $x$, $y \in \{1,\ldots,n\}$ be such that $x < y$. If $x$ is strictly higher than $y$ in $t$ then $\overline{(x\ y)t} \lhd \bar{t}$.*

PROOF. Let $x$ be in row $k$ of $t$ and let $y$ be in row $\ell$ of $t$. By hypothesis, $k < \ell$. Let $z \in \{1,\ldots,n\}$. If $x \leq z < y$ then

$$\begin{aligned}\mathrm{sh}_{\leq z}(\overline{(x\ y)t})_k &= \mathrm{sh}_{\leq z}(\bar{t})_k - 1\\ \mathrm{sh}_{\leq z}(\overline{(x\ y)t})_\ell &= \mathrm{sh}_{\leq z}(\bar{t})_\ell + 1.\end{aligned}$$

Whenever $i \notin \{k,\ell\}$ or $z < x$ or $y \leq z$ we have $\mathrm{sh}_{\leq z}(\overline{(x\ y)t})_i = \mathrm{sh}_{\leq z}(\bar{t})_i$. It easily follows from these equations and the definition of the dominance order for compositions that $\overline{(x\ y)t} \lhd \bar{t}$. □

LEMMA 2.2.4. *Let $t$ be a column standard $\lambda/\mu$-tableau. Then $e(t) = \{t\} + w$, where $w$ is a $\mathbf{Z}$-linear combination of $\lambda/\mu$-tabloids $\{s\}$ such that $\{s\} \lhd \{t\}$.*

PROOF. The proof of Lemma 8.3 in [**33**] still holds, replacing Lemma 3.15 in [**33**] with our Lemma 2.2.3. □

PROOF OF LEMMA 2.2.2. Let $e(t) = \sum_s \alpha_s e(s)$ where the sum is over all standard $\lambda/\mu$-tableaux and $\alpha_s \in \mathbf{Z}$ for each $s$. Let $u$ be a standard tableau maximal in the dominance order such that $\alpha_u \neq 0$. Applying Lemma 2.2.4 to $e(u)$ gives

$$e(u) = \{u\} + w^{\lhd\{u\}},$$

where $w^{\lhd\{u\}}$ is a $\mathbf{Z}$-linear combination of $\lambda/\mu$-tabloids each dominated by $\{u\}$. By Lemma 2.2.4 and the maximality of $u$, there is no other standard $\lambda/\mu$-tableau $s$ with $\alpha_s \neq 0$ such that $e(s)$ has $\{u\}$ as a summand. Therefore the coefficient of $\{u\}$ in $e(t)$ is $\alpha_u$. Applying Lemma 2.2.4, now to $e(t)$, gives

$$e(t) = \{t\} + w^{\lhd\{t\}},$$

where $w^{\lhd\{t\}}$ is a $\mathbf{Z}$-linear combination of $\lambda/\mu$-tabloids each dominated by $\{t\}$. In particular $\{t\} \unrhd \{u\}$, and so we have that $\bar{t} = u$ by the maximality of $u$. Hence

$$e(t) = \alpha_{\bar{t}} e(\bar{t}) + w,$$

where $w$ is a $\mathbf{Z}$-linear combination of standard polytabloids $e(v)$ for standard tableaux $v$ such that $v \lhd \bar{t}$. It follows that $\{t\}$ cannot be a summand of $w$

in the equation immediately above. Since the coefficient of $\{t\}$ in $e(t)$ is 1, we have $\alpha_{\overline{t}} = 1$. $\square$

We isolate the following corollary of Lemma 2.2.2.

COROLLARY 2.2.5. *Let $s$ be a standard $\lambda/\mu$-tableau, and let $u$ be a column standard $\lambda/\mu$-tableau. Suppose that there exists $x \in \{1, 2, \ldots, n\}$ such that the boxes containing $1, 2, \ldots, x-1$ are the same in $s$ and $u$, and $x$ is lower in $u$ than $s$. If*

$$e(u) = \sum \alpha_v e(v),$$

*where the sum is over all standard $\lambda$-tableaux $v$, then $\alpha_s = 0$.*

PROOF. By assumption, $\mathrm{sh}_{\leq z}(s) = \mathrm{sh}_{\leq z}(\overline{u})$ if $1 \leq z < x$. As $x$ is in a lower row in $u$ than in $s$, we have $\mathrm{sh}_{\leq x}(\overline{u}) \ntrianglerighteq \mathrm{sh}_{\leq x}(s)$. Now apply Lemma 2.2.2. $\square$

## 2.3. The Murnaghan–Nakayama rule for border strips

In this section we give a bijective proof that $\chi^{\lambda/\mu}(\rho) = (-1)^{\mathrm{ht}(\lambda/\mu)}$ when $\lambda/\mu$ is a border strip of size $n$ and $\rho$ is the $n$-cycle $(1\ 2\ \ldots\ n)$. This deals with one of the two cases in (2.1). Our proof shows that the matrix representing $\rho$ in the standard basis of $S^{\lambda/\mu}$ has a unique non-zero entry on its diagonal. The relevant standard tableau is defined as follows.

DEFINITION. Let $\lambda/\mu$ be a border strip of size $n$. Say that a box $(i, j) \in [\lambda/\mu]$ is *columnar* if $(i+1, j) \in [\lambda/\mu]$. We define the standard $\lambda/\mu$-tableau $t_{\lambda/\mu}$ as follows:

(i) assign the numbers $\{1, \ldots, z\}$ in ascending order to the $z$ columnar boxes of $\lambda/\mu$, starting with 1 in row 1 and finishing with $z$ in the row above the bottom row;

(ii) then assign the numbers $\{z+1, \ldots, n\}$ in ascending order to the $n - z$ non-columnar boxes, starting with $z+1$ in column 1 and finishing with $n$ in the rightmost column.

For example, $t_{(5,3,3)/(2,2)}$, $t_{(5,3,2)/(2,1)}$ and $t_{(5,1,1)/\varnothing}$ are respectively

$$\begin{array}{ccccc} & & 1 & 6 & 7 \\ & & 2 & & \\ 3 & 4 & 5 & & \end{array}, \qquad \begin{array}{ccccc} & & 1 & 6 & 7 \\ & 2 & 5 & & \\ 3 & 4 & & & \end{array} \quad \text{and} \quad \begin{array}{ccccc} 1 & 4 & 5 & 6 & 7 \\ 2 & & & & \\ 3 & & & & \end{array},$$

where 1 and 2 are the entries in columnar boxes in each case. We remark that there are no columnar boxes if and only if $\lambda/\mu$ is a horizontal strip, as defined in §1.1.4.

As useful pieces of notation, we define $x^-$ and $x^+$ for $x \in \{1, \ldots, n\}$ by $x^- = x - 1$ and

$$x^+ = \begin{cases} x+1 & \text{if } 1 \leq x < n \\ 1 & \text{if } x = n. \end{cases}$$

Thus $\rho x = x^+$ for all $x \in \{1, \ldots, n\}$ and $1^- = 0$. Given a $\lambda/\mu$-tableau $t$, we define $t^+$ by $t^+(i,j) = \bigl(t(i,j)\bigr)^+$. By (1.1), $e(\rho t) = e(t^+)$.

A standard $\lambda/\mu$-tableau $t$ such that $e(t)$ has a non-zero coefficient in the unique expression of $e(t^+)$ as a $\mathbf{Z}$-linear combination of standard polytabloids is said to be *trace-contributing*. Since $\chi^{\lambda/\mu}(\rho)$ is the trace of the matrix representing $\rho$ in the standard basis, it suffices to prove the following proposition.

PROPOSITION 2.3.1. *Let $\lambda/\mu$ be a border strip. Then the unique trace-contributing $\lambda/\mu$-tableau is $t_{\lambda/\mu}$. The coefficient of $e(t_{\lambda/\mu})$ in $e(t^+_{\lambda/\mu})$ is $(-1)^{\mathrm{ht}(\lambda/\mu)}$.*

The proof of Proposition 2.3.1 is by induction on the number of top corner boxes of $\lambda/\mu$, as defined below. The necessary preliminaries are collected below. We then prove the base case, when $\lambda/\mu = (n-\ell, 1^\ell)$ for some $\ell \in \mathbf{N}_0$; this gives a good flavour of the general argument. In the remainder of this section we give the inductive step.

We assume, without loss of generality, that $\mu_1 < \lambda_1$ and $\mu_{\ell(\lambda)} = 0$, so the non-empty rows of $\lambda/\mu$ are $1, \ldots, \ell(\lambda)$ and column 1 of $\lambda/\mu$ is non-empty. We can do this since the character indexed by a skew diagram is equal to the character indexed by the same skew diagram with its empty rows and columns removed.

**2.3.1. Preliminaries for the proof of Proposition 2.3.1.** For $Z \subseteq \{1, \ldots, n\}$ and $t$ a row standard $\lambda/\mu$-tableau we define $\mathrm{sh}_Z(t)$ to be the composition $\beta$ such that

$$\beta_i = \bigl|\{x : x \in \text{row } i \text{ of } t,\, x \in Z\}\bigr|$$

for $1 \le i \le \ell(\lambda)$. Set $\mathrm{sh}_{<y}(t) = \mathrm{sh}_{\{1,\ldots,y^-\}}(t)$. We also use $\mathrm{sh}_{\le y}(t)$, as already defined at the beginning of §2.2.

DEFINITION. Let $\lambda/\mu$ be a border strip. We say that column $j$ of $\lambda/\mu$ is *singleton* if it contains a unique box. We define a *top corner box* to be a box $(i,j) \in [\lambda/\mu]$ such that $(i, j-1), (i-1, j) \notin [\lambda/\mu]$ and a *bottom corner box* to be a box $(i,j) \in [\lambda/\mu]$ such that $(i+1, j), (i, j+1) \notin [\lambda/\mu]$.

LEMMA 2.3.2. *Let $\lambda/\mu$ be a border strip and let $t$ be a $\lambda/\mu$-tableau. If columns $j$ and $j+1$ of $\lambda/\mu$ are singleton, with their unique box in row $i$, then $e(t) = (x\ y)e(t)$ where $x = t(i,j)$ and $y = t(i, j+1)$.*

PROOF. This follows immediately from the Garnir relation (1.4), taking $X = \{x\}$ and $Y = \{y\}$. $\square$

In fact, all the Garnir relations that we use in this section can be reduced to single transpositions. Let $x$ and $y$ be entries in adjacent columns of a column standard tableau, with $x$ left of $y$ and $x > y$. We say that $(x\ y)$ is a *Garnir swap* if at least one column is not singleton, and otherwise that $(x\ y)$ is a *horizontal swap.*

LEMMA 2.3.3. *Let $t$ be a trace-contributing border strip tableau. Then $t$ can be obtained from $\widetilde{t^+}$ by iterated horizontal swaps, Garnir swaps and column straightenings. If in such a sequence* 1 *moves, then* 1 *moves either left or down.*

PROOF. The first claim is immediate from Theorem 1.1.3(i). The second follows from Corollary 2.2.5 taking $x = 1$. □

Given $X \subseteq \{1, 2, \ldots, n\}$, we define $X^+ = \{x^+ : x \in X\}$. We also write $\{b^+, \ldots, c^-\}$ for the set $\{i \in \mathbf{N} : b^+ \le i \le c^-\}$. The following combinatorial result on the map $x \mapsto x^+$ is used several times to restrict the possible entries of trace-contributing tableaux.

LEMMA 2.3.4. *Let $X$ be a set of natural numbers such that $1, n \notin X$. Also suppose that $b, c$ are not contained in $X$. We have $\{b^+\} \cup X^+ = X \cup \{c\}$ if and only if $b^+ = \min X$, $c = \max X^+$ and $X = \{b^+, \ldots, c^-\}$.*

PROOF. Since $\min X \notin X^+$ we have $\min X = b^+$. Similarly, since $\max X^+ \notin X$ we have $\max X^+ = c$. Suppose for a contradiction that $X$ is a proper subset of $\{b^+, \ldots, c^-\}$. Setting

$$d = \min(\{b^+, \ldots, c^-\} \backslash X)$$

we see that since $b^+ = \min X \in X$, we have $d > b^+$. The minimality of $d$ implies that $d^- \in X$ and so $d \in X^+$; since $d < c$ and $\{b^+\} \cup X^+ = X \cup \{c\}$, we have $d \in X$, a contradiction. The converse is obvious. □

Finally, as a notational convention, when we specify a set, we always list the elements in increasing order. In diagrams the symbol $\star$ marks an entry we have no need to specify more explicitly.

**2.3.2. Base case: one top corner box.** In this case $\mu = \varnothing$ and $\lambda = (n-\ell, 1^\ell)$ for some $\ell \in \mathbf{N}_0$. If $\ell = 0$ then there is a unique standard $(n)$-tableau and the result is clear. Suppose that $\ell > 0$ and let $t$ be a standard $(n-\ell, 1^\ell)$-tableau with entries $\{1, y_1, \ldots, y_{\ell-1}, c\}$ in column 1. (By our notational convention, $1 < y_1 < \ldots < y_{\ell-1} < c$.) If $c = n$ then $\widetilde{t^+}$ is standard with first column entries $\{1, 1^+, y_1^+, \ldots, y_{r-1}^+\}$. Hence, in order for $t$ to be trace-contributing, we must have that $c < n$. After a sequence of horizontal swaps applied to $\widetilde{t^+}$ we obtain the tableau shown overleaf.

$$\begin{array}{ccccc} \boxed{1^+} & \boxed{1} & \boxed{\star} & \cdots & \boxed{\star} \\ \boxed{y_1^+} & & & & \\ \vdots & & & & \\ \boxed{y_{\ell-1}^+} & & & & \\ \boxed{c^+} & & & & \end{array}$$

A Garnir swap of 1 with $1^+$ or any $y_i^+$ gives, after column straightening and a sequence of horizontal swaps, a standard tableau having $c^+$ in its bottom left position. We may therefore assume, by Lemma 2.3.3, that 1 is swapped with $c^+$. After column straightening, which introduces the sign $(-1)^\ell$, a sequence of horizontal swaps gives the standard tableau having $\{1, 1^+, y_1^+, \ldots, y_{\ell-1}^+\}$ in its first column. Thus if $t$ is trace-contributing then $\{1^+, y_1^+, \ldots, y_{\ell-1}^+\} = \{y_1, \ldots, y_{\ell-1}, c\}$. By Lemma 2.3.4, $\{y_1, \ldots, y_{\ell-1}, c\} = \{2, \ldots, \ell+1\}$. Therefore $t = t_{(n-\ell,1^\ell)}$ and the coefficient of $e(t_{(n-\ell,1^\ell)})$ in $e(t^+_{(n-\ell,1^\ell)})$ is $(-1)^\ell$, as required.

**2.3.3. Inductive step.** Let $\delta(i) \in \mathbf{N}_0^{\ell(\lambda)}$ denote the composition defined by $\delta(i)_i = 1$ and $\delta(i)_k = 0$ if $k \neq i$.

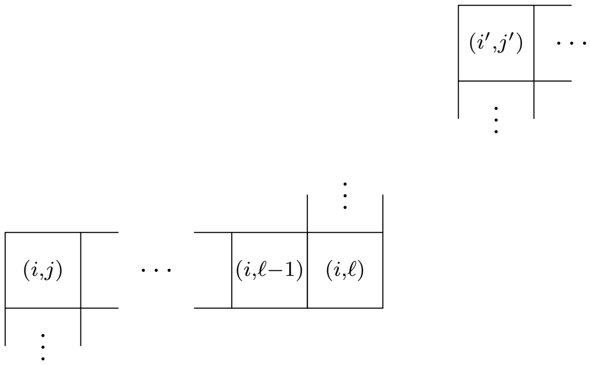
Proposition 2.3.5. *Let $\lambda/\mu$ be a border strip, and let $t$ be a standard $\lambda/\mu$-tableau. Let $c \in \mathbf{N}$ and suppose that* either $c = 1$ or $c > 1$ *and the entries* $1, \ldots, c^-$ *and* $n$ *lie in the same column of* $t$. *Let $(i, j)$ be the box of $t$ containing $c$, and let $(i', j')$ be the box of $\widetilde{t^+}$ containing $c$. If $t$ is a trace-contributing tableau, then $i = i'$.*

Before we continue, we mention that we give an example illustrating the various tableaux in the proof of Proposition 2.3.5 in Example 2.3.6 below.

Proof. By hypothesis, the highest $c^-$ entries in column $j'$ of $t$ and $\widetilde{t^+}$ are $1, \ldots, c^-$. Let $s = \widetilde{t^+}$. Setting $\beta = \mathrm{sh}_{<c}(t) = \mathrm{sh}_{<c}(\overline{s})$ we have $\mathrm{sh}_{\leq c}(t) = \beta + \delta(i)$ and $\mathrm{sh}_{\leq c}(\overline{s}) = \beta + \delta(i')$. By Lemma 2.2.2, the hypothesis that $t$ is trace-contributing implies that $\mathrm{sh}_{\leq c}(\overline{s}) \trianglerighteq \mathrm{sh}_{\leq c}(t)$. Therefore $i \geq i'$.

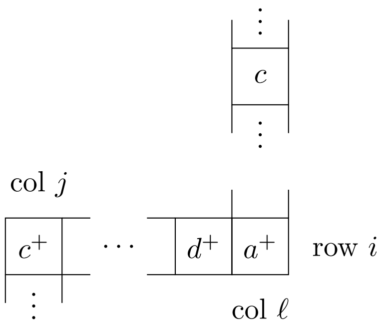
If $j = j'$ then *either* $c = 1$ and 1 is at the top of the column of $t$ which has $n$ at its bottom, *or* $c > 1$ and $c$ is immediately below $c^-$ in both $s$ and $t$. In either case $i = i'$.

We may therefore suppose, for a contradiction, that $i > i'$ and $j < j'$. By hypothesis the box $(i, j)$ of $t$ containing $c$ is the top corner box in row $i$. Let $(i, \ell)$ be the bottom corner box in row $i$; note that $\ell \leq j'$, as shown in the diagram overleaf.




By the hypothesis that $t$ is trace-contributing and Lemma 2.3.3 there is a sequence of horizontal swaps, Garnir swaps, and column straightenings from $\widetilde{t^+}$ to $t$. Suppose that in such a sequence an entry $b < c$ is moved. If $b$ is the first such entry moved in this sequence, and $u$ is the tableau obtained after column straightening, then, by Corollary 2.2.5 applied with $x = b$, the coefficient of $e(t)$ in $e(u)$ is zero. Therefore the entries $\{1, \ldots, c^-\}$ are fixed and $c$ is the smallest number moved. It follows that the only non-standard row in $\widetilde{t^+}$ is the row containing the bottom corner box in column $j'$, and so any such sequence starts with boxes in this row.

Take such a sequence and stop it immediately after the first swap in which $c$ enters row $i$. Let $v$ be the column standard tableau so obtained, and let $u$ be its immediate predecessor. When $c$ enters row $i$ of $v$, it is swapped with the entry, $d^+$ say, in box $(i, \ell - 1)$ of $u$. Indeed $c$ is in column $\ell - 1$ in $v$ since Garnir relations are defined on adjacent columns. Observe that the entries in boxes strictly to the left of column $\ell$ are the same in $\widetilde{t^+}$ and $u$, since no swap in the sequence from $\widetilde{t^+}$ to $u$ involves an entry in these columns. Let $a^+$ be the entry in box $(i, \ell)$ of $u$. Thus the column standard tableau $u$ is as shown below and $v = (\widetilde{c, d^+})u$.




Note that $d^+ > a^+$ since otherwise $u$ is standard with respect to all boxes weakly to the left of column $\ell$, and so $d^+$ cannot be moved in a Garnir swap.

To complete the proof we require the following critical quantity. *Let $r$ be maximal such that entries $c, \ldots, r$ are strictly to the left of column $\ell$ in the original tableau $t$.* If $r = d$ then, since $d > a$, $a$ is strictly to the left of column $\ell$ in $t$; this is impossible since $a^+$ appears in column $\ell$ in $u$. Therefore $r < d$. Since $d$ is in position $(i, \ell - 1)$ of $t$ and $r \geq c$, it follows that $c \neq d$. Moreover, the entries $c^+, \ldots, r^+$ are in the same boxes in $t^+$ and $v$.

*Claim.* We have $v \ntrianglerighteq t$. *Proof of claim.* Let $\mathrm{sh}_{\{c^+,\ldots,r^+\}}(u) = \delta$. By hypothesis and our stopping condition on swaps, if $q \leq r$ then the box of $q^+$ in $u$ is the box of $q$ in $t$. Hence $\mathrm{sh}_{\{c,\ldots,r\}}(t) = \delta$. Since $d > r$ and $d$ is in position $(i, \ell - 1)$ of $t$, we see that $r^+$ is not in row $i$ of $t$. By maximality of $r$, the row of $t$ containing $r^+$ is row $h$ for some $h < i$. Clearly the row of $c$ in $v$ is $i$. Therefore $\mathrm{sh}_{\{c,\ldots,r^+\}}(\overline{v}) = \delta + \delta(i)$ and $\mathrm{sh}_{\{c,\ldots,r^+\}}(t) = \delta + \delta(h)$. Since $1, \ldots, c^-$ are in the same positions in both $v$ and $t$, it follows that

$$\mathrm{sh}_{\leq r^+}(t) \rhd \mathrm{sh}_{\leq r^+}(\overline{v})$$

which implies the claim.

It now follows from Lemma 2.2.2, as before, that $e(t)$ does not appear in $e(v)$, a final contradiction. This completes the proof. □

EXAMPLE 2.3.6. Let $c = 1$. Let $t$, $t^+$, and $v$ be the $(5,4,1)/(3)$-tableaux shown below.

$$t = \begin{array}{|c|c|c|c|c|} \hline & & & 4 & 7 \\ \hline 1 & 3 & 5 & 6 & \\ \hline 2 & & & & \\ \hline \end{array} \qquad t^+ = \begin{array}{|c|c|c|c|c|} \hline & & & 5 & 1 \\ \hline 2 & 4 & 6 & 7 & \\ \hline 3 & & & & \\ \hline \end{array} \qquad v = \begin{array}{|c|c|c|c|c|} \hline & & & 5 & 7 \\ \hline 2 & 4 & 1 & 6 & \\ \hline 3 & & & & \\ \hline \end{array}$$

In the notation of the proof of Proposition 2.3.5, $i = 2$, $\ell = 4$, and $r = 3$.

In the sequence of operations used to straighten $e(\widetilde{t^+})$, the tableau $v$ is the unique successor of $\widetilde{t^+}$ such that 1 enters row 2. Then $\mathrm{sh}_{\leq 3}(t) = \mathrm{sh}_{\leq 3}(\overline{v})$. However $\mathrm{sh}_{\leq 4}(t) = (1,2,1) \rhd (0,3,1) = \mathrm{sh}_{\leq 4}(\overline{v})$, and so $\overline{v} \ntrianglerighteq t$. It follows that $e(t)$ does not appear in $e(v)$. Therefore $t$ is not trace-contributing, as expected from the proposition.

COROLLARY 2.3.7. *If $t$ is a trace-contributing tableau then either* 1 *and* $n$ *are in the same column of $t$, or* 1 *and* $n$ *are in the top row of $t$.*

PROOF. Let 1 and $n$ be in positions $(i, j)$ of $t$ and $(i', j')$ of $t$, respectively. If column $j'$ is singleton then $n$ is the top right entry of $t$ and, taking $c = 1$ in Proposition 2.3.5, we get $i = i'$; thus 1 and $n$ are in the top row of $t$. Otherwise, when we column straighten $t^+$ to obtain $\widetilde{t^+}$, the entry 1 in position $(i', j')$ moves up to position $(i'', j')$ where $i'' < i'$. Again taking $c = 1$ in Proposition 2.3.5, we get $i = i''$. Since $(i'', j')$ is the top corner box in its row, and so is $(i, j)$, we see that $j = j'$. Hence 1 and $n$ are in the same column of $t$. □

PROOF OF PROPOSITION 2.3.1. We now complete the inductive step of the proof.

Suppose that $\lambda/\mu$ has more than one top corner box, and that $t$ is a trace-contributing $\lambda/\mu$-tableau. Let 1 be in position $(i, j)$ of $t$ and in position $(i', j')$ of $\widetilde{t^+}$. By Proposition 2.3.5, we have $i = i'$.

*Case* (1). Suppose that 1 and $n$ lie in the same row of $t$. By Corollary 2.3.7, this is the top row. Let the entries in the top row of $t$ be $\{1, x_1, \ldots, x_{k-1}, n\}$ and the entries in the column of 1 be $\{1, y_1, \ldots, y_{\ell-1}, c\}$.

Straightening the top row of $t^+$ by a sequence of $k-1$ horizontal swaps moves $1^+$ and 1 into adjacent positions, giving the tableau $u$ shown below.

$$\begin{array}{cccccc}
 & \boxed{1^+} & \boxed{1} & \boxed{x_1^+} & \cdots & \boxed{x_{k-1}^+} \\
 & \boxed{y_1^+} & & & & \\
 & \vdots & & & & \\
 & \boxed{y_{\ell-1}^+} & & & & \\
\cdots & \boxed{c^+} & & & &
\end{array}$$

As in the base case, the only Garnir swap that can lead to $t$ is $(1, c^+)$, which introduces the sign $(-1)^\ell$. Let $v = (\widetilde{1, c^+})u$, as shown below.

$$\begin{array}{cccccc}
 & \boxed{1} & \boxed{c^+} & \boxed{x_1^+} & \cdots & \boxed{x_{k-1}^+} \\
 & \boxed{1^+} & & & & \\
 & \boxed{y_1^+} & & & & \\
 & \vdots & & & & \\
\cdots & \boxed{y_{\ell-1}^+} & & & &
\end{array}$$

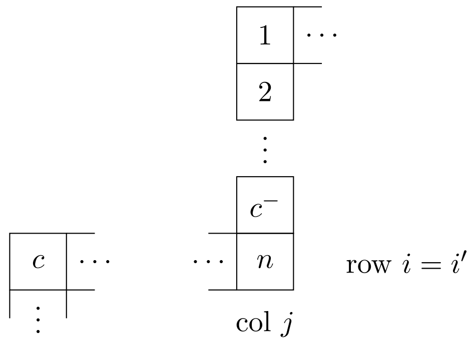
By Lemma 2.3.3 and Corollary 2.2.5, $v$ can be straightened by a sequence of horizontal swaps, Garnir swaps and column straightenings which either fix 1, and so leave invariant the content of its top row, or move 1 into a lower row, giving a tableau, $w$ say, such that, $e(t)$ does not appear in $e(w)$. Since $e(t)$ has a non-zero coefficient in $e(v)$, we have

$$\{c^+, x_1^+, \ldots, x_{k-1}^+\} = \{x_1, \ldots, x_{k-1}, n\}.$$

Lemma 2.3.4 implies that $c^+ = x_1 = n-k+1$, $x_{k-1}^+ = n$ and $\{x_1, \ldots, x_{k-1}\} = \{n-k+1, \ldots, n-1\}$. Thus $t$ and $v$ have top row entries $\{1, n-k+1, \ldots, n\}$.

Let $T$ and $V$ be the tableaux obtained from $t$ and $v$ by deleting all but the top corner box in their top rows. This removes entries $\{n-k+1, \ldots, n\}$. Let $\lambda^\star/\mu$ be the common shape of $T$ and $V$. Observe that $T$ has greatest entry $n-k = c$ in the bottom corner box of its rightmost column and that $V$ is the column straightening of $T^\dagger$, where $\dagger$ is defined as $+$ on tableaux, but replacing $n$ with $n-k$. By induction, $T = t_{\lambda^\star/\mu}$, and since $t$ has $n-k+1, \ldots, n$ in its top row, we have $t = t_{\lambda/\mu}$. Moreover, the coefficient of $e(T)$ in $e(T^\dagger)$ is $(-1)^{\mathrm{ht}(\lambda^\star/\mu)}$, Since $\mathrm{ht}(\lambda^\star/\mu) = \mathrm{ht}(\lambda/\mu)$, the coefficient of $e(t)$ in $e(t^+)$ is $(-1)^{\mathrm{ht}(\lambda/\mu)}$, as required.

*Case* (2). If Case (1) does not apply then, since $i = i'$, 1 and $n$ are in the same column of $t$ and so $j = j'$. Take $c$ maximal such that $1, 2, \ldots, c^-$ are in column $j$ of $t$. Suppose that in column $j$ of $t$, the entry immediately below $c^-$ equals $d$ for some $d < n$. By Proposition 2.3.5, the row of $c$ in $t$ is the same as the row of $c$ in $\widetilde{t^+}$. It follows that $c = d$, which contradicts the maximality of $c$ unless column $j$ of $t$ has entries $1, 2, \ldots, c^-, n$, as shown below.




By Lemma 2.3.3 there is a sequence of horizontal swaps, Garnir swaps and column straightenings from $\widetilde{t^+}$ to $t$. As seen in the proof of Proposition 2.3.5, it follows easily from Lemma 2.2.2 that $1, \ldots, c^-$ do not move. Let $X$ be the set of entries of $t$ lying strictly to the right of column $j$. These entries become $X^+$ in $\widetilde{t^+}$, which is standard with respect to these columns. No permutation in our chosen sequence can involve a entry in one of these columns. Hence $X^+ = X$, and so $X = \varnothing$.

We have shown that $j$ is the rightmost column of $t$, and that $t$ agrees with $t_{\lambda/\mu}$ in this column. Let $T$ be the tableau obtained from $t$ by deleting all but the bottom corner box in column $j$ and subtracting $c^-$ from each remaining entry. Thus the top row of $T$ has entries $1, \ldots, n - c^-$ and $n - c^-$ is its greatest entry. Let $T$ have shape $\lambda^\star/\mu^\star$. By induction, $T = t_{\lambda^\star/\mu^\star}$, and hence $t = t_{\lambda/\mu}$. Let $T^\dagger$ be defined as $T^+$, but replacing $n$ with $n - c^-$. By induction, the coefficient of $e(T)$ in $e(T^\dagger)$, is $(-1)^{\mathrm{ht}(\lambda^\star/\mu^\star)}$. Since $\mathrm{ht}(\lambda^\star/\mu^\star) + c^- = \mathrm{ht}(\lambda/\mu)$, and the sign introduced by column straightening $t^+$ is $(-1)^{c^-}$, the coefficient of $e(t)$ in $e(t^+)$ is $(-1)^{\mathrm{ht}(\lambda/\mu)}$, as required. $\square$

## 2.4. Proof of the Murnaghan–Nakayama rule

Let $\lambda/\mu$ be a skew partition of size $n$ and let $\rho \in S_n$ be an $n$-cycle. Following the outline, to complete the proof of the Murnaghan–Nakayama rule, we must show that $\chi^{\lambda/\mu}(\rho) = 0$ if $\lambda/\mu$ is not a border strip. We require the following two lemmas.

LEMMA 2.4.1. *Let* $0 \le \ell \le n$. *If*

$$\langle \chi^\lambda, \chi^\mu \times 1_{S_\ell} \times \mathrm{sgn}_{S_{n-\ell}} \uparrow_{S_m \times S_\ell \times S_{n-\ell}}^{S_{m+n}} \rangle > 0,$$

*then* $[\lambda/\mu]$ *has no four boxes making the shape* $(2, 2)$.

PROOF. By the versions of Pieri's rule and Young's rule given in §1.1.4, the hypothesis implies that $\lambda$ is obtained from $\mu$ by adding a horizontal strip of size $\ell$ and then a vertical strip of size $n-\ell$. If two boxes from a horizontal strip are added to row $i$ then at most one box can be added below them in row $i+1$ by a vertical strip. The result follows. $\square$

LEMMA 2.4.2. *If $\lambda$ is a partition of $n$ and $\rho$ is an $n$-cycle then $\chi^\lambda(\rho) \neq 0$ if and only if $\lambda = (n-\ell, 1^\ell)$ where $0 \leq \ell < n$.*

PROOF. By a column orthogonality relation

$$\sum_\lambda \chi^\lambda(\rho)^2 = |\mathrm{Cent}_{S_n}(\rho)| = n,$$

where the sum is over all partitions $\lambda$ of $n$. By (2.1) in the case proved in §2.3, we have $\chi^{(n-\ell,1^\ell)}(\rho) = (-1)^{\ell-1}$ for $0 \leq \ell < n$. Therefore the partitions $(n-\ell, 1^\ell)$ give all the non-zero summands. $\square$

PROPOSITION 2.4.3. *Let $\lambda/\mu$ be a skew partition of size $n$ and let $\rho \in S_n$ be an $n$-cycle. If $\lambda/\mu$ is not a border strip then $\chi^{\lambda/\mu}(\rho) = 0$.*

PROOF. If $[\lambda/\mu]$ is disconnected then it is clear from the Standard Basis Theorem (Theorem 1.1.3(ii)) that $S^{\lambda/\mu}$ is isomorphic to a module induced from a proper Young subgroup $S_{n-\ell} \times S_\ell$ of $S_n$. Since no conjugate of $\rho$ lies in this subgroup, we have $\chi^{\lambda/\mu}(\rho) = 0$.

In the remaining case $\lambda/\mu$ has four boxes making the shape $(2,2)$. By either Pieri's rule or Young's rule, we have

$$\langle 1_{S_\ell} \times \mathrm{sgn}_{S_{n-\ell}} \big\uparrow_{S_\ell \times S_{n-\ell}}^{S_n}, \chi^{(n-\ell,1^\ell)} \rangle = 1.$$

Hence

$$\begin{aligned}\langle \chi^\lambda, \chi^\mu \times 1_{S_\ell} \times \mathrm{sgn}_{S_{n-\ell}} \big\uparrow_{S_m \times S_\ell \times S_{n-\ell}}^{S_{m+n}} \rangle &\geq \langle \chi^\lambda, \chi^\mu \times \chi^{(n-\ell,1^\ell)} \big\uparrow_{S_m \times S_n}^{S_{m+n}} \rangle \\ &= \langle \chi^{\lambda/\mu}, \chi^{(n-\ell,1^\ell)} \rangle\end{aligned}$$

where the equality follows from Lemma 1.1.11. By Lemma 2.4.1 the left-hand size is 0. Hence $\langle \chi^{\lambda/\mu}, \chi^{(n-\ell,1^\ell)} \rangle = 0$ for $0 \leq \ell < n$. By Lemma 2.4.2, this implies the result. $\square$

CHAPTER 3

# Restrictions of odd-degree characters

Let $G$ be a finite group, and let $\mathrm{Irr}_{2'}(G)$ be the set of ordinary irreducible characters of $G$ that have odd-degree. McKay conjectured in [**50**] that $|\mathrm{Irr}_{2'}(G)|$ is equal to $|\mathrm{Irr}_{2'}(N_G(P))|$, where $P$ is a Sylow 2-subgroup of $G$. The conjecture has recently been proved by Malle and Späth in [**47**] using the classification of finite simple groups. The McKay Conjecture is a particular example of the more general *local-global conjectures*, which have become of significant interest in the representation theory of finite groups. An aim of these local-global conjectures is to understand the representation theory of $G$ by considering the representation theory of the smaller group $N_G(P)$, where $P$ is a $p$-subgroup of $G$. Another example of a local-global conjecture is the generalisation of the McKay Conjecture to all primes. Whilst this generalisation is easy to state, finding a proof in the odd-prime case is still an open problem. For further examples of local-global conjectures and the progress made towards proving these, see [**46**].

This chapter is motivated by the original statement of the McKay Conjecture when $p = 2$ for the case of the symmetric group of degree a two-power. Indeed, given $n \in \mathbf{N}$, let $P_{2^n}$ be a Sylow 2-subgroup of $S_{2^n}$. It follows from [**64**, Corollary 2] (see also [**11**, Theorem 5.1.2]) that $P_{2^n}$ is self-normalising in $S_{2^n}$. Therefore in this case the McKay Conjecture states that $|\mathrm{Irr}_{2'}(S_{2^n})| = |\mathrm{Irr}_{2'}(P_{2^n})|$. This equality was first proved by Olsson in [**57**].

In [**22**] Giannelli also proves the McKay Conjecture for $S_{2^n}$ by providing a bijection between $\mathrm{Irr}_{2'}(S_{2^n})$ and $\mathrm{Irr}_{2'}(P_{2^n})$. We remark that bijective proofs of local-global conjectures are of interest, as such proofs demonstrate deeper underlying representation-theoretic connections between $G$ and $N_G(P)$. Giannelli's proof uses that every $\chi \in \mathrm{Irr}_{2'}(S_{2^n})$ has a unique degree-one constituent upon restriction to $P_{2^n}$, and that every degree-one $P_{2^n}$-character appears in $\chi\downarrow_{P_{2^n}}$, for some $\chi \in \mathrm{Irr}_{2'}(S_{2^n})$. Observe that the degree of an odd-degree irreducible character of $P_{2^n}$ necessarily divides $|P_{2^n}|$, which is a two-power. It follows that an odd-degree irreducible $P_{2^n}$-character has degree one. Therefore the restriction map

$$\Phi_n : \mathrm{Irr}_{2'}(S_{2^n}) \to \mathrm{Irr}_{2'}(P_{2^n})$$

gives a bijection between $\mathrm{Irr}_{2'}(S_{2^n})$ and $\mathrm{Irr}_{2'}(P_{2^n})$. Furthermore, the map $\Phi_n$ is extended (see [**22**, §3]) to give a bijection between $\mathrm{Irr}_{2'}(S_n)$ and $\mathrm{Irr}_{2'}(N_{S_n}(P))$, where $P$ is a Sylow 2-subgroup of $S_n$ for all $n \in \mathbf{N}$.

Motivated by [**22**] we prove Theorem 3.0.1 below, which gives a bijective proof of the McKay conjecture for certain groups of the form $G \wr S_{2^n}$. The proof of the theorem shows that for the groups $G$ considered, each character in $\mathrm{Irr}_{2'}(G \wr S_{2^n})$ has a unique odd-degree constituent upon restriction to $N_{G\wr S_{2^n}}(P)$, where $P$ is a Sylow 2-subgroup of $G \wr S_{2^n}$ in each case.

THEOREM 3.0.1. *Let $G$ be one of the following groups:*

- *$S_{2^a}$, where $a \in \mathbf{N}$*
- *$C_2^a$, where $a \in \mathbf{N}$*
- *any finite abelian $p$-group, where $p$ is an odd prime.*

*and let $P$ be a Sylow 2-subgroup of $G \wr S_{2^n}$, Given $\chi \in \mathrm{Irr}_{2'}(G \wr S_{2^n})$, the restricted character $\chi{\downarrow_{N_{G\wr S_{2^n}}(P)}}$ has a unique degree-one constituent, denoted $\Phi(\chi)$. Moreover, the map $\chi \mapsto \Phi(\chi)$ is a bijection between $\mathrm{Irr}_{2'}(G \wr S_{2^n})$ and $\mathrm{Irr}(N_{G\wr S_{2^n}}(P))$.*

As well as proving this theorem, we also make the bijection $\Phi_n$ in [**22**] completely explicit by describing the unique degree-one constituent of $\chi{\downarrow_{P_{2^n}}}$, for all $\chi \in \mathrm{Irr}_{2'}(S_{2^n})$. We do this by explicitly constructing the unique one-dimensional $\mathbf{Q}P_{2^n}$-submodule of a chosen $\mathbf{Q}S_{2^n}$-module that affords $\chi$, and then determining the ordinary character of this one-dimensional submodule. Observe that it is possible to work over the rational field $\mathbf{Q}$ when constructing the modules affording the characters in $\mathrm{Lin}(P_{2^n})$ by using the results of §1.4.3 and §3.1.1 (see below). Indeed the construction of $P_{2^n}$ given in §3.1.1 shows that it is isomorphic to

$$(\dots((\underbrace{C_2 \wr C_2) \wr C_2) \dots \wr C_2}_{n \text{ times}}).$$

It therefore follows from the discussion immediately before Theorem 1.4.5 in §1.4.3 that the modules corresponding to the characters in $\mathrm{Lin}(P_{2^n})$ can be realised over $\mathbf{Q}$.

As mentioned above, the bijection $\Phi_n$ between $\mathrm{Irr}_{2'}(S_{2^n})$ and $\mathrm{Irr}_{2'}(P_{2^n})$ relies on the remarkable fact that $\chi \in \mathrm{Irr}_{2'}(S_{2^n})$ has a unique degree-one constituent on restriction to $P_{2^n}$. We end this chapter by considering the irreducible constituents of $\chi{\downarrow_{P_{2^n}}}$ of either degree two, or degree four. In particular we give explicit formulas for the number of such constituents of $\chi{\downarrow_{P_{2^n}}}$ . We also explain why we only count these low degree constituents, and not the irreducible constituents of degree at least 8.

**Outline.** In §3.1 we provide the background required on the set $\mathrm{Irr}_{2'}(G \wr S_{2^n})$, where $G$ is any finite group. We begin by considering the case $1 \wr S_{2^n}$ as this is required to describe the results in the general case. In particular we give a construction of $P_{2^n}$ and the odd-dimensional irreducible $\mathbf{Q}S_{2^n}$-modules in §3.1.1 and §3.1.2, respectively. We then describe $\mathrm{Irr}_{2'}(G \wr S_{2^n})$ in §3.1.3.

These constructions of the Sylow 2-subgroup and the odd-dimensional modules are used again in §3.2, where we make completely explicit the bijection $\Phi_n$ between $\mathrm{Irr}_{2'}(S_{2^n})$ and $\mathrm{Irr}_{2'}(P_{2^n})$. The main results in §3.2 are Propositions 3.2.4 and 3.2.5. Proposition 3.2.4 constructs the unique one-dimensional $\mathbf{Q}P_{2^n}$-submodule of each odd-dimensional irreducible $\mathbf{Q}S_{2^n}$-module. In Proposition 3.2.5 we then show that the one-dimensional $\mathbf{Q}P_{2^n}$-modules that we have constructed are non-isomorphic by considering the action of $P_{2^n}$ on each of these submodules. In the same spirit of considering actions on modules, we end §3.2 with Lemma 3.2.6. This determines the ordinary characters of the one-dimensional $\mathbf{Q}P_{2^n}$-modules that we construct, thereby determining $\Phi_n(\chi)$ for all $\chi \in \mathrm{Irr}_{2'}(S_{2^n})$.

In §3.3 and §3.4 we prove that the restriction map gives a bijection when either $G$ equals $S_{2^a}$, or $G$ is an abelian $p$-group, respectively. We will see in each of these cases that a Sylow 2-subgroup of $G \wr S_{2^n}$ is isomorphic to a Sylow 2-subgroup of a symmetric group of degree a two-power. This is not the case in general for a Sylow 2-subgroup of $C_2^a \wr S_{2^n}$, and so we defer this case to §3.5.

We end this chapter with §3.6, in which we give explicit formulas for the numbers of two-degree and four-degree irreducible constituents of $\chi \downarrow_{P_{2^n}}^{S_{2^n}}$, where $\chi \in \mathrm{Irr}_{2'}(S_{2^n})$.

## 3.1. Odd-degree characters and Sylow 2-subgroups

Throughout this section fix $n \in \mathbf{N}$, and fix a finite group $G$. Following the outline of this chapter, we describe the set $\mathrm{Irr}_{2'}(G \wr S_{2^n})$.

### 3.1.1. A Sylow 2-subgroup of $S_{2^n}$.

Definition. Let $i \in \mathbf{N}$ be such that $1 \leq i \leq n$. Define the element $\sigma_i$ of $S_{2^n}$ by

$$\sigma_i = (1\ 2^{i-1}+1)(2\ 2^{i-1}+2)\dots(2^{i-1}\ 2^i).$$

The subgroup of $S_{2^n}$ generated by the set

$$\{\sigma_i : 1 \leq i \leq n\}$$

is a Sylow 2-subgroup of $S_{2^n}$, and for the remainder of this section $P_{2^n}$ refers to this particular subgroup. Observe that

$$P_{2^n} = (P_{2^{n-1}} \times {}^{\sigma_n}P_{2^{n-1}}) \rtimes \langle \sigma_n \rangle \cong P_{2^{n-1}} \wr C_2,$$

which is a special case of the construction of Sylow $p$-subgroups of symmetric groups given in [**35**, 4.1.20]. We write $Q_n$ for the base group $P_{2^{n-1}} \times {}^{\sigma_n}P_{2^{n-1}}$ of $P_{2^n}$.

**3.1.2. Odd-dimensional irreducible $\mathbf{Q}S_{2^n}$-modules.** It is proved in [**57**, Lemma 4.1] that $\mathrm{Irr}_{2'}(S_{2^n})$ consists precisely of the irreducible characters labelled by the partitions of the form $(2^n-k, 1^k)$, where $0 \leq k < 2^n$. Instead of working with the polytabloid construction for the Specht modules labelled by these partitions given in §1.1.1, we use the construction given by the following lemma.

LEMMA 3.1.1. [**53**, Proposition 2.3(a)] *Let* $0 \leq k < n$. *Then*

$$S^{(n-k,1^k)} \cong \bigwedge^k S^{(n-1,1)},$$

*as* $\mathbf{Q}S_n$*-modules. By definition,* $\bigwedge^0 S^{(n-1,1)}$ *is the trivial* $\mathbf{Q}S_n$*-module.*

We now identify $\bigwedge^k S^{(2^n-1,1)}$ as a submodule of $\bigwedge^k M^{(2^n-1,1)}$. Observe that the permutation module $M^{(2^n-1,1)}$ is isomorphic to the $\mathbf{Q}S_{2^n}$-module with basis

$$\{e_1, \ldots, e_{2^n}\},$$

and action given by $\sigma e_i = e_{\sigma(i)}$, where $1 \leq i \leq 2^n$ and $\sigma \in S_{2^n}$. Given $i \in \{2, \ldots, 2^n\}$, define $w_i = e_i - e_1$. It follows that the Specht module $S^{(2^n-1,1)}$ is isomorphic to the submodule of $M^{(2^n-1,1)}$ with basis equal to the set of $w_i$ such that $2 \leq i \leq 2^n$. Moreover, $\bigwedge^k S^{(2^n-1,1)}$ has a $\mathbf{Q}$-basis given by

$$\{w_{i_1} \wedge \cdots \wedge w_{i_k} : 2 \leq i_1 < \cdots < i_k \leq 2^n\}.$$

Lemma 3.1.3 below gives a method for determining whether or not a vector in $\bigwedge^k M^{(2^n-1,1)}$ is contained in $\bigwedge^k S^{(2^n-1,1)}$. In order to state this lemma, we require the following definition.

DEFINITION. Given $0 \leq k < n$, define the *boundary map*

$$\widehat{\delta}_k : \bigwedge^k M^{(n-1,1)} \to \bigwedge^{k-1} M^{(n-1,1)}$$

by

$$\widehat{\delta}_k(e_{i_1} \wedge \cdots \wedge e_{i_k}) = \sum_{a=1}^{k} (-1)^{a-1} e_{i_1} \wedge \cdots \wedge \widehat{e_{i_a}} \wedge \cdots \wedge e_{i_k},$$

where the hat above the wedge factor $e_{i_a}$ denotes that it is omitted.

REMARK 3.1.2. If we regard $e_1, \ldots, e_n$ as the vertices of an oriented $(n-1)$-simplex $S$, then the wedge product $e_{i_1} \wedge \cdots \wedge e_{i_k}$ can be viewed as the oriented $(k-1)$-simplex lying on $S$ with vertices $e_{i_1}, \ldots, e_{i_k}$. If $k \geq 2$, then $\widehat{\delta}_k$ sends a $(k-1)$-simplex to its boundary of $(k-2)$-simplices, hence it being named the boundary map.

We have the following useful identity, which is Equation (5) in [**23**]. Given $u \in \bigwedge^r M^{(n-1,1)}$ and $v \in \bigwedge^s M^{(n-1,1)}$, where $r, s \in \mathbf{N}$, then

$$\widehat{\delta}_{r+s}(u \wedge v) = \widehat{\delta}_r(u) \wedge v + (-1)^r u \wedge \widehat{\delta}_s(v). \tag{3.1}$$

LEMMA 3.1.3. [**23**, Proposition 5.2] *The chain complex*

$$0 \to \bigwedge^{n} M^{(n-1,1)} \xrightarrow{\widehat{\delta}_n} \cdots \xrightarrow{\widehat{\delta}_r} \bigwedge^{r-1} M^{(n-1,1)} \xrightarrow{\widehat{\delta}_{r-1}} \cdots \to M^{(n-1,1)} \to \mathbf{Q} \to 0,$$

*is exact in all places. Furthermore*

$$\ker \widehat{\delta}_k = \operatorname{im} \widehat{\delta}_{k+1} \cong \bigwedge^{k} S^{(n-1,1)}.$$

It follows from Lemma 3.1.3 that $v \in \bigwedge^k M^{(2^n-1,1)}$ is contained in $\bigwedge^k S^{(2^n-1,1)}$ if and only if $\widehat{\delta}_k(v) = 0$.

**3.1.3. The set** $\mathrm{Irr}_{2'}(G \wr S_{2^n})$**.** Let $\mathrm{Irr}(G) = \{\psi_1, \ldots, \psi_t\}$. Recall that $\mathcal{P}^t(n)$ denotes the set of multi-partitions of $n$ with length at most $t$. Given $(\lambda^1, \ldots, \lambda^t) \in \mathcal{P}^t(2^n)$, define

$$\chi_{\lambda^1,\ldots,\lambda^t} = \left(\boxtimes_{i=1}^t \widetilde{\psi_i}^{\times n_i} \operatorname{Inf}_{S_{n_i}}^{G\wr S_{n_i}} \chi^{\lambda^i}\right)\Big\uparrow_{G\wr S_{(n_1,\ldots,n_t)}}^{G\wr S_{2^n}},$$

where $n_i := |\lambda^i|$ for each $i$.

We remind the reader of the following result from §1.3.5, which will be used in the proof of Lemma 3.1.4 below.

COROLLARY 1.3.19. *Fix $a \in \mathbf{N}_0$ such that $0 \leq a \leq 2^n$. The binomial coefficient $\binom{2^n}{a}$ is odd if and only if either $a = 0$, or $a = 2^n$.*

The following lemma provides a necessary condition for $\chi_{\lambda^1,\ldots,\lambda^t}$ to have odd-degree.

LEMMA 3.1.4. *Suppose that the character $\chi_{\lambda^1,\lambda^2,\ldots,\lambda^t}$ has odd-degree. Then exactly one $\lambda^i$ is non-empty.*

PROOF. Let $d$ be the degree of the character

$$\boxtimes_{i=1}^t \widetilde{\psi_i}^{\times n_i} \operatorname{Inf}_{S_{n_i}}^{G\wr S_{n_i}} \chi^{\lambda^i},$$

where $n_i := |\lambda^i|$ for all $i$. Then $\chi_{\lambda^1,\lambda^2,\ldots,\lambda^t}$ has degree

$$d[S_{2^n} : S_{n_1} \times S_{n_2} \times \cdots \times S_{n_t}] = d\binom{2^n}{n_1, n_2, \ldots, n_t}.$$

We prove that a multinomial coefficient of the form

$$\binom{2^n}{n_1, n_2, \ldots, n_s}$$

is odd only if at most one $n_i$ is non-zero. We proceed by induction on $s$. The base case is when $s = 1$, which is immediate. Suppose that $s > 1$, and that the claim holds inductively. The multinomial coefficient can be written as

$$\binom{2^n}{n_1, n_2, \ldots, n_s} = \binom{2^n}{n_1}\binom{2^n - n_1}{n_2, \ldots, n_s}.$$

If the multinomial coefficient is odd, by Corollary 1.3.19 applied to the first factor on the right hand side either $n_1 = 2^n$, or $n_1 = 0$. In the first case, the lemma is proved. In the second case the inductive hypothesis says that at most one of $n_2, \dots, n_s$ is non-zero, as required.

Since $n \geq 1$, at least one $\lambda^i$ in $\{\lambda^1, \dots, \lambda^t\}$ is non-empty. However, we have assumed that $\chi_{\lambda_1,\dots,\lambda_t}$ has odd-degree, and so the induction in the previous paragraph shows that at most one $\lambda^i$ is non-empty. It follows that exactly one $\lambda^i$ is non-empty, as claimed. $\square$

Remark 3.1.5. It follows from Lemma 3.1.4 that

$$\mathrm{Irr}_{2'}(G \wr S_{2^n}) = \{\widetilde{\psi}^{\times 2^n} \operatorname{Inf}_{S_{2^n}}^{G \wr S_{2^n}} \chi^{(2^n-k,1^k)} : \psi \in \mathrm{Irr}_{2'}(G) \text{ and } 0 \leq k < 2^n\}.$$

## 3.2. One-dimensional submodules

Fix $n \in \mathbf{N}$. Recall from the introduction of this chapter that the odd-dimensional irreducible $\mathbf{Q}P_{2^n}$-modules are precisely the $\mathbf{Q}P_{2^n}$-modules that are one-dimensional. In this section we make the bijection

$$\Phi_n : \mathrm{Irr}_{2'}(S_{2^n}) \xrightarrow{\mathrm{Res}} \mathrm{Irr}(P_{2^n})$$

completely explicit by constructing the unique one-dimensional submodule of $\bigwedge^k S^{(2^n-1,1)}$, for $0 \leq k < 2^n$, and then determining its ordinary character.

In order to state and prove our main propositions, the following preliminaries are required.

Lemma 3.2.1. *Let $G$ be a finite group, and let $G'$ be the commutator subgroup of $G$. The degree-one characters of $G$ are precisely the inflations to $G$ of the irreducible characters of $G/G'$. Moreover, the group* $\mathrm{Lin}(G)$ *is isomorphic to $G/G'$.*

The following proposition describes the abelianisation $P_{2^n}/P'_{2^n}$ of $P_{2^n}$.

Proposition 3.2.2. *There is an isomorphism of groups*

$$P_{2^n}/P'_{2^n} \cong C_2^n.$$

In order to prove this proposition, we require the following result.

Lemma 3.2.3. [**26**, Proposition 3.3] *Let $G$ and $K$ be finite groups. Then*

$$(K \rtimes G)/(K \rtimes G)' \cong (K/K')/\langle {}^g k k^{-1} K' : k \in K, g \in G\rangle \times G/G',$$

*where ${}^g k$ denotes the image of $k$ under the conjugation action of $g$.*

Proof of Proposition 3.2.2. We proceed by induction on $n$. The base case is when $n = 1$, where $P_2 = S_2 \cong C_2$. The result in this case is immediate.

Given $n > 1$, assume that the result holds inductively. Recall that

$$P_{2^n} = P_{2^{n-1}} \times {}^{\sigma_n} P_{2^{n-1}} \rtimes \langle \sigma_n \rangle.$$

Also recall that $Q_n = P_{2^{n-1}} \times ({}^{\sigma_n}P_{2^{n-1}})$, and so $Q'_n = P'_{2^{n-1}} \times ({}^{\sigma_n}P'_{2^{n-1}})$. Lemma 3.2.3 applied to $P_{2^n}$ states that

$$P_{2^n}/P'_{2^n} \cong (Q_n/Q'_n)/\left\langle {}^g kk^{-1}Q'_n : k \in Q_n, g \in \langle\sigma_n\rangle\right\rangle \times \langle\sigma_n\rangle.$$

By the inductive hypothesis $P_{2^{n-1}}/P'_{2^{n-1}} \cong C_2^{n-1}$, and so $Q_n/Q'_n \cong C_2^{2n-2}$. We therefore need to prove that the subgroup

$$K_n := \left\langle {}^g kk^{-1}Q'_n : k \in Q_n, g \in \langle\sigma_n\rangle\right\rangle$$

is isomorphic to $C_2^{n-1}$. As $P_{2^{n-1}}$ is generated by the set $\{\sigma_1, \ldots, \sigma_{n-1}\}$, the subgroup $K_n$ is generated by the set

$$\left\{{}^{\sigma_n}kkQ'_n : k \in \{\sigma_1, \ldots, \sigma_{n-1}, {}^{\sigma_n}\sigma_1, \ldots, {}^{\sigma_n}\sigma_{n-1}\}\right\}.$$

As conjugation by $\sigma_n$ is an involution and ${}^{\sigma_n}\sigma_i$ commutes with $\sigma_i$, this set equals

$$\left\{{}^{\sigma_n}kkQ'_n : k \in \{\sigma_1, \ldots, \sigma_{n-1}\}\right\}.$$

Therefore $K_n$ is generated by $n-1$ elements, each of order two. By definition of the abelianisation of a group, these generators are pairwise commutative, and so $K_n \cong C_2^d$, for some $d \leq n-1$.

Suppose that $d < n-1$. Given $\varnothing \neq J \subseteq \{1, 2, \ldots, n-1\}$, briefly define

$$\sigma_J = \prod_{j \in J} \sigma_j.$$

As $d < n-1$, there exists $J \subseteq \{1, 2, \ldots, n-1\}$, such that ${}^{\sigma_n}(\sigma_J)(\sigma_J)Q'_n = Q'_n$. As the factors $P_{2^{n-1}}$ and ${}^{\sigma_n}P_{2^{n-1}}$ of $Q_n$ commute, we have

$$\sigma_J \in P'_{2^{n-1}}, \text{ and } {}^{\sigma_n}(\sigma_J) \in {}^{\sigma_n}P'_{2^{n-1}}.$$

Then $P_{2^{n-1}}/P'_{2^{n-1}}$, which is generated by

$$\sigma_1 P'_{2^{n-1}}, \ldots, \sigma_{n-1}P'_{2^{n-1}},$$

is isomorphic to a proper subgroup of $C_2^{n-1}$. However this is a contradiction to the inductive hypothesis. Therefore $K_n \cong C_2^{n-1}$, and so

$$P_{2^n}/P'_{2^n} \cong (C_2^{2n-2}/C_2^{n-1}) \times C_2 \cong C_2^n. \qquad \square$$

We now define the vectors $v_{k,n}$, which are the subject of the main propositions in this section. Given $0 \leq k < 2^n$, let $2^{k_1}, \ldots, 2^{k_t}$ be the two-powers appearing with non-zero coefficient in the binary expansion of $k$, where the notation is chosen so that $0 \leq k_1 < \cdots < k_t < n$ Equivalently the two-powers $2^{k_1}, \ldots, 2^{k_t}$ are those that appear with coefficient 1 in the binary expansion of $k$. Define $V_{k,n}$ to be $\bigwedge^k S^{(2^n-1,1)}$ viewed as a $\mathbf{Q}P_{2^n}$-module, and define $v_{k,n}$ as follows:

$$v_{k,n} = \begin{cases} 1, & \text{if } k = 0 \\ (e_1 + \cdots + e_{2^{n-1}}) - (e_{2^{n-1}+1} + \cdots + e_{2^n}), & \text{if } k = 1 \\ v_{k/2,n-1} \wedge \sigma_n v_{k/2,n-1}, & \text{if } k = 2^i, \text{ where } i \in \mathbf{N} \\ v_{2^{k_1},n} \wedge \cdots \wedge v_{2^{k_t},n}, & \text{otherwise.} \end{cases}$$

We remark that the notation is chosen such that $v_{k,n}$ is contained $V_{k,n}$ for all $0 \leq k < 2^n$. Although this is not immediately obvious from the definition of $v_{k,n}$, our first main proposition shows that this is indeed the case.

PROPOSITION 3.2.4. *Let $0 \leq k < 2^n$. Then $\langle v_{k,n} \rangle$ is a one-dimensional submodule of $V_{k,n}$.*

PROPOSITION 3.2.5. *Let $0 \leq k < l < 2^n$. Then $\langle v_{k,n} \rangle$ and $\langle v_{l,n} \rangle$ are not isomorphic as $\mathbf{Q}P_{2^n}$-modules.*

It follows from Proposition 3.2.2 and Proposition 3.2.5 that the set of isomorphism classes of all $\langle v_{k,n} \rangle$ is a complete set of isomorphism classes of one-dimensional $\mathbf{Q}P_{2^n}$-modules. This would also follow from either the bijection $\Phi_n$ between $\mathrm{Irr}_{2'}(S_{2^n})$ and $\mathrm{Irr}_{2'}(P_{2^n})$, or the McKay Conjecture.

PROOF OF PROPOSITION 3.2.4. We prove that $v_{k,n}$ is contained in $V_{k,n}$, and that the subspace $\langle v_{k,n} \rangle$ is closed under the action of $P_{2^n}$. By definition of the $v_{k,n}$, it is sufficient to prove this when either $k = 0$, or $k = 2^i$ for some $1 \leq i < n$.

In the case that $k = 0$, we have $\langle v_{0,n} \rangle = \bigwedge^0 S^{(2^n-1,1)} = V_{0,n}$, which is the trivial $\mathbf{Q}P_{2^n}$-module. Both claims in the previous paragraph therefore hold in this case.

Suppose now that $k = 2^i$, for some $1 \leq i < n$. We proceed by induction on $i$. The base case is when $i = 0$, in which case $k = 1$. Then

$$\widehat{\delta}_1(v_{1,n}) = \widehat{\delta}_1\left((e_1 + \cdots + e_{2^{n-1}}) - (e_{2^{n-1}+1} + \cdots + e_{2^n})\right) = 0,$$

for all $n \in \mathbf{N}$. By the remark immediately after Lemma 3.1.3, $v_{1,n} \in S^{(n-1,1)} = V_{1,n}$, and so $\langle v_{1,n} \rangle$ is a subspace of $V_{1,n}$.

We now show that $\langle v_{1,n} \rangle$ is closed under the action of $\sigma_i$, for $1 \leq i \leq n$. It follows from the definition of $v_{1,n}$ that the generators

$$\sigma_1, \sigma_2, \ldots, \sigma_{n-1},$$

all act trivially on $v_{1,n}$. Furthermore, $\sigma_n$ acts with negative sign on $v_{1,n}$. Therefore $\langle v_{1,n} \rangle$ is closed under the action of $P_{2^n}$.

Suppose now that $i > 0$, and assume inductively that the result holds for all $l < i$. By definition,

$$v_{k,n} = v_{k/2,n-1} \wedge \sigma_n v_{k/2,n-1}.$$

By the inductive hypothesis $\widehat{\delta}_{k/2}(v_{k/2,n-1}) = \widehat{\delta}_{k/2}(\sigma_n v_{k/2,n-1}) = 0$, and so it follows from (3.1) that

$$\begin{aligned}\widehat{\delta}_k(v_{k,n}) &= \widehat{\delta}_k(v_{k/2,n-1} \wedge \sigma_n v_{k/2,n-1})\\ &= \widehat{\delta}_{k/2}(v_{k/2,n-1}) \wedge \sigma_n v_{k/2,n-1} + (-1)^{k/2}\big(v_{k/2,n-1} \wedge \widehat{\delta}_{k/2}(\sigma_n v_{k/2,n-1})\big)\\ &= 0.\end{aligned}$$

Once more applying the remark immediately after Lemma 3.1.3 shows that $\langle v_{k,n}\rangle$ is a subspace of $\bigwedge^k S^{(2^n-1,1)} = V_{k,n}$.

We now consider the action of $\sigma_j$ on $v_{k,n}$ for each $1 \leq j \leq n$. First let $j < n$, and so

$$\sigma_j v_{k,n} = \sigma_j v_{k/2,n-1} \wedge \sigma_j \sigma_n v_{k/2,n-1}.$$

As $\sigma_j$ is a generator for the group $P_{2^{n-1}}$, the inductive hypothesis says that $\langle v_{k/2,n-1}\rangle$ is closed under the action of $\sigma_j$. As $\sigma_j$ has support contained in $\{1, 2, \dots, 2^{n-1}\}$, the action of $\sigma_j$ on $\sigma_n v_{k/2,n-1}$ is trivial. Therefore $\langle v_{k,n}\rangle$ is closed under the action of $\sigma_j$.

Consider now the action of $\sigma_n$ on $v_{k,n}$. We have that

$$\begin{aligned}\sigma_n v_{k,n} &= \sigma_n v_{k/2,n-1} \wedge \sigma_n \sigma_n v_{k/2,n-1}\\ &= \sigma_n v_{k/2,n-1} \wedge v_{k/2,n-1}.\end{aligned}$$

The vector $v_{k/2,n-1}$ is contained in $\bigwedge^{k/2} S^{(2^n-1,1)}$. By the anti-commutativity of the exterior power, it follows that

$$\sigma_n v_{k,n} = (-1)^{k/2} v_{k,n},$$

and so $\langle v_{k,n}\rangle$ is closed under the action of $P_{2^n}$. $\square$

Proof of Proposition 3.2.5. By definition of the $v_{k,n}$, it is sufficient to prove that $\langle v_{2^i,n}\rangle$ and $\langle v_{2^j,n}\rangle$ are non-isomorphic when $i \neq j$. Indeed let $\chi_i$ be the character of $\langle v_{2^i,n}\rangle$, where $0 \leq i < n$. Given $k \in \mathbf{N}$ such that $0 \leq k < 2^n$, let $k = k_0 2^0 + k_1 2^1 + \cdots + k_{n-1}2^{n-1}$ be the binary expansion of $k$. Then by definition $\langle v_{k,n}\rangle$ has ordinary character equal to the product

$$\prod_{i:\ k_i=1} \chi_i.$$

Moreover, $\mathrm{Lin}(P_{2^n}) \cong \langle\chi_0\rangle \times \cdots \times \langle\chi_{n-1}\rangle$ by Proposition 3.2.2. Therefore if $l \neq k$, then there exists some $\chi_i$ that appears as a factor of the ordinary character of $\langle v_{l,n}\rangle$, but not as a factor of the ordinary character of $\langle v_{k,n}\rangle$. It follows that $\langle v_{l,n}\rangle$ and $\langle v_{k,n}\rangle$ are not isomorphic.

We now proceed by induction on $n$. The base case is when $n = 0$, for which the result is immediate.

Suppose now that $n > 0$, and assume that the result holds inductively. We distinguish two cases.

*Case* (1). Given $i, j \in \mathbf{N}$, suppose that $\langle v_{2^i,n}\rangle$ and $\langle v_{2^j,n}\rangle$, are isomorphic as $\mathbf{Q}P_{2^n}$-modules. By definition the $\mathbf{Q}P_{2^{n-1}}$-modules

$$\langle v_{2^{i-1},n-1}\rangle \text{ and } \langle v_{2^{j-1},n-1}\rangle,$$

are isomorphic. By the inductive hypothesis we must have $2^{i-1} = 2^{j-1}$, and so $i = j$.

*Case* (2). Let $i = 0$ and fix $1 \leq j < n$. If $j > 1$, then $\sigma_n$ acts on $v_{2^j,n}$ trivially, whereas $\sigma_n$ acts on $v_{1,n}$ with negative sign. Therefore

$$\langle v_{1,n}\rangle \not\cong \langle v_{l,n}\rangle.$$

If $j = 1$, then $l = 2$, and similarly considering the action of $\sigma_{n-1}$ on $\langle v_{1,n}\rangle$ and $\langle v_{2,n}\rangle$ shows that these two modules are not isomorphic. □

We now determine the ordinary character of $\langle v_{k,n}\rangle$. By construction of the $v_{k,n}$, it is sufficient to do this when $k$ is a 2-power.

As $P_{2^n}$ is generated by the $\sigma_i$, it follows from Lemma 3.2.1 and Proposition 3.2.2 that

$$\mathrm{Lin}(P_{2^n}) \cong \langle \sigma_1 P'_{2^n}\rangle \times \langle \sigma_2 P'_{2^n}\rangle \times \cdots \times \langle \sigma_n P'_{2^n}\rangle.$$

In particular the character of each $\langle v_{2^j,n}\rangle$ is determined by the sign with which each $\sigma_i$ acts on $\langle v_{2^j,n}\rangle$. We determine this sign in the following lemma, and we give an example of this result in Example 3.2.7 below.

LEMMA 3.2.6. *Given $1 \leq i \leq n$ and $0 \leq j < n$, we have*

$$\sigma_i v_{2^j,n} = \begin{cases} v_{2^j,n} & \text{if } i + j \notin \{n, n+1\} \\ -v_{2^j,n} & \text{if } i + j \in \{n, n+1\}. \end{cases}$$

PROOF. By construction the first factor in the wedge product defining $v_{2^j,n}$ equals

$$(e_1 + e_2 + \cdots + e_{2^{n-j-1}}) - (e_{2^{n-j-1}+1} + e_{2^{n-j-1}+2} + \cdots + e_{2^{n-j}}).$$

The generators $\sigma_1, \ldots, \sigma_{n-j-1}$ clearly act trivially on $v_{2^j,n}$, and $\sigma_{n-j}$ acts with negative sign on $v_{2^j,n}$. Given $i > n-j$, the generator $\sigma_i$ acts by transposing the $k^{\text{th}}$ and $(k+i+j-n)^{\text{th}}$ wedge factors of $v_{2^j,n}$ for each $1 \leq k \leq i+j-n$. Therefore in this case $\sigma_i$ transposes an even number of pairs of wedge factors of $v_{2^j,n}$, except when $i + j - n = 1$. It follows from the anti-commutativity of the exterior power that the when $i > n - j$, $\sigma_i$ acts with negative sign on $v_{2^j,n}$ if and only if $i + j - n = 1$.

It follows that if $j > 0$ then only $\sigma_{n-j}$ and $\sigma_{n+1-j}$ act with negative sign on $v_{2^j,n}$. If $j = 0$ then only $\sigma_n$ acts on $v_{1,n}$ with negative sign. □

EXAMPLE 3.2.7. In this example we write $\tau : C_2 \to \{\pm 1\}$ for the non-trivial irreducible character of $C_2$. Let $n = 2$, and so $P_4 = \langle (1\ 2), (1\ 3)(2\ 4)\rangle$.

For each $v_{k,2}$ such that $1 \leq k \leq 3$, we determine $\Phi_2(v_{k,2}) \in \mathrm{Irr}_{2'}(P_4)$. Consider first $v_{1,2} = (e_1 + e_2) - (e_3 + e_4)$. We see that $\sigma_1$ acts with positive sign on $v_{1,2}$, and that $\sigma_2$ acts with negative sign on $v_{1,2}$. Therefore using the notation of §1.2.2, the ordinary character $\chi_1$ of $\langle v_{1,2}\rangle$ equals $\mathrm{Inf}_{C_2}^{P_4}\tau$. Moreover, $\Phi_2(\chi^{(3,1)}) = \mathrm{Inf}_{C_2}^{P_4}\tau$.

By similarly considering the actions of $\sigma_1$ and $\sigma_2$ on

$$v_{2,2} = (e_1 - e_2) \wedge (e_3 - e_4),$$

we see that the ordinary character $\chi_2$ of $\langle v_{2,2}\rangle$ equals $\widetilde{\tau}^{\times 2}\,\mathrm{Inf}_{C_2}^{P_4}\tau$, and so $\Phi_2(\chi^{(2,1^2)}) = \widetilde{\tau}^{\times 2}\,\mathrm{Inf}_{C_2}^{P_4}\tau$.

Furthermore by the construction of the $v_{k,n}$, the ordinary character of $\langle v_{3,2}\rangle$ equals $\widetilde{\tau}^{\times 2}$, and so $\Phi_2(\chi^{(1^4)}) = \widetilde{\tau}^{\times 2}$.

## 3.3. The case $S_{2^a} \wr S_{2^n}$

Given $a, n \in \mathbf{N}$, counting cardinalities shows that $S_{2^a} \wr S_{2^n}$ has a Sylow 2-subgroup isomorphic to $P_{2^a} \wr P_{2^n}$. It is shown in [**35**, 4.1.23] that the imprimitive wreath product is associative in the following sense: for subgroups $G \leq S_b, H \leq S_c, K \leq S_d$, we have

$$(G \wr H) \wr K \cong G \wr (H \wr K).$$

It follows that $P_{2^a} \wr P_{2^n} = P_{2^{a+n}}$. By the remark at the end of §3.1.1, $P_{2^{a+n}}$ is self-normalising in $S_{2^{a+n}}$. Therefore $P_{2^{a+n}}$ also self-normalising in $S_{2^a} \wr S_{2^n}$.

The main result in this section is Proposition 3.3.1.

PROPOSITION 3.3.1. *Let $\chi \in \mathrm{Irr}_{2'}(S_{2^a} \wr S_{2^n})$. Then $\chi\big\downarrow^{S_{2^a}\wr S_{2^n}}_{P_{2^a}\wr P_{2^n}}$ has a unique degree-one constituent, denoted $\Theta(\chi)$. Furthermore*

$$\mathrm{Irr}_{2'}(P_{2^a} \wr P_{2^n}) = \{\Theta(\chi) : \chi \in \mathrm{Irr}_{2'}(S_{2^a} \wr S_{2^n})\},$$

*and the map $\chi \mapsto \Theta(\chi)$ is a bijection.*

In order to prove the proposition, the following easy lemma is required.

LEMMA 3.3.2. *Let $G$ be a finite group, and let $N \lhd G$. Let $N \leq H \leq G$, and let $\chi$ be an ordinary $G/N$-character. Then*

$$(\mathrm{Inf}_{G/N}^{G}\chi)\big\downarrow^{G}_{H} = \mathrm{Inf}_{H/N}^{H}(\chi\big\downarrow^{G/N}_{H/N}).$$

We are now ready to prove Proposition 3.3.1.

PROOF OF PROPOSITION 3.3.1. It follows from Lemma 3.1.1 and Remark 3.1.5 that $\chi \in \mathrm{Irr}_{2'}(S_{2^a} \wr S_{2^n})$ is of the form

$$\chi = \widetilde{\chi^{(2^a-k,1^k)}}^{\times 2^n}\ \mathrm{Inf}_{S_{2^n}}^{S_{2^a}\wr S_{2^n}}\chi^{(2^n-l,1^l)},$$

for some $0 \leq k < 2^a$, and $0 \leq l < 2^n$. We claim that $\chi\big\downarrow_{P_{2^a}\wr P_{2^n}}$ has a unique degree-one constituent. We prove this claim by showing that each of

$$\widetilde{\chi^{(2^a-k,1^k)}}^{\times 2^n} \text{ and } \mathrm{Inf}_{S_{2^n}}^{S_{2^a}\wr S_{2^n}}\chi^{(2^n-l,1^l)}$$

has a unique degree-one constituent upon restriction to $P_{2^a} \wr P_{2^n}$. It follows using the bijection $\Phi_n$ and Lemma 3.3.2 that

$$\big(\mathrm{Inf}_{S_{2^n}}^{S_{2^a}\wr S_{2^n}} \chi^{(2^n-l,1^l)}\big)\big\downarrow^{S_{2^a}\wr S_{2^n}}_{P_{2^a}\wr P_{2^n}} = \mathrm{Inf}_{P_{2^n}}^{P_{2^a}\wr P_{2^n}} \left(\chi^{(2^n-l,1^l)}\big\downarrow^{S_{2^n}}_{P_{2^n}}\right)$$

has a unique degree-one constituent. Let

$$\psi = \widetilde{\chi^{(2^a-k,1^k)}}^{\times 2^n} \big\downarrow^{S_{2^a}\wr S_{2^n}}_{P_{2^a}\wr P_{2^n}}.$$

Any constituent of $\psi$ induced from a proper subgroup of $P_{2^a} \wr P_{2^n}$ has degree strictly greater than 1. It therefore follows from Theorem 1.2.6 that every degree-one constituent of $\psi$ is of the form $\widetilde{\pi}^{\times 2^n}$, for some degree-one constituent $\pi$ of $\chi^{(2^a-k,1^k)} \downarrow^{S_{2^a}}_{P_{2^a}}$. Therefore $\pi$ is the unique degree-one constituent of $\chi^{(2^a-k,1^k)} \downarrow^{S_{2^a}}_{P_{2^a}}$, and so $\chi \downarrow_{P_{2^a}\wr P_{2^n}}$ has a unique degree-one constituent, which equals

$$\widetilde{\pi}^{\times 2^n}\, \mathrm{Inf}_{S_{2^n}}^{S_{2^a}\wr S_{2^n}} \Phi_n(\chi^{(2^n-k,1^k)}).$$

Write $\Theta(\chi)$ for this unique degree-one constituent of $\chi\downarrow^{S_{2^a}\wr S_{2^n}}_{P_{2^a}\wr P_{2^n}}$. It remains to prove that the map $\chi \mapsto \Theta(\chi)$ is a bijection. Let $\chi_1, \chi_2 \in \mathrm{Irr}_{2'}(S_{2^a} \wr S_{2^n})$ be such that

$$\begin{aligned} \chi_1 &= \widetilde{\chi^{(2^a-k_1,1^{k_1})}}^{\times 2^n}\, \mathrm{Inf}_{S_{2^n}}^{S_{2^a}\wr S_{2^n}} \chi^{(2^n-l_1,1^{l_1})}, \\ \chi_2 &= \widetilde{\chi^{(2^a-k_2,1^{k_2})}}^{\times 2^n}\, \mathrm{Inf}_{S_{2^n}}^{S_{2^a}\wr S_{2^n}} \chi^{(2^n-l_2,1^{l_2})}, \end{aligned}$$

where $0 \le k_1, k_2 < 2^a$, and $0 \le l_1, l_2 < 2^n$. Suppose that $\Theta(\chi_1) = \Theta(\chi_2)$. If

$$\mathrm{Inf}_{P_{2^n}}^{P_{2^a}\wr P_{2^n}} \Phi_n(\chi^{(2^n-l_1,1^{l_1})}) = \mathrm{Inf}_{P_{2^n}}^{P_{2^a}\wr P_{2^n}} \Phi_n(\chi^{(2^n-l_2,1^{l_2})}),$$

then $l_1 = l_2$. If

$$\Theta\left(\widetilde{\chi^{(2^a-k_1,1^{k_1})}}^{\times 2^n}\right) = \Theta\left(\widetilde{\chi^{(2^a-k_2,1^{k_2})}}^{\times 2^n}\right),$$

then the action of the subgroup $P_{2^a} \times \{1\} \times \cdots \times \{1\}$ is the same on the representations corresponding to these two characters. It follows that

$$\Phi_n(\chi^{(2^a-k_1,1^{k_1})}) = \Phi_n(\chi^{(2^a-k_2,1^{k_2})}),$$

and so $k_1 = k_2$. Therefore the map $\Theta$ is injective. By Remark 3.1.5 $|\operatorname{Lin}(S_{2^a} \wr S_{2^n})| = 2^{a+n}$, and by Lemma 3.2.1 and Proposition 3.2.2 $|\operatorname{Lin}(P_{2^a} \wr P_{2^n})| = 2^{a+n}$. This proves that

$$\mathrm{Irr}_{2'}(P_{2^a} \wr P_{2^n}) = \{\Theta(\chi) : \chi \in \mathrm{Irr}_{2'}(S_{2^a} \wr S_{2^n})\},$$

and that the map $\Theta$ is a bijection. $\square$

### 3.4. The case $G \wr S_{2^n}$ when $G$ is an abelian $p$-group

Throughout this section, let $G$ be an abelian $p$-group, where $p$ is an odd prime, and fix $n \in \mathbf{N}$. Then $1 \wr P_{2^n}$ is a Sylow 2-subgroup of $G \wr S_{2^n}$, which we denote by $P_{2^n}$ in this section. The main result in this section is the following proposition.

PROPOSITION 3.4.1. *Let* $\chi \in \mathrm{Irr}_{2'}(G \wr S_{2^n})$. *Then* $\chi \downarrow^{G \wr S_{2^n}}_{N_{G \wr S_{2^n}}(P_{2^n})}$ *has a unique degree-one constituent, denoted* $\Theta(\chi)$. *Moreover,*

$$\mathrm{Irr}_{2'}(N_{G \wr S_{2^n}}(P_{2^n})) = \{\Theta(\chi) : \chi \in \mathrm{Irr}_{2'}(G \wr S_{2^n})\},$$

*and the map* $\chi \mapsto \Theta(\chi)$ *is a bijection.*

As $G$ is an abelian group, Remark 3.1.5 states that

$$\mathrm{Irr}_{2'}(G \wr S_{2^n}) = \{\widetilde{\psi}^{\times 2^n} \operatorname{Inf}_{S_{2^n}}^{G \wr S_{2^n}} \chi^{(2^n-k,1^k)} : \psi \in \mathrm{Irr}(G) \text{ and } 0 \leq k < 2^n\}.$$

The following lemma describes the normaliser of $P_{2^n}$ in $G \wr S_{2^n}$.

LEMMA 3.4.2. *Let $G$ be a finite group, and let $Q$ be a transitive subgroup of $S_n$. Then*

$$N_{G \wr S_n}(Q) = \Delta(G) \times N_{S_n}(Q),$$

*where $\Delta(G)$ denotes the diagonal subgroup of $G^n$.*

PROOF. By Lemma 1.2.3, $N_{G \wr S_n}(Q) = C_G(Q) \rtimes N_{S_n}(Q)$. It follows that given $g = (g_1, \ldots, g_n; \sigma) \in N_{G \wr S_n}(Q)$, we have $(g_1, \ldots, g_n; 1) \in C_G(Q)$. Then $g_i = g_j$ for $i$ and $j$ in the same $Q$-orbit. By assumption $Q$ is transitive and so $g_1 = g_2 = \cdots = g_n$. Therefore $C_G(Q) = \Delta(G)$, and so $N_{G \wr S_n}(Q) = \Delta(G) \rtimes N_{S_n}(Q)$. As the place permutation action of $S_n$ on $\Delta(G)$ is trivial, the product is direct. □

It follows from Lemma 3.4.2 that $N_{G \wr S_{2^n}}(P_{2^n}) \cong G \times P_{2^n}$, and so there is a natural correspondence between $\mathrm{Irr}_{2'}(N_{G \wr S_{2^n}}(P_{2^n}))$ and the set

$$\{\psi \times \chi^{(2^n-k,1^k)} : \psi \in \mathrm{Irr}(G) \text{ and } 0 \leq k < 2^n\}.$$

We now prove Proposition 3.4.1.

PROOF OF PROPOSITION 3.4.1. Let $\chi \in \mathrm{Irr}_{2'}(G \wr S_{2^n})$ be such that

$$\chi = \widetilde{\psi}^{\times 2^n} \operatorname{Inf}_{S_{2^n}}^{G \wr S_{2^n}} \chi^{(2^n-k,1^k)},$$

where $\psi \in \mathrm{Irr}(G)$ and $0 \leq k < 2^n$. Then the unique degree-one constituent of $\chi \downarrow_{\Delta(G) \times P_{2^n}}$ equals

$$\widetilde{\psi}^{\times 2^n} \downarrow_{\Delta(G)} \times \Phi_n(\chi^{(2^n-k,1^k)}), \tag{3.2}$$

which we denote by $\Theta(\chi)$. Let $\chi_1, \chi_2 \in \mathrm{Irr}_{2'}(G \wr S_{2^n})$ be such that

$$\chi_1 = \widetilde{\psi_1}^{\times 2^n} \operatorname{Inf}_{S_{2^n}}^{G \wr S_{2^n}} \chi^{(2^n-k_1,1^{k_1})},$$
$$\chi_2 = \widetilde{\psi_2}^{\times 2^n} \operatorname{Inf}_{S_{2^n}}^{G \wr S_{2^n}} \chi^{(2^n-k_2,1^{k_2})},$$

where $\psi_1, \psi_2 \in \mathrm{Irr}(G)$, and $0 \leq k_1, k_2 < 2^n$. It follows from (3.2) that if $\chi_1\downarrow_{\Delta(G)\times P_{2^n}} = \chi_2\downarrow_{\Delta(G)\times P_{2^n}}$, then $k_1 = k_2$ and

$$\widetilde{\psi_1}^{\times 2^n}\downarrow_{\Delta(G)} = \widetilde{\psi_2}^{\times 2^n}\downarrow_{\Delta(G)}.$$

Given $g \in G$, we have that

$$\widetilde{\psi_i}^{\times 2^n}(g,\ldots,g;1) = {\psi_i}^{2^n}(g).$$

As $G$ is abelian, $|\mathrm{Lin}(G)| = |G|$. Moreover $p$ is odd, and so $2^n$ does not divide $|\mathrm{Lin}(G)|$. Therefore the map $\psi \mapsto \psi^{2^n}$ is a bijection. It follows that $\psi_1^{\times 2^n} \neq \psi_2^{\times 2^n}$ implies that $\psi_1 \neq \psi_2$, and so $\Theta$ is injective. As $\mathrm{Irr}_{2'}(G \wr S_{2^n}) = 2^n|G| = \mathrm{Irr}_{2'}(G \times P_{2^n})$,

$$\mathrm{Irr}_{2'}(N_{G\wr S_{2^n}}(P_{2^n})) = \{\Theta(\chi) : \chi \in \mathrm{Irr}_{2'}(G \wr S_{2^n})\},$$

and the map $\Theta$ is a bijection. $\square$

### 3.5. The case $C_2^a \wr S_{2^n}$

Given $a, n \in \mathbf{N}$, counting cardinalities shows that $C_2^a \wr P_{2^n}$ is a Sylow 2-subgroup of $C_2^a \wr S_{2^n}$. We begin with the following lemma, which shows that $N_{C_2^a\wr S_{2^n}}(C_2^a \wr P_{2^n}) = C_2^a \wr P_{2^n}$.

Lemma 3.5.1. *The subgroup $C_2^a \wr P_{2^n}$ is self-normalising in $C_2^a \wr S_{2^n}$.*

Proof. We prove that $N_{C_2^a\wr S_{2^n}}(C_2^a \wr P_{2^n}) \leq C_2^a \wr P_{2^n}$, as the reverse containment is obvious. By definition of the multiplication in the imprimitive wreath product, it is sufficient to prove that if $(1;\tau) \in S_{2^n}$ normalises $C_2^a \wr P_{2^n}$, then $(1;\tau) \in P_{2^n}$. Given $(x;\sigma) \in C_2^a \wr P_{2^n}$, suppose that

$$(1;\tau^{-1})(x;\sigma)(1;\tau) \in C_2^a \wr P_{2^n}.$$

Then $\tau^{-1}\sigma\tau \in P_{2^n}$. This argument holds for all elements in $C_2^a \wr P_{2^n}$, and so $\tau$ normalises $P_{2^n}$. As $P_{2^n}$ is self-normalising in $S_{2^n}$, it follows that $(1;\tau) \in P_{2^n}$, as required. $\square$

The main result in this section is the following proposition.

Proposition 3.5.2. *Let $\chi \in \mathrm{Irr}_{2'}(C_2^a \wr S_{2^n})$. Then $\chi\downarrow^{C_2^a\wr S_{2^n}}_{C_2^a\wr P_{2^n}}$ has a unique degree-one constituent, denoted $\Theta(\chi)$. Furthermore*

$$\mathrm{Irr}_{2'}(C_2^a \wr P_{2^n}) = \{\Theta(\chi) : \chi \in \mathrm{Irr}_{2'}(C_2^a \wr S_{2^n})\},$$

*and the map $\chi \mapsto \Theta(\chi)$ is a bijection.*

As $C_2^a$ is an abelian group, it follows from Remark 3.1.5 that

$$\mathrm{Irr}_{2'}(C_2^a \wr S_{2^n}) = \{\widetilde{\psi}^{\times 2^n}\,\mathrm{Inf}_{S_{2^n}}^{C_2^a\wr S_{2^n}}\chi^{(2^n-k,1^k)} : \psi \in \mathrm{Irr}(C_2^a) \text{ and } 0 \leq k < 2^n\}.$$

As in §3.2 to count $\mathrm{Lin}(C_2^a \wr P_{2^n})$ we describe the group $(C_2^a \wr P_{2^n})/(C_2^a \wr P_{2^n})'$. We do this by proving the following more general result.

LEMMA 3.5.3. *Let $H$ be an abelian group, and let $G \leq S_n$ be transitive. Then*

$$(H \wr G)/(H \wr G)' \cong H \times G/G'.$$

PROOF. Define

$$K = \langle {}^g(1,\dots,h,\dots,1)(1,\dots,h,\dots,1)^{-1} : h \in H, g \in G\rangle.$$

By assumption $H$ is abelian, and so Lemma 3.2.3 gives that

$$(H \wr G)/(H \wr G)' \cong H^n/K \times G/G'.$$

It is therefore sufficient to prove that the quotient group $H^n/K$ is isomorphic to $H$. As $G$ is assumed to be transitive, we have that

$$K = \langle (1,\dots,h,\dots,h^{-1},\dots,1) : h \in H\rangle.$$

As $H$ is abelian, the group $K$ is equal to the subgroup of $H^n$ generated by all elements $(h_1,\dots,h_n)$ such that $h_1 \dots h_n = 1$. Moreover, the set $\{(h,1,\dots,1) : h \in H\}$ is a complete set of coset representatives for $K$ in $H^n$. Indeed the element $(x_1,\dots,x_n) \in H^n$ is contained in $(h,1,\dots,1)K$ if and only if $x_1 \dots x_n = h$. The map $(h,1,\dots,1)K \mapsto h$ now gives the required isomorphism. $\square$

It follows Proposition 3.2.2 and Lemma 3.5.3 that

$$(C_2^a \wr P_{2^n})/(C_2^a \wr P_{2^n})' \cong C_2^a \times C_2^n \cong C_2^{a+n}.$$

We are now ready to prove Proposition 3.5.2.

PROOF OF PROPOSITION 3.5.2. Let $\chi \in \mathrm{Irr}_{2'}(C_2^a \wr S_{2^n})$ be of the form

$$\chi = \widetilde{\psi}^{\times 2^n} \operatorname{Inf}_{S_{2^n}}^{C_2^a \wr S_{2^n}} \chi^{(2^n-k,1^k)},$$

where $\psi \in \mathrm{Irr}(C_2^a)$, and $0 \leq k < 2^n$. To simplify the notation, we write $\gamma$ for $\chi\big\downarrow^{C_2^a \wr S_{2^n}}_{C_2^a \wr P_{2^n}}$. Since $C_2^a$ is abelian,

$$\widetilde{\psi}^{\times 2^n} \operatorname{Inf}_{P_{2^n}}^{C_2^a \wr P_{2^n}} \Phi_n(\chi^{(2^n-k,1^k)})$$

is the unique degree-one constituent of $\gamma$, which we denote by $\Theta(\chi)$. By Theorem 1.2.6 and using that $\Phi_n$ is a bijection

$$\widetilde{\psi_1}^{\times 2^n} \operatorname{Inf}_{P_{2^n}}^{C_2^a \wr P_{2^n}} \Phi_n(\chi^{(2^n-k,1^k)}) = \widetilde{\psi_2}^{\times 2^n} \operatorname{Inf}_{P_{2^n}}^{C_2^a \wr P_{2^n}} \Phi_n(\chi^{(2^n-l,1^l)})$$

if and only if $\psi_1 = \psi_2$ and $k = l$. Therefore the map $\Theta$ is injective. By the sentence immediately after the proof of Lemma 3.5.3 $|\operatorname{Lin}(C_2^a \wr P_{2^n})| = 2^{a+n}$. This proves that

$$\mathrm{Irr}_{2'}(C_{2^a} \wr P_{2^n}) = \{\Theta(\chi) : \chi \in \mathrm{Irr}_{2'}(C_{2^a} \wr S_{2^n})\},$$

and that the map $\Theta$ is a bijection. $\square$

### 3.6. Low degree constituents of $\chi\downarrow_{P_{2^n}}^{S_{2^n}}$.

Given $n \in \mathbf{N}$ and $0 \leq k < 2^n$, let $\chi_n^k$ denote the $S_{2^n}$-character $\chi^{(2^n-k,1^k)}$. Define $\alpha(k,j,n)$ to be the number of irreducible constituents of degree $2^j$ appearing in the restricted character $\chi_n^k\downarrow_{P_{2^n}}$. The bijection $\Phi_n$ between $\mathrm{Irr}_{2'}(S_{2^n})$ and $\mathrm{Irr}_{2'}(P_{2^n})$ shows that $\alpha(k,0,n) = 1$ for all $k,n \in \mathbf{N}$. In this section we provide explicit formulas $\alpha(k,j,n)$, where $0 < j \leq 2$. Using Frobenius reciprocity and Clifford theory, our approach to determining $\alpha(k,j,n)$ is by studying the restriction of $\chi_n^k$ to small index subgroups of $P_{2^n}$.

In order to verify our formulas, we will refer to Tables 1 and 2 on the following page. Table 1 gives the values of $\alpha(k,j,n)$ for $S_8$, with $k$ labelling the rows and $j$ labelling the columns. The entries in the table have been computed using MAGMA ([**4**]). The analogous table for $S_{16}$ is Table 2. Due to the need to consider symmetric groups of exponentially increasing degree, we do not provide the analogous tables for $S_{32}$ onwards. Observe that the partition $(k+1,1^{2^n-k-1})$ is the conjugate partition of $(2^n-k,1^k)$. By the discussion following Theorem 1.1.2, it is therefore sufficient to determine $\alpha(k,j,n)$ for $k \leq 2^{n-1}-1$.

Write $D_n$ for $S_{2^{n-1}} \times S_{2^{n-1}}$, and recall that $Q_n = P_{2^{n-1}} \times {}^{\sigma_n}P_{2^{n-1}}$. Essential to the proofs of the results in this section is Equation (3.3). We remark that this equation follows from Corollary 1.1.10 of Theorem 1.1.4:

$$\chi_n^k\downarrow_{D_n} = \sum_{i=0}^{k}\left(\chi_{n-1}^i \times \chi_{n-1}^{k-i}\right) + \sum_{i=0}^{k-1}\left(\chi_{n-1}^i \times \chi_{n-1}^{k-1-i}\right). \tag{3.3}$$

Our starting point is Lemma 3.6.1 below, which determines $\alpha(k,1,n)$. The proof of this result was communicated to the author during personal communication with Eugenio Giannelli.

Using Theorem 1.2.6 observe that any two-degree irreducible constituent of $\chi_n^k\downarrow_{P_{2^n}}$ is of the form

$$\psi_{\alpha,\beta} := (\alpha \times \beta)\uparrow_{Q_n}^{P_{2^n}},$$

| $\alpha(k,j,3)$ | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 1 | 0 | 0 |
| 1 | 1 | 1 | 1 |
| 2 | 1 | 2 | 4 |
| 3 | 1 | 3 | 7 |

TABLE 1

| $\alpha(k,j,4)$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 1 | 1 | 1 | 0 | 0 |
| 2 | 1 | 2 | 5 | 6 | 2 | 0 |
| 3 | 1 | 3 | 12 | 18 | 12 | 2 |
| 4 | 1 | 4 | 19 | 36 | 36 | 13 |
| 5 | 1 | 5 | 24 | 54 | 72 | 41 |
| 6 | 1 | 6 | 28 | 66 | 114 | 79 |
| 7 | 1 | 7 | 31 | 71 | 148 | 105 |

TABLE 2

where $\alpha, \beta \in \mathrm{Irr}(P_{2^{n-1}})$ are such that $\alpha \neq \beta$ and $\alpha(1) = \beta(1) = 1$.

LEMMA 3.6.1. *Given $n \in \mathbf{N}$ such that $n \geq 2$, fix $0 \leq k \leq 2^{n-1} - 1$. Then $\alpha(k, 1, n) = k$.*

PROOF. Using Frobenius reciprocity observe that $\psi_{\alpha,\beta}$ is a constituent of $\chi_n^k \downarrow_{P_{2^n}}$ if and only if $\alpha \times \beta + \beta \times \alpha$ is a constituent of $\chi_n^k \downarrow_{Q_n}$ . It is therefore sufficient to count the number of constituents of $\chi_n^k \downarrow_{Q_n}$ of the form $\alpha \times \beta + \beta \times \alpha$, where $\alpha \neq \beta$ and $\alpha(1) = \beta(1) = 1$. As $\alpha \times \beta$ is a constituent of $\left(\chi_{n-1}^i \times \chi_{n-1}^{k-i}\right) \downarrow_{Q_n}$ if and only if $\beta \times \alpha$ is a constituent of $(\chi_{n-1}^{k-i} \times \chi_{n-1}^i)\downarrow_{Q_n}$, it suffices to count the number of constituents of

$$\left(\sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} (\chi_{n-1}^i \times \chi_{n-1}^{k-i}) + \sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} (\chi_{n-1}^i \times \chi_{n-1}^{k-1-i})\right) \downarrow_{Q_n}$$

of the form $\alpha \times \beta$ such that $\alpha \neq \beta$ and $\alpha(1) = \beta(1) = 1$.

We now distinguish two cases, determined by $k$.

*Case* (1). If $k$ is even, then the number of constituents of the required form equals the number of degree-one constituents of

$$\left(\sum_{i=0}^{\frac{k}{2}-1} (\chi_{n-1}^i \times \chi_{n-1}^{k-i}) + \sum_{i=0}^{\frac{k}{2}-1} (\chi_{n-1}^i \times \chi_{n-1}^{k-1-i})\right) \downarrow_{Q_n}.$$

Observe that $\Phi_{n-1}(\chi_{n-1}^i) \neq \Phi_{n-1}(\chi_{n-1}^{k-i})$ for each $i$ in the first summation since $\Phi_{n-1}$ is a bijection and $i \neq k - i$. The analogous statement is true for the second summation, and so the number of required constituents equals $k$.

*Case* (2). If $k$ is odd, then we argue in a similar way by counting the number of degree-one constituents of the required form appearing in

$$\left(\sum_{i=0}^{\frac{k-1}{2}} (\chi_{n-1}^i \times \chi_{n-1}^{k-i}) + \sum_{i=0}^{\frac{k-1}{2}-1} (\chi_{n-1}^i \times \chi_{n-1}^{k-1-i})\right) \downarrow_{Q_n}. \qquad \square$$

We now state the following proposition, which determines $\alpha(k, 2, n)$ when $k \leq 2^{n-1} - 1$.

PROPOSITION 3.6.2. *Given $n \in \mathbf{N}$ such that $n \geq 4$, let $0 \leq k \leq 2^{n-1} - 1$. Then $\alpha(k, 2, n)$ equals*

*(1)* $k^2 + \lfloor \frac{k}{2} \rfloor + 2\lfloor \frac{\lceil \frac{k}{2} \rceil}{2} \rfloor$, *if* $k \leq 2^{n-2} - 1$

*(2)* $2^{2n-4} + 2^{n-1}\ell - (\ell + 1)^2 + \lfloor \frac{k}{2} \rfloor + 2\lfloor \frac{\lceil \frac{k}{2} \rceil}{2} \rfloor$, *whenever* $k = 2^{n-2} + \ell$ *for some* $0 \leq \ell < 2^{n-2}$.

In order to prove Proposition 3.6.2, we require Lemma 3.6.3 below. Observe that the proof of Lemma 3.6.1 and the bijection $\Phi_{n-1}$ show that any two-degree irreducible constituent of $\chi_n^k \downarrow_{P_{2^n}}$ appears with multiplicity one.

Then if $\psi$ is a two-degree irreducible constituent of both $\chi_n^k\!\downarrow_{P_{2^n}}$ and $\chi_n^l\!\downarrow_{P_{2^n}}$, then we say that $\chi_n^k$ and $\chi_n^l$ *have* $\psi$ *in common.*

LEMMA 3.6.3. *Let* $1 \le l < k \le 2^{n-1}$. *Then the characters* $\chi_n^k\!\downarrow_{P_{2^n}}$ *and* $\chi_n^l\!\downarrow_{P_{2^n}}$ *have a two-degree irreducible constituent in common if and only if* $k-l=1$. *Moreover,* $\chi_n^{k-1}\!\downarrow_{P_{2^n}}$ *and* $\chi_n^k\!\downarrow_{P_{2^n}}$ *have exactly* $\lfloor \frac{k}{2} \rfloor$ *two-degree irreducible constituents in common.*

PROOF. Suppose that $\psi_{\alpha,\beta}$ is a constituent of both $\chi_n^k\!\downarrow_{P_{2^n}}$ and $\chi_n^l\!\downarrow_{P_{2^n}}$. Then by Frobenius reciprocity $\alpha \times \beta$ is a constituent of both $\chi_n^k\!\downarrow_{Q_n}$ and $\chi_n^l\!\downarrow_{Q_n}$. Recall that $D_n = S_{2^{n-1}} \times S_{2^{n-1}}$. As $\Phi_{n-1}$ is a bijection, we can define $\chi_{n-1}^a = \Phi_{n-1}^{-1}(\alpha)$ and $\chi_{n-1}^b = \Phi_{n-1}^{-1}(\beta)$, where $0 \le a, b < 2^{n-1}$. It follows from the transitivity of restriction that the $D_n$-character $\chi_{n-1}^a \times \chi_{n-1}^b$ is a constituent of both $\chi_n^k\!\downarrow_{D_n}$ and $\chi_n^l\!\downarrow_{D_n}$. By considering (3.3) we see that $\chi_{n-1}^a \times \chi_{n-1}^b$ is a constituent of $\chi_n^k\!\downarrow_{D_n}$ if and only if either $a+b=k$, or $a+b=k-1$. The same argument holds for $\chi_n^l$, and so either $a+b=l$ or $a+b=l-1$. By assumption $k>l$, and so it is necessarily the case that $k-1=a+b=l$. In particular $l=k-1$, which proves the first statement of the lemma.

We now prove the second statement of the lemma. As $1 \le k \le 2^{n-1}-1$, the sequence $(2^{n-1}-(k-1), 1^{k-1})$ is a partition of $2^{n-1}$. The irreducible $D_n$-characters appearing in both $\chi_n^k\!\downarrow_{D_n}$ and $\chi_n^{k-1}\!\downarrow_{D_n}$ are precisely the summands of

$$\sum_{i=0}^{k-1} \left(\chi_{n-1}^i \times \chi_{n-1}^{k-1-i}\right).$$

Every constituent of $(\chi_{n-1}^i \times \chi_{n-1}^{k-1-i} + \chi_{n-1}^{k-1-i} \times \chi_{n-1}^i)\!\downarrow_{Q_n}$ of the form $\alpha \times \beta + \beta \times \alpha$, where $\alpha \ne \beta$ and $\alpha(1) = \beta(1) = 1$, corresponds to a two-degree irreducible constituent appearing in both of $\chi_n^k\!\downarrow_{P_{2^n}}$ and $\chi_n^{k-1}\!\downarrow_{P_{2^n}}$. Equivalently each degree-one constituent of

$$\left( \sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} \left(\chi_{n-1}^i \times \chi_{n-1}^{k-1-i}\right) \right) \downarrow_{Q_n} \tag{3.4}$$

of the form $\alpha \times \beta$, where $\alpha \ne \beta$ and $\alpha(1) = \beta(1) = 1$, corresponds to a two-degree irreducible constituent appearing in both of $\chi_n^k\!\downarrow_{P_{2^n}}$ and $\chi_n^{k-1}\!\downarrow_{P_{2^n}}$. It therefore remains to count the number of degree-one constituents in (3.4). In order to do this, we distinguish two cases.

*Case* (1). Suppose that $k$ is even. Then the sum in (3.4) becomes

$$\left( \left(\chi_{n-1}^0 \times \chi_{n-1}^{k-1}\right) + \cdots + \left(\chi_{n-1}^{\frac{k}{2}-1} \times \chi_{n-1}^{\frac{k}{2}}\right) \right) \downarrow_{Q_n}.$$

In this case there are $\frac{k}{2}$ degree-one constituents of the required form.

*Case* (2). Suppose that $k$ is odd. Then the sum in (3.4) becomes

$$\left( (\chi^0_{n-1} \times \chi^{k-1}_{n-1}) + \cdots + (\chi^{\frac{k-1}{2}-1}_{n-1} \times \chi^{\frac{k-1}{2}+1}_{n-1}) + (\chi^{\frac{k-1}{2}}_{n-1} \times \chi^{\frac{k-1}{2}}_{n-1}) \right) \downarrow_{Q_n}.$$

In this case there are $\frac{k-1}{2}$ degree-one constituents of the required form. $\square$

Proof of Proposition 3.6.2. We begin by considering more closely the four-degree irreducible characters of $P_{2^n}$, each of which has exactly one of the following forms:

(1) $(\vartheta \times \lambda)\big\uparrow^{P_{2^n}}_{Q_n}$
(2) $\widetilde{\mu}^{\times 2}\,\mathrm{Inf}^{P_{2^n}}_{C_2}\,\rho$,

where either $\vartheta, \lambda \in \mathrm{Irr}(P_{2^{n-1}})$ are such that $\vartheta(1) = 2$ and $\lambda(1) = 1$, or $\mu \in \mathrm{Irr}(P_{2^{n-1}})$ is such that $\mu(1) = 2$ and $\rho \in \mathrm{Irr}(C_2)$.

If $(\vartheta \times \lambda)\uparrow^{P_{2^n}}_{Q_n}$ is a four-degree irreducible constituent of $\chi^k_n \downarrow_{P_{2^n}}$, then $\vartheta \times \lambda + \lambda \times \vartheta$ is a constituent of $\chi^k_n \downarrow_{Q_n}$ . We therefore count the number of constituents of the form $\vartheta \times \lambda$ in

$$\left( \sum_{i=0}^{k} \chi^i_{n-1} \times \chi^{k-i}_{n-1} + \sum_{i=0}^{k-1} \chi^i_{n-1} \times \chi^{k-1-i}_{n-1} \right) \downarrow_{Q_n}, \tag{3.5}$$

where $\vartheta(1) = 2$ and $\lambda(1) = 1$. Given $1 \leq i \leq k$, as each of $\chi^{k-i}_{n-1} \downarrow_{P_{2^{n-1}}}$ and $\chi^{k-1-i}_{n-1} \downarrow_{P_{2^{n-1}}}$ has a unique degree-one constituent, it is sufficient to count the number of two-degree irreducible constituents appearing in $\chi^i_{n-1} \downarrow_{P_{2^{n-1}}}$ .

If $\widetilde{\psi}^{\times 2}\,\mathrm{Inf}^{P_{2^n}}_{C_2}\,\rho$ is a four-degree irreducible constituent of $\chi^k_n \downarrow_{P_{2^n}}$, then $\psi \times \psi$ is a constituent of $\chi^k_n \downarrow_{Q_n}$ . We are therefore also required to count the number of constituents of this form that appear in (3.5).

First suppose that $k \leq 2^{n-2} - 1$, as in the first case of the proposition. Consider the constituents in (3.5) of the form $\vartheta \times \lambda$, where $\vartheta(1) = 2$ and $\lambda(1) = 1$. As $k \leq 2^{n-2} - 1$, we have that $\alpha(i, 1, n-1) = i$ for all $0 \leq i \leq k$. The number of constituents of the required form in (3.5) therefore equals

$$\sum_{i=1}^{k} i + \sum_{i=1}^{k-1} i = k^2. \tag{3.6}$$

We now count the number of constituents in (3.5) of the form $\psi \times \psi$ such that $\psi(1) = 2$. As $k \leq 2^{n-1} - 1$, we have that $\frac{k}{2} < 2^{n-2} - 1$. We now distinguish two cases, determined by $k$. If $k$ is even, Lemma 3.6.1 states that

$$(\chi^{\frac{k}{2}}_{n-1} \times \chi^{\frac{k}{2}}_{n-1})\big\downarrow_{Q_n}$$

has exactly $\frac{k}{2}$ constituents of the required form. Also by Lemma 3.6.3, the characters

$$(\chi^{\frac{k}{2}-1}_{n-1} \times \chi^{\frac{k}{2}}_{n-1})\big\downarrow_{Q_n} \text{ and } (\chi^{\frac{k}{2}}_{n-1} \times \chi^{\frac{k}{2}-1}_{n-1})\big\downarrow_{Q_n}$$

each have exactly $\lfloor \frac{(\frac{k}{2})}{2} \rfloor$ irreducible constituents of the form $\psi \times \psi$ such that $\psi(1) = 2$. Summing these values with (3.6) gives the result in this case.

Similarly if $k$ is odd, then Lemma 3.6.3 gives that

$$(\chi_{n-1}^{\frac{k-1}{2}} \times \chi_{n-1}^{\frac{k-1}{2}+1})\downarrow_{Q_n} \text{ and } (\chi_{n-1}^{\frac{k-1}{2}+1} \times \chi_{n-1}^{\frac{k-1}{2}})\downarrow_{Q_n}$$

each have exactly $\lfloor\frac{(\frac{k+1}{2})}{2}\rfloor$ constituents of the form $\psi\times\psi$ such that $\psi(1)=2$. Moreover, Lemma 3.6.1 states that

$$(\chi_{n-1}^{\frac{k-1}{2}} \times \chi_{n-1}^{\frac{k-1}{2}})\downarrow_{Q_n}$$

has exactly $\frac{k-1}{2}$ constituents of this form. Once more summing these values with (3.6) gives the result in this case.

Now let $k = 2^{n-2}+\ell$ for some $\ell \in \mathbf{N}_0$. Then $\chi_n^{2^{n-2}+\ell}\downarrow_{D_n}$ equals

$$\begin{aligned}(3.7)\qquad &\sum_{i=0}^{2^{n-2}-1}\big(\chi_{n-1}^{i}\times\chi_{n-1}^{2^{n-2}+\ell-i}\big) + \sum_{i=0}^{2^{n-2}-1}\big(\chi_{n-1}^{i}\times\chi_{n-1}^{2^{n-2}+\ell-1-i}\big)\\ &+\sum_{i=2^{n-2}}^{2^{n-2}+\ell}\big(\chi_{n-1}^{i}\times\chi_{n-1}^{2^{n-2}+\ell-i}\big) + \sum_{i=2^{n-2}}^{2^{n-2}+\ell-1}\big(\chi_{n-1}^{i}\times\chi_{n-1}^{2^{n-2}+\ell-1-i}\big).\end{aligned}$$

Consider the constituents $\vartheta\times\lambda$, where $\vartheta(1)=2$ and $\lambda(1)=1$, in (3.7). As $\alpha(2^{n-2}+t,1,n-1) = 2^{n-2}-1-t$ for each $0\le t\le 2^{n-2}-1$, the number of constituents of this form appearing in (3.7) is equal to

$$\begin{aligned}&(2^{n-2}-1)(2^{n-2}) + 2\sum_{i=0}^{\ell-1}(2^{n-2}-1-i) + (2^{n-2}-1-\ell)\\ &= (2^{n-2}-1)(2^{n-2}) + 2^{n-1}\ell - \ell(\ell+1) + 2^{n-2} - (\ell+1)\\ (3.8)\qquad &= 2^{2n-4} + 2^{n-1}\ell - (\ell+1)^2.\end{aligned}$$

The same argument as in the case when $k \le 2^{n-1}-1$ shows that the number of constituents in (3.7) of the form $\psi\times\psi$ such that $\psi(1)=2$ equals

$$\lfloor\frac{k}{2}\rfloor + 2\lfloor\frac{\lceil\frac{k}{2}\rceil}{2}\rfloor.$$

Summing this with (3.8) completes the proof in this case. $\square$

The following example verifies the second case of the proposition against the values in Table 2 for $k\in\{4,5,6,7\}$.

EXAMPLE 3.6.4. Let $n=4$.

(1) Let $k=4$, and so $\ell=0$. Then

$$\alpha(4,2,4) = 16+8(0)-1+2+2(1) = 19.$$

(2) Let $k=5$, and so $\ell=1$. Then

$$\alpha(5,2,4) = 16+8(1)-4+2+2(1) = 24.$$

(3) Let $k=6$, and so $\ell=2$. Then

$$\alpha(6,2,4) = 16+8(2)-9+3+2(1) = 28.$$

(4) Let $k = 7$, and so $\ell = 3$. Then

$$\alpha(7,2,4) = 16 + 8(3) - 16 + 3 + 2(2) = 31.$$

We have seen that the two-degree irreducible constituents of $\chi_n^k{\downarrow}_{P_{2^n}}$ each appear with multiplicity one. The following example shows that this is generally not the case for the irreducible constituents of $\chi_n^k{\downarrow}_{P_{2^n}}$ of degree at least four.

EXAMPLE 3.6.5. As in Example 3.2.7, we write $\tau : C_2 \to \{\pm 1\}$ for the non-trivial irreducible character of $C_2$. Let $\lambda = (6,1^2) \vdash 8$. By (3.3)

$$\begin{aligned}\chi^{(6,1^2)}{\big\downarrow}_{S_4\times S_4} &= \big(\chi^{(4)}\times\chi^{(2,1^2)}\big) + \big(\chi^{(3,1)}\times\chi^{(3,1)}\big) + \big(\chi^{(2,1^2)}\times\chi^{(4)}\big)\\ &\quad + \big(\chi^{(4)}\times\chi^{(3,1)}\big) + \big(\chi^{(3,1)}\times\chi^{(4)}\big).\end{aligned}$$

The only two-degree irreducible character of $P_4$ is $(1\times\tau){\uparrow}_{Q_4}^{P_4}$, which occurs with multiplicity one in both $\chi^{(3,1)}{\downarrow}_{P_4}$ and $\chi^{(2,1^2)}{\downarrow}_{P_4}$. Then

$$1_{P_4}\times(1\times\tau)\big\uparrow_{Q_4}^{P_4}$$

is a constituent of $(\chi^{(4)}\times\chi^{(3,1)}){\downarrow}_{P_4}$ and $(\chi^{(4)}\times\chi^{(2,1^2)}){\downarrow}_{P_4}$, with multiplicity one in each case. It follows that the four-degree irreducible $P_8$-character

$$\big(1_{P_4}\times(1\times\tau)\big\uparrow_{Q_4}^{P_4}\big)\big\uparrow_{Q_8}^{P_8}$$

appears with multiplicity two in $\chi^{(6,1^2)}{\downarrow}_{P_8}$.

We remark that using the results in this section, it is possible to determine a formula for $\alpha(k,3,n)$. The key observation is that any 8-degree irreducible character of $P_{2^n}$ has exactly one of the following forms:

(1) $(\vartheta\times\lambda)\big\uparrow_{Q_n}^{P_{2^n}}$
(2) $(\xi\times\psi)\big\uparrow_{Q_n}^{P_{2^n}}$

where either $\vartheta(1) = 4$ and $\lambda(1) = 1$, or $\xi(1) = \psi(1) = 2$ and $\xi \neq \psi$. We can count the constituents of the first form using Proposition 3.6.2. Similarly we can count the constituents of the second form by considering the irreducible constituents of

$$\left(\sum_{i=0}^{\lfloor\frac{k}{2}\rfloor}\big(\chi_{n-1}^i\times\chi_{n-1}^{k-i}\big) + \sum_{i=0}^{\lfloor\frac{k-1}{2}\rfloor}\big(\chi_{n-1}^i\times\chi_{n-1}^{k-1-i}\big)\right){\Big\downarrow}_{Q_n}$$

of the form $\xi\times\psi$ such that $\xi(1) = \psi(1) = 2$ and $\xi\neq\psi$. This, in turn, can be done by counting the number of irreducible constituents of the form $\xi\times\psi$ such that $\xi(1) = \psi(1) = 2$ using Lemma 3.6.1, and taking into account that each constituent of the form $(\xi\times\psi){\uparrow}_{Q_n}^{P_{2^n}}$, where $\xi(1) = \psi(1) = 2$ and $\xi\neq\psi$, appearing in either $\chi_{n-1}^{k/2}\times\chi_{n-1}^{k/2}$ if $k$ is even, or $\chi_{n-1}^{k-1/2}\times\chi_{n-1}^{k-1/2}$ if $k$ is odd, is counted twice. We then subtract the number of constituents of the form $\xi\times\xi$ such that $\xi(1) = 2$ using Lemma 3.6.3.

A natural question to ask is whether it is possible determine $\alpha(k,j,n)$ for $j \geq 4$. As an example, in order to count the irreducible constituents of $\chi{\downarrow_{P_{2^n}}}$ of degree sixteen, we need to determine when $\chi^k_{n-1}{\downarrow_{P_{2^{n-1}}}}$ and $\chi^l_{n-1}{\downarrow_{P_{2^{n-1}}}}$, where $0 \leq k, l < 2^{n-1}$, have a four-degree constituent in common. Moreover, when such a common constituent arises, we require its multiplicity in each of $\chi^k_{n-1}{\downarrow_{P_{2^{n-1}}}}$ and $\chi^l_{n-1}{\downarrow_{P_{2^{n-1}}}}$ . As shown in Proposition 3.6.2, this number will not be a polynomial function in $k$ due to the different formulas for $0 \leq k \leq 2^{n-2} - 1$ and $2^{n-2} \leq k \leq 2^{n-1} - 1$. Also, due to the differences in these two cases, $\alpha(k,4,n)$ depends on four cases for $0 \leq k \leq 2^{n-1} - 1$, namely: $0 \leq k \leq 2^{n-3}-1$, $2^{n-3} \leq k \leq 2^{n-2}-1$, $2^{n-2} \leq k \leq 2^{n-3}+2^{n-2}-1$, and $2^{n-3} + 2^{n-2} \leq k \leq 2^{n-1} - 1$. Furthermore, the formulas for $\alpha(k,j,n)$ that we have given so far depend heavily on each $j$, and so do not appear to generalise to a formula for $\alpha(k,j,n)$ for arbitrary $j$.

CHAPTER 4

# Endomorphism algebras of two-row permutation modules

Fix $n \in \mathbf{N}$, and let $F$ be a field of characteristic $p > 0$. We consider the structure of the $FS_n$-permutation module $M^\lambda$ (defined in §1.1.1), where $\lambda$ is a partition of $n$ with at most two parts. In this case write $\lambda = (\lambda_1, \lambda_2)$, and so $M^{(\lambda_1,\lambda_2)}$ corresponds to the action of $S_n$ on the cosets of the maximal Young subgroup $S_{(\lambda_1,\lambda_2)}$.

The Krull–Schmidt Theorem states that $M^{(\lambda_1,\lambda_2)}$ has a direct sum decomposition into indecomposable $FS_n$-modules, and that these indecomposable summands are unique up to isomorphism. A natural problem to therefore consider is to express $M^{(\lambda_1,\lambda_2)}$ as a direct sum of its indecomposable summands. We will see in §4.1 below that in this special case such a decomposition is unique. Moreover, in the case when $p$ does not divide $n!$, the decomposition of $M^{(\lambda_1,\lambda_2)}$ as a direct sum of its irreducible submodules is known. However when $p$ divides $n!$, expressing $M^{(\lambda_1,\lambda_2)}$ as the direct sum of its indecomposable summands remains a notoriously difficult open problem. A complete solution to this problem was given by Doty, Erdmann and Henke in [**15**] when $p = 2$, and in this chapter we give a complete solution when $p = 3$. We remark that some of our methods for constructing the indecomposable summands of $M^{(\lambda_1,\lambda_2)}$ over a field of characteristic 3 are based on ideas from [**15**]. In §4.1.1 we make clear those ideas that are from [**15**], and those that are new.

## 4.1. Indecomposable summands and endomorphism algebras

Although it is difficult in general to express an $FS_n$-module as a direct sum of its indecomposable summands, we have partial information on the summands of $M^\lambda$, where $\lambda$ is any partition of $n$. Indeed let

$$M^\lambda = \bigoplus_{i=1}^{m} Y_i$$

be a fixed direct sum decomposition of $M^\lambda$ such that each $Y_i$ is indecomposable. It follows from James' Submodule theorem (see Theorem 1.3.20) that there is a unique $Y_i$ containing the Specht module $S^\lambda$. Moreover, it is known (see [**18**, Theorem 1]) that $Y_i$ is unique up to isomorphism. We write $Y^\lambda$ for this summand, and we refer to this module as the *Young module* labelled by $\lambda$. Recall that $\trianglerighteq$ denotes the dominance order for partitions. It is also known ([**18**, Theorem 1]) that $M^\lambda$ is in general isomorphic to a direct sum of Young modules $Y^\mu$ such that $\mu \trianglerighteq \lambda$. We can therefore write

$$M^\lambda \cong Y^\lambda \oplus \bigoplus_{\mu \rhd \lambda} [M^\lambda : Y^\mu] Y^\mu,$$

where $[M^\lambda : Y^\mu]$ denotes the number of indecomposable summands of $M^\lambda$ that are isomorphic to $Y^\mu$. We refer to the multiplicity $[M^\lambda : Y^\mu]$ as a *$p$-Kostka number.*

Although a complete characterisation of the $p$-Kostka numbers appears to be out of reach, they are completely understood when $\lambda$ has at most two parts. Indeed let $\mu = (\mu_1, \mu_2)$ be a partition of $n$ such that $\mu \trianglerighteq \lambda$. Define $m = \lambda_1 - \lambda_2$ and $g = \lambda_2 - \mu_2$. Observe that $m \geq 0$ as $\lambda$ is a partition, and $g \geq 0$ as $\mu \trianglerighteq \lambda$. Henke proved in [**29**] is that the $p$-Kostka number $[M^{(\lambda_1,\lambda_2)} : Y^{(\mu_1,\mu_2)}]$ is non-zero if and only if the binomial coefficient

$$B(m, g) := \binom{m+2g}{g}$$

is non-zero modulo $p$. By Lemma 1.3.18 this is the case if and only if the $p$-ary addition of $m + g$ and $g$ is carry free. Henke's result is proved using a result of Donkin [**13**, (3.6)] based on Klyachko's multiplicity formula [**38**, Corollary 9.2]. In the case that $Y^{(\mu_1,\mu_2)}$ is a summand of $M^{(\lambda_1,\lambda_2)}$, Henke also proved that the corresponding $p$-Kostka number equals one [**29**, Lemma 3.2].

Let $A$ denote the endomorphism algebra of an $FG$-module $M$, and let $e$ be a primitive idempotent in $A$. Recall that $eM$ is an indecomposable summand of $M$, and that every indecomposable summand of $M$ arises in this way. Therefore in this chapter the central object of study is the endomorphism algebra $S_F((\lambda_1, \lambda_2)) := \mathrm{End}_{FS_n}(M^{(\lambda_1,\lambda_2)})$, where $F$ is a field of characteristic 3. In particular we construct a complete set of primitive idempotents in $S_F((\lambda_1, \lambda_2))$.

We now give the presentation of $S_F((\lambda_1, \lambda_2))$ that we use throughout this chapter, which holds over any field. Given $r \in \mathbf{N}$, fix an $r$-dimensional $F$-vector space $E$ with basis $\{v_1, \ldots, v_r\}$. Form the $n$-fold tensor product $E^{\otimes n}$, on which $S_n$ acts by place permutation. We extend this action linearly to the group algebra $FS_n$, and we define the *Schur algebra*

$$S_F(r, n) = \mathrm{End}_{FS_n}(E^{\otimes n}).$$

Instead of using the tabloid construction of $M^\lambda$ given in §1.1.1, we describe a submodule of $E^{\otimes n}$ that is isomorphic to $M^\lambda$. Define

$$I(r, n) = \{(i_1, \ldots, i_n) : i_j \in \{1, 2, \ldots, r\} \text{ for all } j\}.$$

Given a composition $\lambda$ of $n$, we say that $(i_1, \ldots, i_n) \in I(r, n)$ has *weight* $\lambda$ if

$$|\{j : i_j = k\}| = \lambda_k,$$

for all $1 \leq k \leq \ell(\lambda)$. For instance, the elements in $I(2,3)$ of weight $(2,1)$ are

$$(1,1,2), (1,2,1) \text{ and } (2,1,1).$$

Then $M^\lambda$ is isomorphic to the $F$-span of the set

$$\{v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_n} : (i_1, i_2, \ldots, i_n) \text{ has weight } \lambda\}.$$

Observe that this defines $M^\lambda$ when $\lambda$ is a composition of $n$. Moreover, there is a decomposition of $FS_n$-modules

$$E^{\otimes n} = \bigoplus_{\lambda \in \Lambda(r,n)} M^\lambda,$$

where, as usual, $\Lambda(r,n)$ denotes the set of compositions of $n$ with at most $r$ parts.

We work with partitions of $n$ with at most two parts, and so we fix $r = 2$ throughout the rest of this chapter. The main result in [**17**] is an explicit presentation of $S_{\mathbf{Q}}(2,n)$ as a quotient of the universal enveloping algebra $U(\mathfrak{gl}_2)$. This result can be used to give an explicit presentation of the endomorphism algebra $S_F((\lambda_1, \lambda_2))$, which we now describe. Following the notation in [**15**] and [**17**], define $e = e_{21}, f = e_{12}, H_1 = e_{11}$, and $H_2 = e_{22}$, where $e_{ij}$ is the standard matrix unit in the Lie algebra $\mathfrak{gl}_2$. As in [**17**, 3.4], given $\ell \in \mathbf{N}_0$ and an element $T$ in an associative $\mathbf{Q}$-algebra with 1, define

$$T^{(\ell)} = \frac{T^\ell}{\ell!} \text{ and } \binom{T}{\ell} = \frac{T(T-1)\ldots(T-\ell+1)}{\ell!}.$$

Then given $\lambda = (\lambda_1, \lambda_2) \in \Lambda(2,n)$, define

$$1_\lambda = \binom{H_1}{\lambda_1}\binom{H_2}{\lambda_2}.$$

It is proved in [**14**, Lemma 5.3] that $1_\lambda$ is an idempotent in $S_{\mathbf{Q}}(2,n)$, and that $1_\lambda E^{\otimes n} = M^\lambda$. Given $i \in \mathbf{N}_0$, we define

$$b(i) = 1_\lambda f^{(i)} e^{(i)} 1_\lambda.$$

The following lemma completely describes $S_F(\lambda)$ as an associative $F$-algebra. We remark that this lemma is an equivalent restatement of Proposition 3.6 in [**15**], chosen to make it obvious that $S_F(\lambda)$ is commutative.

LEMMA 4.1.1. [**15**, Proposition 3.6] *Given $n \in \mathbf{N}$, let $\lambda = (\lambda_1, \lambda_2) \vdash n$, and define $m = \lambda_1 - \lambda_2$. Then $S_F(\lambda)$ has an $F$-basis given by the set*

$$\{b(i) : 0 \leq i \leq \lambda_2\}.$$

*Moreover, the multiplication of the basis elements is given by the formula*

$$b(i)b(j) = \sum_{h=\max\{i,j\}}^{i+j} \binom{h}{i}\binom{h}{j}\binom{m+i+j}{i+j-h} b(h),$$

*where we set $b(a) = 0$ if $a > \lambda_2$.*

We refer to the basis given in this lemma as the *canonical basis* of $S_F(\lambda)$. We also make some remarks regarding this lemma. The presentation of the Schur algebra in [**17**] is over the field $\mathbf{Q}$. Nevertheless $b(i)$ is well-defined over a field of positive characteristic $p$. Moreover, the structure constants given in Lemma 4.1.1 are integers. Therefore the above multiplication formula holds over a field of characteristic $p$ by reducing the coefficients modulo $p$. Furthermore, the $\mathbf{Q}S_n$-module $M^{(\lambda_1,\lambda_2)}$ is multiplicity free, and so $S_{\mathbf{Q}}((\lambda_1,\lambda_2))$ is a commutative algebra. This implies that $S_F((\lambda_1,\lambda_2))$ is also a commutative algebra, and so its primitive idempotents are unique. Therefore the direct sum decomposition of $M^{(\lambda_1,\lambda_2)}$ into its indecomposable summands is unique, as claimed in the introduction of this chapter. Also a direct computation using the multiplication formula shows that $b(0)$ is the identity in $S_F(\lambda)$, and we write $\mathbf{1}$ for $b(0)$.

We also have the following useful lemma from [**15**], which provides an easy formula for calculating certain products in $S_F((\lambda_1,\lambda_2))$.

LEMMA 4.1.2. [**15**, Lemma 3.7] *Let $p$ be a prime number, and let $i \in \mathbf{N}$ be such that $i$ has $p$-adic expansion $i = i_0p^0 + i_1p^1 + \cdots$. Then $b(i) = \prod_{t\geq 0} b(i_t \cdot p^t)$.*

As a final remark, we note that Lemma 4.1.1 is an example of the various connections between the representation theories of the symmetric group and the general linear group via the Schur algebra. For further details, we refer the reader to [**27**] and [**48**].

**4.1.1. Main results.** The first main result in this chapter is Theorem 4.1.3, which constructs the central primitive idempotents in $S_F(\lambda)$ when $F$ is a field of characteristic 3. Our second main result is Theorem 4.1.4, which determines the Young modules that the primitive idempotents constructed in Theorem 4.1.3 correspond to. This completes the construction of the Young modules $Y^{(\mu_1,\mu_2)}$ over a field of characteristic 3.

We now state the ideas from [**15**] that we use to prove our main results. The basis and corresponding multiplication formula of $S_F(\lambda)$ given above is from [**15**]. Our construction of the primitive idempotents in $S_F(\lambda)$ uses the same idea as [**15**] of giving a correspondence between particular elements of $S_F(\lambda)$ and the binomial coefficients $\binom{a}{b}$ such that $0 \leq b \leq a < p$. The number of binomial coefficients of this form clearly increases with $p$, and so it seems difficult to determine such a correspondence for fields of characteristic $p \geq 5$. It is remarked in [**15**, §1] that explicitly constructing the primitive idempotents appears difficult even when $p = 3$. By proving our main results, we completely solve the problem in this case. We also note that the argument used to prove that the idempotents we construct are primitive is based on the counting argument in [**15**, §2.4]. Moreover, the proof of Theorem 4.1.4

is taken directly from the proof of Theorem 7.1 in [**15**]. We repeat the proof of [**15**, Theorem 7.1] here in order to make this chapter more self-contained.

We now describe where our ideas differ to those in [**15**]. We have seen in Lemma 4.1.1 that the multiplication structure of $S_F(\lambda)$ depends only on $m$, whereas our construction of the primitive idempotents depends on $B(m,g)$. We are therefore required to determine the critical parameter $m$ given $m+2g$ and $g$. An important observation in [**15**] is that if $g$ has binary expansion $g = \sum_{i\geq 0} g_i 2^i$, then $2g$ has binary expansion $2g = \sum_{i\geq 1} g_{i-1}2^i$. Furthermore, the proof of the Idempotent Theorem in [**15**] uses that the sum of any two idempotents is an idempotent over a field of characteristic 2. These observations only hold when $p=2$, and so we take a different approach when proving the analogous results in our case (see §4.4 and §4.5).

Throughout the rest of this section, we assume that $F$ is a field of characteristic 3. We now define the elements $e_{m,g} \in S_F(\lambda)$, which are the subject of Theorem 4.1.3. Let $m, g \in \mathbf{N}_0$ be such that $B(m,g)$ is non-zero modulo 3. Define the index sets

$$\begin{aligned}
I^{(0)}_{m,g} &= \{u : g_u = 0 \text{ and } (m+2g)_u = 0\}\\
J^{(0)}_{m,g} &= \{u : g_u = 1 \text{ and } (m+2g)_u = 2\}\\
I^{(1)}_{m,g} &= \{u : g_u = 0 \text{ and } (m+2g)_u = 1\}\\
J^{(1)}_{m,g} &= \{u : g_u = 2 \text{ and } (m+2g)_u = 2\}\\
I^{(2)}_{m,g} &= \{u : g_u = 0 \text{ and } (m+2g)_u = 2\}\\
J^{(2)}_{m,g} &= \{u : g_u = 1 \text{ and } (m+2g)_u = 1\}.
\end{aligned}$$

The chosen notation for these index sets may not seem intuitive upon first reading, but the results in §4.4 will make this clear.

Define

$$\begin{aligned}
e_{m,g} = &\prod_{u\in I^{(0)}_{m,g}} \mathbf{1} + b(3^u) - b(2\cdot 3^u) \cdot \prod_{u\in J^{(0)}_{m,g}} b(2\cdot 3^u) - b(3^u)\\
&\cdot \prod_{u\in I^{(1)}_{m,g}} \mathbf{1} - b(2\cdot 3^u) \cdot \prod_{u\in J^{(1)}_{m,g}} b(2\cdot 3^u)\\
&\cdot \prod_{u\in I^{(2)}_{m,g}} \mathbf{1} - b(3^u) + b(2\cdot 3^u) \cdot \prod_{u\in J^{(2)}_{m,g}} b(3^u) - b(2\cdot 3^u).
\end{aligned}$$

As stated in Lemma 4.1.1, if $b(a)$ in this product is such that $a > \lambda_2$, then we set $b(a) = 0$. We give an example of $e_{m,g}$ in §4.2. Given $t \in \mathbf{N}_0$, define $(e_{m,g})_{\leq t}$ by taking the products defining $e_{m,g}$ over the $u$ in each index set such that $u \leq t$. Also define $(e_{m,g})_{<t}$ in the analogous way.

We are now ready to state our main theorems, which we do overleaf.

THEOREM 4.1.3. *Given $n \in \mathbf{N}$, let $\lambda = (\lambda_1, \lambda_2) \vdash n$ and $m = \lambda_1 - \lambda_2$. The set of elements $e_{m,g}$, with $B(m, g)$ non-zero modulo 3 and $g \leq \lambda_2$, give a complete set of primitive orthogonal idempotents for $S_F(\lambda)$.*

THEOREM 4.1.4. *Let $\lambda = (\lambda_1, \lambda_2)$ and $\mu = (\mu_1, \mu_2)$ be partitions of $n$ such that $Y^\mu$ is a direct summand of $M^\lambda$. Define*

$$m = \lambda_1 - \lambda_2 \text{ and } g = \lambda_2 - \mu_2.$$

*Then $e_{m,g}$ is the primitive idempotent in $S_F(\lambda)$ such that $e_{m,g}M^\lambda \cong Y^\mu$.*

We provide an outline for the remainder of this chapter.

**Outline.** In §4.2 we introduce some notational conventions that we use. We also explicitly define the elements $e_{m,g}$ using the 3-adic expansion (see §1.3.5 or §4.2) of $B(m, g)$.

In §4.3 we consider more closely the multiplication structure of $S_F(\lambda)$. In particular we define the element $\psi_{m,u} \in S_F(\lambda)$, where $u \in \mathbf{N}_0$. The product of $(e_{m,g})_{<u}$ (defined on the previous page) and $\psi_{m,u}$ is fundamental in the proof of Theorem 4.1.3.

We have seen in Lemma 4.1.1 that the critical parameter in the multiplication formula for $\mathrm{End}_{FS_n}(M^{(\lambda_1,\lambda_2)})$ is $m := \lambda_1 - \lambda_2$. In §4.4 we therefore relate the 3-adic expansion of $B(m, g)$ to the 3-adic expansion of $m$. We see that this depends on the carries in the ternary addition of $m$ and $g$.

In §4.5 we prove Theorem 4.1.3. We prove Proposition 4.5.1, which states that the elements $(e_{m,g})_{\leq u}$ are idempotents for all $u \in \mathbf{N}_0$. Before we prove Proposition 4.5.1, we show how it implies that the elements $e_{m,g}$ are idempotent in $S_F(\lambda)$. The proof of Proposition 4.5.1 is by induction on $u$. We give the base case of this induction in §4.5.1, and we complete the inductive step in §4.5.2. In §4.5.3 we show that the elements $e_{m,g}$ are mutually orthogonal. A simple counting argument then shows that these elements give a complete set of primitive orthogonal idempotents in $S_F(\lambda)$, thereby completing the proof of Theorem 4.1.3.

In §4.5.4 we consider an application of Theorem 4.1.3. In particular we prove that, over a field of characteristic 3, the module $M^{(\lambda_1,\lambda_2)}$ is indecomposable if and only if either $(\lambda_1, \lambda_2) = (n, 0)$, or $(\lambda_1, \lambda_2) = (n - 1, 1)$ and 3 divides $n$.

In §4.6 we prove Theorem 4.1.4. Following the exposition in [**15**], the proof of the theorem is by induction on $n$. Observe that $m$ and $g$ are invariant under adding the partition $(1^2)$ to both $\lambda$ and $\mu$. In the inductive step we therefore prove that if $e_{m,g}M^\lambda \cong Y^\mu$, then $e_{m,g}M^{\lambda+(1^2)} \cong Y^{\mu+(1^2)}$. We remark that this is an algebraic realisation of the column removal phenomenon for the decomposition matrices of symmetric groups proved by James (see [**34**]).

## 4.2. Primitive idempotents and Lucas' Theorem

Let $p$ be a prime number. Given $c \in \mathbf{N}_0$ with $p$-adic expansion $c = \sum_{u=0}^{t} c_u p^u$, we write $c =_p [c_0, c_1, \ldots, c_t]$. Given $s \in \mathbf{N}$, we also write $c_{<s}$ for $\sum_{u=0}^{s-1} c_u p^u$. Given $d =_p [d_0, d_1, \ldots, d_t]$, Lucas' Theorem (Lemma 1.3.17) states that

$$\binom{c}{d} \equiv \prod_{u=0}^{t} \binom{c_u}{d_u} \mod p.$$

Recall that we refer to the factorisation on the right hand side as the $p$-adic expansion of $\binom{c}{d}$. In this chapter we define *factor* $u$ in the $p$-adic expansion of $\binom{c}{d}$ as the binomial coefficient $\binom{c_u}{d_u}$, and we write $\binom{c}{d}_u$ for $\binom{c_u}{d_u}$ for all $0 \le u \le t$. Given $m, g \in \mathbf{N}_0$, we write $B(m,g)_p$ for the $p$-adic expansion of $B(m,g)$.

Recall from Lemma 4.1.1 that $S_F(\lambda)$ has an $F$-basis equal to

$$\{b(i) : 0 \le i \le \lambda_2\},$$

and also recall that $\mathbf{1}$ denotes $b(0) = 1_{S_F(\lambda)}$. Define the order $\le$ on the $b(i)$ by $b(i) \le b(j)$ if and only if $i \le j$.

We remark that we can define $e_{m,g}$ by assigning elements in $S_F(\lambda)$ to all possible factors of $B(m,g)_3$, and then multiplying these elements of $S_F(\lambda)$ according to the factors of $B(m,g)_3$ (see Example 4.2.1 below). The assignment is as follows:

$$\begin{aligned}
\binom{0}{0}_u &\leftrightarrow \mathbf{1} + b(3^u) - b(2\cdot 3^u) & \qquad \binom{2}{1}_u &\leftrightarrow b(2\cdot 3^u) - b(3^u)\\
\binom{1}{0}_u &\leftrightarrow \mathbf{1} - b(2\cdot 3^u) & \qquad \binom{2}{2}_u &\leftrightarrow b(2\cdot 3^u)\\
\binom{2}{0}_u &\leftrightarrow \mathbf{1} - b(3^u) + b(2\cdot 3^u) & \qquad \binom{1}{1}_u &\leftrightarrow b(3^u) - b(2\cdot 3^u),
\end{aligned}$$

and assigning zero to any other factor of $B(m,g)_3$. We define *factor* $u$ of $e_{m,g}$ as the factor of $e_{m,g}$ corresponding to factor $u$ of $B(m,g)_3$. The factors not shown in the above display are precisely those $\binom{c}{d}$ with $0 \le c < d < 3$. If $B(m,g)_3$ has such a factor, then $B(m,g) \equiv 0$ modulo 3. Therefore $e_{m,g}$ is defined and equal to 0 even when it does not correspond to a summand of $M^\lambda$.

EXAMPLE 4.2.1. Let $\lambda = (36, 13)$, and let $\mu = (49, 0)$. Then $m = 23$, $g = 13$, and

$$B(23,13)_3 = \binom{1}{1}\binom{1}{1}\binom{2}{1}\binom{1}{0}\binom{0}{0}\binom{0}{0}\cdots.$$

Therefore $e_{23,13}$ equals

$$(b(1) - b(2))(b(3) - b(6))(b(18) - b(9))(\mathbf{1} - b(54))(\mathbf{1} + b(81) - b(162))\ldots.$$

As $b(a) = 0$ for $a > 13$ in $S_F((36,13))$, only finitely many factors in this infinite product are not equal to $\mathbf{1}$. Then by Lemma 4.1.2

$$\begin{aligned} e_{23,13} &= (b(1) - b(2))(b(3) - b(6))(-b(9)) \\ &= -b(13) + b(14) + b(16) - b(17) \\ &= -b(13) \end{aligned}$$

in $S_F((36,13))$.

## 4.3. Multiplication in $S_F(\lambda)$

Throughout this section fix $m \in \mathbf{N}_0$, and fix a partition $\lambda = (\lambda_1, \lambda_2)$ such that $m = \lambda_1 - \lambda_2$. Observe that factor $u$ of $e_{m,g}$ can be expressed in terms of the elements

$$b(2 \cdot 3^u) - b(3^u) \text{ and } b(2 \cdot 3^u), \tag{4.1}$$

where $u \in \mathbf{N}_0$. In the proof of Theorem 4.1.3, we show that $(e_{m,g})^2_{\leq u} = (e_{m,g})_{\leq u}$. To this end we need to determine the squares of the elements in (4.1). In this section we therefore assume that $\lambda_2 \geq 2 \cdot 3^u$, and we determine the products $b(3^u)^2, b(2 \cdot 3^u)^2$, and $b(3^u)b(2 \cdot 3^u)$ using Lemma 4.1.1.

Definition. Given $u \in \mathbf{N}$, define

$$\psi_{m,u} = \sum_{k=1}^{3^u - 1} \binom{m_{<u}}{3^u - k} b(k).$$

Also define $\psi_{m,0} = 0$.

We remark that our motivation for defining $\psi_{m,u}$ is twofold. The immediate reason is that we can express the products $b(3^u)^2, b(2 \cdot 3^u)^2$, and $b(3^u)b(2 \cdot 3^u)$ in terms of $\psi_{m,u}$. Also, as stated in the outline, the product of $\psi_{m,u}$ with $(e_{m,g})_{<u}$ is fundamental in the proof of Theorem 4.1.3.

Consider first $b(3^u)^2$. Lemma 4.1.1 gives

$$b(3^u)^2 = \sum_{h=3^u}^{2 \cdot 3^u} \binom{h}{3^u}^2 \binom{m + 2 \cdot 3^u}{2 \cdot 3^u - h} b(h).$$

A direct computation using this formula shows that the coefficient of $b(3^u)$ in $b(3^u)^2$ equals $\binom{m_u + 2}{1}$, and that the coefficient of $b(2 \cdot 3^u)$ always equals 1. Also observe that in this sum if $3^u < h < 2 \cdot 3^u$, then we can write $h = 3^u + k$, where $0 < k < 3^u$. Then by Lucas' Theorem, for all such $h$ we have

$$\binom{m + 2 \cdot 3^u}{2 \cdot 3^u - h} \equiv_3 \binom{m_{<u}}{3^u - k} \binom{m_u + 2}{0} \equiv_3 \binom{m_{<u}}{3^u - k},$$

and so using Lemma 4.1.2 we can write

$$b(3^u)^2 = b(3^u) \left[ \binom{m_u + 2}{1} + \psi_{m,u} \right] + b(2 \cdot 3^u). \tag{4.2}$$

Consider now

$$b(2 \cdot 3^u)^2 = \sum_{h=2\cdot 3^u}^{4\cdot 3^u} \binom{h}{2 \cdot 3^u}^2 \binom{m+3^u+3^{u+1}}{4 \cdot 3^u - h} b(h).$$

Observe that if $h \geq 3^{u+1}$ in this sum, then the ternary addition of $2 \cdot 3^u$ and $h - 2 \cdot 3^u$ is not carry free. It follows from Lemma 1.3.18 that $\binom{h}{2\cdot 3^u} \equiv_3 0$. Arguing similarly as above, the coefficient of $b(2 \cdot 3^u)$ in $b(2 \cdot 3^u)^2$ equals $\binom{m_u+1}{2}$. Moreover, if $2 \cdot 3^u < k < 3^{u+1}$, then we can write $h = 2 \cdot 3^u + k$, where $0 < k < 3^u$. Again by Lucas' Theorem, for all such $h$

$$\binom{m+3^u+3^{u+1}}{4 \cdot 3^u - h} = \binom{m+3^u+3^{u+1}}{3^u+3^u-k} \equiv_3 \binom{m_{<u}}{3^u-k}\binom{m_u+1}{1}.$$

Using Lemma 4.1.2 once more we obtain

$$b(2 \cdot 3^u)^2 = b(2 \cdot 3^u)\left[\binom{m_u+1}{2} + \binom{m_u+1}{1}\psi_{m,u}\right]. \tag{4.3}$$

An entirely similar argument gives

$$b(3^u)b(2 \cdot 3^u) = b(2 \cdot 3^u)\left[2\binom{m_u}{1} - \psi_{m,u}\right]. \tag{4.4}$$

If $j$ is maximal such that $b(j)$ appears with non-zero coefficient in one of $b(3^u)^2, b(3^u)b(2 \cdot 3^u)$, or $b(2 \cdot 3^u)^2$, then (4.2), (4.3) and (4.4) show that $j < 3^{u+1}$. We therefore have the following lemma, which will be used in the inductive step of the proof of Proposition 4.5.1.

LEMMA 4.3.1. *Let $u \in \mathbf{N}$ be such that $2 \cdot 3^u \leq \lambda_2$. Then the $F$-span of the set*

$$\{b(k) : k < 3^u\}$$

*is a subalgebra of $S_F(\lambda)$.*

We end this section with the following lemma, which determines when $e_{m,g}$ is non-zero in $S_F(\lambda)$. We remark that the first statement of the lemma can be observed in Example 4.2.1.

LEMMA 4.3.2. *Let $g \in \mathbf{N}_0$ be such that $B(m,g)$ is non-zero modulo 3. Then*

$$e_{m,g} = B(m,g)b(g) + \sum_{i>g} \alpha_i b(i),$$

*for some $\alpha_i \in \mathbf{F}_3$. In particular, $e_{m,g}$ is non-zero in $S_F(\lambda)$ if and only if $g \leq \lambda_2$.*

PROOF. Write $e_{m,g}$ as a linear combination of the canonical basis of $S_F(\lambda)$ given in Lemma 4.1.1. As the index sets defining $e_{m,g}$ are mutually disjoint, Lemma 4.1.2 implies that the smallest term in $e_{m,g}$ is the product of the smallest term in each factor (see §4.2) of $e_{m,g}$. By the construction of $e_{m,g}$ immediately before Lemma 4.1.2, the smallest term in factor $u$ of

$e_{m,g}$ is $b(g_u 3^u)$ with coefficient $\binom{(m+2g)_u}{g_u}$. It follows that the smallest term in $e_{m,g}$ is $\prod_u b(g_u 3^u) = b(g)$ with coefficient $\prod_u \binom{(m+2g)_u}{g_u} \equiv_3 B(m,g)$.

The second statement of the lemma now follows as the largest element in the canonical basis of $S_F(\lambda)$ is $b(\lambda_2)$. $\square$

## 4.4. Analysis of the binomial coefficient $B(m,g)$

Fix a prime number $p$, and let $m, g \in \mathbf{N}_0$ be such that $B(m,g)$ is non-zero modulo $p$. In this section we use the $p$-adic expansion of $B(m,g)$ to understand $m$. We do this using the $p$-ary addition of $m$ and $g$. Before doing this we consider Example 4.4.2 below, which demonstrates the link between $B(m,g)$ and $m$ that occurs in the general case.

For the convenience of the reader, we redefine the carry notation introduced in §1.3.5 in terms of $m$ and $g$. Consider the following representation of the $p$-ary addition of $m$ and $g$:

$$\begin{array}{r|ccccc} m & m_0 & m_1 & \dots & m_u & \dots \\ g & g_0 & g_1 & \dots & g_u & \dots \\ \hline m+g & (m+g)_0 & (m+g)_1 & \dots & (m+g)_u & \dots \end{array},$$

where $m =_p [m_0, m_1, \dots]$, and the analogous statements hold for $g$ and $m+g$. Given $u \in \mathbf{N}_0$, define $x_u \in \{0, 1, 2, \dots, p-1\}$ to be such that

$$m_u + g_u + x_{u-1} = (m+g)_u + p x_u, \tag{4.5}$$

so that $x_u$ is the carry *leaving* column $u$ in this addition. Therefore for all $u \in \mathbf{N}$, $x_{u-1}$ is the carry *entering* column $u$ in this addition. We also define $x_{-1} = 0$.

REMARK 4.4.1. The carries $x_u$ serve two purposes in this chapter. The first, as we will see in this section, is that we can determine $m_u$ using $x_{u-1}$. The second is that the product $(e_{m,g})_{<u}\psi_{m,u}$ can be determined entirely by the carry $x_{u-1}$ (see Lemma 4.5.3). We admit that it remains mysterious to us as to why this product depends only on $x_{u-1}$.

EXAMPLE 4.4.2. Let $\mu \in \mathbf{N}_0$ and $\nu \in \mathbf{N}$ be such that $\nu > \mu$. Let $h \in \mathbf{N}$ be such that $h < p^\mu$ and $\binom{2h}{h}$ is non-zero modulo $p$.

We consider the case when $m = p^\mu$ and $g = p^\nu - p^\mu + h$. Then $x_u = 0$ for $0 \le u \le \mu - 1$, and $x_u = 1$ for $\mu \le u \le \nu - 1$.

Let $h_u$ be the digits in the $p$-adic expansion of $h$. The conditions on $h$ imply that $h_u \le \frac{p-1}{2}$ for all $u$, and $h_u = 0$ for $u \ge \mu$. Then $m + 2g = p^\nu + (p^\nu - p^\mu) + 2h$, and so the $p$-adic expansion of $\binom{m+2g}{g}$ equals

$$\binom{2h_0}{h_0}\binom{2h_1}{h_1}\cdots\binom{2h_{\mu-1}}{h_{\mu-1}}\binom{p-1}{p-1}\cdots\binom{p-1}{p-1}\binom{1}{0},$$

where the rightmost factor appearing is factor $\nu$.

Observe that if $u < \mu$ then $(m+2g)_u - 2g_u = 0 = m_u$. Similarly we have $(m+2g)_\mu - 2g_\mu \equiv_p 1 = m_\mu$. If $u > \mu$ then $(m+2g)_u - 2g_u \equiv_p 1 = m_u + 1$. In all cases we can therefore write

$$(m+2g)_u - 2g_u \equiv_p m_u + x_{u-1}.$$

The final statement in Example 4.4.2 follows from the more general property that $m_u$ is determined by factor $u$ in $B(m, g)_p$ via the carries in the $p$-ary addition of $m$ and $g$. We are able to determine each $m_u$ in this way as the $p$-ary addition of $m + g$ and $g$ is carry free. Motivated by this, we determine the possible values of the carry $x_u$.

LEMMA 4.4.3. *Suppose that in the $p$-ary addition of $m$ and $g$ the carry $x_u$ is non-zero for some $u \in \mathbf{N}_0$. Then $x_u = 1$.*

PROOF. We proceed by induction on $u$. The base case is when $u = 0$. In this case

$$m_0 + g_0 \leq p - 1 + p - 1 = 2p - 2 = p - 2 + p,$$

and so $x_0$ is at most 1.

Let $u > 1$, and assume inductively that $x_{u-1} \leq 1$. Suppose that $x_u \geq 2$. Then

$$2p - 1 \geq m_u + g_u + x_{u-1} = (m+g)_u + px_u \geq 2p,$$

which is a contradiction. □

In the following lemma, we determine the possibilities for $m_u$ given factor $u$ of $B(m, g)_p$.

LEMMA 4.4.4. *Let $a, b \in \mathbf{N}_0$ be such that $0 \leq b \leq a < p$, and let factor $u$ of $B(m, g)_p$ equal $\binom{a}{b}$. Let $z \in \{0, 1, \ldots, p-1\}$ be the unique integer such that $z \equiv_p a - 2b$. Then either $m_u \equiv_p z$ and $x_{u-1} = 0$, or $m_u \equiv_p z - 1$ and $x_{u-1} = 1$. Moreover, $x_u = 1$ if and only if $m_u + g_u + x_{u-1} \geq p$.*

PROOF. It follows from the definition of $B(m, g)_p$ that $(m+2g)_u = a$ and $g_u = b$. As $B(m, g)$ is non-zero modulo $p$, it follows from Lemma 1.3.18 that the $p$-ary addition of $m + g$ and $g$ is carry free. Therefore $(m + g)_u = a - b$, and so by definition of the carries

$$m_u + b + x_{u-1} = a - b + px_u \equiv_p a - b.$$

If $x_{u-1} = 0$, then $m_u \equiv_p a - 2b = z$. Similarly if $x_{u-1} = 1$, then $m_u \equiv_p z - 1$, as required.

The second statement is immediate by definition of the carry $x_u$ and Lemma 4.4.3. □

In particular Lemma 4.4.4 shows that $(m+2g)_u - 2g_u \equiv_p m_u + x_{u-1}$ for all $u \in \mathbf{N}_0$.

## 4.5. The primitive idempotents of $S_F(\lambda)$

Fix $m, g \in \mathbf{N}_0$ such that $B(m,g)$ is non-zero modulo 3, and let $\lambda = (\lambda_1, \lambda_2)$ be such that $m = \lambda_1 - \lambda_2$. Throughout the rest of this chapter, $F$ is assumed to be a field of characteristic 3. We prove the following proposition by filling in the details in the outline.

PROPOSITION 4.5.1. *Fix $u \in \mathbf{N}_0$. Then $(e_{m,g})_{\leq u}$ is an idempotent in $S_F((m + 3^{u+1} - 1, 3^{u+1} - 1))$.*

We remark that Proposition 4.5.1, together with Lemma 4.3.1, implies that $(e_{m,g})_{\leq u}$ is also idempotent in $S_F((m+a, a))$ for all $a \geq 3^{u+1}$.

We prove Proposition 4.5.1 by induction on $u$, in which the base case is $u = 0$. Before we do this, we show how the proposition implies that $e_{m,g}$ is an idempotent in $S_F(\lambda)$. Indeed, by Lemma 4.1.1, $S_F(\lambda)$ has a basis given by the set

$$\{b(i) : 0 \leq i \leq \lambda_2\}.$$

Let $u \in \mathbf{N}_0$ be such that $3^u \leq \lambda_2 < 3^{u+1}$. If $e_{m,g}$ is non-zero in $S_F(\lambda)$, then by our assumption on $B(m,g)$ and Lemma 4.3.2 we have $g \leq \lambda_2$. Therefore $g < 3^{u+1}$, and so by construction, $(e_{m,g})_{\leq u} = e_{m,g}$ when viewed as an element of $S_F(\lambda)$. As the multiplication structure of $S_F(\lambda)$ depends only on $m$, Proposition 4.5.1 gives

$$(e_{m,g})^2 = ((e_{m,g})_{\leq u})^2 = (e_{m,g})_{\leq u} = e_{m,g} \in S_F(\lambda),$$

as required

We now proceed with the proof of Proposition 4.5.1.

**4.5.1. The base case.** By definition $x_{-1} = 0$. In this case Lemma 4.4.4 states that factor 0 of $B(m,g)_3$ equals $\binom{a}{b}$, where $a - 2b \equiv_3 m_0$. We distinguish three cases, determined by $m_0$.

*Case* (1). Suppose that $m_0 = 0$. Then the only possibilities for factor 0 of $B(m,g)_3$ are

$$\binom{0}{0} \text{ or } \binom{2}{1}.$$

By definition $(e_{m,g})_{\leq 0}$ equals either $b(2) - b(1)$ or $1 - b(1) + b(2)$. It is sufficient to prove that $b(2) - b(1)$ is idempotent when $m_0 = 0$. Indeed (4.2), (4.3) and (4.4) applied with $u = 0$ and $m_0 = 0$ give

$$\begin{aligned}(b(2) - b(1))^2 &= b(2)^2 + b(1)b(2) + b(1)^2 \\ &= 0 + 0 + b(2) - b(1) = b(2) - b(1).\end{aligned}$$

*Case* (2). Suppose that $m_0 = 1$. Then the only possibilities for factor 0 of $B(m,g)_3$ are

$$\binom{1}{0} \text{ or } \binom{2}{2},$$

and so $(e_{m,g})_{\leq 0}$ equals either $b(2)$ or $1-b(2)$. Applying (4.3) with $u=0$ and $m_0=1$ shows that $b(2)$ is idempotent in this case.

*Case* (3). Suppose that $m_0=2$. Then the only possibilities for factor 0 of $B(m,g)_3$ are

$$\binom{2}{0} \text{ or } \binom{1}{1},$$

and so $(e_{m,g})_{\leq 0}$ equals either $b(1)-b(2)$ or $1-b(1)+b(2)$. Again (4.2), (4.3) and (4.4) applied with $u=0$ and $m_0=2$ give

$$\begin{aligned}(b(1)-b(2))^2 &= b(1)^2+b(1)b(2)+b(2)^2\\ &= b(1)+b(2)+b(2)+0 \equiv_3 b(1)-b(2),\end{aligned}$$

as required.

**4.5.2. The inductive step.** Throughout this section fix $u \in \mathbf{N}$. It follows from Lemma 4.3.1 that $((e_{m,g})_{\leq u})^2$ is contained in the $F$-span of

$$\{b(i) : i < 3^{u+1}\},$$

and so it is sufficient to prove that $(e_{m,g})_{\leq u}$ is an idempotent in $S_F((m+\lambda_2,\lambda_2))$, where $\lambda_2 < 3^{u+1}$.

Assume inductively that $(e_{m,g})_{\leq t}$ is an idempotent in $S_F(\lambda)$ for all $t<u$. We require the following lemmas.

LEMMA 4.5.2. *Let $t \in \mathbf{N}_0$ be such that $t<u$. Suppose that $v := (e_{m,g})_{\leq t}w$, is an idempotent in $S_F(\lambda)$. Then $vw=v$ and $v(\mathbf{1}-w)=0$.*

PROOF. We have assumed that $(e_{m,g})_{\leq t}$ is an idempotent in $S_F(\lambda)$, and so

$$vw = (e_{m,g})_{\leq t}w^2 = ((e_{m,g})_{\leq t})^2w^2 = v^2 = v,$$

as required. The proof that $v(\mathbf{1}-w)=0$ is entirely similar. □

Recall from §4.4 that $x_t$ denotes the carry leaving column $t$ in the ternary addition of $m$ and $g$, and that

$$\psi_{m,t} = \sum_{k=1}^{3^t-1}\binom{m_{<t}}{3^t-k}b(k),$$

for $t \in \mathbf{N}$ and $\psi_{m,0}=0$.

LEMMA 4.5.3. *Let $t \in \mathbf{N}_0$ be such that $t \leq u$. Then*

$$(e_{m,g})_{<t}\psi_{m,t} = \begin{cases} 0 & \text{if } x_{t-1}=0,\\ (e_{m,g})_{<t} & \text{if } x_{t-1}=1.\end{cases}$$

PROOF. We proceed by induction on $t$. The base case is when $t=0$, where the product defining $(e_{m,g})_{<0}$ is empty. Therefore $(e_{m,g})_{<0}=1$. By definition $x_{-1}=0$ and $\psi_{m,0}=0$, and so the result holds in this case.

Suppose now that $t \geq 1$ and that the result holds for all $s < t$. By Lemma 4.1.2 we can write

$$\begin{aligned}\psi_{m,t} &= \sum_{k=1}^{3^{t-1}-1} \binom{m_{<t}}{3^t - k} b(k) \\ &\quad + b(3^{t-1}) \left[ \binom{m_{t-1}}{2} + \sum_{k=1}^{3^{t-1}-1} \binom{m_{<t}}{3^t - (3^{t-1} + k)} b(k) \right] \\ &\quad + b(2 \cdot 3^{t-1}) \left[ \binom{m_{t-1}}{1} + \sum_{k=1}^{3^{t-1}-1} \binom{m_{<t}}{3^t - (2 \cdot 3^{t-1} + k)} b(k) \right].\end{aligned}$$

For $1 \leq k \leq 3^{t-1} - 1$, Lucas' Theorem implies that

$$\begin{aligned}\binom{m_{<t}}{3^t - k} &= \binom{m_{<t-1} + m_{t-1} \cdot 3^{t-1}}{3^{t-1} - k + 2 \cdot 3^{t-1}} \\ &\equiv_3 \binom{m_{<t-1}}{3^{t-1} - k} \binom{m_{t-1}}{2}.\end{aligned}$$

Applying entirely similar arguments for all $3^{t-1} \leq k \leq 3^t - 1$ shows that

$$\begin{aligned}\psi_{m,t} = \psi_{m,t-1} &\left[ \binom{m_{t-1}}{2} + \binom{m_{t-1}}{1} b(3^{t-1}) + \binom{m_{t-1}}{0} b(2 \cdot 3^{t-1}) \right] \\ &+ \binom{m_{t-1}}{2} b(3^{t-1}) + \binom{m_{t-1}}{1} b(2 \cdot 3^{t-1}).\end{aligned} \tag{4.6}$$

We now distinguish three cases, determined by $m_{t-1}$.

*Case* (1). Suppose that $m_{t-1} = 0$. Then (4.6) becomes

$$\psi_{m,t} = \psi_{m,t-1} b(2 \cdot 3^{t-1}).$$

If $x_{t-2} = 0$, then the first statement of Lemma 4.4.4 implies that factor $t - 1$ of $B(m, g)_3$ equals either $\binom{0}{0}$ or $\binom{2}{1}$. As $x_{t-2} = m_{t-1} = 0$, the second statement of Lemma 4.4.4 gives that $x_{t-1} = 0$. Moreover, the inductive hypothesis of this lemma gives

$$(e_{m,g})_{<t} \psi_{m,t} = (e_{m,g})_{<t-1} \psi_{m,t-1} b(2 \cdot 3^{t-1}) w = 0,$$

where $w$ equals either $\mathbf{1} + b(3^{t-1}) - b(2 \cdot 3^{t-1})$ if factor $t - 1$ equals $\binom{0}{0}$, or $b(2 \cdot 3^{t-1}) - b(3^{t-1})$ if factor $t - 1$ equals $\binom{2}{1}$. The result therefore holds in this case.

If $x_{t-2} = 1$, then the first statement of Lemma 4.4.4 implies that factor $t - 1$ of $B(m, g)_3$ equals either $\binom{1}{0}$ or $\binom{2}{2}$. By construction

$$(e_{m,g})_{<t} = (e_{m,g})_{<t-1} w,$$

where $w$ equals either $\mathbf{1} - b(2 \cdot 3^{t-1})$ if factor $t-1$ equals $\binom{1}{0}$, or $b(2 \cdot 3^{t-1})$ if factor $t-1$ equals $\binom{2}{2}$. Then

$$\begin{aligned}(e_{m,g})_{<t}\psi_{m,t} &= (e_{m,g})_{<t-1}w\psi_{m,t-1}b(2 \cdot 3^{t-1}) \\ &= (e_{m,g})_{<t-1}wb(2 \cdot 3^{t-1}),\end{aligned}$$

where the second equality holds by the inductive hypothesis of this lemma. If factor $t-1$ of $B(m,g)_3$ equals $\binom{1}{0}$, then the second statement of Lemma 4.4.4 applied with $m_{t-1} = 0, g_{t-1} = 0$, and $x_{t-2} = 1$ gives $x_{t-1} = 0$. Moreover, $w = \mathbf{1} - b(2 \cdot 3^{t-1})$ in this case, and so $(e_{m,g})_{<t}\psi_{m,t} = (e_{m,g})_{<t}(\mathbf{1} - w)$. As $v := (e_{m,g})_{<t} = (e_{m,g})_{<t-1}w$ is an idempotent by the inductive hypothesis of Proposition 4.5.1, it follows from Lemma 4.5.2 that

$$(e_{m,g})_{<t}\psi_{m,t} = v(\mathbf{1} - w) = 0.$$

If factor $t-1$ of $B(m,g)_3$ equals $\binom{2}{2}$, then the second statement of Lemma 4.4.4 now applied with $m_{t-1} = 0, g_{t-1} = 2$, and $x_{t-2} = 1$ gives $x_{t-1} = 1$. Moreover, $w = b(2 \cdot 3^{t-1})$ in this case, and so $(e_{m,g})_{<t}\psi_{m,t} = (e_{m,g})_{<t}w$. As $v := (e_{m,g})_{<t} = (e_{m,g})_{<t-1}w$ is an idempotent by the inductive hypothesis of Proposition 4.5.1, it follows from Lemma 4.5.2 that

$$(e_{m,g})_{<t}\psi_{m,t} = vw = v = (e_{m,g})_{<t}.$$

*Case* (2). Suppose that $m_{t-1} = 1$. Then (4.6) becomes

$$\psi_{m,t} = \psi_{m,t-1}(b(3^{t-1}) + b(2 \cdot 3^{t-1})) + b(2 \cdot 3^{t-1}).$$

If $x_{t-2} = 0$, then the first statement of Lemma 4.4.4 implies that factor $t-1$ of $B(m,g)_3$ equals either $\binom{1}{0}$ or $\binom{2}{2}$. Again by the construction of $e_{m,g}$

$$(e_{m,g})_{<t} = (e_{m,g})_{<t-1}w,$$

where $w$ equals either $\mathbf{1} - b(2 \cdot 3^{t-1})$ if factor $t-1$ equals $\binom{1}{0}$, or $b(2 \cdot 3^{t-1})$ if factor $t-1$ equals $\binom{2}{2}$. Moreover, the inductive hypothesis of this lemma implies that

$$(e_{m,g})_{<t}\psi_{m,t} = (e_{m,g})_{<t-1}b(2 \cdot 3^{t-1})w,$$

for both possibilities of $w$. The argument is now the same as when $x_{t-2} = 1$ in Case (1).

If $x_{t-2} = 1$, then the first statement of Lemma 4.4.4 implies that factor $t-1$ of $B(m,g)_3$ equals either $\binom{2}{0}$ or $\binom{1}{1}$. By construction

$$(e_{m,g})_{<t} = (e_{m,g})_{<t-1}w,$$

where $w$ equals either $\mathbf{1} - b(3^{t-1}) + b(2 \cdot 3^{t-1})$ if factor $t-1$ equals $\binom{2}{0}$, or $b(3^{t-1}) - b(2 \cdot 3^{t-1})$ if factor $t-1$ equals $\binom{1}{1}$. Then

$$\begin{aligned}(e_{m,g})_{<t}\psi_{m,t} &= (e_{m,g})_{<t-1}w(\psi_{m,t-1}(b(3^{t-1}) + b(2 \cdot 3^{t-1})) + b(2 \cdot 3^{t-1})) \\ &= (e_{m,g})_{<t-1}w(b(3^{t-1}) - b(2 \cdot 3^{t-1})),\end{aligned}$$

where the second equality holds by the inductive hypothesis of this lemma. If factor $t-1$ of $B(m,g)_3$ equals $\binom{2}{0}$, then the second statement of Lemma 4.4.4 applied with $m_{t-1}=1, g_{t-1}=0$, and $x_{t-2}=1$ gives $x_{t-1}=0$. Moreover, $w=\mathbf{1}-b(3^{t-1})+b(2\cdot 3^{t-1})$, and so $(e_{m,g})_{<t}\psi_{m,t}=(e_{m,g})_{<t}(\mathbf{1}-w)$. As $v:=(e_{m,g})_{<t}=(e_{m,g})_{<t-1}w$ is an idempotent by the inductive hypothesis of Proposition 4.5.1, it follows from Lemma 4.5.2 that

$$(e_{m,g})_{<t}\psi_{m,t}=v(\mathbf{1}-w)=0.$$

If factor $t-1$ of $B(m,g)_3$ equals $\binom{1}{1}$, then the second statement of Lemma 4.4.4 now applied with $m_{t-1}=1, g_{t-1}=1$, and $x_{t-2}=1$ gives $x_{t-1}=1$. Moreover, $w=b(3^{t-1})-b(2\cdot 3^{t-1})$ in this case. As $v:=(e_{m,g})_{<t}=(e_{m,g})_{<t-1}w$ is an idempotent by the inductive hypothesis of Proposition 4.5.1, it follows from Lemma 4.5.2 that

$$(e_{m,g})_{<t}\psi_{m,t}=vw=v=(e_{m,g})_{<t}.$$

*Case* (3). Suppose that $m_{t-1}=2$. Then (4.6) becomes

$$\psi_{m,t}=\psi_{m,t-1}(\mathbf{1}-b(3^{t-1})+b(2\cdot 3^{t-1}))+b(3^{t-1})-b(2\cdot 3^{t-1}).$$

If $x_{t-2}=0$, then the first statement of Lemma 4.4.4 implies that factor $t-1$ of $B(m,g)_3$ equals either $\binom{2}{0}$ or $\binom{1}{1}$. Again by the construction of $e_{m,g}$

$$(e_{m,g})_{<t}=(e_{m,g})_{<t-1}w,$$

where $w$ equals either $\mathbf{1}-b(3^{t-1})+b(2\cdot 3^{t-1})$ if factor $t-1$ equals $\binom{2}{0}$, or $b(3^{t-1})-b(2\cdot 3^{t-1})$ if factor $t-1$ equals $\binom{1}{1}$. The argument is now the same as when $x_{t-2}=1$ in Case (2).

If $x_{t-2}=1$, then the first statement of Lemma 4.4.4 implies that factor $t-1$ of $B(m,g)_3$ equals either $\binom{0}{0}$ or $\binom{2}{1}$. By construction

$$(e_{m,g})_{<t}=(e_{m,g})_{<t-1}w,$$

where $w$ equals either $\mathbf{1}+b(3^{t-1})-b(2\cdot 3^{t-1})$ if factor $t-1$ equals $\binom{0}{0}$, or $b(2\cdot 3^{t-1})-b(3^{t-1})$ if factor $t-1$ equals $\binom{2}{1}$. Then $(e_{m,g})_{<t}\psi_{m,t}$ equals

$$(e_{m,g})_{<t-1}w(\psi_{m,t-1}(\mathbf{1}-b(3^{t-1})+b(2\cdot 3^{t-1}))+b(3^{t-1})-b(2\cdot 3^{t-1})),$$

which by the inductive hypothesis of this lemma equals $(e_{m,g})_{<t-1}w$ for both possibilities of $w$. Therefore $(e_{m,g})_{<t}\psi_{m,t}=(e_{m,g})_{<t}$. As

$$m_{t-1}+x_{t-2}+g_{t-1}=3+g_{t-1}\geq 3,$$

it follows from the second statement of Lemma 4.4.4 that $x_{t-1}=1$ for both possible factors. The result therefore holds in this case. □

We now complete the inductive step of the proof of Proposition 4.5.1.

Proof of the inductive step. Assume that $3^u\leq\lambda_2\leq 2\cdot 3^u$. If $\lambda_2<2\cdot 3^u$, then in the following calculations we regard all terms equal to $b(2\cdot 3^u)$ as zero. We consider each possibility for factor $u$ of $B(m,g)_3$ in turn.

*Case* (1a). Suppose that factor $u$ of $B(m,g)_3$ equals $\binom{2}{1}$. By Lemma 4.4.4 either $m_u = 0$ and $x_{u-1} = 0$, or $m_u = 2$ and $x_{u-1} = 1$. By construction of $e_{m,g}$ and the inductive hypothesis

$$\begin{aligned}(e_{m,g})^2_{\leq u} &= ((e_{m,g})_{<u})^2(b(2\cdot 3^u) - b(3^u))^2\\ &= (e_{m,g})_{<u}(b(2\cdot 3^u)^2 + b(2\cdot 3^u)b(3^u) + b(3^u)^2)\\ &= (e_{m,g})_{<u}b(2\cdot 3^u)\left[\binom{m_u+1}{2} + \binom{m_u+1}{1}\psi_{m,u}\right]\\ &+ (e_{m,g})_{<u}b(2\cdot 3^u)\left[2\binom{m_u}{1} - \psi_{m,u}\right]\\ &+ (e_{m,g})_{<u}\left(b(3^u)\left[\binom{m_u+2}{1} + \psi_{m,u}\right] + b(2\cdot 3^u)\right),\end{aligned}$$

where the final equality holds by (4.2), (4.3) and (4.4). It is now sufficient to prove that

$$(e_{m,g})_{<u}\psi_{m,u} = \begin{cases} 0 & \text{if } m_u = 0 \text{ and } x_{u-1} = 0\\ (e_{m,g})_{<u} & \text{if } m_u = 2 \text{ and } x_{u-1} = 1.\end{cases}$$

This follows from Lemma 4.5.3.

*Case* (1b). Suppose that factor $u$ of $B(m,g)_3$ equals $\binom{0}{0}$. By Lemma 4.4.4 either $m_u = 0$ and $x_{u-1} = 0$, or $m_u = 2$ and $x_{u-1} = 1$. By construction of $e_{m,g}$ and the inductive hypothesis

$$\begin{aligned}(e_{m,g})^2_{\leq u} &= ((e_{m,g})_{<u})^2(\mathbf{1} + b(3^u) - b(2\cdot 3^u))^2\\ &= (e_{m,g})_{<u}(\mathbf{1}+b(3^u)^2+b(2\cdot 3^u)^2-b(3^u)+b(2\cdot 3^u)+b(2\cdot 3^u)b(3^u))\\ &= (e_{m,g})_{<u}(\mathbf{1} - b(3^u) + b(2\cdot 3^u))\\ &+ (e_{m,g})_{<u}b(2\cdot 3^u)\left[\binom{m_u+1}{2} + \binom{m_u+1}{1}\psi_{m,u}\right]\\ &+ (e_{m,g})_{<u}b(2\cdot 3^u)\left[2\binom{m_u}{1} - \psi_{m,u}\right]\\ &+ (e_{m,g})_{<u}\left(b(3^u)\left[\binom{m_u+2}{1} + \psi_{m,u}\right] + b(2\cdot 3^u)\right),\end{aligned}$$

where the final equality holds by (4.2), (4.3) and (4.4). It is now sufficient to prove that

$$(e_{m,g})_{<u}\psi_{m,u} = \begin{cases} 0 & \text{if } m_u = 0 \text{ and } x_{u-1} = 0\\ (e_{m,g})_{<u} & \text{if } m_u = 2 \text{ and } x_{u-1} = 1.\end{cases}$$

Again this follows from Lemma 4.5.3.

*Case* (2a). Suppose that factor $u$ of $B(m,g)_3$ equals $\binom{2}{2}$. By Lemma 4.4.4 either $m_u = 1$ and $x_{u-1} = 0$, or $m_u = 0$ and $x_{u-1} = 1$. By construction of $e_{m,g}$ and the inductive hypothesis

$$\begin{aligned}(e_{m,g})^2_{\leq u} &= (e_{m,g})^2_{<u} b(2\cdot 3^u)^2 \\ &= (e_{m,g})_{<u} b(2\cdot 3^u)\left[\binom{m_u+1}{2} + \binom{m_u+1}{1}\psi_{m,u}\right],\end{aligned}$$

where the final equality holds by (4.3). It is now sufficient to prove that

$$(e_{m,g})_{<u}\psi_{m,u} = \begin{cases} 0 & \text{if } m_u = 1 \text{ and } x_{u-1} = 0, \\ (e_{m,g})_{<u} & \text{if } m_u = 0 \text{ and } x_{u-1} = 1. \end{cases}$$

This follows from Lemma 4.5.3.

*Case* (2b). Suppose that factor $u$ of $B(m,g)_3$ equals $\binom{1}{0}$. By Lemma 4.4.4 either $m_u = 1$ and $x_{u-1} = 0$, or $m_u = 0$ and $x_{u-1} = 1$. By construction of $e_{m,g}$ and the inductive hypothesis

$$\begin{aligned}(e_{m,g})^2_{\leq u} &= (e_{m,g})^2_{<u}(\mathbf{1} - b(2\cdot 3^u))^2 \\ &= (e_{m,g})^2_{<u}(\mathbf{1} + b(2\cdot 3^u) + b(2\cdot 3^u)^2) \\ &= (e_{m,g})_{<u}\left(\mathbf{1} + b(2\cdot 3^u)\left[1 + \binom{m_u+1}{2} + \binom{m_u+1}{1}\psi_{m,u}\right]\right),\end{aligned}$$

where the final equality holds by (4.3). It is now sufficient to prove that

$$(e_{m,g})_{<u}\psi_{m,u} = \begin{cases} 0 & \text{if } m_u = 1 \text{ and } x_{u-1} = 0, \\ (e_{m,g})_{<u} & \text{if } m_u = 0 \text{ and } x_{u-1} = 1. \end{cases}$$

Again this follows from Lemma 4.5.3.

*Case* (3a). Suppose that factor $u$ of $B(m,g)_3$ equals $\binom{1}{1}$. By Lemma 4.4.4 either $m_u = 2$ and $x_{u-1} = 0$, or $m_u = 1$ and $x_{u-1} = 1$. By construction of $e_{m,g}$ and the inductive hypothesis

$$\begin{aligned}(e_{m,g})^2_{\leq u} &= (e_{m,g})^2_{<u}(b(3^u) - b(2\cdot 3^u))^2 \\ &= (e_{m,g})_{<u} b(2\cdot 3^u)\left[\binom{m_u+1}{2} + \binom{m_u+1}{1}\psi_{m,u}\right] \\ &+ (e_{m,g})_{<u} b(2\cdot 3^u)\left[2\binom{m_u}{1} - \psi_{m,u}\right] \\ &+ (e_{m,g})_{<u}\left(b(3^u)\left[\binom{m_u+2}{1} + \psi_{m,u}\right] + b(2\cdot 3^u)\right),\end{aligned}$$

where the final equality holds by (4.2), (4.3) and (4.4). It is now sufficient to prove that

$$(e_{m,g})_{<u}\psi_{m,u} = \begin{cases} 0 & \text{if } m_u = 2 \text{ and } x_{u-1} = 0 \\ (e_{m,g})_{<u} & \text{if } m_u = 1 \text{ and } x_{u-1} = 1. \end{cases}$$

This follows from Lemma 4.5.3.

*Case* (3b). Suppose that factor $u$ of $B(m,g)_3$ equals $\binom{2}{0}$. By Lemma 4.4.4 either $m_u = 2$ and $x_{u-1} = 0$, or $m_u = 1$ and $x_{u-1} = 1$. By construction of $e_{m,g}$ and the inductive hypothesis

$$\begin{aligned}
(e_{m,g})^2_{\leq u} &= (e_{m,g})^2_{<u}(\mathbf{1} - b(3^u) + b(2\cdot 3^u))^2 \\
&= (e_{m,g})_{<u}(\mathbf{1}+b(3^u)^2+b(2\cdot 3^u)^2+b(3^u)-b(2\cdot 3^u)+b(2\cdot 3^u)b(3^u)) \\
&= (e_{m,g})_{<u}(\mathbf{1} + b(3^u) - b(2\cdot 3^u)) \\
&+ (e_{m,g})_{<u}b(2\cdot 3^u)\left[\binom{m_u+1}{2} + \binom{m_u+1}{1}\psi_{m,u}\right] \\
&+ (e_{m,g})_{<u}b(2\cdot 3^u)\left[2\binom{m_u}{1} - \psi_{m,u}\right] \\
&+ (e_{m,g})_{<u}\left(b(3^u)\left[\binom{m_u+2}{1} + \psi_{m,u}\right] + b(2\cdot 3^u)\right),
\end{aligned}$$

where the final equality holds by (4.2), (4.3) and (4.4). It is now sufficient to prove that

$$(e_{m,g})_{<u}\psi_{m,u} = \begin{cases} 0 & \text{if } m_u = 2 \text{ and } x_{u-1} = 0 \\ (e_{m,g})_{<u} & \text{if } m_u = 1 \text{ and } x_{u-1} = 1. \end{cases}$$

This follows from Lemma 4.5.3. □

Remark 4.5.4. Given $t \in \mathbf{N}$, we can generalise the definition of $\psi_{m,t}$ when $p$ is an arbitrary prime. Furthermore, the recursive formula in (4.6) generalises in an entirely similar way. However it is a special feature for $p \in \{2,3\}$ that we can always write either $(e_{m,g})_{<t}\psi_{m,u} = (e_{m,g})_{<t}w$, or $(e_{m,g})_{<t}\psi_{m,u} = (e_{m,g})_{<t}(\mathbf{1} - w)$, where $w$ equals factor $t-1$ of $(e_{m,g})_{<t}$. This is not the case when $p$ is at least 5, and so we cannot apply Lemma 4.5.2 to obtain the analogue of Lemma 4.5.3 in general.

**4.5.3. The elements $e_{m,g}$ are orthogonal and primitive.** Let $g, d \in \mathbf{N}_0$ be such that both $B(m,g)$ and $B(m,d)$ are non-zero modulo 3, and suppose that $g \neq d$. Write

$$\begin{aligned}
g &=_p [g_0, g_1, g_2, \ldots, g_t] \\
d &=_p [d_0, d_1, d_2, \ldots, d_t].
\end{aligned}$$

Let $u$ be minimal such that $g_u \neq d_u$, and so $(m+2g)_{<u} = (m+2d)_{<u}$ and $(e_{m,g})_{<u} = (e_{m,d})_{<u}$. As in §4.4, let $x_{u-1}$ (resp. $y_{u-1}$) denote the carry leaving column $u-1$ in the ternary addition of $m$ and $g$ (resp. $d$), recalling that the columns in both $p$-ary additions are indexed starting from 0. It follows that $x_{u-1} = y_{u-1}$, and so $(m_u, x_{u-1}) = (m_u, y_{u-1})$. By Lemma 4.4.4, factor $u$ of $B(m,g)_3$ equals $\binom{a}{g_u}$ and factor $u$ of $B(m,d)_3$ equals $\binom{c}{d_u}$, where $a - 2g_u \equiv_3 c - 2d_u \equiv_3 m_u + x_{u-1}$. Moreover, these factors are unequal since $g_u \neq d_u$. As there are exactly two choices for a factor $\binom{x}{y}$ such that

$0 \leq y \leq x < 3$ and $x - 2y \equiv_3 m_u + x_{u-1}$, it follows from the construction of $e_{m,g}$ that

$$(e_{m,g})_{\leq u} = (e_{m,g})_{<u}w \text{ and } (e_{m,d})_{\leq u} = (e_{m,d})_{<u}(\mathbf{1} - w),$$

where $w, \mathbf{1} - w$ are as specified in §4.2. By Proposition 4.5.1, $(e_{m,g})_{\leq u}$ and $(e_{m,d})_{\leq u}$ are idempotents in $S_F(\lambda)$, and so it follows from Lemma 4.5.2 that their product is zero. As $S_F(\lambda)$ is commutative, this implies $e_{m,g}e_{m,d} = 0$.

We now count the number of non-zero $e_{m,g}$ in $S_F(\lambda)$. By Lemma 4.3.2, $e_{m,g}$ is non-zero in $S_F(\lambda)$ if and only if $g \leq \lambda_2$. Therefore the number of non-zero $e_{m,g}$ in $S_F(\lambda)$ equals

$$|\{g : g \leq \lambda_2 \text{ and } B(m, g) \text{ is non-zero modulo } 3\}|.$$

By Theorem 3.3 in [**29**] this equals the number of indecomposable summands of $M^\lambda$. It therefore follows that the set of $e_{m,g}$ such that $g \leq \lambda_2$ is a complete set of primitive orthogonal idempotents for $S_F(\lambda)$.

**4.5.4. Indecomposable Young permutation modules.** In this section let $\lambda = (\lambda_1, \lambda_2)$ be a partition of $n$. We use Theorem 4.1.3 to prove that the only indecomposable Young permutation modules $M^\lambda$ in this case are those such that either $\lambda = (n, 0)$, or $\lambda = (n-1, 1)$ and 3 divides $n$. Although it is well-known that $M^\lambda$ is indecomposable in these cases (see for instance [**33**, Example 5.1]), these being the only possible cases is a non-trivial result. Indeed the analogous statement is false over a field of characteristic 2. In that case when $n$ is even, the module $M^{(n/2,n/2)}$ is indecomposable (see [**15**, Example 3.10] or [**39**, Example 3.8]).

It should be noted that the discussion in the previous paragraph and the main result in this section are consistent with Theorem 2 in [**25**]. In particular [**25**, Theorem 2] determines precisely when the $FS_n$-module $M^\lambda$ is indecomposable, where $\lambda$ is any partition of $n$ and $F$ is of strictly positive characteristic.

We now state and prove the main result of this section.

PROPOSITION 4.5.5. *Let $F$ be a field of characteristic* 3, *and let $\lambda = (\lambda_1, \lambda_2)$ be a partition of $n$. Then $M^\lambda$ is indecomposable if and only if either $\lambda = (n, 0)$, or $\lambda = (n-1, 1)$ and* 3 *divides $n$.*

PROOF. We prove that if $M^\lambda$ is indecomposable, then either $\lambda = (n, 0)$, or $\lambda = (n-1, 1)$ and 3 divides $n$. As remarked in the discussion above, the reverse implication is well-known.

As usual define $m = \lambda_1 - \lambda_2$. Also let $m =_3 [m_0, m_1, \ldots, m_t]$. The module $M^\lambda$ is indecomposable if and only if $\mathbf{1}$ is the only non-zero primitive idempotent in $S_F((\lambda_1, \lambda_2))$. Given $g \in \mathbf{N}_0$, Lemma 4.3.2 states that if $B(m, g)$ is non-zero modulo 3, then the smallest term (with respect to the order defined in §4.2) appearing with non-zero coefficient in $e_{m,g}$ is $b(g)$, with coefficient $B(m,g)$. Then by Theorem 4.1.3, $\mathbf{1}$ is the only non-zero primitive idempotent in $S_F((\lambda_1,\lambda_2))$ if and only if exactly one of the following holds:

(i) $B(m,g)$ equals zero modulo 3 for all $g \in \mathbf{N}$,
(ii) if $g \in \mathbf{N}$ is minimal such that $B(m,g)$ is non-zero modulo 3, then $g > \lambda_2$.

We first show that (i) can never occur. Indeed Lucas' Theorem gives that $B(m,1)$ is non-zero modulo 3 if either $m_0 = 0$ or $m_0 = 2$, and $B(m,2)$ is non-zero modulo 3 whenever $m_0 = 1$. Moreover the chosen value of $g \in \mathbf{N}$ in each of these cases is minimal such that $B(m,g)$ is non-zero.

Suppose now that (ii) holds. We distinguish two cases, determined by $m_0$.

*Case* (1). Suppose that either $m_0 = 0$, or $m_0 = 2$. If $m_0 = 0$, then Lemma 4.3.2 gives that the smallest term in $e_{m,1}$ with non-zero coefficient is $b(1)$. Similarly if $m_0 = 2$, then the smallest term in $e_{m,1}$ with non-zero coefficient is $b(1)$. In either case if $e_{m,1}$ is zero in $S_F((\lambda_1,\lambda_2))$, then $\lambda_2 = 0$.

*Case* (2). Suppose that $m_0 = 1$. By Lemma 4.3.2 the smallest term in $e_{m,2}$ with non-zero coefficient is $b(2)$. Therefore if $e_{m,2}$ is zero in $S_F((\lambda_1,\lambda_2))$, then either $\lambda_2 = 0$ or $\lambda_2 = 1$. It remains to prove that if $m_0 = 1$ and $\lambda_2 = 1$, then 3 divides $n$. Indeed if $m_0 = \lambda_2 = 1$, then

$$\begin{aligned} n = 1 + \lambda_1 = 1 + (1+m) &= 1 + 1 + (1 + m_1 3 + \cdots + m_t 3^t) \\ &= 3 + m_1 3 + \cdots + m_t 3^t, \end{aligned}$$

and so 3 divides $n$, as claimed. □

## 4.6. The correspondence between idempotents and Young modules

Throughout this section let $\lambda = (\lambda_1,\lambda_2)$ and $\mu = (\mu_1,\mu_2)$ be partitions of $n$ satisfying the hypothesis of Theorem 4.1.4.

We prove Theorem 4.1.4 by induction on $n$ by following [**15**, §7]. The base cases are $n = 0$ and $n = 1$. In both cases the only possibility is $\lambda = \mu = (n,0)$. Therefore in §4.6.1 we consider the case when $\mu = (n,0)$ and $\lambda \in \Lambda(2,n)$ is arbitrary. We then complete the inductive step in §4.6.2.

**4.6.1. The case $\mu = (n,0)$.** We distinguish two cases determined by the partition $\lambda$.

If $\lambda = (n,0)$, then $M^{(n,0)}$ is indecomposable and the only primitive idempotent in $S_F((n,0))$ is $\mathbf{1}$. In this case $B(m,g) = \binom{n}{0}$, and so

$$B(m,g)_3 = \binom{n_0}{0} \cdots \binom{n_t}{0},$$

where $n =_3 [n_0, \dots, n_t]$. By construction, for some $\alpha_i \in \mathbf{F}_3$,

$$e_{n,0} = \mathbf{1} + \sum_{i>0} \alpha_i b(i) = \mathbf{1} \in S_F((n,0)),$$

as required. Observe that this proves the base cases of the induction.

Recall from §4.1.1 that $1_\lambda$ is defined to be an idempotent in $S_F(2,n)$ such that $1_\lambda E^{\otimes n} = M^\lambda$. If $\lambda = (m+g, g) \vdash n$, then we show that there exist $u, v \in S_F(2,n)$ such that $uv = e_{m,g}$ and $vu = 1_{(n,0)}$. Then $e_{m,g}$ and $1_{(n,0)}$ are idempotents such that $e_{m,g} = u1_{(n,0)}v$ and $1_{(n,0)} = ve_{m,g}u$. It follows from [**60**, (1.1)] that $e_{m,g}M^\lambda = e_{m,g}E^{\otimes n} \cong 1_{(n,0)}E^{\otimes n} = M^{(n,0)} = Y^{(n,0)}$, as required. Now define

$$u = B(m,g)1_\lambda f^{(g)} 1_{(n,0)} \text{ and } v = 1_{(n,0)} e^{(g)} 1_\lambda.$$

In order to calculate $uv$ and $vu$, we follow parts (b) and (c) in the proof of [**15**, Proposition 7.2]. Indeed define the *simple root* $\alpha = (1,-1)$. By Theorem 2.4 in [**16**] if $\nu \in \Lambda(2,n)$, then

$$e\,1_\nu = \begin{cases} 1_{\nu+\alpha}\, e & \text{if } \nu+\alpha \text{ is a composition,} \\ 0 & \text{otherwise} \end{cases}$$

$$f\,1_\nu = \begin{cases} 1_{\nu-\alpha}\, f & \text{if } \nu-\alpha \text{ is a composition,} \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, Proposition 3.6 in [**16**] states that $H_i 1_\lambda = \lambda_i 1_\lambda$ for $i \in \{1,2\}$. Define $h = H_1 - H_2$, and so $h1_\lambda = m\,1_\lambda$. As $(n,0) + (1,-1)$ is not a composition, the above relations give $e^{(a)}\,1_{(n,0)} = 0$ for all $a \in \mathbf{N}$. Also with $\lambda = (m+g, g)$

$$e^{(g)}\,1_\lambda = 1_{(n,0)}\,e^{(g)}, \quad 1_{(n,0)}\,f^{(g)} = f^{(g)}\,1_\lambda, \quad \tbinom{h}{g}\,1_{(n,0)} = \tbinom{n}{g}\,1_{(n,0)}. \tag{4.7}$$

It follows from the relations in (4.7) and Lemma 4.3.2 that

$$\begin{aligned} uv &= B(m,g)\,1_\lambda f^{(g)} 1_{(n,0)} e^{(g)} 1_\lambda \\ &= B(m,g)\,1_\lambda f^{(g)} e^{(g)} 1_\lambda \\ &= B(m,g)\,b(g) = e_{m,g} \in S_F((m+g,g)). \end{aligned}$$

Also it follows from the relations in (4.7) and [**31**, §26.2] that

$$\begin{aligned} vu &= B(m,g)1_{(n,0)} e^{(g)} 1_\lambda f^{(g)} 1_{(n,0)} \\ &= B(m,g)1_{(n,0)} e^{(g)} f^{(g)} 1_{(n,0)} \\ &= B(m,g)1_{(n,0)}[\sum_{j=0}^{g} f^{(g-j)} \tbinom{h-2g+2j}{j} e^{(g-j)}]1_{(n,0)} \\ &= B(m,g)1_{(n,0)}[f^{(0)} \binom{h}{g} e^{(0)}]1_{(n,0)} \\ &= B(m,g)\binom{n}{g} 1_{(n,0)} = (B(m,g))^2 1_{(n,0)} \equiv_3 1_{(n,0)}, \end{aligned}$$

where the final congruence holds as $B(m,g)$ is non-zero modulo 3.

**4.6.2. The inductive step.** Assume throughout this section that the statement of Theorem 4.1.4 holds inductively for all partitions in $\Lambda(2,n)$ for some $n \in \mathbf{N}_0$. Let $\widetilde{\lambda}$ and $\widetilde{\mu}$ be partitions of $n+2$ with at most two parts satisfying the hypothesis of the theorem. The argument for the case when $\widetilde{\mu}_2 = 0$ is given in §4.6.1, so assume that $\widetilde{\mu}_2 > 0$. Then $\widetilde{\lambda} = \lambda + (1^2)$ and $\widetilde{\mu} = \mu + (1^2)$, where $\lambda$ and $\mu$ are the partitions of $n$ such that $m = \lambda_1 - \lambda_2$ and $g = \lambda_2 - \mu_2$. The inductive step is complete once we prove Proposition 4.6.3 below, which is the result of Theorem 7.3 in [**15**]. To this end define the map $j : E^{\otimes n} \to E^{\otimes n+2}$ by

$$x \mapsto (v_1 \otimes v_2 - v_2 \otimes v_1) \otimes x,$$

where we remind the reader that $\{v_1, v_2\}$ is a fixed basis of $E$. Observe that $j$ is injective. Also it follows from the definition of $M^\lambda$ given in §4.1.1 that $j(M^\lambda) \subseteq M^{\lambda+(1^2)}$. We then have the following lemma.

LEMMA 4.6.1. *Given $x \in M^\lambda$, we have $je_{m,g}(x) = e_{m,g}j(x)$.*

PROOF. We prove that $jb(a)(x) = b(a)j(x)$ for all $x \in M^\lambda$ and $a \in \mathbf{N}_0$. Note that on the left hand side of this equality $b(a)$ is viewed as an element of $S_F(\lambda)$, and on the right hand side it is viewed as an element of $S_F(\lambda+(1^2))$.

The Lie algebra action of $e$ on $v_1 \otimes v_2 - v_2 \otimes v_1$ is as follows:

$$\begin{aligned} e(v_1 \otimes v_2 - v_2 \otimes v_1) &= (ev_1 \otimes v_2 + v_1 \otimes ev_2) - (ev_2 \otimes v_1 + v_2 \otimes ev_1) \\ &= v_1 \otimes v_1 - v_1 \otimes v_1 = 0. \end{aligned}$$

Similarly $f(v_1 \otimes v_2 - v_2 \otimes v_1) = 0$, and so $j$ commutes with the action of $e^{(a)}$ and $f^{(a)}$ for all $a \in \mathbf{N}$. Also considering the Lie algebra action of the product $f^{(a)}e^{(a)}$ on $M^\lambda$ and $M^{\lambda+(1^2)}$ shows that $f^{(a)}e^{(a)}$ preserves $M^\lambda$ and $M^{\lambda+(1^2)}$. As $1_\lambda$ and $1_{\lambda+(1^2)}$ are the projections onto $E^{\otimes n}$ corresponding to $M^\lambda$ and $M^{\lambda+(1^2)}$, respectively, it follows that

$$\begin{aligned} j(b(a)x) = j(1_\lambda f^{(a)}e^{(a)}1_\lambda x) &= j(f^{(a)}e^{(a)}x) \\ &= (v_1 \otimes v_2 - v_2 \otimes v_1) \otimes f^{(a)}e^{(a)}x, \end{aligned}$$

and

$$\begin{aligned} b(a)j(x) &= (1_{\lambda+(1^2)}f^{(a)}e^{(a)}1_{\lambda+(1^2)})((v_1 \otimes v_2 - v_2 \otimes v_1) \otimes x) \\ &= 1_{\lambda+(1^2)}(0 + (v_1 \otimes v_2 - v_2 \otimes v_1) \otimes f^{(a)}e^{(a)}x) \\ &= (v_1 \otimes v_2 - v_2 \otimes v_1) \otimes f^{(a)}e^{(a)}x. \end{aligned}$$

Therefore $jb(a)x = b(a)j(x)$, as required. □

Before we state and prove Proposition 4.6.3, we give the following preliminaries.

Given $\boldsymbol{i} = (i_1, i_2, \ldots, i_n) \in I(2,n)$, define $v_{\boldsymbol{i}} = v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_n}$.

THEOREM 4.6.2. [**33**, Theorem 13.13] *Let $\lambda = (\lambda_1, \lambda_2)$ and $\mu = (\mu_1, \mu_2)$ be partitions of $n$ such that $\mu_2 \le \lambda_2$. If $F$ is a field of characteristic not equal to 2 and $\mu_1 \neq \mu_2$, then $\mathrm{Hom}_{FS_n}(S^\mu, M^\lambda)$ is one-dimensional as an $F$-vector space.*

PROPOSITION 4.6.3. *Suppose that $e_{m,g}M^\lambda \cong Y^\mu$. Then $e_{m,g}M^{\lambda+(1^2)} \cong Y^{\mu+(1^2)}$.*

PROOF. It follows from Theorem 4.6.2 that the copy of $S^\mu$ in $M^\lambda$ is unique, and the analogous statement holds for $S^{\mu+(1^2)}$ and $M^{\lambda+(1^2)}$. By the defining property of the Young module $Y^\mu$, it sufficient to prove that if $e_{m,g}(S^\mu) \neq 0$, then $e_{m,g}(S^{\mu+(1^2)}) \neq 0$. We do this relating the tabloid definition of $M^\lambda$ given in §1.1.1 to the definition introduced in this chapter.

Write $u$ for $\mu_2 + 1$. Let $t_1$ and $t_2$ respectively denote the following standard $\mu$ and $\mu + (1^2)$-tableaux:

$$t_1 = \begin{array}{|c|c|c|c|c|c|c|} \hline 3 & 5 & \dots & 2u-1 & 2u+1 & \dots & n+2 \\ \hline 4 & 6 & \dots & 2u & & & \\ \hline \end{array} \qquad t_2 = \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 3 & \dots & 2u-1 & 2u+1 & \dots & n+2 \\ \hline 2 & 4 & \dots & 2u & & & \\ \hline \end{array}.$$

Write $R(t_i)$ for the row stabiliser of each $t_i$. Also write $C(t_i)$ for the column stabiliser of each $t_i$, and write $\{C(t_i)\}^-$ for $\sum_{\pi \in C(t_i)} \mathrm{sgn}(\pi)\pi$. It is easy to see that $\{C(t_2)\}^- = (1 - (1\ 2))\{C(t_1)\}^-$.

Observe that the column stabiliser of $t_1$ is a subgroup of the symmetric group on $\{3, 4, \dots, n+2\}$. Thus given $\sigma \in \mathrm{Sym}(\{3, 4, \dots, n+2\})$, we define $\sigma^\star \in \mathrm{Sym}(\{1, 2, \dots, n\})$ to be the permutation such that $\sigma^\star(\ell) = \sigma(\ell+2) - 2$ for $1 \le \ell \le n$. Then there is a natural action of $\sigma \in \mathrm{Sym}(\{3, 4, \dots, n+2\})$ on $x \in M^\lambda$ given by $\sigma x = \sigma^\star x$.

Let $\omega_1 = \sum v_{\boldsymbol{i}}$, where the sum runs over all $\boldsymbol{i} \in I(2, n)$ such that $\boldsymbol{i}$ has weight $\lambda$ and $i_\rho = 2$ whenever $\rho + 2$ is in the second row of $t_1$. Observe that $\omega_1$ is fixed by $R(t_1)$, and so the polytabloid $e(t_1)$ corresponds to $\varepsilon_{t_1} := \{C(t_1)\}^-\omega_1$. Note that the actions of $R(t_1)$ and $\{C(t_1)\}^-$ on $\omega_1$ are as defined in the previous paragraph. Then $\varepsilon_{t_1}$ generates the unique copy of $S^\mu$ in $M^\lambda$.

Similarly let $\omega_2 = \sum v_{\boldsymbol{i}}$, where the sum runs over all $\boldsymbol{i} \in I(2, n+2)$ such that $\boldsymbol{i}$ has weight $\lambda$ and $i_\rho = 2$ whenever $\rho$ is in the second row of $t_2$. Then $\omega_2$ is fixed by $R(t_2)$, and so $e(t_2)$ corresponds to $\varepsilon_{t_2} = \{C(t_2)\}^-\omega_2$. Note that the actions of $R(t_2)$ and $\{C(t_2)\}^-$ on $\omega_2$ are given by the usual place permutation defined in §4.1.1. Then $\varepsilon_{t_2}$ generates the unique copy of $S^{\mu+(1^2)}$ in $M^{\lambda+(1^2)}$. By definition of $j$

$$\begin{aligned} j(\varepsilon_{t_1}) &= (v_1 \otimes v_2 - v_2 \otimes v_1) \otimes \varepsilon_{t_1} \\ &= \{C(t_1)\}^-((v_1 \otimes v_2 - v_2 \otimes v_1) \otimes \omega_1), \end{aligned}$$

where in the final line the action of $\{C(t_1)\}^-$ is again by the usual place permutation defined in §4.1.1.

Observe that

$$\omega_2 = v_1 \otimes v_2 \otimes \omega_1 + v_2 \otimes v_2 \otimes \omega,$$

for a certain $\omega$. Then since $v_1 \otimes v_2 - v_2 \otimes v_1 = (1 - (1\ 2))(v_1 \otimes v_2)$, we have

$$\begin{aligned} j(\varepsilon_{t_1}) &= \{C_{t_1}\}^-((v_1 \otimes v_2 - v_2 \otimes v_1) \otimes \omega_1) \\ &= (1 - (1\ 2))\{C_{t_1}\}^-(v_1 \otimes v_2 \otimes \omega_1) \\ &= (1 - (1\ 2))\{C_{t_1}\}^-(v_1 \otimes v_2 \otimes \omega_1 + v_2 \otimes v_2 \otimes \omega) \\ &= \{C_{t_2}\}^-\omega_2 = \varepsilon_{t_2}, \end{aligned}$$

where the third equality holds since $(1 - (1\ 2))(v_2 \otimes v_2 \otimes \omega) = 0$.

If $e_{m,g}(S^\mu) \neq 0$, then $e_{m,g}(\varepsilon_{t_1}) \neq 0$. As the map $j$ is injective, it follows from Lemma 4.6.1 that

$$e_{m,g}(\varepsilon_{t_2}) = e_{m,g}(j(\varepsilon_{t_1})) = j(e_{m,g}(\varepsilon_{t_1})) \neq 0,$$

and so $e_{m,g}(S^{\mu+(1^2)}) \neq 0$. Therefore $e_{m,g}(Y^{\mu+(1^2)}) \neq 0$, which completes the proof. $\square$

CHAPTER 5

# Twisted Baddeley modules and decomposition numbers of $C_2 \wr S_n$

Let $F$ be a field of odd prime characteristic $p$, and fix $n \in \mathbf{N}$. Recall that given partitions $\lambda$ and $\nu$ of $n$ such that $\nu$ is $p$-regular, the decomposition number $d_{\lambda\nu}$ equals the number of composition factors of $S^\lambda$, defined over a field of characteristic $p$, that are isomorphic to $D^\nu$. In [**24**, Theorem 1.1] Giannelli and Wildon use the ordinary representation theory of $S_n$ to determine certain decomposition numbers $d_{\lambda\nu}$. They do this by describing the vertices (see §1.3.1) of certain $FS_n$-modules, and in particular showing that these modules are projective. The description of the decomposition numbers then follows using Brauer reciprocity (see §1.3.4). The modules in question are $p$-permutation modules, and so the authors make use of the connections between the Brauer morphism and vertices (see §1.3.3) to show that these modules are projective.

Recall that the decomposition number $d_{\lambda\nu,\mu\widetilde{\nu}}$ of $C_2 \wr S_n$ is defined to be the number of composition factors of $S^{(\lambda,\mu)}$ that are isomorphic to $D^{(\nu,\widetilde{\nu})}$. In this case both $\nu$ and $\widetilde{\nu}$ are necessarily $p$-regular. Motivated by [**24**], we show that certain $p$-permutation $FC_2 \wr S_n$-modules are projective by considering their vertices. We also do this using the Brauer morphism. We then use Brauer reciprocity to understand particular decomposition numbers of $C_2 \wr S_n$.

We remark that it follows from the Morita equivalence between $FC_2 \wr S_n$ and $\bigoplus_{i=0}^n FS_{(i,n-i)}$ given by Proposition 1.4.8 that

$$d_{\lambda\nu,\mu\widetilde{\nu}} = d_{\lambda\nu} d_{\mu\widetilde{\nu}},$$

and so our result on decomposition numbers follows from [**24**, Theorem 1.1]. However, the vertex of a module is a ring-theoretic property, and so cannot be determined by this Morita equivalence. This justifies us taking a longer route to determine the decomposition numbers of $C_2 \wr S_n$.

**Outline.** In §5.1 we state the two main theorems in this chapter. In order to do this, we define involution models for finite groups and the twisted Baddeley module $M_{(2a,b,c)}$, where $2a + b + c = n$.

In §5.2 we give an explicit combinatorial basis for the module $M_{(2a,b,c)}$, specifically in §5.2.1. The basis that we describe is generally not a permutation basis for $M_{(2a,b,c)}$, and so is in general not a $p$-permutation basis for an arbitrary $p$-subgroup of $C_2 \wr S_n$. In §5.2.2 we show how the given basis can be used to construct a $p$-permutation basis of $M_{(2a,b,c)}$ with respect to a given $p$-subgroup of $C_2 \wr S_n$.

In §5.3 we prove Theorem 5.1.1, which is the first main result in this chapter. The proof is technical in areas, and so it is broken down into three steps. We first consider the Brauer correspondent of $M_{(2a,b,c)}$ with respect to a particular cyclic group of order $p$ in $C_2 \wr S_n$, denoted $R_r$, where $rp \leq n$. We decompose $M_{(2a,b,c)}$ as a direct sum of indecomposable $FN_{C_2 \wr S_n}(R_r)$-modules, denoted $N_{(\lambda,t,u)}$, using Clifford theory arguments. We see that each summand $N_{(\lambda,t,u)}$ of $M_{(2a,b,c)}$ has a vertex containing the group $R_{\omega^\star}$ (defined in the first step of the proof). In the second step, we therefore consider the module $N_{(\lambda,t,u)}(R_{\omega^\star})$. We show that $N_{(\lambda,t,u)}(R_{\omega^\star})$ is an indecomposable $N_{C_2 \wr S_n}(R_{\omega^\star})$-module, and we determine its vertex. In the third step, we use the description of the vertices of $N_{(\lambda,t,u)}(R_{\omega^\star})$ to complete the proof of Theorem 5.1.1.

In §5.4 we prove Theorem 5.1.2, which is the second main result in this chapter. We begin by giving details of the correspondence between the blocks of $FC_2 \wr S_n$ and the blocks of $FN_{C_2 \wr S_n}(R_r)$. This result will be essential in the proof of Theorem 5.1.2. We show that every summand of $M_{(2a,b,c)}$ in the block $B((\gamma, w_b(\gamma)), (\delta, w_c(\delta)))$ is projective. We then lift these projective summands of $M_{(2a,b,c)}$ from $\mathbf{F}_p$ to $\mathbf{Z}_p$ using Scott's Lifting Theorem, and thus determine the ordinary characters of these lifted summands. The characterisation of the decomposition numbers then follows from Brauer reciprocity.

## 5.1. An involution model for $C_2 \wr S_n$

We say that a finite group $G$ has an *involution model* if there exists a set of elements $\{e_1, e_2, \ldots, e_t\} \subseteq G$, such that $e_i^2 = 1$ for all $i$, and for each $e_i$ there exists a linear character $\psi_i$ of $C_G(e_i)$ such that

$$\sum_{i=1}^{t} \psi_i^G = \sum_{\psi \in \mathrm{Irr}(G)} \psi.$$

The main result of [**32**] is that the sum of the ordinary characters of the $\mathbf{Q}S_{2m+k}$-modules

$$H^{(2m;k)} := (\mathbf{Q}_{S_2 \wr S_m} \boxtimes \mathrm{sgn}_{S_k}) \big\uparrow_{S_2 \wr S_m \times S_k}^{S_{2m+k}}$$

is an involution model for $S_{2m+k}$. These modules are known as the *twisted Foulkes modules*. In [**2**] Baddeley constructs an explicit involution model for $G \wr S_n$, using a given involution model for $G$. In the case that $G = C_2$, we refer to the modules constructed by Baddeley as *twisted Baddeley modules*, which we now define.

Given $a \in \mathbf{N}$, define $f_a \in C_2 \wr S_{2a}$ to be the permutation equal to

$$(1\ a+1)(2\ a+2)\ldots(a\ 2a)(\overline{1}\ \overline{a+1})(\overline{2}\ \overline{a+2})\ldots(\overline{a}\ \overline{2a}),$$

and let $V_a$ be the centraliser of $f_a$ in $C_2 \wr S_{2a}$. Therefore $V_a$ is equal to

$$\langle (1\ \overline{1})(a+1\ \overline{a+1}), (2\ \overline{2})(a+2\ \overline{a+2}), \ldots, (a\ \overline{a})(2a\ \overline{2a})\rangle \rtimes \xi(S_2 \wr S_a),$$

where $\xi$ is as defined in §1.4.1. Also define $V_\lambda$ to be the subgroup of $V_a$ equal to

$$\langle (1\ \overline{1})(a+1\ \overline{a+1}), (2\ \overline{2})(a+2\ \overline{a+2}), \ldots, (a\ \overline{a})(2a\ \overline{2a})\rangle \rtimes \xi(S_2 \wr S_\lambda),$$

where $\lambda \vdash a$, and $S_\lambda$ is the corresponding Young subgroup of $S_a$.

Recall that $N$ denotes the non-trivial irreducible $FC_2$-module. Given $(a,b,c) \in \mathbf{N}_0^3$ such that $2a+b+c=n$, we define the module

$$M_{(2a,b,c)} = (F\big\uparrow_{V_a}^{C_2\wr S_{2a}} \boxtimes \mathrm{Inf}_{S_b}^{C_2\wr S_b} \mathrm{sgn}_b \boxtimes (\widetilde{N}^{\otimes c} \otimes \mathrm{Inf}_{S_c}^{C_2\wr S_c} \mathrm{sgn}_c))\big\uparrow_{C_2\wr S_{(2a,b,c)}}^{C_2\wr S_n}.$$

Theorem 5.1.1 characterises the vertices of the indecomposable summands of $M_{(2a,b,c)}$. In order to state Theorem 5.1.1, we also require the following notation. Given $r \in \mathbf{N}$ such that $rp \leq n$, define

$$T_r' := \{(\lambda, t, u) : \lambda \in \Lambda(2,s), 2s+t+u = r \text{ and } sp \leq a, tp \leq b, up \leq c\},$$

where we remind the reader that $\Lambda(2,s)$ denotes the set of all compositions of $s$ in at most two parts.

THEOREM 5.1.1. *Let $(a,b,c) \in \mathbf{N}_0^3$ be such that $2a+b+c=n$, and let $U$ be a non-projective indecomposable summand of $M_{(2a,b,c)}$. Then $U$ has a vertex equal to a Sylow $p$-subgroup of*

$$V_{p\lambda} \times C_2 \wr S_{tp} \times C_2 \wr S_{up},$$

*for some $r \in \mathbf{N}$, where $rp \leq n$, and $(\lambda, t, u) \in T_r'$.*

As stated in the introduction of this chapter, Theorem 5.1.1 does not follow from Proposition 1.4.8. In Example 5.3.16 we make this explicit by describing the vertices of the non-projective indecomposable summands of $M_{(54,0,0)}$ over a field of characteristic 3.

In order to state our theorem on the decomposition numbers of $C_2 \wr S_n$, we require the following notation. Given a $p$-core partition $\gamma$ (see §1.4.4) and given $b \in \mathbf{N}_0$, let $w_b(\gamma)$ be the minimum number of border strips of size $p$ such that when added to $\gamma$, we obtain a partition with exactly $b$ odd parts. Let $\mathcal{E}_b(\gamma)$ be the set of all partitions of $|\gamma| + w_b(\gamma)p$ obtained in this way.

THEOREM 5.1.2. *Let $\gamma$ and $\delta$ be $p$-core partitions, and let $b, c \in \mathbf{N}_0$. If $b \geq p$ (resp. $c \geq p$), suppose that $w_{b-p}(\gamma) \neq w_b(\gamma) - 1$ (resp. $w_{c-p}(\delta) \neq w_c(\delta) - 1$). Then there exists a set partition of $\mathcal{E}_b(\gamma) \times \mathcal{E}_c(\delta)$, say $\Lambda_1, \ldots, \Lambda_t$, such that each $\Lambda_i$ has a unique pair $(\nu_i, \widetilde{\nu_i})$ with $\nu_i$ and $\widetilde{\nu_i}$ both maximal in the dominance orders on $\mathcal{E}_b(\gamma)$ and $\mathcal{E}_c(\delta)$, respectively. Moreover, $\nu_i$ and $\widetilde{\nu_i}$ are $p$-regular for each $i$, and the decomposition number $d_{\lambda\nu_i, \mu\widetilde{\nu_i}}$ equals one if $(\lambda, \mu) \in \Lambda_i$, and equals zero otherwise.*

## 5.2. A construction of the twisted Baddeley modules

Let $a, b, c \in \mathbf{N}_0$ be such that $n = 2a + b + c$. In this section we explicitly construct the module $M_{(2a,b,c)}$. We also provide a $p$-permutation basis of $M_{(2a,b,c)}$ with respect to an arbitrary $p$-subgroup of $C_2 \wr S_n$.

**5.2.1. A module isomorphic to $M_{(2a,b,c)}$.** Let $\mathcal{C}_{(2a,b,c)}$ be the set

$$\left\{ \{g, \gamma, \delta\} : \begin{array}{l} g \in C_2 \wr S_n \text{ has cycle type } a \text{ positive 2-cycles} \\ \gamma = (\{i_{a+1}, \overline{i_{a+1}}\}, \ldots, \{i_{a+b}, \overline{i_{a+b}}\}) \\ \delta = ([\, i_{a+b+1}, \overline{i_{a+b+1}}\,], \ldots, [\, i_n, \overline{i_n}\,]) \\ \operatorname{supp}(g) \cup \{i_{a+1}, \overline{i_{a+1}}, \ldots, i_n, \overline{i_n}\} = \{1, \overline{1}, \ldots, n, \overline{n}\} \end{array} \right\},$$

where $[x, \overline{x}] = -[\overline{x}, x]$ as in §1.4.3.

Let $v = \{g, \gamma, \delta\} \in \mathcal{C}_{(2a,b,c)}$ be such that

$$\begin{aligned} \gamma &= (\{i_{a+1}, \overline{i_{a+1}}\}, \ldots, \{i_{a+b}, \overline{i_{a+b}}\}) \\ \delta &= ([\, i_{a+b+1}, \overline{i_{a+b+1}}\,], \ldots, [\, i_n, \overline{i_n}\,]). \end{aligned}$$

Define

$$\begin{aligned} \mathcal{S}(v) &= \operatorname{supp}(g) \cap \{1, 2, \ldots, n\} \\ \mathcal{T}(v) &= \{i_{a+1}, \ldots, i_{a+b}\} \\ \mathcal{U}(v) &= \{i_{a+b+1}, \ldots, i_n\}. \end{aligned}$$

As $2a + b + c = n$, these sets are mutually disjoint.

There is an action of $h \in C_2 \wr S_n$ on $v$ given by $hv = \{{}^h g, h\gamma, h\delta\}$. With $\mathcal{D}_{(2a,b,c)}$ defined to be $F$-span of the set

$$\{v - h \operatorname{sgn}\big(\widehat{h}\big) v : v \in \mathcal{C}_{(2a,b,c)}, h \in C_2 \wr S_{\mathcal{T}(v)} \times C_2 \wr S_{\mathcal{U}(v)}\},$$

we have the following lemma.

Lemma 5.2.1. *The vector space $F\mathcal{D}_{(2a,b,c)}$ is an $FC_2 \wr S_n$-submodule of $F\mathcal{C}_{(2a,b,c)}$.*

Proof. We show that $F\mathcal{D}_{(2a,b,c)}$ is closed under the action of $C_2 \wr S_n$. Fix $k \in C_2 \wr S_n$ and $v - \operatorname{sgn}\big(\widehat{h}\big) hv \in \mathcal{D}_{(2a,b,c)}$, where $h \in C_2 \wr S_{\mathcal{T}(v)} \times C_2 \wr S_{\mathcal{U}(v)}$. With $h' := {}^k h$, it follows that

$$\begin{aligned} k(v - \operatorname{sgn}(\widehat{h}) hv) &= kv - \operatorname{sgn}(\,\widehat{h}\,) khv \\ &= kv - \operatorname{sgn}(\,\widehat{h}\,) h'(kv) \\ &= kv - \operatorname{sgn}(\,\widehat{h'}\,) h'(kv), \end{aligned}$$

where the third equality holds as $h$ and $h'$ are conjugate in $C_2 \wr S_n$. By definition of $\mathcal{T}(v)$, there an equality $\mathcal{T}(kv) = \{\widehat{k}x : x \in \mathcal{T}(v)\}$, and the analogous equality holds for $\mathcal{U}(v)$. The lemma is now proved as $\operatorname{supp}(h') = \{kx : x \in \operatorname{supp}(h)\}$, and so $h' \in C_2 \wr S_{\mathcal{T}(kv)} \times C_2 \wr S_{\mathcal{U}(kv)}$. $\square$

Let $v = \{g, \gamma, \delta\} + \mathcal{D}_{(2a,b,c)}$ be such that

$$\begin{aligned}\gamma &= (\{i_{a+1}, \overline{i_{a+1}}\}, \ldots, \{i_{a+b}, \overline{i_{a+b}}\})\\ \delta &= ([i_{a+b+1}, \overline{i_{a+b+1}}], \ldots, [i_n, \overline{i_n}]),\end{aligned}$$

where $i_{a+1}, \ldots, i_n \in \{1, 2, \ldots, n\}$, with $i_{a+1} < \cdots < i_{a+b}$ and $i_{a+b+1} < \cdots < i_n$. Write $\mathcal{B}_{(2a,b,c)}$ for the set of all $v + \mathcal{D}_{(2a,b,c)}$ of this form. It follows from Lemma 5.2.1 that $\mathcal{B}_{(2a,b,c)}$ is a basis of $F\mathcal{C}_{(2a,b,c)}/\mathcal{D}_{(2a,b,c)}$. We use this basis in the following lemma to show that the quotient module $F\mathcal{C}_{(2a,b,c)}/\mathcal{D}_{(2a,b,c)}$ is isomorphic to $M_{(2a,b,c)}$ as an $FC_2 \wr S_n$-module. To simplify the notation, we write $(g, \gamma, \delta)$ for $\{g, \gamma, \delta\} + \mathcal{D}_{(2a,b,c)} \in \mathcal{B}_{(2a,b,c)}$.

LEMMA 5.2.2. *The $F$-span of $\mathcal{B}_{(2a,b,c)}$ is isomorphic to $M_{(2a,b,c)}$ as an $FC_2 \wr S_n$-module.*

PROOF. Recall that $f_a$ is the element equal to

$$(1\ a+1)(2\ a+2)\ldots(a\ 2a)(\overline{1}\ \overline{a+1})(\overline{2}\ \overline{a+2})\ldots(\overline{a}\ \overline{2a}),$$

with centraliser $V_a$ in $C_2 \wr S_{2a}$. It follows that the module $F\big\uparrow_{V_a}^{C_2\wr S_{2a}}$ has an $F$-basis given by the elements in the conjugacy class of $f_a$ in $C_2 \wr S_{2a}$. Let

$$\begin{aligned}\gamma &= (\{2a+1, \overline{2a+1}\}, \ldots, \{2a+b, \overline{2a+b}\})\\ \delta &= ([2a+b+1, \overline{2a+b+1}], \ldots, [n, \overline{n}]),\end{aligned}$$

and define $S$ to be the $F$-span of $\{({}^g f_a, \gamma, \delta) : g \in C_2 \wr S_{2a}\}$. Then $S$ is isomorphic, as an $F[C_2 \wr (S_{\{1,2,\ldots,2a\}} \times S_{\{2a+1,\ldots,2a+b\}} \times S_{\{2a+b+1,\ldots,n\}})]$-module, to

$$F\big\uparrow_{V_a}^{C_2\wr S_{2a}} \boxtimes (\mathrm{Inf}_{S_b}^{C_2\wr S_b} \mathrm{sgn}_b) \boxtimes (\widetilde{N}^{\otimes c} \otimes \mathrm{Inf}_{S_c}^{C_2\wr S_c} \mathrm{sgn}_c),$$

where we remind the reader that $N$ denotes the non-trivial one-dimensional $FC_2$-module. Let

$$w = (h, (\{j_{a+1}, \overline{j_{a+1}}\}, \ldots, \{j_{a+b}, \overline{j_{a+b}}\}), ([\,j_{a+b+1}, \overline{j_{a+b+1}}\,], \ldots, [\,j_n, \overline{j_n}\,])),$$

be a vector in $\mathcal{B}_{(2a,b,c)}$. As the natural action of $C_2 \wr S_n$ on its blocks

$$\{1, \overline{1}\}, \{2, \overline{2}\}, \ldots, \{n, \overline{n}\},$$

is transitive, there exists $\sigma \in C_2 \wr S_n$ such that ${}^\sigma f_a = h$, and ${}^\sigma k = j_k$ for all $k \in \{a+1, \ldots, n\}$. It follows that $\sigma\overline{v} = \pm\overline{w}$, and so $FS$ generates $F\mathcal{C}_{(2a,b,c)}/\mathcal{D}_{(2a,b,c)}$. Recall that $F\mathcal{C}_{(2a,b,c)}/\mathcal{D}_{(2a,b,c)}$ has a basis indexed by elements of the form $({}^g f_a, \tilde{\gamma}, \tilde{\delta})$. By the remark following Lemma 1.4.2, there are

$$\frac{2^n n!}{4^a a! 2^{b+c}(b+c)!}$$

conjugates of $f_a$ in $C_2 \wr S_n$. Given any such conjugate there are

$$\binom{b+c}{b}$$

ways to choose the support of $\widetilde{\gamma}$, which then determines $\widetilde{\gamma}$ and $\widetilde{\delta}$ completely. Therefore

$$\begin{aligned}\dim_F M_{(2a,b,c)} &= \frac{2^n n!}{4^a a! 2^{b+c}(b+c)!} \times \binom{b+c}{b} \\ &= \frac{2^{2a} \times (2a)!}{4^a a!} \times \frac{n!}{(2a)!b!c!} \\ &= \dim_F FS \times [C_2 \wr S_n : C_2 \wr S_{(2a,b,c)}].\end{aligned}$$

The result now follows by applying Lemma 1.3.2. □

Consider now the module $M_{(2a,0,0)} \cong F \uparrow_{V_a}^{C_2 \wr S_{2a}}$, which is a permutation module and therefore a $p$-permutation module. The modules $M_{(0,b,0)} = \operatorname{Inf}_{S_b}^{C_2 \wr S_b} \operatorname{sgn}_{S_b}$ and $M_{(0,0,c)} = \widetilde{N}^{\otimes c} \otimes \operatorname{Inf}_{S_c}^{C_2 \wr S_c} \operatorname{sgn}_{S_c}$ are one-dimensional modules. Therefore the action of any $p$-subgroup of $C_2 \wr S_b$ or $C_2 \wr S_c$ on $M_{(0,b,0)}$ or $M_{(0,0,c)}$, respectively, is trivial. It follows that both $M_{(0,b,0)}$ and $M_{(0,0,c)}$ are $p$-permutation modules. By definition

$$M_{(2a,b,c)} \cong \left(M_{(2a,0,0)} \boxtimes M_{(0,b,0)} \boxtimes M_{(0,0,c)}\right) \uparrow_{C_2 \wr S_{(2a,b,c)}}^{C_2 \wr S_n},$$

and so part (2) of Proposition 1.3.8 gives that $M_{(2a,b,c)}$ is a $p$-permutation module.

**5.2.2. A $p$-permutation basis of $M_{(2a,b,c)}$.** In this section we assume that $Q$ is a $p$-group contained in the top group $T_n$ of $C_2 \wr S_n$. Also given $(g, \gamma, \delta) \in \mathcal{B}_{(2a,b,c)}$ such that

$$\begin{aligned}\gamma &= (\{i_{a+1}, \overline{i_{a+1}}\}, \ldots, \{i_{a+b}, \overline{i_{a+b}}\}) \\ \delta &= ([\, i_{a+b+1}, \overline{i_{a+b+1}}\,], \ldots, [\, i_n, \overline{i_n}\,]),\end{aligned}$$

define $\vartheta((g, \gamma, \delta)) = (g, \gamma', \delta')$ where

$$\begin{aligned}\gamma' &= \{\{i_{a+1}, \overline{i_{a+1}}\}, \ldots, \{i_{a+b}, \overline{i_{a+b}}\}\} \\ \delta' &= \{[\, i_{a+b+1}, \overline{i_{a+b+1}}\,], \ldots, [\, i_n, \overline{i_n}\,]\}.\end{aligned}$$

LEMMA 5.2.3. *Let $Q$ be a $p$-subgroup of $T_n$. Then*

*(1) there is a choice of sign $s_v$ for each $v \in \mathcal{B}_{(2a,b,c)}$ such that*

$$\{s_v v : v \in \mathcal{B}_{(2a,b,c)}\}$$

*is a $p$-permutation basis of $M_{(2a,b,c)}$ with respect to $Q$,*

*(2) the element $v$ is fixed by $Q$ if and only $\vartheta(v)$ is fixed by $Q$. In this case, $s_v = 1$.*

PROOF. Let $H_{(2a,b,c)}$ be the set

$$\left\{\vartheta(v) : v \in \mathcal{B}_{(2a,b,c)}\right\}.$$

It is clear that there exists a natural bijection between $H_{(2a,b,c)}$ and $\mathcal{B}_{(2a,b,c)}$. Since $Q \leq T_n$, there is a natural action of $Q$ on $H_{(2a,b,c)}$.

Let $\vartheta(v_1), \vartheta(v_2), \ldots, \vartheta(v_l)$ be representatives for the $Q$-orbits on $H_{(2a,b,c)}$. Given $\vartheta(v) \in H_{(2a,b,c)}$, there exists a unique $k$ such that $\vartheta(v_k) = g\vartheta(v)$ for some $g \in Q$. Then $gv$ and $v_k$ are equal up to some ordering of the elements in their respective $b$-tuples and $c$-tuples. Therefore $v_k = s_v gv$, for some $s_v \in \{-1, +1\}$.

Suppose that there exists some other $\widetilde{g} \in Q$ such that $\vartheta(v_k) = \widetilde{g}\vartheta(v)$. Then $\pm v = \widetilde{g}^{-1}gv$, and so the $F$-span of $v$ is a one-dimensional module for the cyclic group generated by $\widetilde{g}^{-1}g$. The only such module is the trivial module, and so $gv = \widetilde{g}v$. The sign $s_v$ is therefore well-defined.

In order to complete the proof of the first part of the lemma, we need to check that the set

$$\{s_v v : v \in \mathcal{B}_{(2a,b,c)}\}$$

is a $p$-permutation basis for $M_{(2a,b,c)}$ with respect to $Q$. Suppose that $h \in Q$ is such that $s_v hv = \pm s_w w$, for $v$ and $w$ in $\mathcal{B}_{(2a,b,c)}$. Then $s_v v$ and $\pm s_w w$ lie in the same $Q$-orbit, and so there exists some $k$ such that $s_v v = gv_k$, and $\pm s_w w = \widetilde{g}v_k$. Therefore $\widetilde{g}^{-1}hgv_k = \pm v_k$. Arguing as before shows that the sign on the right hand side is positive, and so the first part of the lemma is proved.

For the second part of the lemma, if

$$\vartheta(v) := (g, \{\{i_{a+1}, \overline{i_{a+1}}\}, \ldots, \{i_{a+b}, \overline{i_{a+b}}\}\}, \{[\, i_{a+b+1}, \overline{i_{a+b+1}}\,], \ldots, [\, i_n, \overline{i_n}\,]\})$$

is fixed by $Q$, then $hv = \pm v$ for all $h \in Q$. Therefore the $F$-span of $v$ is a one-dimensional $Q$-module, and so $v$ is also fixed by $Q$ as required. Moreover, as $\vartheta(v)$ is its own $Q$-orbit representative, we have that $s_v = 1$. $\square$

## 5.3. The vertices of the summands of $M_{(2a,b,c)}$

Let $U$ be a non-projective indecomposable summand of $M_{(2a,b,c)}$. The vertex of $U$ is therefore non-trivial, and so it contains a conjugate of the cyclic group $C_p$ (viewed as a subgroup of $C_2 \wr S_n$). By the discussion in §1.4.1, any copy of $C_p$ in $C_2 \wr S_n$ is conjugate to

$$R_r := \langle \sigma_1 \sigma_2 \ldots \sigma_r \rangle,$$

where $\sigma_j := ((j-1)p + 1 \;\ldots\; jp)(\overline{(j-1)p+1} \;\ldots\; \overline{jp})$, for some $rp \le n$. It follows that $U(R_r) \neq 0$, and so in the first step of the proof of Theorem 5.1.1, we completely determine the indecomposable summands of $M_{(2a,b,c)}(R_r)$. In order to do this, we first describe the group $N_{C_2 \wr S_n}(R_r)$.

**5.3.1. The normaliser of $R_r$.** It is clear that there is a factorisation

$$N_{C_2 \wr S_n}(R_r) = N_{C_2 \wr S_{rp}}(R_r) \times C_2 \wr S_{\{rp+1,\ldots,n\}}, \tag{5.1}$$

and so it suffices to describe the group $N_{C_2 \wr S_{rp}}(R_r)$.

Let $j \in \mathbf{N}$ be such that $j \le r$. Define

$$\tau_j = ((j-1)p + 1 \; \overline{(j-1)p+1}) \ldots (jp \; \overline{jp}).$$

The subgroup $\langle \tau_1, \tau_2, \ldots, \tau_r \rangle$ is the full centraliser of $R_r$ in the subgroup $B_{rp}$ (see §1.4) of $C_2 \wr S_{rp}$.

Let $i \in \mathbf{N}$ be such that $i < r$. Define

$$\rho_i = ((i-1)p+1 \ ip+1)(\overline{(i-1)p+1} \ \overline{ip+1}) \ldots (ip \ (i+1)p)(\overline{ip} \ \overline{(i+1)p}).$$

We note that

$$^{\rho_i}\sigma_j = \begin{cases} \sigma_{j+1} & j = i \\ \sigma_{j-1} & j = i+1 \\ \sigma_j & j \notin \{i, i+1\}. \end{cases}$$

Let $x$ be a fixed primitive root modulo $p$. Given $i \in \mathbf{N}$, let $j$ be the unique natural number such that $(j-1)p < i \leq jp$. We define $z_r \in C_2 \wr S_{rp}$ to be the permutation such that $z_r(\bar{i}\,) = \overline{z_r(i)}$ and

$$z_r(i) = x(i-1) + 1 - i_k p,$$

where $i_k$ is the unique non-negative integer such that $(j-1)p < x(i-1) + 1 - i_k p \leq jp$ for all $i$. We give an example of $z_r$ in the case when $p = 3$.

Example 5.3.1. Let $p = 3$, and let $x = 2$. Then

$$z_r = (2\ 3)(\overline{2}\ \overline{3})(5\ 6)(\overline{5}\ \overline{6}) \ldots (3r-1\ 3r)(\overline{3r-1}\ \overline{3r}),$$

and observe that $^{z_r}(\sigma_1\sigma_2\ldots\sigma_r) = (\sigma_1\sigma_2\ldots\sigma_r)^2$.

For all $1 \leq i \leq r$, the element $z_r$ commutes with $\tau_i$, and $^{z_r}\sigma_i = \sigma_i^x$. As $R_r \leq T_n$, applying Lemma 1.2.3 gives the following result.

Lemma 5.3.2. *The normaliser subgroup $N_{C_2 \wr S_{rp}}(R_r)$ is generated by the set*

$$\{\tau_i, \sigma_i, \rho_i : 1 \leq i \leq r-1\} \cup \{\tau_r, \sigma_r\} \cup \{z_r\}.$$

*Furthermore, this set without the element $z_r$ generates the centraliser subgroup $C_{C_2 \wr S_{rp}}(R_r)$.*

Observe that there are isomorphisms of abstract groups $N_{C_2 \wr S_{rp}}(R_r) \cong (C_{2p} \wr S_r) \rtimes C_{p-1}$, and $C_{C_2 \wr S_{rp}}(R_r) \cong C_{2p} \wr S_r$.

**5.3.2. The proof of Theorem 5.1.1.** We are now ready to proceed with the first step of the proof.

**First step: The Brauer correspondent $M_{(2a,b,c)}(R_r)$.** Fix $r \in \mathbf{N}$ such that $rp \leq n$. Define

$$T^r = \{(2s, t, u) \in \mathbf{N}_0^3 : 2s + t + u = r, sp \leq a, tp \leq b, up \leq c\}.$$

By the first part of Lemma 5.2.3, for each $v \in \mathcal{B}_{(2a,b,c)}$, there exists $s_v \in \{-1, 1\}$ such that $\{s_v v : v \in \mathcal{B}_{(2a,b,c)}\}$ is a $p$-permutation basis of $M_{(2a,b,c)}$ with respect to $R_r$. Moreover, by the second part of Lemma 5.2.3 we can take $s_v = 1$ for all $v \in \mathcal{B}^{R_r}_{(2sp,tp,up)}$.

Given $(2s,t,u) \in T^r$, define $\mathcal{A}_{(2s,t,u)}$ to be the set

$$\left\{ v : \begin{array}{l} v \in \mathcal{B}^{R_r}_{(2a,b,c)} \\ \mathcal{S}(v) \text{ contains exactly } 2s \text{ orbits of } \widehat{R_r} \text{ of length } p \\ \mathcal{T}(v) \text{ contains exactly } t \text{ orbits of } \widehat{R_r} \text{ of length } p \\ \mathcal{U}(v) \text{ contains exactly } u \text{ orbits of } \widehat{R_r} \text{ of length } p \end{array} \right\}.$$

LEMMA 5.3.3. *There is a decomposition of $FN_{C_2 \wr S_n}(R_r)$-modules given by the direct sum*

$$M_{(2a,b,c)}(R_r) = \bigoplus_{(2s,t,u)} \langle \mathcal{A}_{(2s,t,u)} \rangle,$$

*where the sum runs over all $(2s,t,u) \in T^r$.*

PROOF. Given $v \in \mathcal{A}_{(2s,t,u)}$, let

$$v = (g, (\{i_{a+1}, \overline{i_{a+1}}\}, \ldots, \{i_{a+b}, \overline{i_{a+b}}\}), ([i_{a+b+1}, \overline{i_{a+b+1}}], \ldots, [i_n, \overline{i_n}])).$$

We first prove that the number of $\widehat{R_r}$-orbits contained in $\mathcal{S}(v)$ must be even. If $v \in \mathcal{B}^{R_r}_{(2a,b,c)}$, then $g \in C_{C_2 \wr S_n}(R_r)$. Therefore $g$ permutes the $R_r$-orbits as blocks for its action, and the same is true for $\widehat{g}$ and $\widehat{R_r}$. As $\widehat{g}$ has order 2 and $p$ is odd, the number of $\widehat{R_r}$-orbits contained in $\mathcal{S}(v)$ is necessarily even. Given $h \in N_{C_2 \wr S_n}(R_r)$, let $hv = \pm\tilde{v}$, where

$$\tilde{v} = ({}^h g, (\{j_{a+1}, \overline{j_{a+1}}\}, \ldots, \{j_{a+b}, \overline{j_{a+b}}\}), ([j_{a+b+1}, \overline{j_{a+b+1}}], \ldots, [j_n, \overline{j_n}])).$$

The $\widehat{R_r}$-orbits contained in $\mathcal{S}(\tilde{v})$ are exactly the conjugates by $\widehat{h}$ of the $\widehat{R_r}$-orbits contained in $\mathcal{S}(v)$. The same argument holds for $\mathcal{T}(\tilde{v})$ and $\mathcal{U}(\tilde{v})$, and so $\tilde{v} \in \langle \mathcal{A}_{(2s,t,u)} \rangle$. It follows that $\langle \mathcal{A}_{(2s,t,u)} \rangle$ is a submodule of $M_{(2a,b,c)}(R_r)$. The lemma now follows as $\mathcal{B}_{(2a,b,c)} = \bigcup \mathcal{A}_{(2s,t,u)}$. □

In the following lemma, we factorise the module $\langle \mathcal{A}_{(2s,t,u)} \rangle$ as an outer tensor product of modules, compatible with the factorisation of $N_{C_2 \wr S_n}(R_r)$ in (5.1). By doing this, we see that in order to understand $M_{(2a,b,c)}(R_r)$, it is sufficient to understand the module $M_{(2sp,tp,up)}(R_r)$, where $(2s,t,u) \in T^r$.

LEMMA 5.3.4. *There is an isomorphism*

$$\langle \mathcal{A}_{(2s,t,u)} \rangle \cong M_{(2sp,tp,up)}(R_r) \boxtimes M_{(2(a-sp),b-tp,c-up)},$$

*of $F[N_{C_2 \wr S_{rp}}(R_r) \times C_2 \wr S_{\{rp+1,\ldots,n\}}]$-modules.*

PROOF. Let $\mathcal{B}^+_{(2(a-sp),b-tp,c-up)}$ denote the set consisting of the elements in $\mathcal{B}_{(2(a-sp),b-tp,c-up)}$, each shifted appropriately by $rp$ or $\overline{rp}$. The $F$-span of $\mathcal{B}^+_{(2(a-sp),b-tp,c-up)}$ is therefore an $F[C_2 \wr S_{\{rp+1,\ldots,n\}}]$-module isomorphic to $M_{(2(a-sp),b-tp,c-up)}$.

Let $v \in \mathcal{A}_{(2s,t,u)}$ be such that

$$v = (g, (\{i_{a+1}, \overline{i_{a+1}}\}, \ldots, \{i_{a+b}, \overline{i_{a+b}}\}), ([i_{a+b+1}, \overline{i_{a+b+1}}], \ldots, [i_n, \overline{i_n}])),$$

where $\mathcal{S}(v) = \{i_1, \ldots, i_a\}$ and the notation is chosen so that

$$\{i_1, \ldots, i_{2sp}\} \cup \{i_{a+1}, \ldots, i_{a+tp}\} \cup \{i_{a+b+1}, \ldots, i_{a+b+up}\} = \{1, 2, \ldots, rp\}.$$

Let $v_1 \in \mathcal{B}^{R_r}_{(2sp,tp,up)}$ be the unique element such that

$$\begin{aligned}
\mathcal{S}(v_1) &= \mathcal{S}(v) \cap \{1, 2, \ldots, rp\} \\
\mathcal{T}(v_1) &= \mathcal{T}(v) \cap \{1, 2, \ldots, rp\} \\
\mathcal{U}(v_1) &= \mathcal{U}(v) \cap \{1, 2, \ldots, rp\}.
\end{aligned}$$

By construction, the $p$-element $\sigma_1\sigma_2 \ldots \sigma_r$ has support $\{1, \overline{1}, \ldots, rp, \overline{rp}\}$, and so $v$ is fixed by $R_r$ if and only if $v_1$ is fixed by $R_r$. Let $v_2 \in \mathcal{B}^{+}_{(2(a-sp),b-tp,c-up)}$ be such that

$$\begin{aligned}
\mathcal{S}(v_2) &= \mathcal{S}(v) \backslash \mathcal{S}(v_1) \\
\mathcal{T}(v_2) &= \mathcal{T}(v) \backslash \mathcal{T}(v_1) \\
\mathcal{U}(v_2) &= \mathcal{U}(v) \backslash \mathcal{U}(v_1).
\end{aligned}$$

It follows that there is a natural bijection $\Theta$ between $\mathcal{B}^{R_r}_{(2a,b,c)}$ and

$$\mathcal{B}^{R_r}_{(2sp,tp,up)} \times \mathcal{B}^{+}_{(2(a-sp),b-tp,c-up)},$$

defined by $\Theta(v) = v_1 \otimes v_2$.

We now show that $\Theta$ is an $F[N_{C_2 \wr S_n}(R_r)]$-module homomorphism. Given $g \in N_{C_2 \wr S_n}(R_r)$ and $v \in \mathcal{B}_{(2a,b,c)}$, let $v^\star \in \mathcal{B}_{(2a,b,c)}$ be such that the entries in its $b$-tuple and $c$-tuple are those of $gv$ in ascending order (with respect to the orders in §1.4.3). Let $h \in C_2 \wr S_{\mathcal{T}(gv)} \times C_2 \wr S_{\mathcal{U}(gv)}$ be the unique permutation such that $v^\star = hgv$. As $g \in N_{C_2 \wr S_{rp}}(R_r) \times C_2 \wr S_{\{rp+1,\ldots,n\}}$, it permutes the elements in the sets $\{1, 2, \ldots, rp, \overline{1}, \ldots, \overline{rp}\}$ and $\{rp+1, \ldots, n, \overline{rp+1}, \ldots, \overline{n}\}$ separately. It follows that there is a factorisation $h = h_1 h_2$, where $h_1 \in C_2 \wr S_{\{1,2,\ldots,rp\}}$ and $h_2 \in C_2 \wr S_{\{rp+1,\ldots,n\}}$. Therefore

$$\begin{aligned}
\Theta(gv) &= \Theta(\operatorname{sgn}(\,\widehat{h}\,)v^\star) \\
&= \operatorname{sgn}(\,\widehat{h}\,)(v_1^\star \otimes v_2^\star) \\
&= \operatorname{sgn}(\,\widehat{h}\,)\operatorname{sgn}(\,\widehat{h_1}\,)\operatorname{sgn}(\,\widehat{h_2}\,)(v_1 \otimes v_2) \\
&= v_1 \otimes v_2 = g\Theta(v),
\end{aligned}$$

and so the result is proved. □

In order to express $M_{(2sp,tp,up)}(R_r)$ as a sum of indecomposable modules, we first write $M_{(2sp,tp,up)}(R_r)$ as a direct sum of $FN_{C_2 \wr S_{rp}}(R_r)$-modules $N_{(\lambda,t,u)}$ (defined below), before showing that each of these modules is indecomposable. We require a deeper understanding of the fixed points in $\mathcal{B}^{R_r}_{(2sp,tp,up)}$ before we can define $N_{(\lambda,t,u)}$. To do this we consider the example $M_{(2p,0,0)}(R_r)$ for all $r \in \mathbf{N}$. This illustrative example will also be used when describing $\mathcal{B}^{R_r}_{(2sp,tp,up)}$ in the general case.

EXAMPLE 5.3.5. The $FC_2 \wr S_{2p}$-module $M_{(2p,0,0)}$ is a permutation module, with permutation basis given by the set

$$\mathcal{B}_p := \{{}^h f_p : h \in C_2 \wr S_{2p}\}.$$

The set $T^j$ is empty for all $j \in \mathbf{N}$ such that $j \neq 2$, and so we consider $M_{(2p,0,0)}(R_2)$. Rewrite $\sigma_1\sigma_2$ as follows:

$$\sigma_1\sigma_2 = (1\ 2\ \dots\ p)(\overline{1}\ \overline{2}\ \dots\ \overline{p})(1^*\ 2^* \dots\ p^*)(\overline{1^*}\ \overline{2^*} \dots\ \overline{p^*}),$$

where $x^* := x + p$ for $1 \leq x \leq p$.

Let $g \in \mathcal{B}_p$ be fixed by $R_2$. If $g(1) = x$, then for $1 \leq i \leq p-1$,

$$g(i+1) = (\sigma_1\sigma_2)^i(x), \tag{5.2}$$

and so $g$ is completely determined by $g(1)$.

Suppose that $x \in \{2, \overline{2}, \dots, p, \overline{p}\}$. If $x \in \{2, \dots, p\}$, it follows from (5.2) that $g(x) = 2x-1 \mod p$. As $p$ is odd, we cannot have that $g(x) = 1$, and so $g$ does not have order 2. It follows that $g$ cannot be a conjugate of $f_p$, which is a contradiction. An entirely similar argument shows that $x \notin \{\overline{2}, \dots, \overline{p}\}$.

There are now precisely $2p$ possible choices for $x$, each of which completely determines $g$. Therfore the module $M_{(2p,0,0)}(R_2)$ has dimension $2p$.

Fix $(2s, t, u) \in T^r$, and let $k = t + u$. We define $\Omega^{(2s;k)}$ to be the set of elements of the form

$$\{\{i_1, i'_1\}, \dots, \{i_s, i'_s\}, \{j_1, \dots, j_k\}\},$$

where $\{i_1, i'_1, \dots, i_s, i'_s, j_1, \dots, j_k\} = \{1, 2, \dots, r\}$. Let $c_{s,k} = |\Omega^{(2s;k)}|$. Given $\omega \in \Omega^{(2s;k)}$ of the above form, define

$$R_\omega = \langle \sigma_{i_1}\sigma_{i'_1}\rangle \times \dots \times \langle \sigma_{i_s}\sigma_{i'_s}\rangle \times \langle \sigma_{j_1}\rangle \times \dots \times \langle \sigma_{j_k}\rangle.$$

and write $\mathcal{B}(\omega)$ for $\mathcal{B}^{R_\omega}_{(2sp,tp,up)}$.

LEMMA 5.3.6. *Given* $v \in \mathcal{B}^{R_r}_{(2sp,tp,up)}$, *there exists a unique* $\omega \in \Omega^{(2s;k)}$ *such that* $v \in \mathcal{B}(\omega)$.

PROOF. By the second part of Lemma 5.2.3, the vector $v \in \mathcal{B}_{(2sp,tp,up)}$ is fixed by $R_r$ if and only if $\vartheta(v)$ is fixed by $R_r$. Let $v$ be such that $\vartheta(v) = (g, \gamma, \delta)$ where

$$\gamma = \{\{i_{2sp+1}, \overline{i_{2sp+1}}\}, \dots, \{i_{(2s+t)p}, \overline{i_{(2s+t)p}}\}\}$$
$$\delta = \{[i_{(2s+t)p+1}, \overline{i_{(2s+t)p+1}}], \dots, [i_{rp}, \overline{i_{rp}}]\}.$$

By definition there is a factorisation $g = g_1 \dots g_s$, where each $g_j$ has cycle type $p$ positive 2-cycles. For each $1 \leq j \leq s$, let $\{i_1, \dots, i_{2s}\}$ be such that $\mathrm{supp}(\sigma_{i_{2j-1}}\sigma_{i_{2j}}) = \mathrm{supp}(g_j)$. It follows from Example 5.3.5 that $g$ commutes with $R_r$ if and only if $g$ commutes with $\sigma_{i_{2j-1}}\sigma_{i_{2j}}$ for each $j$.

Let $\{j_1, \dots, j_t\}$ be such that

$$\mathrm{supp}(\sigma_{j_1} \dots \sigma_{j_t}) = \{i_{2sp+1}, \overline{i_{2sp+1}}, \dots, i_{(2s+t)p}, \overline{i_{(2s+t)p}}\}.$$

As $\gamma$ is fixed by $R_r$, the set $\mathcal{T}(v)$ is equal to a union of $\widehat{R_r}$-orbits. The orbits of $\widehat{R_r}$ are equal to precisely the orbits of $\widehat{\sigma_{j_i}}$ for each $1 \leq i \leq r$. Therefore $\gamma$ is fixed by $R_r$ if and only if it is fixed by the group $\langle \sigma_{j_1} \rangle \times \cdots \times \langle \sigma_{j_t} \rangle$.

Similarly if $\delta$ is such that

$$\operatorname{supp}(\sigma_{k_1} \dots \sigma_{k_u}) = \{i_{(2s+t)p+1}, \overline{i_{(2s+t)p+1}}, \dots, i_{rp}, \overline{i_{rp}}\},$$

then $\delta$ is fixed by the group $\langle \sigma_{k_1} \rangle \times \cdots \times \langle \sigma_{k_u} \rangle$.

Therefore if $v$ is fixed by $R_r$, then $v$ is fixed by $R_\omega$, where

$$\omega := \{\{i_1, i_2\}, \dots, \{i_{2s-1}, i_{2s}\}, \{j_1, \dots, j_t, k_1, \dots, k_u\}\}.$$

Moreover, the uniqueness of $\omega$ follows as it is determined by the fixed sets $\operatorname{supp}(g), \operatorname{supp}(\gamma)$, and $\operatorname{supp}(\delta)$. $\square$

Given $\varnothing \neq E \subseteq \{1, 2, \dots, r\}$, define $\tau_E = \prod_{e \in E} \tau_e$. If $E$ is empty, then set $\tau_E = 1$.

Definition. Fix $y \in \{-1, 1\}^s$. Given $(g, \gamma, \delta) \in \mathcal{B}^{R_r}_{(2sp,tp,up)}$, let

$$\omega = \{\{i_1, i'_1\}, \dots, \{i_s, i'_s\}, \{j_1, \dots, j_k\}\}$$

be the unique element of $\Omega^{(2s;k)}$ such that $(g, \gamma, \delta) \in \mathcal{B}(\omega)$. Define

$$(y(g), \gamma, \delta) = \sum_{E \subseteq \{i_1, \dots, i_s\}} (\prod_{e \in E} y_e)({}^{\tau_E} g, \gamma, \delta).$$

It follows from Example 5.3.5 and Lemma 5.3.6 that $\tau_{i_j}$ and $\tau_{i'_j}$ act in the same way on $g$, and so $(y(g), \gamma, \delta)$ is well-defined.

Example 5.3.7. Let $r = 4$, and consider the element $f_6 \in \mathcal{B}_{(12,0,0)}$, where we remind the reader that

$$f_6 = (1\ 7)(2\ 8)(3\ 9)(4\ 10)(5\ 11)(6\ 12)(\overline{1}\ \overline{7})(\overline{2}\ \overline{8})(\overline{3}\ \overline{9})(\overline{4}\ \overline{10})(\overline{5}\ \overline{11})(\overline{6}\ \overline{12}).$$

Moreover, $f_6$ is contained in $B(\omega)$, where $\omega = \{\{1, 3\}, \{2, 4\}\}$. Define $x, y \in \{-1, 1\}^2$ as follows: $x = (1, -1)$ and $y = (-1, 1)$. Then

$$\begin{aligned} x(f_6) &= f_6 + {}^{\tau_1} f_6 - {}^{\tau_2} f_6 - {}^{\tau_1 \tau_2} f_6 \\ y(f_6) &= f_6 - {}^{\tau_1} f_6 + {}^{\tau_2} f_6 - {}^{\tau_1 \tau_2} f_6. \end{aligned}$$

Given $y \in \{-1, 1\}^s$, if $\lambda \in \Lambda(2, s)$ is such that $\lambda_1$ (resp. $\lambda_2$) equals the number of $y_i$ equal to $+1$ (resp. $-1$), then we say that $y$ has *weight* $\lambda$.

We now define $N_{(\lambda,t,u)}$ to be the $F$-span of

$$\{(y(g), \gamma, \delta) : (g, \gamma, \delta) \in \mathcal{B}^{R_r}_{(2sp,tp,up)} \text{ and } y \text{ has weight } \lambda\}. \tag{5.3}$$

It is clear that

$$M_{(2sp,tp,up)}(R_r) = \bigoplus_{\lambda \in \Lambda(2,s)} N_{(\lambda,t,u)}, \tag{5.4}$$

is an equality of vector spaces. We give an example of $N_{(\lambda,t,u)}$ in Example 5.3.13 below.

Observe that $N_{C_2 \wr S_{rp}}(R_r)$ permutes the $R_r$-orbits as blocks for its action. It follows that $N_{C_2 \wr S_{rp}}(R_r)$ acts on the set of subgroups of the form $R_\omega$ by conjugation. We have seen in the proof of Lemma 5.3.6 that the $R_r$-orbits are the same as the orbits of the subgroup

$$C := \langle \sigma_1 \rangle \times \cdots \times \langle \sigma_r \rangle,$$

and we write $\mathcal{O}_i$ for the union of the non-trivial orbits of $\langle \sigma_i \rangle$.

LEMMA 5.3.8. *Given $\omega, \widetilde{\omega} \in \Omega^{(2s;k)}$, let $h \in N_{C_2 \wr S_{rp}}(R_r)$ be such that ${}^h R_\omega = R_{\widetilde{\omega}}$. Then $h(y(g), \gamma, \delta)$ is contained in the $F$-span of $\mathcal{B}(\widetilde{\omega})$.*

PROOF. Given $1 \le i \le r$, let $\tilde{i}$ be such that $h\mathcal{O}_i = \mathcal{O}_{\tilde{i}}$. It follows from the definition of $(y(g), \gamma, \delta)$ that

$$\begin{aligned} h(y(g), \gamma, \delta) &= \sum_{E \subseteq \{i_1, \ldots, i_s\}} (\prod_{e \in E} y_e) h({}^{\tau_E} g, \gamma, \delta) \\ &= \sum_{E \subseteq \{i_1, \ldots, i_s\}} (\prod_{e \in E} y_e)(({}^{h\tau_E} g), h\gamma, h\delta) \\ &= \sum_{\widetilde{E} \subseteq \{\widetilde{i_1}, \ldots, \widetilde{i_s}\}} (\prod_{e \in E} y_e)({}^{\tau_{\widetilde{E}}}({}^h g), h\gamma, h\delta) \\ &= (\widetilde{y}({}^h g), h\gamma, h\delta), \end{aligned}$$

where $\widetilde{E} := \{\tilde{i} : i \in E\}$ and $\widetilde{y}_{\tilde{i}} := y_i$ for all $i \in \{i_1, i_2, \ldots, i_s\}$. The lemma is proved once we show that $({}^h g, h\gamma, h\delta)$ is fixed by $R_{\widetilde{\omega}}$. As ${}^h\sigma_i = \sigma_{\tilde{i}}$ for all $1 \le i \le r$, for $1 \le j \le s$

$${}^{\sigma_{\widetilde{i_j}} \sigma_{\widetilde{i'_j}}}({}^h g) = {}^h\left({}^{\sigma_{i_j} \sigma_{i'_j}} g\right) = {}^h g.$$

An entirely similar argument shows that $\sigma_{\widetilde{i_j}} h\gamma = h\gamma$ for $s < j \le s + t$, and that $\sigma_{\widetilde{i_j}} h\delta = h\delta$ for $s + t < j \le r$. $\square$

COROLLARY 5.3.9. *Let $h \in N_{C_2 \wr S_{rp}}(R_r)$ be such that ${}^h\tau_i = \tau_i$ and ${}^h\sigma_i = \sigma_i^x$, for $1 \le i \le r$ and some $x \in \mathbf{N}$. If $(g, \gamma, \delta) \in \mathcal{B}(\omega)$, then $h(g, \gamma, \delta)$ is contained in the $F$-span of $B(\omega)$. In particular if $h = \tau_{i_j}$ for some $j \in \{1, 2, \ldots, s\}$, then $h(y(g), \gamma, \delta) = y_j(y(g), \gamma, \delta)$.*

PROOF. For the first statement, observe that ${}^h\mathcal{O}_i = \mathcal{O}_i$ for $1 \le i \le r$. Now apply Lemma 5.3.8.

For the second statement observe that if $h = \tau_{i_j}$ for some $j \in \{1, \ldots, s\}$, then

$$\begin{aligned} \tau_{i_j}(y(g), \gamma, \delta) &= \sum_{E \subseteq \{i_1, \ldots, i_s\}} (\prod_{e \in E} y_e)({}^{\tau_{i_j} \tau_E} g, \gamma, \delta) \\ &= y_j \sum_{E \subseteq \{i_1, \ldots, i_s\}} (\prod_{e \in E} y_e y_j)({}^{\tau_{i_j} \tau_E} g, \gamma, \delta) = y_j(y(g), \gamma, \delta). \quad \square \end{aligned}$$

It follows from Lemma 5.3.8 that each $N_{(\lambda,t,u)}$ is an $FN_{C_2\wr S_{rp}}(R_r)$-module, and so the decomposition of $M_{(2sp,tp,up)}(R_r)$ in (5.4) is as a direct sum of $FN_{C_2\wr S_{rp}}(R_r)$-modules.

Write $K_r$ for $C_{C_2\wr S_{rp}}(R_r)$. In order to prove that each $N_{(\lambda,t,u)}$ is an indecomposable $FN_{C_2\wr S_{rp}}(R_r)$-module, we show that it is an indecomposable an $FK_r$-module. We do this by filling in the details of the following sketch.

Given $1 \leq i \leq r$, define $D_i = \langle \sigma_i, \tau_i \rangle$, and so $D_1 \times \cdots \times D_r$ is a normal subgroup in $K_r$. We define an $F[D_1 \times \cdots \times D_r]$-module $N_y^{\omega^\star}$, and in Lemma 5.3.10 we determine its inertial group $Y_{(\lambda,t,u)}$ in $K_r$. Using Lemma 5.3.11 we determine the dimension of $N_{(\lambda,t,u)}$. In Lemma 5.3.12 we show that $N_y := N_y^{\omega^\star} \uparrow_{X_{(\lambda,t,u)}}^{Y_{(\lambda,t,u)}}$ is indecomposable, where $X_{(\lambda,t,u)}$ is the largest subgroup in $Y_{(\lambda,t,u)}$ that $N_y^{\omega^\star}$ can be extended to. We then also prove that $N_{(\lambda,t,u)} \cong N_y \uparrow_{Y_{(\lambda,t,u)}}^{K_r}$. It follows using Proposition 1.2.5 that $N_{(\lambda,t,u)}$ is an indecomposable $FK_r$-module.

Define $\omega^\star = \{\{1, s+1\}, \ldots, \{s, 2s\}, \{2s+1, \ldots, r\}\} \in \Omega^{(2s;k)}$. Let $v^\star := (f_{sp}, \gamma^\star, \delta^\star) \in B(\omega^\star)$ be such that

$$\mathcal{T}(v^\star) = \operatorname{supp}(\sigma_{2s+1} \ldots \sigma_{2s+t}) \cap \{1, 2, \ldots, n\}$$
$$\mathcal{U}(v^\star) = \operatorname{supp}(\sigma_{2s+t+1} \ldots \sigma_r) \cap \{1, 2, \ldots, n\}.$$

Given $\lambda \in \Lambda(2, s)$, define $y_\lambda \in \{-1, 1\}^s$ to be the tuple of weight $\lambda$ such that

$$(y_\lambda)_i = \begin{cases} 1 & \text{if } 1 \leq i \leq \lambda_1 \\ -1 & \text{if } \lambda_1 + 1 \leq i \leq s. \end{cases}$$

We now define $N_y^{\omega^\star}$ to be the $F$-span of

$$\{(y_\lambda(g), \gamma^\star, \delta^\star) : (g, \gamma^\star, \delta^\star) \in B(\omega^\star)\}.$$

Also let $X_{(\lambda,t,u)}$ be the subgroup of $K_r$ generated by the set

$$\{\sigma_i, \tau_i : 1 \leq i \leq r\} \cup \{\rho_1^{\rho_2\rho_3\cdots\rho_s}\} \cup \{\rho_i\rho_{i+s} : 1 \leq i \leq s-1 \text{ and } i \neq \lambda_1\}$$
$$\cup \{\rho_i : 2s+1 \leq i < r, i \neq 2s+t\},$$

and let $Y_{(\lambda,t,u)}$ be the subgroup of $K_r$ generated by the set

$$\{\sigma_i, \tau_i : 1 \leq i \leq r\} \cup \{\rho_i : 1 \leq i \leq r-1 \text{ and } i \notin \{2\lambda_1, 2s, 2s+t\}\}.$$

Similar to the remark following Lemma 5.3.2, there are isomorphisms of abstract groups $X_{(\lambda,t,u)} \cong C_{2p} \wr ((S_2 \wr S_\lambda) \times S_t \times S_u)$, and $Y_{(\lambda,t,u)} \cong C_{2p} \wr (S_{2\lambda} \times S_t \times S_u)$.

LEMMA 5.3.10. *The vector space $N_y^{\omega^\star}$ is an $F[D_1 \times \cdots \times D_r]$-module, with inertial group $Y_{(\lambda,t,u)}$ in $K_r$. Moreover, we can extend $N_y^{\omega^\star}$ to a module for $FX_{(\lambda,t,u)}$.*

PROOF. That $N_y^{\omega^\star}$ is an $F[D_1 \times \cdots \times D_r]$-module follows by applying the first statement of Corollary 5.3.9.

Write $T$ for the inertial group of $N_y^{\omega^\star}$, which permutes the groups $D_i$ by conjugation. The permutations $\sigma_1, \ldots, \sigma_{2s}$ act freely on $N_y^{\omega^\star}$, whereas $\sigma_{2s+1}, \ldots, \sigma_r$ all act trivially on $N_y^{\omega^\star}$. Therefore $T$ must be contained in the subgroup of $K_r$ that permutes the groups $D_1, \ldots, D_{2s}$ amongst themselves and the groups $D_{2s+1}, \ldots, D_r$ amongst themselves.

For $2s < i \leq 2s+t$, the action of $\tau_i$ on $(y_\lambda(g), \gamma^\star, \delta^\star)$ is determined by its action on

$$(\{(i-1)p+1, \overline{(i-1)p+1}\}, \ldots, \{ip, \overline{ip}\}).$$

Therefore $\tau_i$ acts trivially in this case. Similarly for $2s+t < i \leq r$, the action of $\tau_i$ on $(y_\lambda(g), \gamma^\star, \delta^\star)$ is determined by its action on

$$([(i-1)p+1, \overline{(i-1)p+1}\,], \ldots, [ip, \overline{ip}]).$$

It follows that $\tau_i$ acts with sign $(-1)^p$, which is negative as $p$ is odd. Therefore $T$ must be contained in the subgroup of $K_r$ that permutes the subgroups $D_{2s+1}, \ldots, D_{2s+t}$ amongst themselves, and the subgroups $D_{2s+t+1}, \ldots, D_r$ amongst themselves.

It follows from the second statement of Corollary 5.3.9 that $T$ must permute the groups $D_1, \ldots, D_{\lambda_1}, D_{s+1}, \ldots, D_{\lambda_1+s}$ amongst themselves, and the same is true for the groups $D_{\lambda_1+1}, \ldots, D_{2s}, D_{\lambda_1+s+1}, \ldots, D_{2s}$. This shows that $T$ is contained in $Y_{(\lambda,t,u)}$. Moreover, if $h \in Y_{(\lambda,t,u)}$, then ${}^h(N_y^{\omega^\star}) \cong N_y^{\omega^\star}$. Therefore $Y_{(\lambda,t,u)}$ is contained in $T$, which proves the second statement of the lemma.

For the final statement, it remains to prove that $N_y^{\omega^\star}$ is closed under the action of

$$Z \cup \{\rho_i : 2s+1 \leq i < r, i \neq 2s+t\},$$

where $Z = \{\rho_1^{\rho_2\rho_3\ldots\rho_s}\} \cup \{\rho_i\rho_{i+s} : 1 \leq i \leq s-1 \text{ and } i \neq \lambda_1\}$. It is sufficient to prove that each of

- $({}^z(y(g)), \gamma^\star, \delta^\star)$, where $z \in Z$
- $(y(g), \rho_i\gamma^\star, \delta^\star)$, where $2s+1 \leq i < 2s+t$
- $(y(g), \gamma^\star, \rho_i\delta^\star)$, where $2s+t < i < r$,

is contained in $N_y^{\omega^\star}$.

First consider $\rho_i\gamma^\star$, where $2s+1 \leq i < 2s+t$. In this case $\rho_i$ permutes precisely those orbits of $R_{\omega^\star}$ with support equal to the support of $\gamma^\star$. Therefore $\rho_i\gamma^\star = \pm\gamma^\star$. The same argument shows that $\rho_i\delta^\star = \pm\delta^\star$ for $2s+t < i < r$.

Given $z \in \{\rho_1^{\rho_2\rho_3\ldots\rho_s}\} \cup \{\rho_i\rho_{i+s} : 1 \leq i \leq s-1 \text{ and } i \neq \lambda_1\}$, we have

$$\begin{aligned} z(y(g), \gamma^\star, \delta^\star) &= \sum_{E \subseteq \{1,\ldots,s\}} (\prod_{e \in E} y_e)({}^{z\tau_E}g, \gamma^\star, \delta^\star) \\ &= \sum_{E' \subseteq \{1,\ldots,s\}} (\prod_{e \in E} y_e)({}^{\tau_{E'}}({}^z g), \gamma^\star, \delta^\star), \end{aligned}$$

where $E' = E$ if $z = \rho_1^{\rho_2\rho_3\cdots\rho_s}$, otherwise $E'$ is the subset of $\{1,\dots,s\}$ obtained from $E$ by swapping $i$ and $i+1$ for some $1 \le i \le s-1$. In particular $i$ is such that $y_i = y_{i+1}$, and so in either case it follows that $z(y(g),\gamma,\delta) = (y({}^zg),\gamma,\delta)$. The lemma is proved once we show that $({}^zg,\gamma,\delta) \in B(\omega^\star)$. This follows from the first statement of Corollary 5.3.9 as $z$ centralises $R_{\omega^\star}$. $\square$

Before we state and prove our next lemma, we remind the reader that $k = t + u$.

LEMMA 5.3.11. *The module $M_{(2sp,tp,up)}(R_r)$ has dimension equal to*

$$(2p)^s \times \binom{k}{t} \times c_{s,k}.$$

PROOF. By Lemma 5.3.6 every element in $\mathcal{B}^{R_r}_{(2sp,tp,up)}$ is fixed by $R_\omega$, for a unique $\omega \in \Omega^{(2s;k)}$. We therefore count the size of $\mathcal{B}(\omega)$ for each $\omega$. Fix $\omega \in \Omega^{(2s;k)}$, and write $\omega = \{\{i_1,i'_1\},\dots,\{i_s,i'_s\},\{j_1,\dots,j_k\}\}$.

Let $(g,\gamma,\delta) \in \mathcal{B}(\omega)$. Then we can write $g = g_1 \dots g_s$, where each $g_j$ has cycle type $p$ positive 2-cycles and $g_j$ is fixed by $\sigma_{i_j}\sigma_{i'_j}$ for each $1 \le j \le s$. By Example 5.3.5, each $\sigma_{i_j}\sigma_{i'_j}$ has $2p$ fixed points in $\mathcal{B}_{(2p,0,0)}$. Therefore there are $(2p)^s$ choices for $g$ in this case.

Let $\gamma := (\{\gamma_1,\overline{\gamma_1}\},\{\gamma_2,\overline{\gamma_2}\},\dots,\{\gamma_{tp},\overline{\gamma_{tp}}\})$ be such that $\gamma_1 < \gamma_2 < \cdots < \gamma_{tp}$ and $\mathrm{supp}(\sigma_{j_1}\dots\sigma_{j_t}) = \{\gamma_1,\overline{\gamma_1},\dots,\gamma_{tp},\overline{\gamma_{tp}}\}$. Then $\gamma$ is the unique element of this form with support not disjoint to $\sigma_{j_1}\dots\sigma_{j_t}$ that is fixed by $\sigma_{j_1}\dots\sigma_{j_t}$. Similarly, we define $\delta = ([\delta_1,\overline{\delta_1}],\dots,[\delta_{up},\overline{\delta_{up}}])$ to be such that $\delta_1 < \delta_2 < \cdots < \delta_{up}$ and $\{\delta_1,\overline{\delta_1},\dots,\delta_{tp},\overline{\delta_{tp}}\} = \mathrm{supp}(\sigma_{j_{t+1}}\dots\sigma_{j_k})$. Then $\delta$ is the unique element with support not disjoint to $\sigma_{j_{t+1}}\dots\sigma_{j_k}$ that is fixed by $\sigma_{j_{t+1}}\dots\sigma_{j_k}$.

As there are $\binom{k}{t}$ ways to choose $j_1,j_2,\dots,j_t$, there are $(2p)^s \times \binom{k}{t}$ fixed points of $R_\omega$ in $\mathcal{B}_{(2sp,tp,up)}$. The statement of the lemma now follows by definition of $c_{s,k}$. $\square$

We now prove that $N_{(\lambda,t,u)}$ is indecomposable by filling in the sketch after Corollary 5.3.9. We give an example of the 'induction' procedure in the proof in Example 5.3.13.

LEMMA 5.3.12. *The module $N_{(\lambda,t,u)}$ is an indecomposable $FK_r$-module.*

PROOF. Define $\Omega^{(2\lambda;k)}$ to be the subset of $\Omega^{(2s;k)}$ consisting precisely of the $\{\{i_1,i'_1\},\dots,\{i_{\lambda_1},i'_{\lambda_1}\},\{i_{\lambda_1+1},i_{\lambda_1+1}\},\dots,\{i_s,i'_s\},\{j_1,\dots,j_k\}\}$ such that

$$\begin{aligned}
\{i_1,i'_1,\dots,i_{\lambda_1},i'_{\lambda_1}\} &= \{1,\dots,\lambda_1,s+1,\dots,s+\lambda_1\}\\
\{i_{\lambda_1+1},i'_{\lambda_1+1},\dots,i_s,i'_s\} &= \{\lambda_1+1,\dots,s,s+\lambda_1+1,\dots,2s\}\\
\{j_1,\dots,j_k\} &= \{2s+1,\dots,r\},
\end{aligned}$$

and define $c_{\lambda,k} = |\Omega^{(2\lambda;k)}|$. The module $N_y := N_y^{\omega^\star}\big\uparrow^{Y_{(\lambda,t,u)}}_{X_{(\lambda,t,u)}}$ has a basis given by the set

$$\{(y(g),\gamma^\star,\delta^\star) : (g,\gamma^\star,\delta^\star) \in \mathcal{B}(\omega), \omega \in \Omega^{(2\lambda;k)}\}.$$

Therefore $N_y(R_{\omega^\star})$ and $N_y^{\omega^\star}$ are equal as vector spaces. By the second paragraph in the proof of Lemma 5.3.11, there are $(2p)^s$ choices for $g$ in $(g, \gamma^\star, \delta^\star)$. By definition, $(y(g), \gamma^\star, \delta^\star)$ is the alternating sum of exactly $2^s$ elements. Moreover, given $E \subseteq \{i_1, \ldots, i_s\}$, the second statement of Corollary 5.3.9 implies that $(y(g), \gamma^\star, \delta^\star)$ and $\tau_E(y(g), \gamma^\star, \delta^\star)$ are equal up to a sign. As there are $2^s$ choices for $E$, it follows that $N_y^{\omega^\star}$ has dimension $p^s$.

Recall that $C = \langle \sigma_1, \ldots, \sigma_r\rangle$. The group $R_{\omega^\star}$ acts trivially on $N_y^{\omega^\star}$, and so by Lemma 1.3.2

$$N_y(R_{\omega^\star})\big\downarrow_C \cong F\big\uparrow_{R_{\omega^\star}}^C.$$

Therefore $N_y(R_{\omega^\star})$ is an indecomposable $FC$-module, and so $N_y(R_{\omega^\star})$ is an indecomposable $FN_{Y_{(\lambda,t,u)}}(R_{\omega^\star})$-module. It follows that there exists a unique summand of $N_y$ with vertex containing $R_{\omega^\star}$. Let $W$ be a non-zero indecomposable summand of $N_y$. As

$$N_y\big\downarrow_C \cong \bigoplus_{\omega\in\Omega^{(2\lambda;k)}} F\big\uparrow_{R_\omega}^C,$$

the Krull–Schmidt Theorem implies that each indecomposable summand of $W\!\downarrow_C$ is isomorphic to $F\!\uparrow_{R_{\omega^\star}}^C$. Therefore $W(R_{\omega^\star}) \neq 0$, and so Lemma 1.3.9 states that $W$ has a vertex containing $R_{\omega^\star}$. As $W$ was an arbitrarily chosen summand of $N_y$, it must be the case that $N_y$ is indecomposable.

Let $(\widetilde{y}(g), \tilde{\gamma}, \tilde{\delta}) \in N_{(\lambda,t,u)}$ be such that $(g, \tilde{\gamma}, \tilde{\delta}) \in B(\widetilde{\omega})$. As $\widetilde{y}$ has weight $\lambda$ and $K_r$ permutes the $R_r$-orbits transitively, it follows from Lemma 5.3.8 that there exists $\rho \in \langle \rho_1, \ldots, \rho_{r-1}\rangle$ such that $\pm(\widetilde{y}(g), \tilde{\gamma}, \tilde{\delta}) = \rho(y({}^{\rho^{-1}}g), \gamma, \delta)$, where $(y(g^{\rho^{-1}}), \gamma, \delta) \in N_y$. Therefore $N_y$ generates $N_{(\lambda,t,u)}$ as an $FK_r$-module.

By definition there are $c_{s,k}$ choices for $\omega \in \Omega^{(2s;k)}$, and there are $\binom{s}{\lambda_1}$ choices for $y \in \{-1,1\}^s$ of weight $\lambda$. Therefore $N_{(\lambda,t,u)}$ has dimension

$$c_{s,k} \times \binom{s}{\lambda_1} \times p^s \times \binom{k}{t}.$$

As $N_y$ has dimension $c_{\lambda,k} \times p^s$, applying Lemma 1.3.2 gives

$$N_{(\lambda,t,u)} \cong N_y\big\uparrow_{Y_{(\lambda,t,u)}}^{K_r}.$$

Lemma 5.3.10 states that $Y_{(\lambda,t,u)}$ is the inertial group of the $F[D_1\times\cdots\times D_r]$-module $N_y^{\omega^\star}$. As $N_y^{\omega^\star}$ is extended from $D_1\times\cdots\times D_r$ to $X_{(\lambda,t,u)}$, we have that $N_y\downarrow_{D_1\times\cdots\times D_r}$ is isomorphic to a direct sum of $[Y_{(\lambda,t,u)} : X_{(\lambda,t,u)}]$ copies of $N_y^{\omega^\star}$. Therefore Proposition 1.2.5 implies that $N_{(\lambda,t,u)}$ is an indecomposable $FK_r$-module. $\square$

Example 5.3.13. Let $F$ be a field of characteristic 3, and let $r = 4$. Write $K_4$ for $C_{C_2\wr S_{12}}(R_4)$. In this example we describe the indecomposable summand $N_{((1^2),0,0)}$ of $M_{(12,0,0)}(R_4)$.

Recall that $K_4$ is isomorphic to $C_6 \wr S_4$, with base group

$$D := \langle \sigma_i, \tau_i : 1 \leq i \leq 4\rangle.$$

Let $y = (1,-1)$, which has weight $(1^2)$, and define $\omega = \{\{1,3\},\{2,4\}\}$. Observe that $f_6 \in \mathcal{B}^{R_\omega}_{(12,0,0)}$. Therefore in this case

$$N^\omega_y = \langle g + {}^{\tau_1}g - {}^{\tau_2}g - {}^{\tau_1\tau_2}g : g \in \mathcal{B}^{R_\omega}_{(12,0,0)}\rangle_F,$$

which is closed under the action of $D$, but not under the action of $K_4$. Indeed take

$$\rho_1 := (1\ 4)(2\ 5)(3\ 6)(\overline{1}\ \overline{4})(\overline{2}\ \overline{5})(\overline{3}\ \overline{6}) \in K_4,$$

and observe that

$$(5.5)\quad {}^{\rho_1}(f_6 + {}^{\tau_1}f_6 - {}^{\tau_2}f_6 - {}^{\tau_1\tau_2}f_6) = ({}^{\rho_1}f_6) - {}^{\tau_1}({}^{\rho_1}f_6) + {}^{\tau_2}({}^{\rho_1}f_6) - {}^{\tau_1\tau_2}({}^{\rho_1}f_6),$$

on which $\tau_2$ acts with positive sign. However, $\tau_2$ acts with negative sign on all elements of $N^\omega_y$. Furthermore, ${}^{\rho_1}f_6 \in \mathcal{B}(\widetilde{\omega})$, where $\widetilde{\omega} = \{\{1,4\},\{2,3\}\}$.

By considering the actions on $N^\omega_y$ of

$$(1\ 7)(2\ 8)(3\ 9)(\overline{1}\ \overline{7})(\overline{2}\ \overline{8})(\overline{3}\ \overline{9}) \text{ and } (4\ 10)(5\ 11)(6\ 12)(\overline{4}\ \overline{10})(\overline{5}\ \overline{11})(\overline{6}\ \overline{12}),$$

we see that the inertial group $I$ of $N^\omega_y$ is isomorphic to $C_6 \wr S_{2(1^2)}$. Moreover, $N^\omega_y$ is an $FI$-module. The final sentence in the previous paragraph shows that the $FK_4$-module generated by $N^\omega_y$ equals

$$\langle g + {}^{\tau_1}g - {}^{\tau_2}g - {}^{\tau_1\tau_2}g,\ g - {}^{\tau_1}g + {}^{\tau_2}g - {}^{\tau_1\tau_2}g : g \in \mathcal{B}^{R_4}_{(12,0,0)}\rangle_F,$$

which is isomorphic to $N^\omega_y \uparrow^{K_4}_I$ by Lemma 1.3.2. By definition this equals $N_{((1^2),0,0)}$, as expected from Lemma 5.3.12.

By Lemma 1.3.11 the modules $M_{(2sp,tp,up)}(R_r)$ and $N_{(\lambda,t,u)}$, for any $\lambda \in \Lambda(2,s)$, are $p$-permutation $FN_{C_2\wr S_{rp}}(R_r)$-modules. We briefly write $J_r$ for $N_{C_2\wr S_{rp}}(R_r)$. As $R_r$ is a normal subgroup of $R_{\omega^\star}$, it follows from Lemma 1.3.11 that

$$M_{(2sp,tp,up)}(R_r)(R_{\omega^\star}) \cong M_{(2sp,tp,up)}(R_{\omega^\star}),$$

as $FN_{J_r}(R_{\omega^\star})$-modules. Then Lemma 5.3.12 implies that

$$M_{(2sp,tp,up)}(R_{\omega^\star}) \cong \bigoplus_{\lambda\in\Lambda(2,s)} N_{(\lambda,t,u)}(R_{\omega^\star}),$$

as $FN_{J_r}(R_{\omega^\star})$-modules. Moreover, for all $\lambda \in \Lambda(2,s)$, the basis defining $N_{(\lambda,t,u)}$ in (5.3) is a $p$-permutation basis of $N_{(\lambda,t,u)}$ with respect to $R_{\omega^\star}$.

Recall that $U$ is a non-projective indecomposable summand of $M_{(2a,b,c)}$. It follows from the proof of Lemma 5.3.6 that each $N_{(\lambda,t,u)}(R_{\omega^\star}) \neq 0$, and so by the Krull–Schmidt Theorem $U(R_{\omega^\star}) \neq 0$. By Lemma 1.3.9 every non-projective indecomposable summand of $M_{(2sp,tp,up)}$ therefore has a vertex containing $R_{\omega^\star}$.

In the second step of the proof of Theorem 5.1.1, we consider the module $N_{(\lambda,t,u)}(R_{\omega^\star})$ in order to understand $U(R_{\omega^\star})$.

**Second step: The vertices of $N_{(\lambda,t,u)}(R_{\omega^\star})$.** In this step we show that $N_{(\lambda,t,u)}(R_{\omega^\star})$ is an indecomposable $FC_{K_r}(R_{\omega^\star})$-module, where we remind the reader that $K_r = C_{C_2 \wr S_{rp}}(R_r)$. It follows that $N_{(\lambda,t,u)}(R_{\omega^\star})$ is an indecomposable $FN_{C_2\wr S_{rp}}(R_{\omega^\star})$-module, and in Lemma 5.3.15 we determine its vertex.

Observe that the group $C_{K_r}(R_{\omega^\star})$ is generated by the set

$$\{\sigma_i, \tau_i : 1 \leq i \leq r\} \cup \{\rho_1^{\rho_2\rho_3\cdots\rho_s}\} \cup \{\rho_i\rho_{i+s} : 1 \leq i \leq s-1 \text{ and } i \neq \lambda_1\} \\ \cup \{\rho_i : 2s+1 \leq i \leq r\},$$

and so there is an inclusion $X_{(\lambda,t,u)} \leq C_{K_r}(R_{\omega^\star})$.

LEMMA 5.3.14. *Let $\lambda \in \Lambda(2,s)$. Then $N_{(\lambda,t,u)}(R_{\omega^\star})$ is an indecomposable $FN_{K_r}(R_{\omega^\star})$-module.*

PROOF. By definition $R_{\omega^\star}$ acts trivially on $N_y^{\omega^\star}$, and so it follows from Lemma 1.3.2 that $N_y^{\omega^\star}\downarrow_C \cong F\uparrow_{R_{\omega^\star}}^C$ . This is indecomposable as an $FC$-module, and so $N_y^{\omega^\star}$ is an indecomposable $FX_{(\lambda,t,u)}$-module.

Fix $(\widetilde{y}(g), \tilde{\gamma}, \tilde{\delta}) \in N_{(\lambda,t,u)}(R_{\omega^\star})$. As $C_{K_r}(R_{\omega^\star})$ permutes the $R_{\omega^\star}$-orbits of a fixed size transitively amongst themselves, it follows from Lemma 5.3.8 that there exists some

$$\rho \in \langle \rho_1^{\rho_2\rho_3\cdots\rho_s}, \rho_1\rho_{s+1}, \ldots, \rho_{s-1}\rho_{2s-1}, \rho_{2s+1}, \ldots, \rho_{r-1}\rangle$$

such that $\pm(\widetilde{y}(g), \tilde{\gamma}, \tilde{\delta}) = \rho(y({}^{\rho^{-1}}g), \gamma, \delta)$, where $(y({}^{\rho^{-1}}g), \gamma, \delta) \in N_y^{\omega^\star}$. Therefore $N_y^{\omega^\star}$ generates $N_{(\lambda,t,u)}(R_{\omega^\star})$ as an $FC_{K_r}(R_{\omega^\star})$-module. As there are exactly $\binom{s}{\lambda_1}$ tuples of weight $\lambda$ in $\{-1,1\}^s$, Corollary 5.3.9 and Lemma 5.3.11 imply that the module $N_{(\lambda,t,u)}(R_{\omega^\star})$ has dimension

$$\binom{s}{\lambda_1} \times [S_{t+u} : S_t \times S_u] \times p^s.$$

By Lemma 1.3.2 we therefore have that

$$N_{(\lambda,t,u)}(R_{\omega^\star})\big\downarrow_{C_{K_r}(R_{\omega^\star})} \cong N_y^{\omega^\star}\big\uparrow_{X_{(\lambda,t,u)}}^{C_{K_r}(R_{\omega^\star})}.$$

Using Lemma 5.3.10 we see that the inertial group of $N_y^{\omega^\star}$ in $C_{K_r}(R_{\omega^\star})$ is equal to $X_{(\lambda,t,u)}$. It follows from Proposition 1.2.5 that $N_{(\lambda,t,u)}(R_{\omega^\star})$ is an indecomposable $FC_{K_r}(R_{\omega^\star})$-module. $\square$

Given $X \subset \{1, 2, \ldots, sp\}$, let $C_2 \wr S_X$ be as in §1.4.1. Also, given $x \in \{1, 2, \ldots, sp\}$, define $x^* = x + sp$. We remark that this definition of $x^*$ agrees with that of $x^*$ in Example 5.3.5, which considers the case when $s = 1$. Given $g \in C_2 \wr S_{\{1,2,\ldots,sp\}}$, let $g^*$ be the permutation in $C_2 \wr S_{\{sp+1,\ldots,2sp\}}$ such that $g^*(i^*) = (g(i))^*$.

Also given $\lambda \in \Lambda(2,s)$, we define $J$ to be the group consisting of all elements $gg^*$ such that $g$ is contained in a Sylow $p$-subgroup of $C_2 \wr S_{\{1,\ldots,p\lambda_1\}} \times C_2 \wr S_{\{p\lambda_1+1,\ldots,sp\}}$ with base group $\langle \sigma_1, \ldots, \sigma_s\rangle$. Let $J^+$ be a Sylow $p$-subgroup of $C_2 \wr S_{\{2sp+1,\ldots,(2s+t)p\}} \times C_2 \wr S_{\{(2s+t)p+1,\ldots,rp\}}$ with base group $\langle \sigma_{2s+1}, \ldots, \sigma_r\rangle$. We define $Q_{(\lambda,t,u)} = J \times J^+$.

By construction, $R_{\omega^\star} \trianglelefteq Q_{(\lambda,t,u)}$, and so $Q_{(\lambda,t,u)} \leq N_{C_2\wr S_{rp}}(R_{\omega^\star})$. By Lemma 1.3.11 and Lemma 5.2.3, there exists a choice of signs $s_v \in \{-1,1\}$ such that

$$\{s_v v : v \in \mathcal{B}^{R_r}_{(2sp,tp,up)}\}$$

is a $p$-permutation basis for $M_{(2sp,tp,up)}(R_{\omega^\star})$ with respect to $Q_{(\lambda,t,u)}$. Given $v := (g,\gamma,\delta) \in \mathcal{B}(\omega^\star)$, let $(h,\tilde{\gamma},\tilde{\delta})$ be a representative for the $Q_{(\lambda,t,u)}$-orbit containing $v$. It follows that for all $E \subseteq \{1,2,\ldots,s\}$, the representative for the $Q_{(\lambda,t,u)}$-orbit containing $({}^{\tau_E}g,\gamma,\delta)$ can be chosen to be of the form $(h',\tilde{\gamma},\tilde{\delta})$. For distinct summands $w$ and $\widetilde{w}$ of $(y(g),\gamma,\delta)$, it follows that $s_w = s_{\widetilde{w}}$. We can therefore write $s_{(g,\gamma,\delta)}$ in the place of $s_w$ for all such $w$, and then

$$\{s_{(g,\gamma,\delta)}(y(g),\gamma,\delta) : (g,\gamma,\delta) \in \mathcal{B}(\omega^\star)\}$$

is a $p$-permutation basis of $N_{(\lambda,t,u)}(R_{\omega^\star})$ with respect to $Q_{(\lambda,t,u)}$.

LEMMA 5.3.15. *The module $N_{(\lambda,t,u)}(R_{\omega^\star})$ has a vertex equal to $Q_{(\lambda,t,u)}$.*

PROOF. Let $y = y_\lambda$. The element $(f_{sp},\gamma^\star,\delta^\star)$ is a fixed point of $Q_{(\lambda,t,u)}$. As $Q_{(\lambda,t,u)} \leq X_{(\lambda,t,u)}$, the element $(y(f_{sp}),\gamma^\star,\delta^\star)$ is also a fixed point of $Q_{(\lambda,t,u)}$. Therefore $N_{(\lambda,t,u)}(R_{\omega^\star})$ has a vertex containing $Q_{(\lambda,t,u)}$.

The element $y(f_{sp})$ is an alternating sum of elements conjugate to $f_{sp}$ in $C_2 \wr S_{rp}$, and so any element in $N_{C_2\wr S_{rp}}(R_r)$ that fixes $y(f_{sp})$ under the conjugacy action must be contained in $V_{sp}$. Indeed suppose that there exists $h \in Q_{(\lambda,t,u)}$ such that $h \notin V_{sp}$. Therefore by definition of $y(g)$, it must be the case that $\tau_S h \in V_{sp}$ for some $S \subset \{1,2,\ldots,s\}$. However $\tau_S$ transposes the $R_{2s}$-orbits $\{(j-1)p+1,\ldots,jp\}$ and $\{\overline{(j-1)p+1},\ldots,\overline{jp}\}$ for each $j \in S$, and fixes all other $R_{2s}$-orbits. As $p$ is odd, $h$ must act trivially on these orbits. The only such elements in $N_{C_2\wr S_{rp}}(R_r)$ are also contained in $V_{sp}$, a contradiction.

Since $Q_{(\lambda,t,u)}$ is the largest $p$-subgroup that is contained in both $X_{(\lambda,t,u)}$ and $V_{sp} \times C_2 \wr S_{tp} \times C_2 \wr S_{up}$, the result follows from Lemma 1.3.9. $\square$

**Third step: Proof of Theorem 5.1.1.** Given $r \in \mathbf{N}$ such that $rp \leq n$, recall that

$$T'_r = \{(\lambda,t,u) : \lambda \in \Lambda(2,s), 2s+t+u = r \text{ and } sp \leq a, tp \leq b, up \leq c\}.$$

We now complete the proof of Theorem 5.1.1. We restate the result for the reader's convenience.

THEOREM 5.1.1. *Let $(a,b,c) \in \mathbf{N}_0^3$ be such that $2a+b+c = n$, and let $U$ be a non-projective indecomposable summand of $M_{(2a,b,c)}$. Then $U$ has a vertex equal to a Sylow $p$-subgroup of*

$$V_{p\lambda} \times C_2 \wr S_{tp} \times C_2 \wr S_{up},$$

*for some $r \in \mathbf{N}$, where $rp \leq n$, and $(\lambda,t,u) \in T'_r$.*

Proof of Theorem 5.1.1. Let $r \in \mathbf{N}$ be maximal such that $R_r$ is contained in a vertex of $U$. By Lemma 5.3.4, Lemma 5.3.12 and the Krull–Schmidt Theorem, there is a subset $T \subset T'_r$, and for each $(\lambda, t, u) \in T$ a summand $W_{(\lambda,t,u)}$ of $M_{(2(a-sp),b-tp,c-up)}$, such that

$$U(R_r) \cong \bigoplus_{(\lambda,t,u)\in T} N_{(\lambda,t,u)} \boxtimes W_{(\lambda,t,u)},$$

where $s = |\lambda|$.

By Lemma 5.3.15 $N_{(\lambda,t,u)}$ has a vertex equal to $Q_{(\lambda,t,u)}$. Let $(\lambda, t, u) \in T$ be such that $s := |\lambda|$ is minimal. Suppose there exists $(2\widetilde{s}, \widetilde{t}, \widetilde{u}) \in T^r$ such that $\widetilde{s} > s$. Given $\widetilde{\lambda} \in \Lambda(2, \widetilde{s})$ and $\omega \in \Omega^{(2s,t+u)}$, the vertex $Q_{(\widetilde{\lambda},\widetilde{t},\widetilde{u})}$ of $N_{(\widetilde{\lambda},\widetilde{t},\widetilde{u})}$ cannot contain a conjugate of $R_\omega$, and so $N_{(\widetilde{\lambda},\widetilde{t},\widetilde{u})}(R_\omega) = 0$.

We therefore consider $U(R_\omega)$ when $\omega \in \Omega^{(2s;t+u)}$. By Lemma 1.3.11 there is an isomorphism $U(R_\omega) \cong U(R_r)(R_\omega)$ and so there exists a subset $S$ of $T$ such that

$$U(R_\omega) \cong \bigoplus_{(\lambda,t,u)\in S} N_{(\lambda,t,u)}(R_\omega) \boxtimes W_{(\lambda,t,u)},$$

where $|\lambda| = s$. Let $Q_{(\lambda,t,u)}$ be maximal such that $(\lambda, t, u) \in S$. Another application of Lemma 1.3.11 gives

$$\begin{aligned} U(Q_{(\lambda,t,u)}) &\cong U(R_\omega)(Q_{(\lambda,t,u)}) \\ &= \textstyle\bigoplus_{(\lambda,\widetilde{t},\widetilde{u})} N_{(\lambda,\widetilde{t},\widetilde{u})}(R_\omega)(Q_{(\lambda,\widetilde{t},\widetilde{u})}) \boxtimes W_{(\lambda,\widetilde{t},\widetilde{u})}, \end{aligned} \tag{5.6}$$

where that the sum runs over the $(\lambda, \widetilde{t}, \widetilde{u}) \in S$ such that $Q_{(\lambda,\tilde{t},\tilde{u})}$ is a conjugate of $Q_{(\lambda,t,u)}$. By Lemma 5.3.15 $N_{(\widetilde{\lambda},\widetilde{t},\widetilde{u})}(R_\omega)(Q_{(\lambda,t,u)}) \neq 0$, and so Lemma 1.3.9 gives that $Q_{(\lambda,t,u)}$ is contained in some conjugate of $Q_{(\widetilde{\lambda},\widetilde{t},\widetilde{u})}$. If $Q_{(\widetilde{\lambda},\widetilde{t},\widetilde{u})}$ is not a conjugate of $Q_{(\lambda,t,u)}$, then $Q_{(\lambda,t,u)}$ is strictly contained in the appropriate conjugate of $Q_{(\lambda,t,u)}$, but this is a contradiction to the maximality of $Q_{(\lambda,t,u)}$.

As $N_{(\lambda,t,u)}(R_\omega)(Q_{(\lambda,t,u)}) \neq 0$, it follows from Lemma 1.3.9 that $U$ has a vertex $Q$ containing $Q_{(\lambda,t,u)}$. Suppose that $Q$ strictly contains $Q_{(\lambda,t,u)}$. Since $Q$ is a $p$-group, there exists some $g \in N_Q(Q_{(\lambda,t,u)})$ such that $g \notin Q_{(\lambda,t,u)}$. The orbits of $Q_{(\lambda,t,u)}$ have length at least $p$ on $\{1, \overline{1}, \ldots, rp, \overline{rp}\}$, whereas the orbits of $Q_{(\lambda,t,u)}$ on

$$\{rp + 1, \overline{rp + 1}, \ldots, n, \overline{n}\}$$

have length 1. As $g$ cannot permute an element in an orbit of length strictly greater than 1 with elements in an orbit of length 1, we can write $g = hh^+$, where $h \in N_{C_2 \wr S_{rp}}(Q_{(\lambda,t,u)})$ and $h^+ \in C_2 \wr S_{\{rp+1,\ldots,n\}}$. The only elements in $Q_{(\lambda,t,u)}$ with cycle type either one positive $p$-cycle, or two positive $p$-cycles are those contained in $R_\omega$. Therefore $N_{C_2\wr S_{rp}}(Q_{(\lambda,t,u)}) \leq N_{C_2\wr S_{rp}}(R_\omega)$, and so $\langle Q_{(\lambda,t,u)}, h\rangle \leq N_L(Q_{(\lambda,t,u)})$, where $L := N_{C_2\wr S_{rp}}(R_\omega)$.

Let $\mathcal{C}$ be a $p$-permutation basis of $N_{(\lambda,t,u)}(R_\omega)$ with respect to $\langle Q_{(\lambda,t,u)}, g\rangle$. By Lemma 1.3.9 the group $\langle Q_{(\lambda,t,u)}, g\rangle$ has a fixed point in $\mathcal{C}$. It follows from (5.6) that there exists some $N_{(\lambda,t,u)}(R_\omega)$ with vertex containing $\langle Q_{(\lambda,t,u)}, h\rangle$. However, we have already seen that $N_{(\lambda,t,u)}(R_\omega)$ has vertex equal to $Q_{(\lambda,t,u)}$.

Therefore $h \in Q_{(\lambda,t,u)}$, and so $h^+$ is a non-identity $p$-element of $Q$. It follows that some power of $h^+$ is a product of positive $p$-cycles with support outside $\{1, \overline{1}, \dots, rp, \overline{rp}\}$. This contradicts the hypothesis that $r$ is maximal, and so the theorem is proved. $\square$

EXAMPLE 5.3.16. Let $p = 3$. The module $M_{(54,0,0)}$ is spanned by the conjugates of

$$f_{27} := (1\ 28)(2\ 29)\dots(27\ 54)(\overline{1}\ \overline{28})(\overline{2}\ \overline{29})\dots(\overline{27}\ \overline{54})$$

in $C_2 \wr S_{54}$. In the notation of Theorem 1.1, we have that $r = 9$ and $T'_9 = \Lambda(2,9)$. By Theorem 1.1, any non-projective indecomposable summand of $M_{(54,0,0)}$ has a vertex containing a Sylow 3-subgroup of $V_{3\lambda}$, for some $\lambda \in \Lambda(2,9)$. In fact we can say more: for every $\lambda \in \Lambda(2,9)$, a Sylow 3-subgroup of $V_{3\lambda}$ contains a conjugate of a Sylow 3-subgroup of $V_{3(5,4)}$, chosen with the permutations $\sigma_1\sigma_{10}, \dots, \sigma_9\sigma_{18}$ in its center.

## 5.4. Decomposition numbers of $C_2 \wr S_n$

In this section we prove Theorem 5.1.2. We assume that $M_{(2a,b,c)}$ is defined over the field $\mathbf{F}_p$, since the results in this section then follow by change of scalars. We define $\chi_{(2a,b,c)}$ to be the ordinary character of $M_{(2a,b,c)}$. Given $(\lambda, \mu) \in \mathcal{P}^2(n)$, recall that we write $\chi^{(\lambda,\mu)}$ for the ordinary character of the hyperoctahedral Specht module $S^{(\lambda,\mu)}$. In the following lemma we decompose the character $\chi_{(2a,b,c)}$ into its irreducible constituents.

LEMMA 5.4.1. *Let $n = 2a + b + c$. The constituents of the character $\chi_{(2a,b,c)}$ are precisely those $\chi^{(\lambda,\mu)}$ such that $(\lambda, \mu) \in \mathcal{P}^2(n)$ and $\lambda$ has exactly $b$ odd parts, and $\mu$ has exactly $c$ odd parts. Moreover each constituent appears with multiplicity one.*

PROOF. This follows from Propositions 1 and 2 in [**2**], and by multiplying through by the ordinary character of the module $\mathrm{Inf}_{S_n}^{C_2 \wr S_n} \mathrm{sgn}_{S_n}$. $\square$

In order to prove Theorem 5.1.2, we need to understand how the blocks of $FC_2 \wr S_n$ correspond to the blocks of $FN_{C_2 \wr S_n}(R_r)$. We therefore require a description of the blocks of $FN_{C_2 \wr S_n}(R_r)$, which we give in the following section.

### 5.4.1. The blocks of $FN_{C_2 \wr S_n}(R_r)$.

Recall from (5.1) that

$$N_{C_2 \wr S_n}(R_r) \cong N_{C_2 \wr S_{rp}}(R_r) \times C_2 \wr S_{\{rp+1,\dots,n\}}.$$

It follows that the blocks of $FN_{C_2 \wr S_n}(R_r)$ are of the form

$$b \otimes B((\gamma, \widetilde{v}), (\delta, \widetilde{w})),$$

where $b$ is a block of $FN_{C_2 \wr S_{rp}}(R_r)$, and $\gamma, \delta$ are $p$-core partitions such that $|\gamma| + \widetilde{v}p + |\delta| + \widetilde{w}p = n - rp$. It therefore suffices to describe the blocks of $FN_{C_2 \wr S_{rp}}(R_r)$.

PROPOSITION 5.4.2. *The blocks of $FN_{C_2\wr S_{rp}}(R_r)$ are labelled by pairs $(\widetilde{v}, \widetilde{w})$ such that $\widetilde{v} + \widetilde{w} = r$. Moreover, the $FN_{C_2\wr S_{rp}}(R_r)$-module $M$ lies in the block labelled by $(\widetilde{v}, \widetilde{w})$ if and only if exactly $\widetilde{v}$ factors of $C_2^r$ act on $M$ by positive sign.*

PROOF. Using the presentation of $N_{C_2\wr S_{rp}}(R_r)$ given in §5.3.1, we see that $N_{C_2\wr S_{rp}}(R_r) \cong C_2^r \rtimes N_{S_{rp}}(R_r)$, where $C_2^r = \langle \tau_1, \ldots, \tau_r \rangle$ in this case. Let $\chi_{\widetilde{v}} \in \mathrm{Lin}(C_2^r)$ be such that

$$\chi_{\widetilde{v}}(\tau_1) = \cdots = \chi_{\widetilde{v}}(\tau_{\widetilde{v}}) = 1$$
$$\chi_{\widetilde{v}}(\tau_{\widetilde{v}+1}) = \cdots = \chi_{\widetilde{v}}(\tau_r) = -1.$$

The stabiliser of $\chi_{\widetilde{v}}$ in $N_{S_{rp}}(R_r)$ is isomorphic to $C_{S_{(\widetilde{v},\widetilde{w})p}}(R_r) \rtimes C_{p-1}$, which has a unique block by Lemma 2.6 in [**9**]. All statements of the result now follow from Theorem 1.4.7. □

We write $b(\widetilde{v}, \widetilde{w})$ for the block of $N_{C_2\wr S_{rp}}(R_r)$ labelled by $(\widetilde{v}, \widetilde{w})$.

REMARK 5.4.3. Recall that in the first step of the proof of Theorem 5.1.1 we wrote $M_{(2sp,tp,up)}(R_r)$ as a direct sum of indecomposable $FN_{C_2\wr S_{rp}}(R_r)$-modules $N_{(\lambda,t,u)}$. It follows from Proposition 5.4.2 that this is in fact a decomposition of $M_{(2sp,tp,up)}(R_r)$ into its block components. In particular the second statement of Proposition 5.4.2 implies that the module $N_{(\lambda,t,u)}$ lies in the block $b(2\lambda_1 + t, 2\lambda_2 + u)$ of $FN_{C_2\wr S_{rp}}(R_r)$.

Given a $p$-core partition $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_t)$ and $v \in \mathbf{N}_0$, we define $\gamma + vp$ to be the partition

$$(\gamma_1 + vp, \gamma_2, \ldots, \gamma_t).$$

It is proved in [**65**, Lemma 7.1] that the Specht module $S^{\gamma+vp}$ is always a $p$-permutation $FS_{|\gamma|+vp}$-module. Fix a $p$-core partition $\delta$ and $w \in \mathbf{N}_0$ such that $|\gamma| + vp + |\delta| + wp = n$. Then part (2) of Proposition 1.3.8 gives that $S^{(\gamma+vp,\delta+wp)}$ is a $p$-permutation $FC_2 \wr S_n$-module.

PROPOSITION 5.4.4. *Fix $v, w \in \mathbf{N}_0$. Let $\widetilde{v}, \widetilde{w} \in \mathbf{N}_0$ be such that $\widetilde{v} \leq v, \widetilde{w} \leq w$, and $\widetilde{v} + \widetilde{w} = r$. The $FN_{C_2\wr S_n}(R_r)$-module $S^{(\gamma+vp,\delta+wp)}(R_r)$ contains a summand lying in the block*

$$b(\widetilde{v}, \widetilde{w}) \otimes B((\gamma, v - \widetilde{v}), (\delta, w - \widetilde{w})).$$

*Moreover the $FN_{C_2\wr S_n}(R_r)$-blocks $b$ such that $b^{C_2\wr S_n} = B((\gamma, v), (\delta, w))$ are precisely those of this form.*

We prove this proposition by applying Lemma 1.3.12 to the module $S^{(\gamma+vp,\delta+wp)}(R_r)$. We first consider $S^{(\gamma+vp,\delta+wp)}(Q)$ in a more general case. Indeed fix a $p$-subgroup $Q$ of $C_2 \wr S_n$ contained, up to conjugacy, in $C_2 \wr S_{(|\delta|+vp,|\gamma|+wp)}$. Also let $Q$ have support size $2rp$ when viewed as a subgroup of $S_{2n}$. Define $U_Q$ to be the kernel of the Brauer morphism from

$(S^{(\gamma+vp,\delta+wp)})^Q$ to $S^{(\gamma+vp,\delta+wp)}(Q)$. We now describe a polytabloid $e_{t_\star}$ that is not contained in $U_Q$. The following preliminaries are required.

Given a $(\gamma + vp, \delta + wp)$-tableau $t$, let $\widehat{t^+}$ denote the tableau obtained by replacing each entry $\{x, \overline{x}\}$ in $t^+$ with $x$, and define $\widehat{t^-}$ in the analogous way. We also require the dominance order on row-standard tableaux, which is defined in §2.2.

Define $t_\star$ to be the tableau such that $\widehat{t_\star}^+$ is the greatest $\gamma+vp$-tableau in the dominance order with entries in $\{1, 2, \dots, |\gamma|+vp\}$, and $\widehat{t_\star}^-$ is the greatest $\delta + wp$-tableau in the dominance order with entries in $\{|\gamma| + vp + 1, \dots, n\}$.

Lemma 5.4.5. *The polytabloid $e(t_\star)$ is not contained in $U_Q$.*

Proof. Let $t = t_\star$. By definition of the Brauer morphism, we have that $U_Q$ is contained in the subspace

$$V := \langle e(s) + ge(s) + \cdots + g^{p-1}e(s) : s \text{ a standard tableau, } g \in Q\rangle,$$

of $S^{(\gamma+vp,\delta+wp)}$, and so it is sufficient to prove that $e(t) \notin V$.

Suppose, for a contradiction, that $e(t) \in V$. Then there exists some $0 \leq i \leq p-1$ such that $e(t)$ has non-zero coefficient in the expression of $g^ie(s)$ as a linear combination of standard polytabloids. We assume that every $g \in Q$ factorises as $g = g_+g_-$, for some $g_+ \in C_2 \wr S_{\{\gamma_1+1,\dots,\gamma_1+vp\}}$ and $g_- \in C_2 \wr S_{\{\gamma+vp+\delta_1+1,\dots,\gamma+vp+\delta_1+wp\}}$.

Using the bilinearity of the outer tensor product, the polytabloid $e_{t^+}$ has non-zero coefficient in the expression of $(g_+)^ie(s^+)$ as a linear combination of standard polytabloids. The analogous statement also holds for $e(t^-)$ and $(g_-)^ie(s^-)$ The action of $Q$ on $e(t^+)$ (resp. $e(t^-)$) is equivalent to the action of $\widehat{Q}$ on $e(\widehat{t^+})$ (resp. $e(\widehat{t^-})$). Therefore it suffices to prove that the polytabloid corresponding to $\widehat{t^+}$ is not contained in the kernel of the Brauer morphism from $(S^{\gamma+vp})^{\widehat{Q}}$ to $S^{\gamma+vp}(\widehat{Q})$, and that the analogous property holds for the polytabloid corresponding to $\widehat{t^-}$. This follows from Lemma 5.2 in [**65**]. $\square$

Before we prove Proposition 5.4.4, we introduce one more piece of notation. Given partitions $(\lambda, \mu) \in \mathcal{P}^2(n)$, we define

$$M^{(\lambda,\mu)} = (\mathrm{Inf}_{S_{|\lambda|}}^{C_2\wr S_{|\lambda|}} M^\lambda \boxtimes \widetilde{N}^{\otimes|\mu|} \otimes \mathrm{Inf}_{S_{|\mu|}}^{C_2\wr S_{|\mu|}} M^\mu)\big\uparrow_{C_2\wr S_{(|\lambda|,|\mu|)}}^{C_2\wr S_n}.$$

Proof of Proposition 5.4.4. Let $R_{(\widetilde{v},\widetilde{w})}$ be the conjugate of $R_r$ contained in the top group $T_n$ with support such that exactly $\widetilde{v}$ non-trivial orbits of $\widehat{R}_{(\widetilde{v},\widetilde{w})}$ are contained at the end of the first row of $\widehat{t_\star^+}$, and exactly $\widetilde{w}$ non-trivial orbits of $\widehat{R}_{(\widetilde{v},\widetilde{w})}$ are contained at the end of the first row of $\widehat{t_\star^-}$.

By Lemma 5.4.5, the polytabloid $e(t_\star)$ is not contained in $U_{R_{(\widetilde{v},\widetilde{w})}}$. Therefore the submodule of $S^{(\gamma+vp,\delta+wp)}(R_{(\widetilde{v},\widetilde{w})})$ generated by $e(t_\star)$, denoted $W$, is non-zero.

Let $s_\star$ be the $(\gamma+(v-\widetilde{v})p, \delta+(w-\widetilde{w})p)$-tableau such that $\widehat{s_\star^+}$ and $\widehat{s_\star^-}$ are the greatest $\gamma+(v-\widetilde{v})p$-tableau and $\delta+(w-\widetilde{w})p$-tableau in the dominance orders on the tableaux with entries

$$\{1, 2, \ldots, |\gamma| + vp\}\backslash \mathrm{supp}(\widehat{R}_{(\widetilde{v},\widetilde{w})})$$
$$\{|\gamma| + vp + 1, |\gamma| + vp + 2, \ldots, n\}\backslash \mathrm{supp}(\widehat{R}_{(\widetilde{v},\widetilde{w})}),$$

respectively. Let $s$ be the $(\widetilde{v}p, \widetilde{w}p)$-tableau with entries in the row of length $\widetilde{v}p$ agreeing with those at the end of the first row of $t_\star^+$, and with entries in the row of length $\widetilde{w}p$ agreeing with those at the end of the first row of $t_\star^-$. The extension of the map $\{s\} \otimes e(s_\star) \mapsto e(t_\star) + U$, denoted $\vartheta$, is an $F[N_{C_2 \wr S_{rp}}(R_{(\widetilde{v},\widetilde{w})}) \times C_2 \wr S_{n-rp}]$-module homomorphism from

$$M := M^{((\widetilde{v}p),(\widetilde{w}p))}(R_{(\widetilde{v},\widetilde{w})}) \boxtimes S^{(\gamma+(v-\widetilde{v})p,\delta+(w-\widetilde{w})p)},$$

to $W$. The extension of the map $e(t) + U \mapsto \{s\} \otimes e(s_\star)$, denoted $\phi$, is a well-defined morphism of $F[N_{C_2 \wr S_{rp}}(R_{(\widetilde{v},\widetilde{w})}) \times C_2 \wr S_{n-rp}]$-modules such that $\phi\vartheta = \mathrm{id}_M$. Therefore $S^{(\gamma+vp,\delta+wp)}(R_{(\widetilde{v},\widetilde{w})})$ has a submodule isomorphic to $M$. By Proposition 1.4.8 and Proposition 5.4.2 $M$ lies in the block

$$b := b(\widetilde{v}, \widetilde{w}) \otimes B((\gamma, v - \widetilde{v}), (\delta, w - \widetilde{w})),$$

and so there exists a summand of $S^{(\gamma+vp,\delta+wp)}(R_{(\widetilde{v},\widetilde{w})})$ lying in this block, which proves the first statement of the proposition. That $B((\gamma, v), (\delta, w))$ corresponds to $b$ now follows immediately from Lemma 1.3.12.

Observe that we have shown if

$$(b(v', w') \otimes B((\gamma', v''), (\delta, w'')))^{C_2 \wr S_n} = B((\gamma, v), (\delta, w)),$$

then $v' + v'' = v$ and $w' + w'' = w$. In particular $v' \leq v$ and $w' \leq w$. Moreover $\gamma' = \gamma$ and $\delta' = \delta$. This completes the proof of the proposition. $\square$

The following example makes explicit the proof of Proposition 5.4.4.

EXAMPLE 5.4.6. Let $p = 3$, $n = 13$, and $r = 2$. Define the 3-core partitions $\gamma = (2)$ and $\delta = (1^2)$. We consider the $FN_{C_2 \wr S_{13}}(R_2)$-module $S := S^{((8),(4,1))}(R_2)$. By Proposition 1.4.8, $S^{((8),(4,1))}$ lies in the block

$$B := B(((2), 2), ((1^2), 1)).$$

In this case $t_\star$ is $((8), (4, 1))$-tableau equal to

$$\begin{array}{cccccccccccc} & & & & & & & & [9,\overline{9}] & [10,\overline{10}] & [11,\overline{11}] & [12,\overline{12}] \\ & & & & & & & & [13,\overline{13}] & & & \\ \{1,\overline{1}\} & \{2,\overline{2}\} & \{3,\overline{3}\} & \{4,\overline{4}\} & \{5,\overline{5}\} & \{6,\overline{6}\} & \{7,\overline{7}\} & \{8,\overline{8}\} & & & & \end{array},$$

where the shaded boxes correspond to the parts added to the 3-cores. By definition the conjugate $R_{(2,0)}$ of $R_2$ equals $\langle(3\,4\,5)(\overline{3}\,\overline{4}\,\overline{5})(6\,7\,8)(\overline{6}\,\overline{7}\,\overline{8})\rangle$. Then $e(t_\star)$ generates an $F[N_{C_2 \wr S_6}(R_2) \times C_2 \wr S_7]$-submodule of $S^{((8),(4,1))}(R_{(2,0)})$ isomorphic to $F \boxtimes S^{((2),(4,1))}$, which lies in the block

$$b_1 := b(2, 0) \otimes B(((2), 0), ((1^2), 1)).$$

Consider now the conjugate $R_{(1,1)}$ of $R_2$, which by definition equals $\langle(6\ 7\ 8)(\overline{6}\ \overline{7}\ \overline{8})(10\ 11\ 12)(\overline{10}\ \overline{11}\ \overline{12})\rangle$. Observe that $R_{(1,1)}$ is conjugate to $R_2$ in $C_2 \wr S_{13}$. Then $e(t_\star)$ generates an $F[N_{C_2\wr S_6}(R_2) \times C_2 \wr S_7]$-submodule of $S^{((8),(4,1))}(R_{(1,1)})$ isomorphic to

$$M^{(3,3)}(R_{(1,1)}) \boxtimes S^{((5),(1^2))},$$

which lies in the block

$$b_2 := b(1,1) \otimes B(((2),1),((1^2),0)).$$

Since $S \cong S^{((8),(4,1))}(R_{(2,0)}) \cong S^{((8),(4,1))}(R_{(1,1)})$, it follows that $S$ has indecomposable summands $U$ and $V$ respectively lying in the blocks $b_1$ and $b_2$. Therefore $b_1^{C_2\wr S_n} = b_2^{C_2\wr S_n} = B$, as expected from the proof of Proposition 5.4.4.

**5.4.2. Proof of Theorem 5.1.2.** Fix $a, b, c \in \mathbf{N}_0$ such that $n = 2a + b + c$. Following the outline of this chapter, we prove Theorem 5.1.2 using Scott's Lifting Theorem and Brauer reciprocity. In order to do this, we determine certain projective summands of the module $M_{(2a,b,c)}$. We remind the reader that given $b \in \mathbf{N}_0$ and a $p$-core partition $\gamma$, we define $w_b(\gamma)$ to be the minimal number of $p$-hooks such that when added to $\gamma$, we obtain a partition with exactly $b$ odd-parts.

Proposition 5.4.7. *Let $b, c \in \mathbf{N}_0$. Given $p$-core partitions $\gamma$ and $\delta$, let $n = |\gamma| + w_b(\gamma)p + |\delta| + w_c(\delta)p$. Suppose that if $b, c \geq p$, then $w_{b-p}(\gamma) \neq w_b(\gamma) - 1$ and $w_{c-p}(\delta) \neq w_c(\delta) - 1$. Then every summand of $M_{(2a,b,c)}$ in the block $B((\gamma, w_b(\gamma)), (\delta, w_c(\delta)))$ is projective.*

Proof. Suppose that there exists a non-projective indecomposable summand $U$ of $M_{(2a,b,c)}$ in the block $B((\gamma, w_b(\gamma)), (\delta, w_c(\delta)))$. It follows from Theorem 5.1.1 that $U$ has a vertex equal to a Sylow $p$-subgroup $Q_{(\lambda,t,u)}$ of

$$V_{p\lambda} \times C_2 \wr S_{tp} \times C_2 \wr S_{up},$$

where $\lambda = (\lambda_1, \lambda_2) \vdash s$ and $sp \leq a$, $tp \leq b$, $up \leq c$.

Let $r = 2s + t + u$, and so $R_r \leq Q_{(\lambda,t,u)}$. It follows from Lemma 5.3.4 and Lemma 5.3.11 that

$$M_{(2a,b,c)}(R_r) \cong \bigoplus N_{(\lambda,t,u)} \boxtimes M_{(2(a-|\lambda|p),b-tp,c-up)},$$

where the sum runs over all $(\lambda, t, u) \in T'_r$. By the Krull–Schmidt Theorem we have

$$(N_{(\lambda,t,u)} \boxtimes W) \mid U(R_r),$$

for some indecomposable summand $W$ of $M_{(2(a-|\lambda|p),b-tp,c-up)}$. By Lemma 1.3.12 the block $B((\gamma, w_b(\gamma)), (\delta, w_c(\delta)))$ therefore corresponds to the block containing $N_{(\lambda,t,u)} \boxtimes W$. The second statement of Proposition 5.4.4 then implies that $W$ lies in a block of the form

$$B := B((\gamma, w_b(\gamma) - i), (\delta, w_c(\delta) - (r - i))),$$

for some $0 \leq i \leq r$. By Lemma 5.4.1 therefore there exists $S^{(\lambda',\mu')}$ lying in $B$ such that $\lambda'$ has exactly $b - tp$ odd parts, and $\mu'$ has exactly $c - up$ odd parts. Adding $tp$ parts of size 1 to $\lambda'$ results in a partition $\lambda$ with $p$-core $\gamma$, weight $w_b(\gamma) - i + t$ and exactly $b$ odd parts. Similarly adding $up$ parts of size 1 to $\mu'$ results in a partition $\mu$ with $p$-core $\delta$, weight $w_c(\delta) - (r - i) + u$ and exactly $c$ odd parts. This contradicts the minimality of either $w_b(\gamma)$ or $w_c(\delta)$ unless $(t, u) = (i, r - i)$.

When $(t, u) = (i, r - i)$, we distinguish two cases. First suppose that $i \neq 0$. Then adding $(i-1)p$ parts of size 1 to $\lambda'$ results in a partition with $p$-core $\gamma$, weight $w_b(\gamma)-1$ and $b-p$ odd parts. Therefore $w_{b-p}(\gamma) = w_b(\gamma)-1$, contradicting the hypothesis of the theorem. In the case that $i = 0$, we argue in a similar way by adding $(r - 1)p$ parts of size 1 to $\mu'$, and contradicting the hypothesis that $w_{c-p}(\delta) \neq w_c(\delta) - 1$. $\square$

Given $p$-regular partitions $\nu_i$ and $\widetilde{\nu_i}$, recall from Theorem 1.3.15 that there exists a projective indecomposable module corresponding to the irreducible module $D^{(\nu_i,\widetilde{\nu_i})}$. We denote this module by $P^{(\nu_i,\widetilde{\nu_i})}$ for the remainder of this section. Also let $P^{(\nu_i,\widetilde{\nu_i})}_{\mathbf{Z}_p}$ denote the module such that

$$P^{(\nu_i,\widetilde{\nu_i})}_{\mathbf{Z}_p} \otimes_{\mathbf{Z}_p} \mathbf{F}_p = P^{(\nu_i,\widetilde{\nu_i})},$$

which exists by Scott's Lifting Theorem (see Theorem 1.3.13). Using Brauer reciprocity (see Theorem 1.3.16) the ordinary character of $P^{(\nu_i,\widetilde{\nu_i})}_{\mathbf{Z}_p}$ is

$$\psi^{(\nu_i,\widetilde{\nu_i})} = \sum_{\lambda,\mu} d_{\lambda\nu_i,\mu\widetilde{\nu_i}} \chi^{(\lambda,\mu)},$$

where we refer the reader to §1.4.4 for the definition of the decomposition number $d_{\lambda\nu_i,\mu\widetilde{\nu_i}}$. Observe that the sum can be taken over the $(\lambda,\mu) \in \mathcal{P}^2(n)$ such that $|\nu_i| = |\lambda|$ with $\lambda \trianglelefteq \nu_i$, and $|\mu| = |\widetilde{\nu_i}|$ with $\mu \trianglelefteq \widetilde{\nu_i}$. Indeed suppose that $D^{(\nu_i,\widetilde{\nu_i})}$ is a composition factor of $S^{(\lambda,\mu)}$, and so by Proposition 1.4.8 $\operatorname{Inf} D^{\nu_i} \boxtimes (\widetilde{N}^{\otimes|\widetilde{\nu_i}|} \otimes \operatorname{Inf} D^{\widetilde{\nu_i}})$ is a composition factor of

$$\operatorname{Inf} S^{\lambda} \boxtimes (\widetilde{N}^{\otimes|\mu|} \otimes \operatorname{Inf} S^{\mu}).$$

The claim now follows from [**33**, Corollary 12.2].

Proposition 5.4.8. *Fix $b, c \in \mathbf{N}_0$. Given $p$-core partitions $\gamma$ and $\delta$, let $n = |\gamma| + w_b(\gamma)p + |\delta| + w_c(\delta)p$. Suppose that if $b, c \geq p$, then $w_{b-p}(\gamma) \neq w_b(\gamma) - 1$ and $w_{c-p}(\delta) \neq w_c(\delta) - 1$. Let $\lambda$ and $\mu$ be maximal partitions in $\mathcal{E}_b(\gamma)$ and $\mathcal{E}_c(\delta)$, respectively. Then $\lambda$ and $\mu$ are both $p$-regular.*

Proof. It follows from Proposition 5.4.7 that every summand of the module $M_{(2a,b,c)}$ in the block $B((\gamma, w_b(\gamma)), (\delta, w_c(\delta)))$ is projective. Moreover, by Lemma 5.4.1 there exists a summand of $M_{(2a,b,c)}$ in this block.

Let

$$P^{(\nu_1,\widetilde{\nu_1})}_{\mathbf{F}_p}, \ldots, P^{(\nu_t,\widetilde{\nu_t})}_{\mathbf{F}_p}$$

be the summands of $M_{(2a,b,c)}$ in the block $B((\gamma, w_b(\gamma)), (\delta, w_c(\delta)))$. Let $M$ denote $M_{(2a,b,c)}$ when defined over $\mathbf{Z}_p$. It follows from Scott's Lifting Theorem that the summands of $M_{(2a,b,c)}$ can be lifted to summands of $M$. The ordinary character of the summand of $M_{(2a,b,c)}$ in $B((\gamma, w_b(\gamma))(\delta, w_c(\delta)))$ is equal to $\psi^{(\nu_1,\widetilde{\nu_1})} + \cdots + \psi^{(\nu_t,\widetilde{\nu_t})}$. It follows from Lemma 5.4.1 that

$$\psi^{(\nu_1,\widetilde{\nu_1})} + \cdots + \psi^{(\nu_t,\widetilde{\nu_t})} = \sum_{(\lambda',\mu')} \chi^{(\lambda',\mu')}, \tag{5.7}$$

where the sum is over all $(\lambda', \mu') \in \mathcal{E}_b(\gamma) \times \mathcal{E}_c(\delta)$. By Brauer reciprocity the constituents $\chi^{(\lambda',\mu')}$ of $\psi^{(\nu_i,\widetilde{\nu_i})}$ are such that $\lambda' \trianglelefteq \nu_i$ and $\mu' \trianglelefteq \widetilde{\nu_i}$ for each $i$. As $\lambda$ and $\mu$ are maximal, $(\nu_i, \widetilde{\nu_i}) = (\lambda, \mu)$ for exactly one $i$, and so the result is proved. □

Each pair of maximal partitions in $\mathcal{E}_b(\gamma) \times \mathcal{E}_c(\delta)$ therefore labels a summand of $M_{(2a,b,c)}$ lying in the block $B((\gamma, w_b(\gamma)), (\delta, w_c(\delta)))$; moreover, every such summand is labelled by a pair of this form. We now prove Theorem 5.1.2.

Proof of Theorem 5.1.2. Let $P_{\mathbf{F}_p}^{(\nu_1,\widetilde{\nu_1})}, \ldots, P_{\mathbf{F}_p}^{(\nu_c,\widetilde{\nu_c})}$ be the summands of $M_{(2a,b,c)}$ lying in the block $B((\gamma, w_b(\gamma)), (\delta, w_c(\delta)))$, all of which are projective. It follows from (5.7) that there exists a set partition $\Lambda_1, \ldots, \Lambda_t$ of $\mathcal{E}_b(\gamma) \times \mathcal{E}_c(\delta)$ such that $(\nu_i, \widetilde{\nu_i}) \in \Lambda_i$ for each $i$ and

$$\psi^{(\nu_i,\widetilde{\nu_i})} = \sum_{(\lambda',\mu') \in \Lambda_i} \chi^{(\lambda',\mu')}.$$

The statement of the theorem now follows by another application of Brauer reciprocity. □

CHAPTER 6

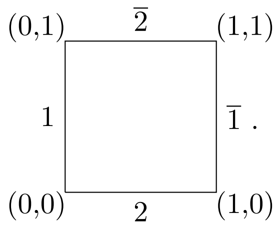
# Cubic singular homology and representations of $C_2 \wr S_n$

Throughout this chapter fix $n \in \mathbf{N}_0$. Let $I$ denote the closed unit interval $[0,1]$, and define the $n$-hypercube to be $I^n$. The hyperoctahedral group $C_2 \wr S_n$ arises naturally as the group of symmetries of the $n$-hypercube, and in this chapter we consider $C_2 \wr S_n$ in this context. Recall that if $n \geq 1$, then we view $C_2 \wr S_n$ as the subgroup of $\mathrm{Sym}(\{1, 2, \ldots, n, \overline{1}, \overline{2}, \ldots, \overline{n}\})$ generated by the set

$$\{(1\ \overline{1}), (1\ 2)(\overline{1}\ \overline{2}), (1\ 2 \ldots n)(\overline{1}\ \overline{2} \ldots \overline{n})\}.$$

We then define face $i$ (resp. $\overline{i}$) to be the $(n-1)$-hypercube with $x_i = 0$ (resp. $x_i = 1$) for all $1 \leq i \leq n$. If $n = 0$, then $C_2 \wr S_0$ is viewed to be the trivial symmetric group, and the 0-hypercube is a point. Therefore in all cases we regard a symmetry of the $n$-hypercube as a permutation of its $2n$ faces.

Example 6.0.1. Let $n = 2$. The 2-hypercube is a square, and we label its faces as follows:




Therefore the reflection through faces 2 and $\overline{2}$ is given by the transposition $(1\ \overline{1})$, and the anticlockwise rotation through the centre of the square is given by $(1\ 2\ \overline{1}\ \overline{2})$. Observe that these two elements generate $C_2 \wr S_2$.

For our purposes, it is more convenient to redefine the $n$-hypercube using a more abstract notation, and then interpret this abstract definition in the usual geometric setting of the $n$-hypercube as $[0,1]^n$.

Indeed we redefine the $n$-hypercube to be the set

$$\{\{1, \overline{1}\}, \ldots, \{n, \overline{n}\}\}.$$

This is acted on, trivially, by $C_2 \wr S_n$ as follows:

$$\sigma\{\{1, \overline{1}\}, \ldots, \{n, \overline{n}\}\} = \{\{\sigma(1), \sigma(\overline{1})\}, \ldots, \{\sigma(n), \sigma(\overline{n})\}\}.$$

The geometric interpretation of this trivial action is that $C_2 \wr S_n$ is the symmetry group of the $n$-hypercube.

Given $0 \leq i < n$, we define an *$i$-hypercube lying on the $n$-hypercube* to be an element of the form

$$\{\{\{a_1, \overline{a_1}\}, \ldots, \{a_i, \overline{a_i}\}\}, \{a_{i+1}, \ldots, a_n\}\}$$

such that

$$\{a_1, \ldots, a_i\} \subset \{1, 2, \ldots, n\},$$
$$\{a_{i+1}, \ldots, a_n\} \subset \{1, 2, \ldots, n, \overline{1}, \overline{2}, \ldots, \overline{n}\},$$

and $\{a_1, \ldots, a_i, \overline{a_1}, \ldots, \overline{a_i}\} \cap \{a_{i+1}, \ldots, a_n\} = \varnothing$. This element has the geometric interpretation as the intersection of the faces of the $n$-hypercube labelled by $a_{i+1}, \ldots, a_n$. Then there is an action of $\sigma \in C_2 \wr S_n$ on the set of $i$-hypercubes lying on the $n$-hypercube given by

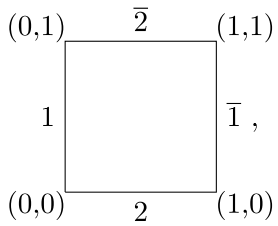
$$\begin{aligned}&\sigma\{\{\{a_1, \overline{a_1}\}, \ldots, \{a_i, \overline{a_i}\}\}, \{a_{i+1}, \ldots, a_n\}\} \\ &\quad = \{\{\{\sigma(a_1), \sigma(\overline{a_1})\}, \ldots, \{\sigma(a_i), \sigma(\overline{a_i})\}\}, \{\sigma(a_{i+1}), \ldots, \sigma(a_n)\}\}.\end{aligned}$$

We pause to give an example of these definitions and their geometric interpretations.

Example 6.0.2. Let $n = 2$. Then according to our definition, the 2-hypercube is equal to the set $\{\{1, \overline{1}\}, \{2, \overline{2}\}\}$. Moroever, setting $i = 0$, we see that the 0-hypercubes lying on the 2-hypercube are

$$(\varnothing, \{1, 2\}), (\varnothing, \{1, \overline{2}\}), (\varnothing, \{\overline{1}, 2\}) \text{ and } (\varnothing, \{\overline{1}, \overline{2}\}).$$

Returning to the usual geometric construction of the 2-hypercube as the square:




we see that the vertex of the square labelled $(0, 0)$ is the intersection of the 1-hypercubes (lines) labelled by 1 and 2. Therefore the vertex $(0, 0)$ corresponds to the 0-hypercube $(\varnothing, \{1, 2\})$ lying on the 2-hypercube. The complete correspondence between the vertices of the square and the 0-hypercubes lying on the 2-hypercube is as follows:

$$(0, 0) \leftrightarrow (\varnothing, \{1, 2\})$$
$$(0, 1) \leftrightarrow (\varnothing, \{1, \overline{2}\})$$
$$(1, 0) \leftrightarrow (\varnothing, \{\overline{1}, 2\})$$
$$(1, 1) \leftrightarrow (\varnothing, \{\overline{1}, \overline{2}\}).$$

We now define the boundary map $\delta_n$ on the $n$-hypercube, which is of central interest in this chapter. For ease of notation, we briefly write $T$ for the $n$-hypercube

$$\{\{1,\overline{1}\},\ldots,\{n,\overline{n}\}\}.$$

If $n \geq 1$, then define

$$\begin{aligned}\delta_n(T) = \sum_{i=1}^{n}(-1)^{i-1}\big(&\{(\{1,\overline{1}\},\ldots\widehat{\{i,\overline{i}\}},\ldots,\{n,\overline{n}\}),\{i\}\}\\ &-\{(\{1,\overline{1}\},\ldots\widehat{\{i,\overline{i}\}},\ldots,\{n,\overline{n}\}),\{\overline{i}\}\}\big),\end{aligned}$$

where the hat over $\{i,\overline{i}\}$ in each term indicates that it is omitted. Therefore $\delta_n(T)$ is formally contained in the vector space spanned by the set of $(n-1)$-hypercubes lying on $T$. In the case that $n = 0$, then $\delta_n(T) := \varnothing$. We refer to the map $\delta_n$ as the *boundary map*, since it sends an $n$-hypercube to its boundary of $(n-1)$-hypercubes for $n \geq 1$.

Given $0 \leq i \leq n$, we have seen that $C_2 \wr S_n$ permutes the set of $i$-hypercubes lying on the $n$-hypercube. It follows that the $\mathbf{Q}$-span of this set is a $\mathbf{Q}C_2 \wr S_n$-permutation module. We can define the boundary map $\delta_i$ for general $i$, which sends an $i$-hypercube lying on $I^n$ to its boundary of $(i-1)$-hypercubes. However, in general the map $\delta_i$ does not commute with the permutation action of $\mathbf{Q}C_2 \wr S_n$ on the set of $i$-hypercubes lying on $I^n$, as illustrated in the example below.

EXAMPLE 6.0.3. Let $n = 2$. By definition $\delta_2(\{\{1,\overline{1}\},\{2,\overline{2}\}\})$ equals

$$((\{2,\overline{2}\}),\{1\}) - ((\{2,\overline{2}\}),\{\overline{1}\}) - ((\{1,\overline{1}\}),\{2\}) + ((\{1,\overline{1}\}),\{\overline{2}\}).$$

Consider the permutation $(1\ \overline{1}) \in C_2 \wr S_2$, which acts trivially on the 2-hypercube. However $(1\ \overline{1})\delta_2(\{\{1,\overline{1}\},\{2,\overline{2}\}\})$ equals

$$-((\{2,\overline{2}\}),\{1\}) + ((\{2,\overline{2}\}),\{\overline{1}\}) - ((\{1,\overline{1}\}),\{2\}) + ((\{1,\overline{1}\}),\{\overline{2}\}),$$

and so $\delta_2$ is not a $\mathbf{Q}C_2 \wr S_n$-module homomorphism.

In order to overcome this obstacle, we define the oriented $n$-hypercube. We remind the reader that given $0 \leq x \leq n$, the $\mathbf{Q}$-span of $[x,\overline{x}]$ is isomorphic to the non-trivial irreducible $\mathbf{Q}\,\mathrm{Sym}(\{x,\overline{x}\})$-module $N$. Also recall that $\widehat{g}$ denotes the image of $g \in C_2 \wr S_n$ under the canonical surjection $C_2 \wr S_n \twoheadrightarrow S_n$. We then define the *oriented $n$-hypercube* to be the tuple

$$([1,\overline{1}],\ldots,[n,\overline{n}]).$$

We also define the one-dimensional $\mathbf{Q}C_2 \wr S_n$-module $U_n$ to be the $\mathbf{Q}$-span of the oriented $n$-hypercube, on which each of $(1\ \overline{1}),\ldots,(n\ \overline{n})$ acts by negative sign, and each $g \in T_n$ acts by $\mathrm{sgn}(\widehat{g})$.

As we have done so above, we discuss the geometric interpretation of the oriented $n$-hypercube.

EXAMPLE 6.0.4. Let $n = 2$, and so by definition the oriented 2-hypercube equals

$$([1, \overline{1}], [2, \overline{2}]).$$

Then $[1, \overline{1}]$ has the geometric interpretation of directing the first interval in $[0, 1] \times [0, 1]$ from 0 to 1, which corresponds to the arrows pointing right in the following figure:

$$\begin{tikzcd}
(0,1) \arrow[r, "\overline{2}"] & (1,1) \\
(0,0) \arrow[u, "1"] \arrow[r, "2"'] & (1,0) \arrow[u, "\overline{1}."']
\end{tikzcd}$$

Similarly $[2, \overline{2}]$ has the geometric interpretation of directing the second interval in $[0, 1] \times [0, 1]$ from 0 to 1, which corresponds to the upwards pointing arrows in the above figure.

Moreover, the lines on the oriented 2-hypercube are also oriented 1-hypercubes. Indeed, the line labelled 1 in the above figure is an oriented 1-hypercube with faces (vertices) $(\varnothing, \{1, 2\})$ and $(\varnothing, \{1, \overline{2}\})$.

Given $0 \leq i < n$, define $\mathcal{C}_i$ to be set of all elements of the form

$$\{([a_1, \overline{a_1}], \ldots, [a_i, \overline{a_i}]), \{a_{i+1}, \ldots, a_n\}\}$$

such that

$$\{a_1, \ldots, a_i\} \subset \{1, 2, \ldots, n\},$$
$$\{a_{i+1}, \ldots, a_n\} \subset \{1, 2, \ldots, n, \overline{1}, \overline{2}, \ldots, \overline{n}\},$$

and $\{a_1, \ldots, a_i, \overline{a_1}, \ldots, \overline{a_i}\} \cap \{a_{i+1}, \ldots, a_n\} = \varnothing$. Then we define an *oriented $i$-hypercube lying on the $n$-hypercube* to be an element of $\mathcal{C}_i$. Then the $\mathbf{Q}$-span of $\mathcal{C}_i$ is a $\mathbf{Q}C_2 \wr S_n$-module, with action given by

$$\begin{aligned}
&\sigma\{([a_1, \overline{a_1}], \ldots, [a_i, \overline{a_i}]), \{a_{i+1}, \ldots, a_n\}\} \\
&= \{([\sigma(a_1), \sigma(\overline{a_1})], \ldots, [\sigma(a_i), \sigma(\overline{a_i})]), \{\sigma(a_{i+1}), \ldots, \sigma(a_n)\}\}.
\end{aligned}$$

Given $v = (([a_1, \overline{a_1}], \ldots, [a_i, \overline{a_i}]), \{a_{i+1}, \ldots, a_n\}) \in \mathcal{C}_i$, define

$$\mathcal{S}(v) = \{a_1, \ldots, a_i\}.$$

Also define $\mathcal{D}_i$ to be $\mathbf{Q}$-span of the set

$$\{v - h \operatorname{sgn}(\widehat{h}) v : v \in \mathcal{C}_i, h \in C_2 \wr S_{\mathcal{S}(v)}\}.$$

LEMMA 6.0.5. *The vector space $F\mathcal{D}_i$ is an $FC_2 \wr S_n$-submodule of $F\mathcal{C}_i$.*

PROOF. We show that $F\mathcal{D}_i$ is closed under the action of $C_2 \wr S_n$. Fix $k \in C_2 \wr S_n$ and $v - \operatorname{sgn}(\widehat{h})hv \in \mathcal{D}_i$, where $h \in C_2 \wr S_{\mathcal{S}(v)}$. With $h' := {}^k h$, it follows that

$$\begin{aligned} k(v - \operatorname{sgn}(\widehat{h})hv) &= kv - \operatorname{sgn}(\widehat{h})khv \\ &= kv - \operatorname{sgn}(\widehat{h})h'(kv) \\ &= kv - \operatorname{sgn}(\widehat{h'})h'(kv), \end{aligned}$$

where the third equality holds since $h$ and $h'$ are conjugate in $C_2 \wr S_n$. By definition of $\mathcal{S}(v)$, there an equality $\mathcal{S}(kv) = \{\widehat{k}x : x \in \mathcal{S}(v)\}$. The lemma is now proved as $\operatorname{supp}(h') = \{kx : x \in \operatorname{supp}(h)\}$, and so $h' \in C_2 \wr S_{\mathcal{S}(kv)}$. $\square$

Let $\{([a_1, \overline{a_1}], \ldots, [a_i, \overline{a_i}]), \{a_{i+1}, \ldots, a_n\}\} + \mathcal{D}_i$ be such that $a_1 < \cdots < a_i$. It follows from Lemma 6.0.5 that the set of all $v + \mathcal{D}_i$ of this form is a basis of the quotient module $F\mathcal{C}_i/\mathcal{D}_i$. To simplify the notation, we write $(([a_1, \overline{a_1}], \ldots, [a_i, \overline{a_i}]), \{a_{i+1}, \ldots, a_n\})$ for

$$\{([a_1, \overline{a_1}], \ldots, [a_i, \overline{a_i}]), \{a_{i+1}, \ldots, a_n\}\} + \mathcal{D}_i.$$

and we also write $U_i$ for $F\mathcal{C}_i/\mathcal{D}_i$. From now on, an oriented $i$-hypercube refers to the element $(([a_1, \overline{a_1}], \ldots, [a_i, \overline{a_i}]), \{a_{i+1}, \ldots, a_n\})$ in the quotient module $U_i$. This has the geometric interpretation as the intersection of the oriented faces ($(n-1)$-hypercubes) of the $n$-hypercube labelled by $a_{i+1}, \ldots, a_n$.

We now redefine $\delta_n$ to be the map

$$\begin{aligned} \delta_n(T) = \sum_{i=1}^{n} (-1)^{i-1} \big( &(([1, \overline{1}], \ldots, \widehat{[i, \overline{i}]}, \ldots, [n, \overline{n}]), \{i\}) \\ &- (([1, \overline{1}], \ldots, \widehat{[i, \overline{i}]}, \ldots, [n, \overline{n}]), \{\overline{i}\}) \big), \end{aligned}$$

and we define $\delta_i$ to be the linear extension of the map that sends

$$(([x_1, \overline{x_1}], \ldots, [x_i, \overline{x_i}]), \{x_{i+1}, \ldots, x_n\})$$

to

$$\begin{aligned} \sum_{j=1}^{n} (-1)^{j-1} \Big( &(([x_1, \overline{x_1}], \ldots, \widehat{[x_j, \overline{x_j}]}, \ldots, [x_i, \overline{x_i}]), \{x_j, x_{i+1}, \ldots, x_n\}) \\ &- (([x_1, \overline{x_1}], \ldots, \widehat{[x_j, \overline{x_j}]}, \ldots, [x_i, \overline{x_i}]), \{\overline{x_j}, x_{i+1}, \ldots, x_n\}) \Big). \end{aligned}$$

Under the setup of oriented hypercubes, the map $\delta_i : U_i \to U_{i-1}$ is a $\mathbf{Q}C_2 \wr S_n$-module homomorphism for all $0 \leq i < n$. Moreover, a short calculation shows that $\delta_i \delta_{i+1} = 0$ for $0 \leq i < n$, and in this chapter we prove the following theorem.

THEOREM 6.0.6. *The chain complex*

$$U_n \xrightarrow{\delta_n} U_{n-1} \xrightarrow{\delta_{n-1}} \cdots \xrightarrow{\delta_2} U_1 \xrightarrow{\delta_1} U_0 \xrightarrow{\delta_0} \mathbf{Q} \tag{6.1}$$

*is exact in all places.*

We mention an important motivation for our study of the chain complex (6.1). Recall from §3.1.2 that there exists a chain complex of $\mathbf{Q}S_n$-modules

$$\bigwedge^{n} M^{(n-1,1)} \xrightarrow{\widehat{\delta}_n} \cdots \xrightarrow{\widehat{\delta}_r} \bigwedge^{r-1} M^{(n-1,1)} \xrightarrow{\widehat{\delta}_{r-1}} \cdots \to M^{(n-1,1)} \to \mathbf{Q} \to 0.$$

The module $\bigwedge^r M^{(n-1,1)}$ has the geometric interpretation as the $\mathbf{Q}$-span of the set of all oriented $r$-simplices lying on the oriented $n$-simplex. It follows that the map $\widehat{\delta}_r$ sends an $r$-simplex to its boundary of $(r-1)$-simplices. We remark that we require oriented simplices so that we also have $\widehat{\delta}_{r-1}\widehat{\delta}_r = 0$. Then the chain complex displayed above is exact in all places. For a proof of this result, we refer the reader to [**66**, §2]. Therefore Theorem 6.0.6 is a natural generalisation from the representation theory of $S_n$ to that of $C_2 \wr S_n$. In fact our proof of Theorem 6.0.6 reduces to the analogous result in simplicial homology.

It should also be noted that cubic homology is of interest in topology (see for instance [**28**], [**37**], [**49**]). Indeed, as remarked in [**28**, page 97], the homotopy group $\pi_n(X)$ of a CW-complex $X$ is defined in terms of maps from the $n$-hypercube $I^n$ into $X$. As well as in this fundamental definition, cubic homology is preferred by some topologists due to it being more suitable than simplex homology in certain applications. For examples of these applications, we refer the reader to [**37**, §1].

Let $F$ be a field of characteristic 2. Given $0 \le i \le n$, define $\Omega_i$ to be the set of all subsets of $\{1, 2, \ldots, n\}$ of size $i$. The main object of study in [**66**] is the *multistep map*

$$\widehat{\psi}_i^{(t)} : F\Omega_i \to F\Omega_{i-t}$$
$$X \mapsto \sum_{\substack{Y \subset X \\ |Y| = i-t}} Y.$$

Observe that when $t = 1$, the boundary map $\widehat{\delta}_i$ coincides with the multistep map $\widehat{\psi}_i^{(1)}$, and so the multistep map is also a natural generalisation of the boundary map. We define and consider the analogous multistep map $\psi_i^{(t)}$ for the $i$-hypercube over a field of characteristic 2 (see §6.2), which provides the analogous generalisation of the boundary map $\delta_i$. In particular we give various differences between the cubic and simplicial homologies in characteristic 2, thereby demonstrating that there are notable differences between the representation theories of $S_n$ and $C_2 \wr S_n$.

**Outline.** In §6.1.1 we define a module $M_i$ that is isomorphic to $U_i$, and we give the definition of the boundary map in terms of a fixed basis $\mathcal{B}_i$ (defined in §6.1.2) of $M_i$. Our reason for working with the module $M_i$ is that it is easier to reduce from the cubic to the simplicial homology using this setup. In §6.1.2 we prove Theorem 6.0.6, and in §6.1.3 we show how

Theorem 6.0.6 implies that the $p$-modular reduction of (6.1) is also exact, where $p$ is a positive prime number.

In §6.2 we define the multistep map $\psi_i^{(t)}$, and we show that $\psi_i^{(t)}\psi_{i+t}^{(t)} = 0$. Therefore the maps $\psi_i^{(t)}$ give rise to a chain complex, which we present in (6.6). In Lemma 6.2.1 we show that the module $U_i$ is indecomposable over a field of characteristic 2 for all $i$. It follows that (6.6) can never be split exact. Lemma 6.2.1 is just one example of the differences between the homologies induced by the maps $\psi_i^{(t)}$ and $\widehat{\psi}_i^{(t)}$ over a field of characteristic 2. As might be expected, there are further differences in this case, some of which we illustrate in §6.2.

We conclude this chapter with §6.3, which explains why we only generalise the boundary map $\delta_i$ to the multistep map $\psi_i^{(t)}$ over a field of characteristic 2. We remark that we do this using the analogous result for $\widehat{\psi}_t^{(t)}$ from [**66**] and the Morita equivalence between $FC_2 \wr S_n$ and $\bigoplus_{i=0}^n FS_{(i,n-i)}$ (see Proposition 1.4.8).

## 6.1. The boundary maps $\delta_i$

Following the outline of this chapter we define the module $M_i$ for all $0 \leq i \leq n$. Firstly define the module $M$ to be the $2n$-dimensional $\mathbf{Q}C_2 \wr S_n$ permutation module with basis

$$\{e_1, e_{\overline{1}}, \ldots, e_n, e_{\overline{n}}\},$$

and action given by the linear extension of

$$\sigma e_i = e_{\sigma(i)},$$

for all $i \in \{1, 2, \ldots, n, \overline{1}, \ldots, \overline{n}\}$ and $\sigma \in C_2 \wr S_n$.

Given $x \in \{1, 2, \ldots, n\}$, define

$$\begin{aligned} e_x^+ &= e_x + e_{\overline{x}} \\ e_x^- &= e_x - e_{\overline{x}}, \end{aligned}$$

and briefly define $U$ to be the $\mathbf{Q}C_2 \wr S_n$-module spanned by

$$\{e_1^-, e_2^-, \ldots, e_n^-\}.$$

Given $0 \leq i \leq n$, define $\mathcal{B}_i$ to be the set of elements of the form

$$e_{x_1}^- \wedge \cdots \wedge e_{x_i}^- \otimes e_{x_{i+1}} e_{x_{i+2}} \ldots e_{x_n} \in \bigwedge^i U \otimes \operatorname{Sym}^{n-i} M,$$

such that

$$\{x_1, \overline{x_1}, \ldots, x_i, \overline{x_i}\} \cap \{x_{i+1}, x_{i+2}, \ldots, x_n\} = \varnothing$$

and $x_{i+j} \notin \{x_{i+k}, \overline{x_{i+k}}\}$ for all $j$ and $k$. Observe that it is entirely possible to have $x_{i+1}, \ldots, x_n \in \{\overline{1}, \ldots, \overline{n}\}$ according to this definition. Then $M_i$ is defined to be the $\mathbf{Q}C_2 \wr S_n$-module spanned by $\mathcal{B}_i$. We give an example of $\mathcal{B}_1$ and $\mathcal{B}_2$ when $n = 3$ in Example 6.1.3.

**6.1.1. The modules $M_i$ and $U_i$ are isomorphic.** Fix $0 \leq i \leq n$. We show that the ordinary characters of $M_i$ and $U_i$ are equal, from which it follows that they are isomorphic as $\mathbf{Q}C_2 \wr S_n$-modules. Define

$$H_i = (C_2 \wr S_i) \times 1 \wr S_{\{i+1,\dots,n\}}.$$

We remind the reader that we write $\chi^{(\lambda,\mu)}$ for the ordinary irreducible character of $C_2 \wr S_n$ labelled by $(\lambda, \mu) \in \mathcal{P}^2(n)$ (see Theorem 1.4.5).

LEMMA 6.1.1. *The module $M_i$ has ordinary character equal to*

$$(\chi^{(\varnothing,(1^i))} \times 1_{1 \wr S_{\{i+1,\dots,n\}}})\big\uparrow_{H_i}^{C_2 \wr S_n}.$$

PROOF. Consider the vector

$$e_1^- \wedge \cdots \wedge e_i^- \otimes e_{i+1}e_{i+2}\dots e_n,$$

which spans a one-dimensional $\mathbf{Q}H_i$-module with ordinary character

$$\chi^{(\varnothing,(1^i))} \times 1_{1 \wr S_{\{i+1,\dots,n\}}}.$$

Moreover, this vector generates $M_i$ as a $\mathbf{Q}C_2 \wr S_n$-module.

Given

$$e_{x_1}^- \wedge \cdots \wedge e_{x_i}^- \otimes e_{x_{i+1}}e_{x_{i+2}}\dots e_{x_n} \in \mathcal{B}_i,$$

there are $\binom{n}{i}$ ways to choose the set $\{x_1, \dots, x_i\}$. With $x_1, \dots, x_i$ fixed, there are $2^{n-i}$ ways to choose the elements of the set $\{x_{i+1}, \dots, x_n\}$, and so

$$\dim_{\mathbf{Q}} M_i = 2^{n-i}\binom{n}{i} = [C_2 \wr S_n : H_i].$$

The result now follows from Lemma 1.3.2. $\square$

By definition of the action of $C_2 \wr S_n$ on $U_n$, we see that $U_n$ has ordinary character $\chi^{(\varnothing,(1^n))}$. We use this to determine the ordinary character of $U_i$ in general.

LEMMA 6.1.2. *The $\mathbf{Q}C_2 \wr S_n$-module $U_i$ has character equal to*

$$(\chi^{(\varnothing,(1^i))} \times 1 \wr S_{\{i+1,\dots,n\}})\big\uparrow_{H_i}^{C_2 \wr S_n},$$

*and so there is an isomorphism $M_i \cong U_i$.*

PROOF. Consider the oriented $i$-hypercube in $U_i$ given by

$$(([1,\bar{1}], \dots, [i,\bar{i}]), \{i+1, \dots, n\}),$$

which we denote by $x$. By the remark immediately before the statement of this lemma, the vector space spanned by $x$ is a one-dimensional module for $C_2 \wr S_i$ with character $\chi^{(\varnothing,(1^i))}$. Furthermore, the subgroup $1 \wr S_{\{i+1,\dots,n\}}$ acts trivially on $x$. Therefore the $\mathbf{Q}$-span of $x$ is a one-dimensional $\mathbf{Q}H_i$-module with ordinary character equal to

$$\chi^{(\varnothing,(1^i))} \times 1_{1 \wr S_{\{i+1,\dots,n\}}}.$$

Since $C_2 \wr S_n$ acts transitively on the set of $i$-dimensional hypercubes lying on the $n$-hypercube, $x$ generates $U_i$ as a $\mathbf{Q}C_2 \wr S_n$-module. As in the proof of Lemma 6.1.1, given

$$(([x_1, \overline{x_1}], \ldots, [x_i, \overline{x_i}]), \{x_{i+1}, \ldots, x_n\})$$

in the defining basis of $U_i$, there are $\binom{n}{i}$ ways to choose the set $\{x_1, \ldots, x_i\}$. With $x_1, \ldots, x_i$ fixed, there are $2^{n-i}$ ways to choose the elements of the set $\{x_{i+1}, \ldots, x_n\}$, and so

$$\dim_{\mathbf{Q}} M_i = 2^{n-i}\binom{n}{i} = [C_2 \wr S_n : H_i],$$

and so the first statement of the lemma follows once more from Lemma 1.3.2.

The second statement is now immediate by Lemma 6.1.1. $\square$

We remark that it is possible to make the isomorphism between $M_i$ and $U_i$ given by Lemma 6.1.1 and Lemma 6.1.2 explicit. Indeed we identify the vector

$$e_{x_1}^- \wedge \cdots \wedge e_{x_i}^- \otimes e_{x_{i+1}} e_{x_{i+2}} \ldots e_{x_n} \in \mathcal{B}_i$$

with the oriented $i$-hypercube

$$(([x_1, \overline{x_1}], \ldots, [x_i, \overline{x_i}]), \{x_{i+1}, \ldots, x_n\}).$$

We therefore have the geometric interpretation of

$$e_{x_1}^- \wedge \cdots \wedge e_{x_i}^- \otimes e_{x_{i+1}} e_{x_{i+2}} \ldots e_{x_n}$$

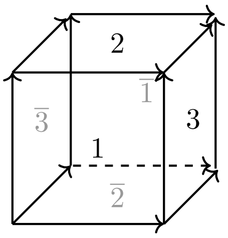
as the intersection of the faces labelled $x_{i+1}, \ldots, x_n$.

We demonstrate this identification in the following example, which also gives an explicit construction of the sets $\mathcal{B}_1$ and $\mathcal{B}_2$ when $n = 3$.

EXAMPLE 6.1.3. Let $n = 3$. By definition

$$\mathcal{B}_2 = \left\{ \begin{array}{l} e_1^- \wedge e_2^- \otimes e_3,\ e_1^- \wedge e_2^- \otimes e_{\overline{3}} \\ e_1^- \wedge e_3^- \otimes e_2,\ e_1^- \wedge e_3^- \otimes e_{\overline{2}} \\ e_2^- \wedge e_3^- \otimes e_1,\ e_2^- \wedge e_3^- \otimes e_{\overline{1}} \end{array} \right\},$$

and the faces of the 3-hypercube are labelled as follows:



,

where the arrows demonstrate that each factor of

$$[0, 1] \times [0, 1] \times [0, 1]$$

is directed from 0 to 1. Then, for instance, the vector

$$e_{\overline{2}}^- \wedge e_{\overline{3}}^- \otimes e_1$$

in $\mathcal{B}_2$ corresponds to the face labelled 1.

Similarly, by definition

$$\mathcal{B}_1 = \left\{ \begin{array}{l} e_{\overline{1}}^- \otimes e_2 e_3,\; e_{\overline{2}}^- \otimes e_1 e_3,\; e_{\overline{3}}^- \otimes e_1 e_2 \\ e_{\overline{1}}^- \otimes e_{\overline{2}} e_3,\; e_{\overline{2}}^- \otimes e_{\overline{1}} e_3,\; e_{\overline{3}}^- \otimes e_{\overline{1}} e_2 \\ e_{\overline{1}}^- \otimes e_2 e_{\overline{3}},\; e_{\overline{2}}^- \otimes e_1 e_{\overline{3}},\; e_{\overline{3}}^- \otimes e_1 e_{\overline{2}} \\ e_{\overline{1}}^- \otimes e_{\overline{2}} e_{\overline{3}},\; e_{\overline{2}}^- \otimes e_{\overline{1}} e_{\overline{3}},\; e_{\overline{3}}^- \otimes e_{\overline{1}} e_{\overline{2}} \end{array} \right\}.$$

Let $\ell_{\overline{1},\overline{2}}$ denote the line given by the intersection of the faces $\overline{1}$ and $\overline{2}$, indicated by the dashed arrow in the figure on the previous page. Then $e_{\overline{3}}^- \otimes e_{\overline{1}} e_{\overline{2}} \in \mathcal{B}_1$ can be identified with $\ell_{\overline{1},\overline{2}}$.

**6.1.2. The boundary maps in characteristic 0.** Using the correspondence between $i$-hypercubes and elements in $\mathcal{B}_i$, we redefine the boundary map $\delta_i : M_i \to M_{i-1}$ to be the linear extension of the map that sends the basis vector

$$e_{x_1}^- \wedge \cdots \wedge e_{x_i}^- \otimes e_{x_{i+1}} e_{x_{i+2}} \dots e_{x_n}$$

to

$$\sum_{j=1}^{i} (-1)^{j-1} e_{x_1}^- \wedge \cdots \wedge \widehat{e_{x_j}^-} \wedge \cdots \wedge e_{x_i}^- \otimes e_{x_j}^- e_{x_{i+1}} e_{x_{i+2}} \dots e_{x_n},$$

where $1 \le i \le n$. Also given $e_{x_1} \dots e_{x_n} \in \mathcal{B}_0$, we define

$$\delta_0(e_{x_1} \dots e_{x_n}) = \varnothing.$$

For the remainder of this section we study the chain complex

$$M_n \xrightarrow{\delta_n} M_{n-1} \xrightarrow{\delta_{n-1}} \cdots \xrightarrow{\delta_2} M_1 \xrightarrow{\delta_1} M_0 \xrightarrow{\delta_0} \mathbf{Q}. \tag{6.2}$$

Our main result is the following proposition, which shows that this chain complex is exact in all places, and therefore proves Theorem 6.0.6.

PROPOSITION 6.1.4. *The chain complex*

$$M_n \xrightarrow{\delta_n} M_{n-1} \xrightarrow{\delta_{n-1}} \cdots \xrightarrow{\delta_2} M_1 \xrightarrow{\delta_1} M_0 \xrightarrow{\delta_0} \mathbf{Q},$$

*is exact in all places.*

As mentioned in the outline, we prove this result by reducing to the homology of the simplex. We therefore remind the reader of the following result from §3.1.2, which is required in the proof of Proposition 6.1.4.

LEMMA 3.1.3. *The chain complex*

$$0 \to \bigwedge^{n} M^{(n-1,1)} \xrightarrow{\widehat{\delta}_n} \cdots \xrightarrow{\widehat{\delta}_r} \bigwedge^{r-1} M^{(n-1,1)} \xrightarrow{\widehat{\delta}_{r-1}} \cdots \to M^{(n-1,1)} \to \mathbf{Q} \to 0,$$

*is exact in all places. Furthermore*

$$\ker \widehat{\delta}_k = \operatorname{im} \widehat{\delta}_{k+1} \cong \bigwedge^k S^{(n-1,1)}.$$

Proof of Proposition 6.1.4. We distinguish two cases, determined by $i$.

*Case* (1). Suppose that $i = 0$. For each $0 \leq j \leq n$, the module $M_0$ has a submodule spanned by the set

$$\{e^-_{x_1} e^-_{x_2} \dots e^-_{x_j} e^+_{x_{j+1}} \dots e^+_{x_n} : \{x_1, \dots, x_n\} = \{1, \dots, n\}\},$$

which we denote by $W_j$. Since each $W_j$ has dimension $\binom{n}{j}$, counting dimensions shows that there is direct sum decomposition

$$M_0 = \bigoplus_{i=0}^{n} W_j$$

of $\mathbf{Q}C_2 \wr S_n$-modules. Moreover, Lemma 1.3.2 shows that for each $0 \leq j \leq n$ there is an isomorphism $W_j \cong S^{((n-j),(j))}$.

We have that $W_j$ is contained in $\ker(\delta_0)$ if and only if $j \neq 0$. Furthermore,

$$e^-_{x_1} e^-_{x_2} \dots e^-_{x_j} e^+_{x_{j+1}} \dots e^+_{x_n} = \delta_1(e^-_{x_1} \otimes e^-_{x_2} \dots e^-_{x_j} e^+_{x_{j+1}} \dots e^+_{x_n}).$$

It follows that

$$\ker(\delta_0) = \bigoplus_{j=1}^{n} W_j = \operatorname{im}(\delta_1).$$

*Case* (2). Suppose that $i > 0$. We start by writing the ordinary character of $M_i$ as a sum of its irreducible constituents. By Lemma 6.1.2 and the transitivity of induction, the ordinary character of $M_i$ equals

$$\begin{aligned}
(\chi^{(\varnothing,(1^i))} \times 1_{S_{n-i}})\big\uparrow_{H_i}^{C_2 \wr S_n} &= (\chi^{(\varnothing,(1^i))} \times 1_{S_{n-i}})\big\uparrow_{H_i}^{C_2 \wr S_{(i,n-i)}}\big\uparrow_{C_2 \wr S_{(i,n-i)}}^{C_2 \wr S_n} \\
&= (\chi^{(\varnothing,(1^i))} \times \sum_{j=0}^{n-i} \chi^{((n-i-j),(j))})\big\uparrow_{C_2 \wr S_{(i,n-i)}}^{C_2 \wr S_n},
\end{aligned}$$

where the final equality holds by the previous case. If $j \geq 1$, then by the transitivity of induction and Young's rule (see Theorem 1.1.12)

$$\begin{aligned}
&(\chi^{(\varnothing,(1^i))} \times \chi^{((n-i-j),(j))})\big\uparrow_{C_2 \wr S_{(i,n-i)}}^{C_2 \wr S_n} \\
&\quad = (\chi^{(\varnothing,(1^i))} \times \chi^{((n-i-j),\varnothing)} \times \chi^{(\varnothing,(j))})\big\uparrow_{C_2 \wr S_{(i,n-i-j,j)}}^{C_2 \wr S_n} \\
&\quad = \chi^{((n-i-j),(j,1^i))} + \chi^{((n-i-j),(j+1,1^{i-1}))}.
\end{aligned}$$

Similarly if $j = 0$, then

$$(\chi^{(\varnothing,(1^i))} \times \chi^{((n-i),\varnothing)})\big\uparrow_{C_2 \wr S_{(i,n-i)}}^{C_2 \wr S_n} = \chi^{((n-i),(1^i))}.$$

It follows that the ordinary character of $M_i$ equals

$$\chi^{((n-i),(1^i))} + \sum_{j=1}^{n-i} \big(\chi^{((n-i-j),(j,1^i))} + \chi^{((n-i-j),(j+1,1^{i-1}))}\big).$$

Fix $j \in \mathbf{N}_0$ such that $j \leq n-i$. We consider the action of $\delta_i$ on the unique submodule of $M_i$ isomorphic to

$$S^{((n-i-j),(j,1^i))} \oplus S^{((n-i-j),(j+1,1^{i-1}))},$$

disregarding the second summand when $j = 0$. Define $V_{i,j}$ to be the $\mathbf{Q}$-span of the set of vectors of the form

$$e^-_{x_1} \wedge \cdots \wedge e^-_{x_i} \otimes e^-_{x_{i+1}} \ldots e^-_{x_{i+j}} e^+_{i+j+1} \ldots e^+_n,$$

where $x_\ell \in \{1, 2, \ldots, i+j\}$ for all $1 \leq \ell \leq i+j$. With $M_{i,j}$ defined to be the $\mathbf{Q}C_2 \wr S_n$-module generated by $V_{i,j}$, there is a direct sum decomposition

$$M_i = \bigoplus_{j=0}^{i} M_{i,j} \tag{6.3}$$

of $\mathbf{Q}C_2 \wr S_n$-modules. Moreover, $V_{i,j}$ is a $\mathbf{Q}[C_2 \wr S_{(i+j,n-i-j)}]$-module with ordinary character equal to

$$(\chi^{(\varnothing,(1^i))} \times \chi^{(\varnothing,(j))})\big\uparrow^{C_2 \wr S_{i+j}}_{C_2 \wr S_{(i,j)}} \times \chi^{((n-i-j),\varnothing)},$$

which, by the same application of Young's rule, equals

$$(\chi^{(\varnothing,(j,1^i))} + \chi^{(\varnothing,(j+1,1^{i-1}))}) \times \chi^{((n-i-j),\varnothing)}.$$

It follows that $M_{i,j}$ is the unique summand of $M_i$ isomorphic to

$$S^{((n-i-j),(j,1^i))} \oplus S^{((n-i-j),(j+1,1^{i-1}))}, \tag{6.4}$$

where we once more disregard the second term if $j = 0$. Define the map

$$\begin{aligned} \vartheta_{i,j} : V_{i,j} &\to \bigwedge^i M^{(i+j-1,1)} \\ e^-_{x_1} \wedge \cdots \wedge e^-_{x_i} \otimes e^-_{x_{i+1}} \ldots e^-_{x_{i+j}} e^+_{i+j+1} \ldots e^+_n &\mapsto v_{x_1} \wedge \cdots \wedge v_{x_i}, \end{aligned}$$

where $\{v_1, \ldots, v_{i+j}\}$ is the natural basis of the $\mathbf{Q}S_{i+j}$-module $M^{(i+j-1,1)}$. Let $K_{i+j}$ be the subgroup of $C_2 \wr S_n$ generated by the set

$$\{(1\ 2)(\overline{1}\ \overline{2}), (1\ 2 \ldots i+j)(\overline{1}\ \overline{2} \ldots \overline{i+j})\},$$

which is isomorphic to $S_{i+j}$. Then $\bigwedge^i M^{(i+j-1,1)}$ is a $\mathbf{Q}K_{i+j}$-module, with action given by

$$\sigma(v_{x_1} \wedge \cdots \wedge v_{x_i}) = \widehat{\sigma}(v_{x_1} \wedge \cdots \wedge v_{x_i}),$$

where, as usual, $\widehat{\sigma}$ denotes the image of $\sigma \in C_2 \wr S_n$ under the natural surjection $C_2 \wr S_n \twoheadrightarrow S_n$. With this action, $\vartheta_{i,j}$ is a $\mathbf{Q}K_{i+j}$-module isomorphism. Moreover, the square

$$\begin{array}{ccc} V_{i,j} & \xrightarrow{\quad\delta_i\quad} & V_{i-1,j+1} \\ {\scriptstyle\vartheta_{i,j}}\big\downarrow & & \big\downarrow{\scriptstyle\vartheta_{i-1,j+1}} \\ \bigwedge^i M^{(i+j-1,1)} & \xrightarrow[\widehat{\delta}_{i,j}]{} & \bigwedge^{i-1} M^{(i+j-1,1)} \end{array}$$

is commutative. Indeed, with

$$x := e^-_{x_1} \wedge \cdots \wedge e^-_{x_i} \otimes e^-_{x_{i+1}} \ldots e^-_{x_{i+j}} e^+_{i+j+1} \ldots e^+_n$$

we have that

$$\vartheta_{i-1,j+1}\delta_i(x) = \sum_{j=1}^{i} (-1)^{j-1} v_{x_1} \wedge \cdots \wedge \widehat{v_{x_j}} \wedge \cdots \wedge v_{x_i} = \widehat{\delta}_{i,j}\vartheta_{i,j}(x).$$

Since $j > 0$, there is an isomorphism of $\mathbf{Q}K_{i+j}$-modules

$$V_{i,j} \cong S^{(j,1^i)} \oplus S^{(j+1,1^{i-1})}.$$

Moreover, by Lemma 3.1.3 $\widehat{\delta_{i,j}}(S^{(j,1^i)}) = 0$ and $\widehat{\delta_{i,j}}(S^{(j+1,1^{i-1})}) \neq 0$. Since $\vartheta_{i,j}$ is an isomorphism, we have that

$$\begin{aligned} \delta_i(V_{i,j}) &\cong \vartheta^{-1}_{i-1,j+1}\widehat{\delta}_{i,j}\vartheta_{i,j}(V_{i,j}) \\ &\cong \vartheta^{-1}_{i-1,j+1}\widehat{\delta}_{i,j}(S^{(j,1^i)} \oplus S^{(j+1,1^{i-1})}) \\ &= \vartheta^{-1}_{i-1,j+1}(S^{(j+1,1^{i-1})}) \cong S^{(\varnothing,(j+1,1^{i-1}))} \boxtimes S^{((n-i-j),\varnothing)}. \end{aligned}$$

Therefore there is an isomorphism of $\mathbf{Q}[C_2 \wr S_{(n-i-j,i+j)}]$-modules

$$\ker(\delta_i) \cap V_{i,j} \cong S^{(\varnothing,(j,1^i))} \boxtimes S^{((n-i-j),\varnothing)} \cong \operatorname{im}(\delta_{i+1}) \cap V_{i,j},$$

and so

$$\ker(\delta_i) \cap M_{i,j} \cong S^{((n-i-j),(j,1^i))} \cong \operatorname{im}(\delta_{i+1}) \cap M_{i,j}.$$

Furthermore, if $j = 0$, then

$$\ker(\delta_i) \cap M_{i,j} = \{0\} = \operatorname{im}(\delta_{i+1}) \cap M_{i,j}.$$

It follows from (6.4) that

$$\ker(\delta_i) \cong \bigoplus_{j=1}^{n-i} S^{((n-i-j),(j,1^i))} \cong \operatorname{im}(\delta_{i+1}).$$

Since the ordinary character of $M_i$ is multiplicity free for $0 \leq i \leq n$, we have $\ker(\delta_i) = \operatorname{im}(\delta_{i+1})$ in all cases, as required. $\square$

The proof of Proposition 6.1.4 shows that $\ker(\delta_i) \cap M_{i,j}$ is the $\mathbf{Q}C_2 \wr S_n$-module generated by $\vartheta^{-1}_{i,j}(\ker(\widehat{\delta}_{i,j}))$. Equation (4) in [**23**] states that the set

$$D_{i,j} := \{\delta(e_1 \wedge e_{a_1} \wedge \cdots \wedge e_{a_j}) : 1 < a_1 < \cdots < a_j \leq n\}$$

is a basis for $\bigwedge^j S^{(n-1,1)}$. Recall that given $H \leq \mathrm{Sym}(\{1,\dots,n\})$, we write $\xi(H)$ for the subgroup of $T_n$ consisting precisely of the permutations $\sigma\overline{\sigma}$ such that $\sigma \in H$, where $\overline{\sigma} \in \mathrm{Sym}(\{\overline{1},\dots,\overline{n}\})$ is such that $\overline{\sigma}(\overline{i}) = \overline{(\sigma(i))}$. Then the set

$$E_{i,j} := \{\sigma\vartheta_{i,j}^{-1}(v) : v \in D_{i,j}, \sigma \in T_n \backslash \xi(S_{(i+j,n-i-j)})\}$$

is a basis of $\ker(\delta_i) \cap M_{i,j}$, and so $\bigcup_{j=0}^{i} E_{i,j}$ is a basis of $\ker(\delta_i)$.

EXAMPLE 6.1.5. Let $n = 4$. We give a basis of $\ker(\delta_2)$ in $M_2$, by giving a basis of $\mathrm{im}(\delta_3)$ in $M_2$.

The vector space $V_{2,0}$ is the $\mathbf{Q}C_2 \wr S_{(2,2)}$-module generated by the vector

$$e_1^- \wedge e_2^- \otimes e_3^+ e_4^+,$$

and $V_{2,1}$ is the $\mathbf{Q}C_2 \wr S_{(3,1)}$-module generated by the vector

$$e_1^- \wedge e_2^- \otimes e_3^- e_4^+.$$

Furthermore, $V_{2,2}$ is the $\mathbf{Q}C_2 \wr S_4$-module generated by the vector

$$e_1^- \wedge e_2^- \otimes e_3^- e_4^-,$$

and $V_{2,2} = M_{2,2}$. Therefore there is an equality

$$M_2 = M_{2,0} \oplus M_{2,1} \oplus M_{2,2} \tag{6.5}$$

of $\mathbf{Q}C_2 \wr S_4$-modules. Since $M_{(2,0)} \cap \mathrm{im}(\delta_3) = 0$, it is sufficient to determine a basis of $\mathrm{im}(\delta_3) \cap M_{2,j}$ for each $j \geq 1$.

Consider first the case when $j = 1$. Then $V_{2,1}$ is isomorphic to $\bigwedge^2 M^{(2,1)}$ as a $\mathbf{Q}K_3$-module. Observe that

$$\widehat{\delta}_{3,1}(e_1 \wedge e_2 \wedge e_3) = e_1 \wedge e_2 - e_1 \wedge e_3 + e_2 \wedge e_3$$

spans $\mathrm{im}(\widehat{\delta}_{3,1})$ in $\bigwedge^2 M^{(2,1)}$. The inverse image of $\widehat{\delta}_{3,1}(e_1 \wedge e_2 \wedge e_3)$ under the map $\vartheta_{2,1}$ is

$$e_1^- \wedge e_2^- \otimes e_3^- e_4^+ - e_1^- \wedge e_3^- \otimes e_2^- e_4^+ + e_2^- \wedge e_3^- \otimes e_1^- e_4^+.$$

Therefore $\mathrm{im}(\delta_3) \cap M_{2,1}$ has a basis equal to

$$E_{2,1} = \left\{ \begin{array}{l} e_1^- \wedge e_2^- \otimes e_3^- e_4^+ - e_1^- \wedge e_3^- \otimes e_2^- e_4^+ + e_2^- \wedge e_3^- \otimes e_1^- e_4^+, \\ e_4^- \wedge e_2^- \otimes e_3^- e_1^+ - e_4^- \wedge e_3^- \otimes e_2^- e_1^+ + e_2^- \wedge e_3^- \otimes e_4^- e_1^+, \\ e_1^- \wedge e_4^- \otimes e_3^- e_2^+ - e_1^- \wedge e_3^- \otimes e_4^- e_2^+ + e_4^- \wedge e_3^- \otimes e_1^- e_2^+, \\ e_1^- \wedge e_2^- \otimes e_4^- e_3^+ - e_1^- \wedge e_4^- \otimes e_2^- e_3^+ + e_2^- \wedge e_4^- \otimes e_1^- e_3^+ \end{array} \right\}.$$

Now let $j = 2$. Then $V_{2,2}$ is isomorphic to $\bigwedge^2 M^{(3,1)}$ as a $\mathbf{Q}K_4$-module. We see that

$$\{\widehat{\delta}_{3,2}(e_1 \wedge e_a \wedge e_b) : (a,b) \in \{(2,3),(2,4),(3,4)\}\}$$

is a basis of $\mathrm{im}(\widehat{\delta}_{3,2})$ in $\bigwedge^2 M^{(3,1)}$. The inverse image of this set under the map $\vartheta_{2,2}$ is equal to

$$E_{2,2} = \left\{ \begin{array}{l} e_1^- \wedge e_2^- \otimes e_3^- e_4^- - e_1^- \wedge e_3^- \otimes e_2^- e_4^- + e_2^- \wedge e_3^- \otimes e_1^- e_4^- , \\ e_1^- \wedge e_2^- \otimes e_3^- e_4^- - e_1^- \wedge e_4^- \otimes e_2^- e_3^- + e_2^- \wedge e_4^- \otimes e_1^- e_3^- , \\ e_1^- \wedge e_3^- \otimes e_2^- e_4^- - e_1^- \wedge e_4^- \otimes e_2^- e_3^- + e_3^- \wedge e_4^- \otimes e_1^- e_2^- \end{array} \right\}.$$

Since $V_{2,2} = M_{2,2}$, the set $E_{2,2}$ is a basis of $\mathrm{im}(\delta_3) \cap M_{2,2}$, and $E_{2,1} \cup E_{2,2}$ is a basis of $\mathrm{im}(\delta_3) = \ker(\delta_2)$.

**6.1.3. The boundary map in positive characteristic.** In this section assume that $F = \mathbf{F}_p$, where $p$ is a prime number. Proposition 2.5.2 in [**3**] states that if

$$0 \to M' \to M \to M'' \to 0$$

is a short exact sequence of $\mathbf{Z}_p$-modules, then there is a long exact sequence

$$\ldots \to \mathrm{Tor}_1^{\mathbf{Z}_p}(F, M) \to \mathrm{Tor}_1^{\mathbf{Z}_p}(F, M'') \to \widehat{M'} \to \widehat{M} \to \widehat{M''} \to 0,$$

where $\widehat{X}$ denotes the $p$-modular reduction of the module $X$ (defined in §1.3.4). In particular, if $M''$ is a free $\mathbf{Z}_p$-module, then the quotient $M/M'$ has no torsion. Equivalently $\mathrm{Tor}_1^{\mathbf{Z}_p}(F, M'') = 0$, and so the sequence

$$0 \to \widehat{M'} \to \widehat{M} \to \widehat{M''} \to 0$$

is exact.

Specialising to our case, given $0 \leq i \leq n$ and taking $M$ equal to $M_i$ defined over the ring of $p$-adic integers $\mathbf{Z}_p$, Proposition 6.1.4 gives that there is a short exact sequence of $\mathbf{Z}_p$-modules

$$0 \to \ker \delta_i \hookrightarrow M_i \twoheadrightarrow \mathrm{im}\, \delta_i \to 0.$$

By the discussion after the proof of Proposition 6.1.4, the module $\mathrm{im}\, \delta_i$ has a $\mathbf{Z}_p$-basis, and so it has no torsion. The discussion in the previous paragraph therefore gives the following lemma.

LEMMA 6.1.6. *Given a prime number $p$ and $0 \leq i \leq n$, let $\delta_{i,F}$ denote the $p$-modular reduction of the map $\delta_i$. Then the chain complex*

$$\widehat{M_n} \xrightarrow{\delta_{n,F}} \widehat{M_{n-1}} \xrightarrow{\delta_{n-1,F}} \cdots \xrightarrow{\delta_{2,F}} \widehat{M_1} \xrightarrow{\delta_{1,F}} \widehat{M_0} \xrightarrow{\delta_{0,F}} F,$$

*is exact in all places.*

## 6.2. The multistep maps $\psi_i^{(t)}$

In this section let $F$ be a field of characteristic 2. In this case all irreducible $FC_2 \wr S_n$-modules contain $FC_2^n$ in their kernel. Therefore the 2-modular reduction $\widehat{M_i}$ of $M_i$ is isomorphic to the permutation module

$$F\big\uparrow_{H_i}^{C_2 \wr S_n}.$$

We consider the the multistep map $\psi_i^{(t)} : \widehat{M}_i \to \widehat{M}_{i-t}$ in this section. In order to define this map the following preliminaries are required. Given $X \subseteq \{1, 2, \ldots, n\}$, define $\overline{X} = \{\overline{x} : x \in X\}$. Let $M$ denote the natural $FC_2 \wr S_n$ permutation module with basis $\{e_1, e_{\overline{1}}, \ldots, e_n, e_{\overline{n}}\}$. Define

$$e_X^+ = \prod_{x \in X} e_x^+ \in \operatorname{Sym}^{|X|} M$$

for $X \subseteq \{1, 2, \ldots, n\}$. Then $\widehat{M}_i$ has a basis given by all elements of the form

$$e_X^+ \otimes e_{x_{i+1}} \ldots e_{x_n} \in \operatorname{Sym}^i M \otimes \operatorname{Sym}^{n-i} M,$$

where $X \subset \{1, 2, \ldots, n\}$ is such that $|X| = i$, and the subset

$$\{x_{i+1}, x_{i+2}, \ldots, x_n\}$$

of $\{1, 2, \ldots, n, \overline{1}, \overline{2}, \ldots, \overline{n}\}$ is such that

$$(X \cup \overline{X}) \cap \{x_{i+1}, x_{i+2}, \ldots, x_n\} = \varnothing,$$

and $x_{i+j} \neq \overline{x_{i+k}}$ for all $j$ and $k$. We denote this basis by $\mathcal{B}_i$.

Fix $t \in \mathbf{N}$ such that $t \leq n$. Given $i \geq t$, define the *multistep map*

$$\psi_i^{(t)} : \widehat{M}_i \to \widehat{M}_{i-t}$$
$$e_X^+ \otimes e_{x_{i+1}} \ldots e_{x_n} \mapsto \sum e_Y^+ \otimes e_{X \setminus Y}^+ e_{x_{i+1}} \ldots e_{x_n},$$

where the sum runs over all $Y \subset X$ such that $|Y| = i - t$.

Suppose that $i \geq 2t$. Observe that

$$\begin{aligned}\psi_{i-t}^{(t)} \psi_i^{(t)}(e_X^+ \otimes e_{x_{i+1}} \ldots e_{x_n}) &= \psi_{i-t}^{(t)}\Big( \sum_{\substack{Y \subset X \\ |Y| = i-t}} e_Y^+ \otimes e_{X \setminus Y}^+ e_{x_{i+1}} \ldots e_{x_n}\Big) \\ &= \sum_{\substack{Z \subset Y \subset X \\ |Z| = i-2t}} e_Z^+ \otimes e_{Y \setminus Z}^+ e_{X \setminus Y}^+ e_{x_{i+1}} \ldots e_{x_n}.\end{aligned}$$

As argued in [**66**, §1], given a fixed $Z \subset X$ such that $|Z| = i - 2t$, there are $\binom{2t}{t}$ choices for $Y \subset X$ such that $|Y| = i - t$ and $Z \subset Y \subset X$. It therefore follows from Lucas' Theorem (see Lemma 1.3.17) that $\psi_{i-t}^{(t)} \psi_i^{(t)} = 0$, and so we can ask when the sequence

$$\widehat{M}_{i+t} \xrightarrow{\psi_{i+t}^{(t)}} \widehat{M}_i \xrightarrow{\psi_i^{(t)}} \widehat{M}_{i-t} \tag{6.6}$$

is exact. Observe that $\psi_i^{(1)}$ is the 2-modular reduction of the boundary map $\delta_i$, and so it follows from Lemma 6.1.6 that (6.6) is exact when $t = 1$.

Over the rational field, we have seen that the homology of the chain complex

$$M_n \xrightarrow{\delta_n} M_{n-1} \xrightarrow{\delta_{n-1}} \cdots \xrightarrow{\delta_2} M_1 \xrightarrow{\delta_1} M_0 \xrightarrow{\delta_0} \mathbf{Q}$$

can be reduced to that of the chain complex

$$\bigwedge^n M^{(n-1,1)} \xrightarrow{\widehat{\delta}_n} \bigwedge^{n-1} M^{(n-1,1)} \xrightarrow{\widehat{\delta}_{n-1}} \cdots \xrightarrow{\widehat{\delta}_2} \bigwedge^1 M^{(n-1,1)} \xrightarrow{\widehat{\delta}_1} \mathbf{Q}.$$

In this section we show that there are various differences between the cubic and simplicial multistep homologies over a field of characteristic 2, and so there does not appear to be a reduction analogous to the characteristic 0 case.

**6.2.1. Non split-exactness.** Let $2^\tau$ be the least two-power appearing the binary expansion of $t$. Suppose that *either* $n = 2a + t$, for some $a < 2^\tau$; *or* $t$ is a two-power and $n \equiv 2a + t \mod 2t$ for some $a \in \mathbf{N}$. If $c \in \mathbf{N}_0$ is maximal such that $a + ct \leq n$, then Theorem 1.3 in [**66**] states that the chain complex

$$0 \to F\Omega_{a+ct} \xrightarrow{\widehat{\psi}^{(t)}_{a+ct}} F\Omega_{a+(c-1)t} \xrightarrow{\widehat{\psi}^{(t)}_{a+(c-1)t}} \cdots \xrightarrow{\widehat{\psi}^{(t)}_{a+2t}} F\Omega_{a+t} \xrightarrow{\widehat{\psi}^{(t)}_{a+t}} F\Omega_a \to 0$$

is exact in every degree if and only if it is split-exact in every degree. The following lemma shows that (6.6) can never be split exact.

Lemma 6.2.1. *The module $\widehat{M_i}$ is indecomposable for all $0 \leq i \leq n$.*

Proof. It follows from Mackey's Theorem that

$$\begin{aligned} \widehat{M_i}\big\downarrow_{C_2^n} &= F\big\uparrow_{H_i}^{C_2 \wr S_n}\big\downarrow_{C_2^n} \\ &\cong \bigoplus_{g \in H_i \backslash C_2 \wr S_n / C_2^n} F\big\downarrow_{{}^g(H_i) \cap C_2^n}\big\uparrow^{C_2^n}. \end{aligned}$$

We require a set of $(H_i, C_2^n)$-double coset representatives. Since $F$ is a field of characteristic 2, the permutation basis $\mathcal{B}_i$ of $M_i$ corresponds to a set of $H_i$-coset representatives. It is clear that

$$\begin{aligned} v_1 &:= e_X^+ \otimes e_{x_{i+1}} \ldots e_{x_n} \\ v_2 &:= e_Y^+ \otimes e_{y_{i+1}} \ldots e_{y_n} \end{aligned}$$

in $\mathcal{B}_i$ lie in the same $C_2^n$-orbit whenever $X = Y$. Moreover, given $x \in X$, the element $(x\ \overline{x})$ acts trivially on $e_X^+ \otimes e_{x_{i+1}} \ldots e_{x_n}$. Therefore if $v_1$ and $v_2$ lie in the same $C_2^n$-orbit, then $X = Y$.

We have shown that $v_1$ and $v_2$ in the previous paragraph lie in the $C_2^n$-orbit if and only if $X = Y$. It follows that we can take a set of $S_{(i,n-i)}$-coset representatives in $S_n$ as the $(H_i, C_2^n)$-double coset representatives. Moreover, we can take $S_n$ to be the top group $T_n$, and we can assume that $S_{(i,n-i)}$ is contained in $T_n$. For each $g \in H_i \backslash C_2 \wr S_n / C_2^n$ in this case it follows that ${}^g(H_i) \cap C_2^n = {}^g(C_2^i)$. Therefore

$$\begin{aligned} \widehat{M_i}\big\downarrow_{C_2^n} &\cong \bigoplus_{g \in H_i \backslash C_2 \wr S_n / C_2^n} F\big\uparrow_{{}^g(C_2^i)}^{C_2^n} \\ &\cong \bigoplus_{g \in H_i \backslash C_2 \wr S_n / C_2^n} {}^g(F\big\uparrow_{C_2^i}^{C_2^n}). \end{aligned}$$

Let $U$ be a non-zero summand of $\widehat{M_i}$. Observe that by the first statement of Lemma 1.3.5 each summand in the final line is indecomposable. By the Krull–Schmidt Theorem the module $U\downarrow_{C_2^n}$ therefore has a summand isomorphic to ${}^{g}(F\uparrow_{C_2^i}^{C_2^n})$, for some $g \in H_i\backslash C_2 \wr S_n/C_2^n$. Since $U$ is closed under the conjugacy action of $C_2 \wr S_n$, it follows that $U\downarrow_{C_2^n}$ has a summand isomorphic to ${}^{g'}(F\uparrow_{C_2^i}^{C_2^n})$ for all $g' \in S_n\backslash S_{(i,n-i)}$. Therefore

$$\widehat{M}_i\big\downarrow_{C_2^n} \cong U\big\downarrow_{C_2^n},$$

and in particular $\dim_F U = \dim_F \widehat{M}_i$. It follows that $U = \widehat{M}_i$, and so the result is proved. □

**6.2.2. Non-exact sequences.** The second main theorem in [**66**] gives a complete description of when the sequence

$$F\Omega_{i+t} \xrightarrow{\widehat{\psi}_{i+t}^{(t)}} F\Omega_i \xrightarrow{\widehat{\psi}_i^{(t)}} F\Omega_{i-t} \tag{6.7}$$

is exact. Let $2^\tau$ be the least two-power appearing in the binary expansion of $t$. Then [**66**, Theorem 1.2] states that (6.7) is exact if and only if exactly one of the following conditions holds:

(i) $t = 1$;
(ii) $i < 2^\tau$ and $i + t \leq n - i$ or $n - i < 2^\tau$ and $n - i + t \leq i$;
(iii) $t$ is a two-power and $n \geq 2i + t$ or $n \leq 2i - t$.

In particular for the $i \in \mathbf{N}_0$ either in the first case of (ii) or in (iii), [**66**, Theorem 1.2] shows that there exists a large enough $n$ such that (6.7) is exact. In the following example we show that the analogous result does not hold for (6.6) in general.

EXAMPLE 6.2.2. Suppose that $t > 1$ and $i = t$. Assume that $n \geq t + 1$. In this case

$$e_1^+ \dots e_t^+ \otimes e_{t+1}^+ e_{t+2} \dots e_n + e_1^+ \dots e_{t-1}^+ e_{t+1}^+ \otimes e_t^+ e_{t+2} \dots e_n$$

is contained in $\ker(\psi_t^{(t)})$. This vector is clearly not contained in $\operatorname{im}(\psi_{2t}^{(t)})$, and so the sequence

$$\widehat{M}_{2t} \xrightarrow{\psi_{2t}^{(t)}} \widehat{M}_t \xrightarrow{\psi_t^{(t)}} \widehat{M}_0$$

is never exact.

**6.2.3. Composition factors modulo 2.** Implicit in the proof of [**66**, Theorem 1.2] is that (6.7) is exact if and only if every composition factor of $F\Omega_i$ is a composition factor of the direct sum of modules $F\Omega_{k+i} \oplus F\Omega_{k-i}$. In the following example we show that this is not the case for (6.6) in general.

EXAMPLE 6.2.3. Let $n = 5$, and consider the sequence

$$\widehat{M}_5 \xrightarrow{\psi_5^{(2)}} \widehat{M}_3 \xrightarrow{\psi_3^{(2)}} \widehat{M}_1.$$

By definition of the multistep map $\psi_3^{(2)}$, the vector

$$e_2^+ e_4^+ e_5^+ \otimes e_1^+ e_3^+ + e_1^+ e_4^+ e_5^+ \otimes e_2^+ e_3^+ + e_2^+ e_3^+ e_5^+ \otimes e_1^+ e_4^+ + e_1^+ e_3^+ e_5^+ \otimes e_2^+ e_4^+.$$

is contained in $\ker(\psi_3^{(2)}) \setminus \operatorname{im}(\psi_5^{(2)})$, and so this sequence is not exact. However computations in MAGMA ([**4**]) show that the composition factors of the permutation module $\widehat{M_3}$ are

- $\operatorname{Inf}_{S_5}^{C_2 \wr S_5} D^{(5)}$ with multiplicity 8;
- $\operatorname{Inf}_{S_5}^{C_2 \wr S_5} D^{(4,1)}$ with multiplicity 4;
- and $\operatorname{Inf}_{S_5}^{C_2 \wr S_5} D^{(3,2)}$ with multiplicity 4.

Moreover, the composition factors of the permutation module $\widehat{M_1}$ are

- $\operatorname{Inf}_{S_5}^{C_2 \wr S_5} D^{(5)}$ with multiplicity 16;
- $\operatorname{Inf}_{S_5}^{C_2 \wr S_5} D^{(4,1)}$ with multiplicity 8;
- and $\operatorname{Inf}_{S_5}^{C_2 \wr S_5} D^{(3,2)}$ with multiplicity 8.

Therefore every composition factor of $\widehat{M_3}$ appears in $\widehat{M_5} \oplus \widehat{M_1}$, even though the sequence in question is not exact.

REMARK 6.2.4. The differences demonstrated in this section show that there does not appear to be a reduction from the cubic homology to the simplicial case in characteristic 2. Also it should be noted that several results in [**66**] are proved using the duality between the homologies of the sequences

$$F\Omega_{i+t} \xrightarrow{\widehat{\psi}_{i+t}^{(t)}} F\Omega_i \xrightarrow{\widehat{\psi}_i^{(t)}} F\Omega_{i-t}$$

$$F\Omega_{n-i+t} \xrightarrow{\widehat{\psi}_{n-i+t}^{(t)}} F\Omega_{n-i} \xrightarrow{\widehat{\psi}_{n-i}^{(t)}} F\Omega_{n-i-t}.$$

This duality holds since $F\Omega_i$ and $F\Omega_{n-i}$ are isomorphic as $FS_n$-modules for all $0 \leq i \leq n$. However $\widehat{M_i}$ and $\widehat{M_{n-i}}$ are not isomorphic as $FC_2 \wr S_n$-modules, except for the case when $n$ is even and $i = n - i = \frac{n}{2}$. Therefore there does not appear to be a natural choice for a sequence whose homology is dual to that of (6.6) in our case.

## 6.3. Multi-step maps in fields of characteristic $p \neq 2$

In this section let $F$ be a field of characteristic $p \neq 2$. Recall that $N$ denotes the non-trivial irreducible $FC_2$-module. We once more write $\widehat{M_i}$ for the $p$-modular reduction of the $\mathbf{Q}C_2 \wr S_n$-module $M_i$. The main result in this section is Proposition 6.3.1 below, which shows that there are no non-zero $FC_2 \wr S_n$-module homomorphisms between $\widehat{M_i}$ and $\widehat{M_{i-t}}$ when $t \geq 2$. This shows that it is only possible to define multistep maps over fields of characteristic 2.

PROPOSITION 6.3.1. *Let $i, t \in \mathbf{N}_0$ be such that $i \geq t \geq 2$, and let $F$ be a field of characteristic not equal to 2. Then* $\operatorname{Hom}_{FC_2 \wr S_n}(\widehat{M_i}, \widehat{M_{i-t}}) = 0$.

We remark that we prove Proposition 6.3.1 by reducing the argument to the case of the simplex, which we are able to do since $p \neq 2$. The following preliminaries are required.

LEMMA 6.3.2. *Fix $i, j \in \mathbf{N}_0$, and let $F$ be a field of characteristic not equal to 2. Then there is an isomorphism of $FS_{i+j}$-modules*

$$(S^{(1^i)} \boxtimes S^{(j)}) \uparrow_{S_i \times S_j}^{S_{i+j}} \cong \bigwedge^i M^{(i+j-1,1)}.$$

PROOF. Let the set

$$\{e_1, \ldots, e_{i+j}\},$$

be the natural basis for $M^{(i+j-1,1)}$. We then have that

$$\mathcal{B} := \{e_{x_1} \wedge \cdots \wedge e_{x_i} : 1 \leq x_1 < x_2 < \cdots < x_i \leq i+j\}$$

is a basis for $\bigwedge^i M^{(i+j-1,1)}$. Using the anti-commutativity of the exterior power, the $F$-span of the vector

$$v := e_1 \wedge e_2 \wedge \cdots \wedge e_i$$

is isomorphic to $S^{(1^i)} \boxtimes S^{(j)}$ as an $FS_{(i,j)}$-module.

Let

$$w := e_{x_1} \wedge e_{x_2} \wedge \cdots \wedge e_{x_j},$$

be a vector in $\mathcal{B}$. Also let $\sigma \in S_n$ be any permutation such that $\sigma(t) = x_t$ for all $1 \leq t \leq i$, and so $\sigma v = w$. Since $w$ was chosen arbitrarily in $\mathcal{B}$, it follows that $\langle v \rangle$ generates $\bigwedge^i M^{(i+j-1,1)}$ as an $FS_{i+j}$-module. Since

$$\dim_F \bigwedge^i M^{(i+j-1,1)} = [S_{i+j} : S_{(i,j)}],$$

the result follows from Lemma 1.3.2. □

The following lemma is a corollary of Proposition 2.1 in [**66**].

LEMMA 6.3.3. *Let $k, l \in \mathbf{N}_0$ be such that $0 \leq k, l \leq n$. If $|k-l| \geq 2$, then $\mathrm{Hom}_{FS_n}(\bigwedge^k M, \bigwedge^l M) = 0$.*

PROOF. By definition there is an isomorphism of $FC_2 \wr S_n$-modules

$$\widehat{M_i} \cong (\widetilde{N}^{\otimes i} \operatorname{Inf}_{S_i}^{C_2 \wr S_i} S^{(1^i)} \boxtimes F) \big\uparrow_{H_i}^{C_2 \wr S_n},$$

and we begin by determining the indecomposable summands of $\widehat{M_i}$. First observe that by the transitivity of induction

$$\widehat{M_i} \cong (\widetilde{N}^{\otimes i} \operatorname{Inf}_{S_i}^{C_2 \wr S_i} S^{(1^i)} \boxtimes F \big\uparrow_{S_{n-i}}^{C_2 \wr S_{n-i}}) \big\uparrow_{C_2 \wr S_{(i,n-i)}}^{C_2 \wr S_n}.$$

As $F$ is a field of characteristic $p \neq 2$, the argument in the first case in the proof of Proposition 6.1.4 still holds, and so there is an isomorphism

$$F \big\uparrow_{S_{n-i}}^{C_2 \wr S_{n-i}} \cong \bigoplus_{j=0}^{n-i} S^{((n-i-j),(j))}.$$

Therefore

$$
\begin{aligned}
\widehat{M_i} &\cong \big(\widetilde{N}^{\otimes i} \operatorname{Inf}_{S_i}^{C_2 \wr S_i} S^{(1^i)} \boxtimes \bigoplus_{j=0}^{n-i} S^{((n-i-j),(j))}\big)\big\uparrow_{C_2 \wr S_{(i,n-i)}}^{C_2 \wr S_n} \\
&\cong \bigoplus_{j=0}^{n-i} \Big(\operatorname{Inf} S^{(n-i-j)} \boxtimes \widetilde{N}^{\otimes i+j} \operatorname{Inf} \Big( (S^{(1^i)} \boxtimes S^{(j)})\big\uparrow_{S_{(i,j)}}^{S_{i+j}} \Big)\Big) \big\uparrow_{C_2 \wr S_{(n-i-j,i+j)}}^{C_2 \wr S_n} \\
&\cong \bigoplus_{j=0}^{n-i} \left( \operatorname{Inf} S^{(n-i-j)} \boxtimes \widetilde{N}^{\otimes i+j} \operatorname{Inf} \bigwedge^{i} M^{(i+j-1,1)} \right) \big\uparrow_{C_2 \wr S_{(n-i-j,i+j)}}^{C_2 \wr S_n}, \qquad (6.8)
\end{aligned}
$$

where the final isomorphism holds by Lemma 6.3.2.

It follows from Proposition 1.2.5 that no two summands in (6.8) are isomorphic. Write $T_{i,j}$ for the unique summand of $\widehat{M_i}$ isomorphic to

$$
\left( \operatorname{Inf} S^{(n-i-j)} \boxtimes \widetilde{N}^{\otimes i+j} \operatorname{Inf} \bigwedge^{i} M^{(i+j-1,1)} \right) \big\uparrow_{C_2 \wr S_{(n-i-j,i+j)}}^{C_2 \wr S_n}.
$$

Proposition 1.4.8 implies that $T_{i,j}$ has no composition factors in common with any summand of $\widehat{M}_{i-t}$ other than $T_{i-t,j+t}$. Therefore it is sufficient to show that

$$
\operatorname{Hom}_{FC_2 \wr S_n}(T_{i,j}, T_{i-t,j+t}) = 0.
$$

Suppose, for a contradiction, that there exists a non-zero module homomorphism $\vartheta \in \operatorname{Hom}_{FC_2 \wr S_n}(T_{i,j}, T_{i-t,j+t})$. It follows from the proof of Proposition 1.4.8 that there exists

$$
0 \neq \vartheta' \in \operatorname{Hom}_{FS_{i+j}}\Big(\bigwedge^{i} M^{(i+j-1,1)}, \bigwedge^{i-t} M^{(i+j-1,1)}\Big).
$$

This is a contradiction to Lemma 6.3.3. $\square$

CHAPTER 7

# Generalisations of Foulkes Characters

In §5 and §6 we considered the representation theory of $C_2 \wr S_n$. Given $m, n \in \mathbf{N}$, in this chapter we consider characters related to an important open problem in the ordinary representation theory of $S_m \wr S_n$, known as Foulkes' Conjecture. In order to state the conjecture, we define a *Foulkes character* to be a character of the form

$$\varphi_{(m)}^{(n)} := 1_{S_m \wr S_n} \big\uparrow^{S_{mn}}.$$

Conjecture 7.0.1 (Foulkes' Conjecture, 1950). *Let $m, n \in \mathbf{N}$ be such that $m < n$, and let $\lambda \vdash mn$. Then*

$$\langle \varphi_{(m)}^{(n)}, \chi^\lambda \rangle \geq \langle \varphi_{(n)}^{(m)}, \chi^\lambda \rangle.$$

Although a proof of the conjecture is yet to be found in general, it has been proved in some special cases. In [**6**] Brion proved that Foulkes' conjecture is true when $n$ is very large relative to $m$ using connections between the representation theories of the symmetric group and the general linear group. As remarked in [**6**], the proof is non-constructive in the sense that it does not give a lower bound for $n$. Nevertheless Brion later found a lower bound for $n$ in terms of $m$ in [**7**]. In [**12**] Dent and Siemons proved the conjecture when $m = 3$ by proving that

$$\dim_{\mathbf{C}} \operatorname{Hom}_{\mathbf{C}S_{3n}}(S^\lambda, \varphi_{(3)}^{(n)}) \geq \dim_{\mathbf{C}} \operatorname{Hom}_{\mathbf{C}S_{3n}}(S^\lambda, \varphi_{(n)}^{(3)}),$$

for all $n \geq 3$ and $\lambda \vdash 3n$. Using [**12**] and a conjecture of Howe in [**30**], McKay proved Foulkes' Conjecture when $m = 4$ in [**51**]. Cheung, Ikenmeyer and Mkrtchyan proved the conjecture when $m = 5$ in [**10**] using a theorem of McKay. Most relevant to this chapter, and indeed our primary motivation, is Theorem 1.5 in [**19**], which provides a recursive formula for computing the constituents of a Foulkes character. This recursive formula is used to verify Foulkes' Conjecture for $m, n \in \mathbf{N}$ such that $m + n \leq 19$. This first main result in this section is a generalisation of the recursive formula in [**19**, Theorem 1.5] to certain *plethysms*, which by definition are characters of the form

$$\varphi_\vartheta^\nu = \left( \widetilde{\chi^\vartheta}^{\times n} \operatorname{Inf}_{S_n}^{S_m \wr S_n} \chi^\nu \right) \big\uparrow_{S_m \wr S_n}^{S_{mn}},$$

where $\vartheta \vdash m$ and $\nu \vdash n$. For definitions of the notations in the above display, we refer the reader to §1.2. Observe that taking $\vartheta = (m)$ and $\nu = (n)$ in the definition of the plethysm gives the Foulkes character $\varphi_{(m)}^{(n)}$.

The multiplicities of the irreducible constituents of $\varphi^\nu_\vartheta$ are known as *plethysm coefficients*. Stanley identifies determining a combinatorial description of the plethysm coefficients in his list of major open problems in algebraic combinatorics (see [**62**, Problem 9]). Our first main result in this chapter provides a recursive formula for the plethysm coefficients corresponding to $\varphi^\nu_{(a,1^b)}$, for $a, b \in \mathbf{N}_0$ such that $a + b = m$.

In §7.2 we continue the theme of generalising the Foulkes characters. It is known that the Foulkes character $\varphi^{(n)}_{(2)}$ is multiplicity free and equal to

$$\sum_{\lambda \vdash n} \chi^{2\lambda},$$

where $2\lambda$ denotes the partition of $2n$ equal to $(2\lambda_1, 2\lambda_2, \ldots, 2\lambda_{\ell(\lambda)})$. This decomposition can be proved, see for instance [**35**, Theorem 5.4.23] or [**32**, Lemma 1], by showing that $\varphi^{(n)}_{(2)}$ satisfies the following two conditions:

(U1) the constituents of $\chi\downarrow_{S_{2n-1}}$ are the $\chi^\mu$ such that $\mu$ has exactly one odd part, each appearing with multiplicity one,

(U2) $\chi^{(2n)}$ is a constituent of $\chi$.

It is then proved that any character satisfying these two conditions must equal $\sum_{\lambda \vdash n} \chi^{2\lambda}$.

Motivated by the remarkable fact that there is a unique $S_{2n}$-character satisfying conditions (U1) and (U2), we prove that there is a unique $S_{2n}$-character $\chi$ satisfying the following conditions:

(U1) the constituents of $\chi\downarrow_{S_{2n-1}}$ are the $\chi^\mu$ such that $\mu$ has exactly one odd part, each appearing with multiplicity one,

(U2′) $\chi^{(2n)}$ is not a constituent of $\chi$.

We remark that the method of comparing coefficients in restricted characters used in §7.2 is an example of Littlewood's 'third method' for decomposing plethysms (see [**43**, page 349]). This method is in fact that used in the proof of [**32**, Lemma 1] to decompose the Foulkes character $\varphi^{(n)}_{(2)}$. Moreover, Littlewood's 'third method' can also be used to decompose the Foulkes character $\varphi^{(n)}_{(3)}$. However, the method cannot be used in general for decomposing $\varphi^{(n)}_{(m)}$ when $m \geq 4$.

## 7.1. Recursive formulas

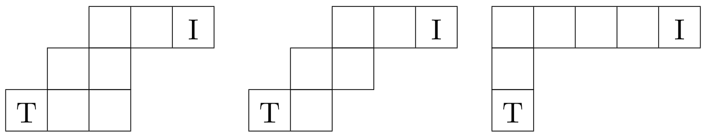
In this section we provide a recursive formula for computing the plethysm coefficients of $\varphi^\lambda_{(a,1^b)}$, where $a, b \in \mathbf{N}_0$ are such that $a + b = m$, and $\lambda \vdash n$ is arbitrary. This recursive formula depends on a certain combinatorial object, known as an $(a, 1^b)$-like border strip $n$-diagram, which was introduced in [**19**]. In order to state the definition of this combinatorial object, the following preliminaries from [**19**] are required.

We start by reminding the reader that given a skew diagram $[\lambda/\mu]$, we define $\mathrm{ht}(\lambda/\mu)$ to be one less the number of non-empty rows of $[\lambda/\mu]$. Also recall that we refer to a border strip as a skew partition whose Young diagram is connected with no four boxes forming the Young diagram $[(2, 2)]$. We define a *border strip diagram* to be the Young diagram of a border strip.

Definition. Given partitions $\lambda$ and $\mu$ such that $\mu \subseteq \lambda$, let $\kappa = [\lambda/\mu]$. Then define the *initial box* of $\kappa$ to be the box $(i_\kappa, j_\kappa)$ in $\kappa$ such that, for all $i \leq i_\kappa$ and $j \geq j_\kappa$, if $(i, j) \in \kappa$ then $i = i_\kappa$ and $j = j_\kappa$.

Similarly define the *terminal box* of $\kappa$ to be the box $(k_\kappa, l_\kappa)$ in $\kappa$ such that, for all $k \geq k_\kappa$ and $l \leq l_\kappa$, if $(k, l) \in \kappa$ then $k = k_\kappa$ and $l = l_\kappa$.

For example, the following are the skew diagrams of $[(5, 3, 3)/(2, 1)]$, $[(5, 3, 2)/(2, 1)]$ and $[(5, 1, 1)/\varnothing]$, respectively.




In each case the entries in the initial and terminal boxes are I and T, respectively.

As remarked in [**19**, page 24] skew diagrams are convex, and so their initial and terminal boxes always exist.

Definition. Define a *border strip $n$-diagram* $D$ to be a skew diagram such that $D$ is a disjoint union of finitely many border strip diagrams, each of size $n$. Moreover $D$ is a *horizontal border strip $n$-diagram* if, for every initial box $(i_\rho, j_\rho)$ of each $\rho \in D$, we have $(i, j_\rho) \notin D$ for all $i < i_\rho$. Similarly, $D$ is a *vertical border strip $n$-diagram* if, for every terminal box $(k_\rho, l_\rho)$ of each $\rho \in D$, we have $(k_\rho, l) \notin D$ for all $l < l_\rho$.

Definition. Given partitions $\lambda$ and $\mu$ such that $\mu \subset \lambda$, suppose that $[\lambda/\mu]$ is a border strip $n$-diagram. We define the *$n$-sign* of $[\lambda/\mu]$, denoted $\varepsilon_n(\lambda/\mu)$, to be $(-1)^h$, where $h$ is the sum of the heights of the border strip diagrams forming $[\lambda/\mu]$.

We remark that there may be more than one choice for a border strip $n$-diagram of fixed shape $\lambda/\mu$. Nevertheless, as remarked after Definition 3.2 in [**19**], the $n$-sign of $[\lambda/\mu]$ is well-defined.

We are now ready to define an $(a, 1^b)$-like border strip $n$-diagram.

Definition. Given partitions $\lambda$ and $\mu$ such that $\mu \subseteq \lambda$, let $\kappa = [\lambda/\mu]$ and $mn = |\lambda/\mu|$. Let $a, b \in \mathbf{N}$ be such that $a + b = m$, and let $D$ and $E$ be two border strip $n$-diagrams such that $\kappa = (D \cup E)$. We say that the pair $(D, E)$ is an *$(a, 1^b)$-like border strip $n$-diagram of shape $\kappa$* if

(1) $|D| = a$ and $|E| = b + 1$,

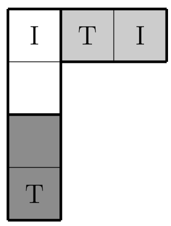
(2) $D \cap E = \{\sigma\}$, where $\sigma$ is the border strip diagram of size $n$ that contains the initial box of $\kappa$,
(3) $D$ is a horizontal border strip $n$-diagram, and $E$ is a vertical border strip $n$-diagram,
(4) there do not exist disjoint $\rho_D \in D$ and $\rho_E \in E$ with boxes $(z_1, z_2) \in \rho_D$ and $(w_1, w_2) \in \rho_E$ such that $w_1 < z_1$ and $w_2 < z_2$.

EXAMPLE 7.1.1. Let $\lambda = (3, 1^3)$, $\mu = \varnothing$, and $n = 2$. Then $[(3, 1^3)]$ is a border strip 2-diagram, uniquely formed by the three border strip diagrams

$$\{(1,1),(2,1)\},\ \{(1,2),(1,3)\},\ \text{and } \{(3,1),(4,1)\}.$$

As can be seen from the diagram below, as a border strip 2-diagram $[(3, 1^3)]$ has 2-sign $(-1)^{1+0+1} = 1$.

The unique $(2, 1)$-like border strip 2-diagram of shape $(3, 1^3)$ is




.

The border strip diagrams in the horizontal 2-diagram are shown in white and light grey, with their initial boxes labelled I. Similarly the border strip diagrams in the vertical 2-diagram are shown in light grey and dark grey, with their terminal boxes labelled T.

Given a skew shape $\kappa$, we write $\mathcal{B}^{\kappa}_{a,b}$ for the set of $(a, 1^b)$-like border strip $n$-diagrams of shape $\kappa$.

We are now ready to state the main result of this section.

THEOREM 7.1.2. *Let $m, n \in \mathbf{N}$. Let $a, b \in \mathbf{N}_0$ be such that $a + b = m$, and let $\nu \vdash n$. If $\lambda \vdash mn$, then*

$$\left\langle \varphi^{\nu}_{(a,1^b)}, \chi^{\lambda} \right\rangle = \frac{1}{n} \sum_{j=1}^{n} \sum_{\mu \subset \lambda} \varepsilon_j(\lambda/\mu) |\mathcal{B}^{\lambda/\mu}_{a,b}| \sum (-1)^{\mathrm{ht}(\nu/\rho)} \left\langle \varphi^{\rho}_{(a,1^b)}, \chi^{\mu} \right\rangle ,$$

*where the third sum runs over all $\rho \subset \nu$ such that $|\rho| = n - j$ and $\nu/\rho$ is a border strip.*

The main tool that we use to prove our recursive formula is the deflation map of $S_m \wr S_n$-characters, introduced in [**19**], which we now define.

DEFINITION. Let $m, n \in \mathbf{N}$, and let $\vartheta$ be an irreducible $S_m$-character. Let $\xi$ be an irreducible $S_m \wr S_n$-character. Then

$$\mathrm{Def}^{\vartheta}_{S_n} \xi = \begin{cases} \chi^{\nu} & \text{if } \xi = \widetilde{\vartheta^{\times n}} \operatorname{Inf}^{S_m \wr S_n}_{S_n} \chi^{\nu}, \text{ for some partition } \nu \text{ of } n \\ 0 & \text{otherwise.} \end{cases}$$

We extend this map linearly to the integral span of the irreducible $S_n$-characters, and we write $\mathrm{Def}^{\vartheta}_{S_n}$ for this morphism.

Furthermore, given $\chi \in \mathrm{Irr}(S_{mn})$, we define

$$\mathrm{Defres}^{\vartheta}_{S_n}(\chi) = \mathrm{Def}^{\vartheta}_{S_n}(\chi\big\downarrow^{S_{mn}}_{S_m \wr S_n}),$$

which we once more extend linearly to the integral span of the irreducible $S_{mn}$-characters.

The following lemma allows us calculate inner products of $S_m \wr S_n$-characters using inner products of $S_n$-characters via the deflation map.

LEMMA 7.1.3. *Let $m, n \in \mathbf{N}$. Let $\vartheta$ be an irreducible character of $S_n$, let $\chi$ be a character of $S_n$, and let $\psi$ be a character of $S_m \wr S_n$. Then*

$$\langle \mathrm{Def}^{\vartheta}_{S_n} \psi, \chi \rangle = \langle \psi, \tilde{\vartheta}^{\times n} \mathrm{Inf}^{S_m \wr S_n}_{S_n} \chi \rangle.$$

PROOF. Given $\lambda \vdash n$, write $a_\lambda$ for the multiplicity of $\widetilde{\vartheta}^{\times n} \mathrm{Inf}^{S_m \wr S_n}_{S_n} \chi^\lambda$ in $\psi$. By definition of the deflation map

$$\langle \mathrm{Def}^{\vartheta}_{S_n} \psi, \chi \rangle = \langle \sum_{\lambda \vdash n} a_\lambda \chi^\lambda, \chi \rangle. \tag{7.1}$$

Let $\chi = \sum_{\lambda \vdash n} b_\lambda \chi^\lambda$, and so the right hand side of (7.1) equals $\sum_{\lambda \vdash n} a_\lambda b_\lambda$.

We now consider the inner product $\langle \psi, \widetilde{\vartheta}^{\times n} \mathrm{Inf}^{S_m \wr S_n}_{S_n} \chi \rangle$. Since inflation is an exact functor, this inner product becomes

$$\langle \psi, \sum_{\lambda \vdash n} b_\lambda \widetilde{\vartheta}^{\times n} \mathrm{Inf}^{S_m \wr S_n}_{S_n} \chi^\lambda \rangle.$$

As $\widetilde{\vartheta}^{\times n} \mathrm{Inf}^{S_m \wr S_n}_{S_n} \chi^\lambda$ is an irreducible character of $S_m \wr S_n$, this inner product also equals $\sum_{\lambda \vdash n} a_\lambda b_\lambda$, and so the lemma is proved. $\square$

We also require the following results from [**19**].

PROPOSITION 7.1.4. *Given $m, n \in \mathbf{N}$, let $\lambda/\mu$ be a skew partition of $mn$. Let $\chi$ be an irreducible character of $S_m$. Let $g \in S_n$ be such that $g = xh$, where $x \in S_\ell$ and $h \in S_{n-\ell}$, for some $1 \le \ell \le n$. Then*

$$\big(\mathrm{Defres}^{\chi}_{S_n} \chi^{\lambda/\mu}\big)(g) = \sum_{\tau} \big(\mathrm{Defres}^{\chi}_{S_\ell} \chi^{\tau/\mu}\big)(x)\big(\mathrm{Defres}^{\chi}_{S_{n-\ell}} \chi^{\lambda/\tau}\big)(h)$$

*where the sum is over all partitions $\tau$ such that $\mu \subseteq \tau \subseteq \lambda$ and $|\tau/\mu| = m\ell$.*

THEOREM 7.1.5. *Given $m, n \in \mathbf{N}$, let $\lambda/\mu$ be a skew partition of $mn$. Let $a, b \in \mathbf{N}$ be such that $a + b = m$. Let $g \in S_n$ be an $n$-cycle. Then*

$$\big(\mathrm{Defres}^{\chi^{(a,1^b)}}_{S_n} \chi^{\lambda/\mu}\big)(g) = \varepsilon_n(\lambda/\mu)|\mathcal{B}^{\lambda/\mu}_{a,b}|.$$

We are now ready to prove Theorem 7.1.2.

Proof of Theorem 7.1.2. It follows from Frobenius reciprocity and Lemma 7.1.3 that

$$\begin{aligned}\left\langle \varphi^{\nu}_{(a,1^b)}, \chi^{\lambda} \right\rangle &= \left\langle \chi^{\nu}, \mathrm{Defres}^{\chi^{(a,1^b)}}_{S_n} \chi^{\lambda} \right\rangle \\ &= \frac{1}{n!} \sum_{g \in S_n} (\mathrm{Defres}^{\chi^{(a,1^b)}}_{S_n} \chi^{\lambda})(g)\chi^{\nu}(g).\end{aligned}$$

We can write $g \in S_n$ as a product of a $j$-cycle containing 1, say $x$, and some $h \in S_{n-j}$ acting on the remaining numbers. The number of possible such $j$-cycles is $(n-1)!/(n-j)!$. Therefore

$$\left\langle \varphi^{\nu}_{(a,1^b)}, \chi^{\lambda} \right\rangle = \frac{1}{n!} \sum_{j=1}^{n} \frac{(n-1)!}{(n-j)!} \sum_{h \in S_{n-j}} (\mathrm{Defres}^{\chi^{(a,1^b)}}_{S_n} \chi^{\lambda})(xh)\chi^{\nu}(xh).$$

By Proposition 7.1.4, we have

$$(\mathrm{Defres}^{\chi^{(a,1^b)}}_{S_n} \chi^{\lambda})(xh) = \sum (\mathrm{Defres}^{\chi^{(a,1^b)}}_{S_j} \chi^{\lambda/\mu})(x)(\mathrm{Defres}^{\chi^{(a,1^b)}}_{S_{n-j}} \chi^{\mu})(h),$$

where the sum runs over all $\mu \subset \lambda$ of size $m(n-j)$. As $x$ is a $j$-cycle, Theorem 7.1.5 gives

$$\mathrm{Defres}^{\chi^{(a,1^b)}}_{S_j} \chi^{\lambda/\mu}(x) = \varepsilon_j(\lambda/\mu)|\mathcal{B}^{\lambda/\mu}_{a,b}|.$$

Now consider $\chi^{\nu}(xh)$, which by the Murnaghan-Nakayama rule (see Theorem 2.1.1) equals

$$\sum (-1)^{\mathrm{ht}(\nu/\rho)} \chi^{\rho}(h),$$

where the sum runs over all $\rho \subseteq \nu$ such that $|\rho| = n-j$ and $\nu/\rho$ is a border strip. It follows that $\langle \varphi^{\nu}_{(a,1^b)}, \chi^{\lambda} \rangle$ equals

$$\begin{aligned}&\frac{1}{n} \sum_{j=1}^{n} \frac{1}{(n-j)!} \sum_{h \in S_{n-j}} \sum_{\mu \subset \lambda} \varepsilon_j(\lambda/\mu)|\mathcal{B}^{\lambda/\mu}_{a,b}|(\mathrm{Defres}^{\chi^{(a,1^b)}}_{S_{n-j}} \chi^{\mu})(h) \sum (-1)^{\mathrm{ht}(\nu/\rho)} \chi^{\rho}(h), \\ &= \frac{1}{n} \sum_{j=1}^{n} \sum_{\mu \subset \lambda} \varepsilon_j(\lambda/\mu)|\mathcal{B}^{\lambda/\mu}_{a,b}| \sum (-1)^{\mathrm{ht}(\nu/\rho)} \frac{1}{(n-j)!} \sum_{h \in S_{n-j}} (\mathrm{Defres}^{\chi^{(a,1^b)}}_{S_{n-j}} \chi^{\mu})(h)\chi^{\rho}(h), \\ &= \frac{1}{n} \sum_{j=1}^{n} \sum_{\mu \subset \lambda} \varepsilon_j(\lambda/\mu)|\mathcal{B}^{\lambda/\mu}_{a,b}| \sum (-1)^{\mathrm{ht}(\nu/\rho)} \left\langle \varphi^{\rho}_{(a,1^b)}, \chi^{\mu} \right\rangle.\end{aligned}$$

where the third sum on the final line runs over all $\rho \subseteq \nu$ such that $|\rho| = n-j$ and $\nu/\rho$ is a border strip. □

Corollary 7.1.6. *Let $m, n \in \mathbf{N}$. Let $a, b \in \mathbf{N}_0$ be such that $a + b = m$, and let $\lambda \vdash mn$. Then*

$$\left\langle \varphi^{(n)}_{(a,1^b)}, \chi^{\lambda} \right\rangle = \frac{1}{n} \sum_{j=1}^{n} \sum_{\mu \subset \lambda} \varepsilon_j(\lambda/\mu)|\mathcal{B}^{\lambda/\mu}_{a,b}| \left\langle \varphi^{(n-j)}_{(a,1^b)}, \chi^{\mu} \right\rangle.$$

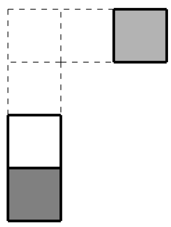
Proof. Observe that, for all $1 \leq j \leq n$, the only subpartition of $(n)$ of size $n-j$ is $(n-j)$. Moreover, $[(n)\backslash(n-j)]$ is a border strip diagram of height zero, and so applying Theorem 7.1.2 gives the result. □

EXAMPLE 7.1.7. Let $m = 3$, and let $n = 2$. We determine the multiplicity

$$\langle \varphi^{(2)}_{(2,1)}, \chi^{(3,1^3)} \rangle.$$

We consider the subpartitions of $(3, 1^3)$ of sizes $3(2 - j)$ for each $1 \leq j \leq 2$ in turn.

In the case that $j = 1$, every partition of 3 is a subpartition $(3, 1^3)$. Moreover, in this case $\varphi^{(1)}_{(2,1)}$ is the irreducible $S_3$-character $\chi^{(2,1)}$. As $\langle \chi^{(2,1)}, \chi^{\mu} \rangle$ is non-zero if and only if $\mu = (2, 1)$, it suffices to count $|\mathcal{B}^{(3,1^3)/(2,1)}_{2,1}|$. There is a unique $(2, 1)$-like border strip 1-diagram of shape $(3, 1^3)/(2, 1)$, given by



.

The boxes forming the horizontal 1-diagram are shown in white and light grey, and the boxes forming the vertical 1-diagram are shown in light grey and dark grey. The boxes formed by dashed lines indicate those removed from $[(3, 1^3)]$ to form the skew diagram $[(3, 1^3)/(2, 1)]$. It follows that

$$\varepsilon_1((3, 1^3)/(2, 1)) = (-1)^{0+0+0} = 1.$$

In the case that $j = 2$, the empty partition $\varnothing$ is the only subpartition of $(3, 1^3)$ of size 0. In this case $\varphi^{\varnothing}_{(2,1)}$ is the trivial $S_0$-character. The unique $(2, 1)$-like border strip 2-diagram of shape $(3, 1^3)/\varnothing = (3, 1^3)$ is shown in Example 7.1.1, and $\varepsilon_2((3, 1^3)/\varnothing) = 1$ in this case. Applying Theorem 7.1.2 (or Corollary 7.1.6) shows that

$$\langle \varphi^{(2,1)}_{(2)}, \chi^{(3,1^3)} \rangle = \frac{1}{2}\big(1 \cdot 1 \cdot \langle \varphi^{(1)}_{(2,1)}, \chi^{(2,1)} \rangle + 1 \cdot 1 \cdot \langle \varphi^{\varnothing}_{(2,1)}, \chi^{\varnothing} \rangle\big) = 1.$$

## 7.2. A unique restriction

Throughout this section fix $n \in \mathbf{N}$. The main result of this section is the following result.

THEOREM 7.2.1. *There is a unique $S_{2n}$-character $\chi$ such that*

(U1) *the constituents of $\chi \downarrow_{S_{2n-1}}$ are the $\chi^{\mu}$ such that $\mu$ exactly one odd part, each appearing with multiplicity one,*

(U2′) *$\chi^{(2n)}$ is not a constituent of $\chi$.*

We prove the result by giving a complete description of the irreducible constituents of any $S_{2n}$-character $\chi$ satisfying conditions (U1) and (U2′). We see that these constituents are completely determined by the two conditions, which proves the theorem. The only prerequisites for the proof are the following definition and lemma. The latter is known as the branching rule for restriction, which is an immediate corollary of Theorem 1.1.4.

DEFINITION. Let $\lambda$ be a partition. We define a *corner box* to be a box $(i,j) \in [\lambda]$ such that $(i+1,j),(i,j+1) \notin [\lambda]$.

LEMMA 7.2.2 (Branching rule for restriction). *Let $\lambda \vdash n$. Then*

$$\chi^\lambda \big\downarrow_{S_{n-1}} = \sum \chi^\mu,$$

*where the sum runs over all partitions $\mu$ of $n-1$ such that $[\mu]$ is obtained by removing a corner box from $[\lambda]$.*

We are now ready to prove Theorem 7.2.1.

PROOF OF THEOREM 7.2.1. Let $\chi$ be an $S_{2n}$-character satisfying the hypothesis of the proposition. As $\chi\downarrow_{S_{2n-1}}$ is multiplicity free, it follows that $\chi$ is necessarily multiplicity free. Given a partition $\lambda$ of $2n$, we determine precisely when $\chi^\lambda$ is a constituent of $\chi$. We distinguish two cases, determined by the number of parts $\ell(\lambda)$ of $\lambda$.

*Case* (1). Suppose that $\ell(\lambda) > 2$. Since $2n$ is even, if $\lambda$ has an odd part $\lambda_i$, then there exists some $j \neq i$ such that $\lambda_j$ is odd. Define

$$\lambda' := (\lambda_1, \dots, \lambda_k - 1, \dots, \lambda_{\ell(\lambda)}),$$

where $k \notin \{i,j\}$ and $(k, \lambda_k)$ is a corner box in $[\lambda]$. Observe that such a corner box exists as $\ell(\lambda) > 2$ and $2n$ is even. As every constituent of $\chi^\lambda \downarrow_{S_{2n-1}}$ appears in $\chi\downarrow_{S_{2n-1}}$, it follows from the branching rule for restriction that $\chi^{\lambda'}$ appears in $\chi\downarrow_{S_{2n-1}}$ . However $\lambda'$ has at least two odd parts, $\lambda_i$ and $\lambda_j$, which is a contradiction to condition (U1). It follows that if $\chi^\lambda$ is a constituent of $\chi$ in this case, then $\lambda$ can have no odd parts.

However every partition $\mu$ of $2n-1$ with exactly one odd-part is such that $\chi^\mu$ is a constituent of $\chi\downarrow_{S_{2n-1}}$ . It follows from the branching rule for restriction that for every partition $\lambda$ of $2n$ with strictly more than two parts and all parts even, $\chi^\lambda$ is necessarily constituent a of $\chi$.

*Case* (2). Suppose now that $\ell(\lambda) \leq 2$. We prove by induction on $l$ that $\chi^{(2n-l,l)}$ is a constituent of $\chi$, for all $l$ odd.

The base case is when $l = 1$. As $\chi^{(2n-1)}$ is a constituent of $\chi\downarrow_{S_{2n-1}}$ and $\chi^{(2n)}$ is not a constituent of $\chi$, the branching rule for restriction gives that $\chi^{(2n-1,1)}$ is a constituent of $\chi$.

Suppose that $l$ is odd and that $l > 1$, and assume inductively that $\chi^{(2n-j,j)}$ is a constituent of $\chi$ for all $j < l$ such that $j$ is odd. As $\chi^{(2n-l+2,l-2)}$ is a constituent of $\chi$, the branching rule for restriction gives that

$$\chi^{(2n-l+2,l-2)} \big\downarrow_{S_{2n-1}} = \chi^{(2n-l+2,l-3)} + \chi^{(2n-l+1,l-2)}$$

is a constituent of $\chi\downarrow_{S_{2n-1}}$. As $\chi^{(2n-l,l-1)}$ is a constituent of $\chi\downarrow_{S_{2n-1}}$, it must be that $\chi^{\mu}$ is a constituent of $\chi$, where $\mu$ is one of the following partitions:

$$(2n-l, l-1, 1), (2n-l+1, l-1), (2n-l, l).$$

By the previous case, the first of these partitions cannot index a constituent of $\chi$. If $\chi^{(2n-l+1,l-1)}$ is a constituent of $\chi$, then $\chi^{(2n-l+1,l-2)}$ is a constituent of $\chi\downarrow_{S_{2n-1}}$ with multiplicity strictly greater than 1. This contradicts the condition (U1), and so we must have that $\chi^{(2n-l,l)}$ is a constituent of $\chi$. This completes the inductive step.

Combining the two cases shows that the constituents of $\chi$ are the even partitions of $2n$ with strictly more than two parts, and the two-part partitions of $2n$ with both parts odd. Therefore conditions (U1) and (U2′) determine $\chi$ uniquely, and so the theorem is proved. □

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