Birkhoff Completeness for Hybrid-Dynamic First-Order Logic
– extended version* –

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Abstract. Hybrid-dynamic first-order logic is a kind of modal logic obtained by enriching many-sorted first-order logic with features that are common to hybrid and to dynamic logics. This provides us with a logical system with an increased expressive power thanks to a number of distinctive attributes: first, the possible worlds of Kripke structures, as well as the nominals used to identify them, are endowed with an algebraic structure; second, we distinguish between rigid symbols, which have the same interpretation across possible worlds – and thus provide support for the standard rigid quantification in modal logic – and flexible symbols, whose interpretation may vary; third, we use modal operators over dynamic-logic actions, which are defined as regular expressions over binary nominal relations. In this context, we propose a general notion of hybrid-dynamic Horn clause and develop a proof calculus for the Horn-clause fragment of hybrid-dynamic first-order logic. We investigate soundness and compactness properties for the syntactic entailment system that corresponds to this proof calculus, and prove a Birkhoff-completeness result for hybrid-dynamic first-order logic.

1 Introduction

The dynamic-reconfiguration paradigm is a most promising approach in the development of highly complex and integrated systems of interacting 'components', which now often evolve dynamically, at run time, in response to internal or external stimuli. More than ever, we are witnessing a continuous increase in the number of applications with reconfigurable features, many of which have aspects that are safety- or security-critical. This calls for suitable formal-specification and verification technologies, and there is already a significant body of research on this topic; hybrid(ized) logics [2,17], first-order dynamic logic [15], and modal μ-calculus [14] are three prominent examples, among many others.

* This paper is an extended version of [10] (presented at TABLEAUX 2019); it includes appendices with proofs of the lemmas and propositions supporting the main results.

The application domain of the work reported in this contribution refers to a broad range of reconfigurable systems whose states or configurations can be presented explicitly, based on some kind of context-independent data types, and for which we distinguish the computations performed at the local/configuration level from the dynamic evolution of the configurations. This suggests a two-layered approach to the design and analysis of reconfigurable systems, involving *a local view*, which amounts to describing the structural properties of configurations, and *a global view*, which corresponds to a specialized language for specifying and reasoning about the way system configurations evolve.

In this paper, we develop sound and complete proof calculi for a new modal logic (recently proposed in [11]) that provides support for the reconfiguration paradigm. The logic, named hybrid-dynamic first-order logic, is obtained by enriching first-order logic (FOL) – regarded as a parameter for the whole construction – with both hybrid and dynamic features. This means that we model reconfigurable systems as Kripke structures (or transition systems), where:

– from a local perspective, we consider a dedicated FOL-signature for configurations, and hence capture configurations as first-order structures; and
– from a global perspective, we consider a second FOL-signature for the possible worlds of the Kripke structure; the terms over that signature are *nominals* used to identify configurations, and the binary nominal relations are regarded as *modalities*, which capture the transitions between configurations.

Sentences are build from equations and relational atoms over the two first-order signatures mentioned above (one pertaining to data, and the other to possible worlds) by using Boolean connectives, quantifiers, standard hybrid-logic operators such as *retrieve* and *store*, and dynamic-logic operators such as *necessity* over structured actions, which are defined as regular expressions over modalities. In practice, actions are used to capture specific patterns of reconfigurability.

The construction is reminiscent of the hybridization of institutions from [17,7] and of the hybrid-dynamic logics presented in [1,16], but it departs fundamentally from any of those studies due to the fact that the possible worlds of the Kripke structures that we consider here have an algebraic structure. This special feature of the logic that we put forward is extremely important for dealing with reconfigurable systems whose states are obtained from initial configurations by applying constructor operations; see, e.g. [12]. In this context, we advance a general notion of Horn clause, which allows the use of implications, universal quantifiers, as well as the hybrid- and dynamic-logic operators listed above.

Besides the fact that it relies on an algebraic structure for possible worlds, the notion of Horn clause that we use in this paper also allows structured actions for (a) the conditions of logical implication, and (b) the arguments of the *necessity* operator. This feature distinguishes the present work from [8], where the first author reported a Birkhoff completeness result for hybrid logics. That is, the Horn clauses that we study in this paper are strictly, and significantly, more expressive than those considered before; this poses a series of new challenges in developing a completeness result. We show that any set of Horn clauses has an initial model despite the fact that the structured actions alone do not have this property. In addition, we provide proof rules to reason formally about the properties of those Kripke structures that are specified using Horn clauses. To conclude, the main result of the paper is a completeness theorem for the Horn-clause fragment of hybrid-dynamic first-order logic.

A brief comparison with the work recently reported in [11] is also in order: both papers deal with properties of hybrid-dynamic first-order logic (with [11] being the contribution in which we introduced the logic); and in both papers we examine Horn clauses; but the results that we develop are complementary: in [11], we focused on an initiality result and on Herbrand's theorem, whereas here we advance proof calculi for the logic. This latter endeavour is much more complex, because it deals with syntactic entailment instead of semantic entailment.

The paper is structured as follows: Section 2 is devoted to the definition of hybrid-dynamic first-order logic. Then, in Section 3, we discuss entailment systems and present the problem we aim to solve. Once the preliminaries are set, we proceed in a layered fashion, in the sense that we consider progressively more complex entailment relations, which are adequate for different fragments of hybrid-dynamic first-order logic. In Section 4 we study completeness for the atomic fragment of the logic. Building on that result, in Section 5 we develop a quasi-completeness result for entailments where the left-hand side is an arbitrary set of Horn clauses, but the right-hand side is only an atomic sentence or an action relation. Finally, in Section 6, we generalize completeness to the full Horn-clause fragment of hybrid-dynamic first-order logic. Proofs of the lemmas and propositions that support the main results can be found in the Appendices A–D.

2 Hybrid-Dynamic First-Order Logic

The hybrid-dynamic first-order logic with user-defined sharing[3] (HDFOLS) that we examine in this work is based on ideas that are similar to those used to define hybrid first-order logic [2] and hybrid first-order logic with rigid symbols [7,5].

We present HDFOLS from an institutional perspective [13], meaning that we focus on signatures and signature morphisms (though, for the purpose of this paper, inclusions would suffice), Kripke structures and homomorphisms, sentences, and the (local) satisfaction relation and condition that relate the syntax and the semantics of the logic. However, other than the notations used, the text requires no prior knowledge of institution theory, and should be accessible to readers with a general background in modal logic and first-order model theory. In order to establish some of the notations used in the rest of the paper, we briefly recall the notion of (many-sorted) first-order signature: a FOL-*signature* is a triple (S, F, P), where S is a set of *sorts*, F is a family $\{F_{ar \to s}\}_{ar \in S^*, s \in S}$ of sets of *operation symbols*, indexed by *arities* $ar \in S^*$ and *sorts* $s \in S$, and P is family $\{P_{ar}\}_{ar \in S^*}$ of sets of *relation symbols*, indexed by *arities* $ar \in S^*$.

Signatures. The *signatures* of HDFOLS are tuples $\Delta = (\Sigma^n, \Sigma^r \subseteq \Sigma)$, where:

[3] This last attribute is meant to indicate the fact that users have control over the symbols that should be interpreted the same across the worlds of a Kripke structure.

1. $\Sigma^n = (S^n, F^n, P^n)$ is a FOL-signature of *nominals* such that $S^n = \{\star\}$,
2. $\Sigma^r = (S^r, F^r, P^r)$ is a FOL-signature of so-called *rigid symbols*, and
3. $\Sigma = (S, F, P)$ is a FOL-signature of both *rigid* and *flexible* symbols.

We let $S^f = S \backslash S^r$, and F^f and P^f be the sub-families of F and P that consist of *flexible symbols* (obtained by removing rigid symbols). In general, we denote by Δ or Δ' signatures of the form $(\Sigma^n, \Sigma^r \subseteq \Sigma)$ or $(\Sigma'^n, \Sigma'^r \subseteq \Sigma')$, respectively.

Signature morphisms. A *signature morphism* $\varphi\colon \Delta \to \Delta'$ consists of a pair of FOL-signature morphisms $\varphi^n\colon \Sigma^n \to \Sigma'^n$ and $\varphi\colon \Sigma \to \Sigma'$ such that $\varphi(\Sigma^r) \subseteq \Sigma'^r$.

Kripke structures. The *models* of a signature Δ are pairs (W, M), where:

1. W is a Σ^n-model, for which we denote by $|W|$ the carrier set of the sort $\star$;
2. $M = \{M_w\}_{w\in|W|}$ is a family of Σ-models, indexed by *worlds* $w \in |W|$, such that the rigid symbols[4] have the same interpretation across possible worlds; i.e., $M_{w_1,\varsigma} = M_{w_2,\varsigma}$ for all worlds $w_1, w_2 \in |W|$ and all symbols ς in Σ^r.

Kripke homomorphisms. A *morphism* $h\colon (W, M) \to (W', M')$ is also a pair $(W \overset{h}{\to} W', \{M_w \overset{h_w}{\to} M'_{h(w)}\}_{w\in|W|})$ consisting of first-order homomorphisms such that $h_{w_1,s} = h_{w_2,s}$ for all possible worlds $w_1, w_2 \in |W|$ and all rigid sorts $s \in S^r$.

Actions. As in dynamic logic, HDFOLS supports structured actions obtained from atoms using sequential composition, union, and iteration. The set A^n of *actions* over Σ^n is defined in an inductive fashion, according to the grammar: $\mathfrak{a} ::= \lambda \mid \mathfrak{a} \mathbin{⨾} \mathfrak{a} \mid \mathfrak{a} \cup \mathfrak{a} \mid \mathfrak{a}^*$, where $\lambda \in P^n_{\star\star}$ is a binary nominal relation. Given a natural number $n > 0$, we denote by $\mathfrak{a}^n$ the composition $\mathfrak{a} \mathbin{⨾} \cdots \mathbin{⨾} \mathfrak{a}$ (where the action $\mathfrak{a}$ occurs n times); and we let $\mathfrak{a}^0(k_1, k_2)$ denote the equation $k_1 = k_2$.

Actions are interpreted in Kripke structures as *accessibility relations* between possible worlds. This is done by extending the interpretation of binary modalities (from $P^n_{\star\star}$): $W_{\mathfrak{a}_1 ⨾ \mathfrak{a}_2} = W_{\mathfrak{a}_1} \mathbin{⨾} W_{\mathfrak{a}_2}$ (diagrammatic composition of relations), $W_{\mathfrak{a}_1\cup\mathfrak{a}_2} = W_{\mathfrak{a}_1} \cup W_{\mathfrak{a}_2}$ (union), and $W_{\mathfrak{a}*} = (W_{\mathfrak{a}})^*$ (reflexive *&* transitive closure).

Hybrid terms. For every Σ^n-model W, the family $T^W = \{T^W_w\}_{w\in|W|}$ of sets of hybrid terms over W is defined inductively according to the following rules:

$$(1)\ \frac{w_0 \in |W| \quad \overline{\tau} \in T^W_{w_0,ar}}{\sigma(\overline{\tau}) \in T^W_{w,s}}\ [\ \sigma \in F^r_{ar\to s}\ ] \qquad (2)\ \frac{w_0 \in |W| \quad \overline{\tau} \in T^W_{w_0,ar}}{\sigma(w_0;\overline{\tau}) \in T^W_{w,s}}\ [\ \sigma \in F^f_{ar\to s},\ s \in S^r\ ] \qquad (3)\ \frac{w \in |W| \quad \overline{\tau} \in T^W_{w,ar}}{\sigma(w;\overline{\tau}) \in T^W_{w,s}}\ [\ \sigma \in F^f_{ar\to s},\ s \in S^f\ ]$$

Notice that flexible operation symbols receive a possible world $w \in |W|$ as a parameter, while rigid operation symbols keep their initial arity. It is easy to check that the hybrid terms of rigid sorts are shared across the worlds.

Fact 1. $T^W_{w_1,s} = T^W_{w_2,s}$ for all possible worlds $w_1, w_2 \in |W|$ and all sorts $s \in S^r$.

Given a world $w \in |W|$, the S-sorted set T^W_w can be regarded as a Σ-model by interpreting every rigid operation symbol $\sigma\colon ar \to s$ as the function that maps (tuples of) hybrid terms $\overline{\tau} \in T^W_{w,ar}$ to $\sigma(\overline{\tau}) \in T^W_{w,s}$, every flexible operation symbol $\sigma\colon ar \to s$ as the function that maps hybrid terms $\overline{\tau} \in T^W_{w,ar}$ to $\sigma(w;\overline{\tau}) \in T^W_{w,s}$, and every relation symbol (either rigid or flexible) as the empty set.

[4] By *symbol* we usually refer to sorts as well, not only to operation/relation symbols.

Lemma 2 (Hybrid-term model and its freeness). *For every Σ^n-model W, (W, T^W) is a Δ-model. Moreover, for any Δ-model (W', M') and first-order Σ^n-homomorphism $f \colon W \to W'$, there exists a unique Δ-homomorphism $h \colon (W, T^W) \to (W', M')$ that agrees with f on W.* $\square$

Standard term model. When W is the first-order term model T_{Σ^n}, by Lemma 2 we obtain the standard hybrid-term model over Δ, denoted $(T_{\Sigma^n}, \{T_k^\Delta\}_{k \in T_{\Sigma^n}})$.

The initiality of the standard term model provides a straightforward interpretation of hybrid terms in Δ-models (W, M): for every hybrid term $t \in T_k^\Delta$, we denote by $(W, M)_t$ or $M_{h(k),t}$ the image of t under the function h_k, where h is the unique homomorphism $(T_{\Sigma^n}, T^\Delta) \to (W, M)$.

Reachable hybrid-term models. We say that a first-order Σ^n-model W is *reachable* if the unique homomorphism $T_{\Sigma^n} \to W$ is surjective. In a similar manner, for HDFOLS, we say that a Δ-model (W, M) is *reachable* if the unique homomorphism $h \colon (T_{\Sigma^n}, T^\Delta) \to (W, M)$ is (componentwise) surjective. In order to avoid naming the homomorphism, we make the following notation.

Notation 3. If a Δ-model (W, M) is reachable, then we may denote by $[\_]$ the unique homomorphism $(T_{\Sigma^n}, T^\Delta) \to (W, M)$ given by the initiality of (T_{Σ^n}, T^Δ).

Proposition 4 (Reachability of hybrid term models). *If W is a* reachable *first-order model of Σ^n, then (W, T^W) is* reachable *for the signature Δ.* $\square$

Sentences. The *atomic sentences* ρ defined over a signature Δ are given by:

$$\rho ::= k_1 = k_2 \mid \lambda(\overline{k'}) \mid t_1 =_{k,s} t_2 \mid \varpi(\bar{t}) \mid \pi(k; \bar{t})$$

where $k, k_i \in T_{\Sigma^n}$ are nominal terms, $\overline{k'}$ is a tuple of terms corresponding to the arity of $\lambda \in P^n$, $t_i \in T_{k,s}^\Delta$ and $\bar{t} \in T_{k,ar}^\Delta$ are (tuples of) hybrid terms,[5] $\varpi \in P_{ar}^r$, and $\pi \in P_{ar}^f$. We refer to these sentences, in order, as *nominal equations*, *nominal relations*, *hybrid equations*, *rigid hybrid relations*, and *non-rigid/flexible hybrid relations*, respectively. When there is no danger of confusion, we may drop one or both subscripts k, s from the notation $t_1 =_{k,s} t_2$. *Full sentences* over Δ are built from atomic sentences according to the following grammar:

$$\gamma ::= \rho \mid \mathfrak{a}(k_1, k_2) \mid @_k \gamma \mid \neg\gamma \mid \bigwedge \Gamma \mid {\downarrow} z \cdot \gamma' \mid \forall X \cdot \gamma'' \mid [\mathfrak{a}]\gamma \mid (o)\,\gamma$$

where $k, k_i \in T_{\Sigma^n}$ are nominal terms, $\mathfrak{a} \in A^n$ is an action, Γ is a finite set of sentences, z is a nominal variable, γ' is a sentence over the signature $\Delta[z]$ obtained by adding z as a new constant to F^n, X is a set of nominal and/or rigid variables, γ'' is a a sentence over the signature $\Delta[X]$ obtained by adding the elements of X as new constants to F^n and F^r, and $o \in F^n_{\star \to \star}$. Other than the first two kinds of sentences (*atoms* and *action relations*), we refer to the sentence-building operators, in order, as *retrieve*, *negation*, *conjunction*, *store*, *universal quantification*, *necessity*, and *next*, respectively. Notice that *necessity* and *next* are parameterized by actions and by unary nominal operations, respectively.

[5] Note that, by Fact 1, if the arity ar is rigid, then the sets $\{T_{k,ar}^\Delta\}_{k \in T_{\Sigma^n}}$ coincide.

We denote by $\mathtt{Sen}^{\mathsf{HDFOLS}}(\Delta)$ the set of all HDFOLS-sentences over Δ.

The local satisfaction relation. Given a Δ-model (W, M) and a world $w \in |W|$, we define the *satisfaction of Δ-sentences at w* by structural induction as follows:

1. *For atomic sentences*:
- $(W, M) \models^w k_1 = k_2$ iff $W_{k_1} = W_{k_2}$ for all nominal equations $k_1 = k_2$;
- $(W, M) \models^w \lambda(\bar{k})$ iff $W_{\bar{k}} \in W_\lambda$ for all nominal relations $\lambda(\bar{k})$;
- $(W, M) \models^w t_1 =_k t_2$ iff $M_{W_k, t_1} = M_{W_k, t_2}$ for all hybrid equations $t_1 =_k t_2$;
- $(W, M) \models^w \varpi(\bar{t})$ iff $(W, M)_{\bar{t}} \in M_{w,\varpi}$ for all rigid relations $\varpi(\bar{t})$;
- $(W, M) \models^w \pi(k; \bar{t})$ iff $(W, M)_{\bar{t}} \in M_{W_k,\pi}$ for flexible relations $\pi(k; \bar{t})$.
2. *For full sentences*:
- $(W, M) \models^w \mathfrak{a}(k_1, k_2)$ iff $(W_{k_1}, W_{k_2}) \in W_{\mathfrak{a}}$ for all action relations $\mathfrak{a}(k_1, k_2)$;
- $(W, M) \models^w @_k\, \gamma$ iff $(W, M) \models^{w'} \gamma$, where $w' = W_k$;
- $(W, M) \models^w \neg\gamma$ iff $(W, M) \not\models^w \gamma$;
- $(W, M) \models^w \bigwedge \Gamma$ iff $(W, M) \models^w \gamma$ for all $\gamma \in \Gamma$;
- $(W, M) \models^w {\downarrow} z \cdot \gamma$ iff $(W, M)^{z \leftarrow w} \models^w \gamma$, where $(W, M)^{z \leftarrow w}$ is the unique $\Delta[z]$-expansion[6] of (W, M) that interprets the variable z as w;
- $(W, M) \models^w \forall X \cdot \gamma$ iff $(W', M') \models^w \gamma$ for all $\Delta[X]$-exp.[6] (W', M') of (W, M);
- $(W, M) \models^w [\mathfrak{a}]\gamma$ iff $(W, M) \models^{w'} \gamma$ for all $w' \in |W|$ such that $(w, w') \in W_{\mathfrak{a}}$;
- $(W, M) \models^w (o)\, \gamma$ iff $(W, M) \models^{w'} \gamma$, where $w' = W_o(w)$.

Fact 5. The following two properties can be checked with ease:

1. The satisfaction of atoms and of action relations ρ does not depend on the possible worlds: $(W, M) \models^w \rho$ iff $(W, M) \models^{w'} \rho$ for all $w, w' \in |W|$.
2. The satisfaction of atoms and of action relations ρ is preserved by homomorphisms: if $(W, M) \models \rho$ and $h \colon (W, M) \to (W', M')$ then $(W', M') \models \rho$.

To state the *satisfaction condition* – and thus finalize the presentation of HDFOLS – let us first notice that every signature morphism $\varphi \colon \Delta \to \Delta'$ induces appropriate *translations of sentences* and *reductions of models*, as follows: every Δ-sentence γ is translated to a Δ'-sentence $\varphi(\gamma)$ by replacing (usually in an inductive manner) the symbols in Δ with symbols from Δ' according to φ; and every Δ'-model (W', M') is reduced to a Δ-model $(W', M')\upharpoonright_\varphi$ that interprets every symbol x in Δ as $(W', M')_{\varphi(x)}$. When φ is an inclusion, we usually denote $(W', M')\upharpoonright_\varphi$ by $(W', M')\upharpoonright_\Delta$ – in this case, the model reduct simply forgets the interpretation of those symbols in Δ' that do not belong to Δ.

The following satisfaction condition can be proved by induction on the structure of Δ-sentences. Its argument is essentially identical to those developed for several other variants of hybrid logic presented in the literature (see, e.g. [5]).

Proposition 6 (Local satisfaction condition for signature morphisms). *For every signature morphism $\varphi \colon \Delta \to \Delta'$, Δ'-model (W', M'), world $w' \in |W'|$, and Δ-sentence γ, we have: $(W', M') \models^w \varphi(\gamma)$ iff $(W', M')\upharpoonright_\varphi \models^w \gamma$.*[7] □

[6] In general, by a $\Delta[X]$-expansion of (W, M) we understand a $\Delta[X]$-model (W', M') that interprets all symbols in Δ in the same way as (W, M).

[7] By the definition of reducts, (W', M') and $(W', M')\upharpoonright_\varphi$ have the same possible worlds.

Substitutions. Consider two signature extensions $\Delta[X]$ and $\Delta[Y]$ with sets of variables, and let $X = X^n \cup X^r$ and $Y = Y^n \cup Y^r$ be the partitions of X and Y into sets of nominal variables and rigid variables. A *Δ-substitution* $\theta\colon X \to Y$ consists of a pair of functions $\theta^n\colon X^n \to T_{\Sigma^n[Y^n]}$ and $\theta^r\colon X^r \to T_k^{\Delta[Y]}$, where k is a nominal term – note that, since the sorts of the hybrid variables are rigid, by Fact 1, it does not matter which nominal term k we choose.

Similarly to signature morphisms, Δ-substitutions $\theta\colon X \to Y$ determine translations of $\Delta[X]$-sentences into $\Delta[Y]$-sentences, and reductions of $\Delta[Y]$-models to $\Delta[X]$-models. The proofs of the next two propositions are similar to the ones given in [9] for hybrid substitutions.

Proposition 7 (Local satisfaction condition for substitutions). *For every Δ-substitution $\theta\colon X \to Y$, every $\Delta[Y]$-model (W, M), world $w \in |W|$, and $\Delta[X]$-sentence γ, we have: $(W, M) \models^w \theta(\gamma)$ iff $(W, M)\upharpoonright_\theta \models^w \gamma$.* □

Fact 8. Let $\theta_{z\leftarrow k}\colon \{z\} \to \varnothing$ be the substitution that maps the nominal variable z to the term k. Then $(W, M)\upharpoonright_{\theta_{z\leftarrow k}} = (W, M)^{z\leftarrow k}$ for every model (W, M).

Propositions 7 and 9 (below) have an important technical role in the Birkhoff completeness proofs presented in the later sections of the paper.

Proposition 9 (Subst. generated by expansions of reachable models). *If (W, M) is reachable, then for every $\Delta[X]$-expansion (W', M') of (W, M) there exists a Δ-substitution $\theta\colon X \to \varnothing$ such that $(W, M)\upharpoonright_\theta = (W', M')$.* □

Expressive power. Fact 5 highlights one of the main distinguishing features of HDFOLS: the satisfaction of atomic sentences, whether they involve flexible symbols or not, does not depend on the possible world where the sentences are evaluated. This contrasts the standard approach in hybrid logic, where each nominal is regarded as an atomic sentence satisfied precisely at the world that corresponds to the interpretation of that nominal. In HDFOLS, the dependence of the satisfaction of sentences on possible worlds is explicit rather than implicit, and is achieved through the *store* operator. Following the lines of [9, Section 4.3], one can show that even without considering action relations, HDFOLS is strictly more expressive than other standard hybrid logics constructed from the same base logic such as the hybrid first-order logic with rigid symbols [7,5].

3 Entailment

Let Γ and Γ' be two sets of sentences over a signature Δ. We say that Γ *semantically entails* Γ', or that Γ' is a *semantic consequence* of Γ, and we write $\Gamma \models_\Delta \Gamma'$, when every Δ-model that satisfies Γ satisfies Γ' too. When the set Γ' is a singleton $\{\gamma\}$, we simplify the notation to $\Gamma \models_\Delta \gamma$. Moreover, we usually drop the subscript Δ when the signature can be easily inferred from the context.

Horn clauses. The problem we propose to address in this paper is that of finding a suitable syntactic characterisation of *entailments* of the form $\Gamma \models \gamma$, where both Γ and γ correspond to the Horn-clause fragment of HDFOLS.

By *Horn clause*, we mean a sentence obtained from atomic sentences by repeated applications of the following sentence-building operators, in any order: (a) *retrieve* (b) *implication* such that the condition is a conjunction of atomic sentences or action relations, (c) *store*, (d) *universal quantification*, (e) *necessity*, and (f) *next*. We denote by HDCLS the Horn-clause fragment of HDFOLS, and by $\mathtt{Sen}^{\mathsf{HDCLS}}(\Delta)$ the set of all Horn clauses over the signature Δ.

In the next sections, we develop a series of *syntactic entailment relations*, whose corresponding entailments are denoted by $\Gamma \vdash \gamma$. All of them are *sound*, in the sense that $\Gamma \vdash \gamma$ implies $\Gamma \models \gamma$; and some are also *compact*, which means that, whenever $\Gamma \vdash \gamma$, there exists a finite subset $\Gamma_f \subseteq \Gamma$ such that $\Gamma_f \vdash \gamma$.

As in previous studies on Birkhoff completeness [4,8], we follow a layered approach. This means that we distinguish the atomic layer of HDCLS from the layer of general Horn clauses. The former is intrinsically dependent on the details of HDCLS, whereas the latter is in essence logic-independent, and can easily be adapted to other hybrid-dynamic logics, not necessarily based on first-order logic. The same ideas apply, for example, to hybrid-dynamic propositional logic.

Nominal replacement. In order to capture syntactically relations between hybrid terms that correspond to different nominals, we introduce a way to replace nominals with nominals within hybrid terms. Given two nominals k_1 and k_2, let $f\colon T_{\Sigma^n} \to T_{\Sigma^n}$ be the function that maps k_1 to k_2 and leaves the other nominals unchanged. We define the family $\{\delta_{k_1/k_2,k}\colon T^{\Delta}_{k} \to T^{\Delta}_{f(k)}\}_{k \in T_{\Sigma^n}}$ by induction:

1. $\delta_{k_1/k_2,k}(\sigma(\bar{t})) = \sigma(\delta_{k_1/k_2,k_0}(\bar{t}))$ when $\sigma \in F^{r}_{ar \to s}$ and $\bar{t} \in T^{\Delta}_{k_0,ar}$;
2. $\delta_{k_1/k_2,k}(\sigma(k_0;\bar{t})) = \sigma(f(k_0);\delta_{k_1/k_2,k_0}(\bar{t}))$ when $\sigma \in F^{f}_{ar \to s}$, $s \in S^{r}$, $\bar{t} \in T^{\Delta}_{k_0,ar}$;
3. $\delta_{k_1/k_2,k}(\sigma(k;\bar{t})) = \sigma(f(k);\delta_{k_1/k_2,k}(\bar{t}))$ when $\sigma \in F^{f}_{ar \to s}$, $s \in S^{f}$, and $\bar{t} \in T^{\Delta}_{k,ar}$.

We usually drop the subscript k, and denote the map $\delta_{k_1/k_2,k}$ simply by δ_{k_1/k_2}.

4 Atomic completeness

In this section, we focus on a completeness result for the atomic fragment of HDCLS. There are two major advancements that distinguish the work presented herein from previous contributions (see, e.g. [8]): (a) the state space of every Kripke model is equipped with a full algebraic structure, and (b) the signatures can have flexible sorts – instead of being restricted to rigid sorts only.

To start, let $\vdash$ be the syntactic entailment relation generated by the rules listed in Figure 1. The following soundness and compactness result can be proved in essentially the same way as in [8]. In particular, the compactness property follows from the fact that all rules have a finite number of premises.

Proposition 10 (Atomic soundness & compactness). *The atomic syntactic entailment relation $\vdash$ is both sound and compact.* □

As it is often the case, completeness is much more difficult to prove, and relies on a number of preliminary results. For the developments presented in this section, we make use of a specific notion of congruence on a Kripke structure.

$$(\mathsf{R}^n)\ \frac{}{\Gamma \vdash k = k} \qquad (\mathsf{S}^n)\ \frac{\Gamma \vdash k_1 = k_2}{\Gamma \vdash k_2 = k_1} \qquad (\mathsf{T}^n)\ \frac{\Gamma \vdash k_1 = k_2 \quad \Gamma \vdash k_2 = k_3}{\Gamma \vdash k_1 = k_3}$$

$$(\mathsf{F}^n)^a\ \frac{\Gamma \vdash \overline{k_1} = \overline{k_2}}{\Gamma \vdash o(\overline{k_1}) = o(\overline{k_2})} \quad (\mathsf{P}^n)\ \frac{\Gamma \vdash \lambda(\overline{k_1}) \ \ \Gamma \vdash \overline{k_1} = \overline{k_2}}{\Gamma \vdash \lambda(\overline{k_2})} \quad (\mathsf{W}^h)\ \frac{\Gamma \vdash k = k'}{\Gamma \vdash t =_{k,s} \delta_{k'/k}(t)}\ [\, s \in S^r \,]$$

$$(\mathsf{W}^r)\ \frac{\Gamma \vdash t_1 =_{k_1,s} t_2}{\Gamma \vdash t_1 =_{k_2,s} t_2}\ [\, s \in S^r \,] \qquad (\mathsf{W}^f)\ \frac{\Gamma \vdash k = k' \quad \Gamma \vdash t_1 =_k t_2}{\Gamma \vdash \delta_{k/k'}(t_1) =_{k'} \delta_{k/k'}(t_2)}$$

$$(\mathsf{R}^h)\ \frac{}{\Gamma \vdash t = t} \qquad (\mathsf{S}^h)\ \frac{\Gamma \vdash t_1 = t_2}{\Gamma \vdash t_2 = t_1} \qquad (\mathsf{T}^h)\ \frac{\Gamma \vdash t_1 =_{k,s} t_2 \quad \Gamma \vdash t_2 =_{k,s} t_3}{\Gamma \vdash t_1 =_{k,s} t_3}$$

$$(\mathsf{F}^r)\ \frac{\Gamma \vdash \overline{t_1} =_{k,ar} \overline{t_2}}{\Gamma \vdash \sigma(\overline{t_1}) =_{k,s} \sigma(\overline{t_2})}\ [\, \sigma \in F^r_{ar \to s} \,] \qquad (\mathsf{F}^f)\ \frac{\Gamma \vdash \overline{t_1} =_{k,ar} \overline{t_2}}{\Gamma \vdash \sigma(k; \overline{t_1}) =_{k,s} \sigma(k; \overline{t_2})}\ [\, \sigma \in F^f_{ar \to s} \,]$$

$$(\mathsf{P}^r)\ \frac{\Gamma \vdash \overline{t_1} =_k \overline{t_2} \ \ \Gamma \vdash \pi(\overline{t_1})}{\Gamma \vdash \pi(\overline{t_2})}\ [\, \pi \in P^r \,] \qquad (\mathsf{P}^f)\ \frac{\Gamma \vdash \overline{t_1} =_k \overline{t_2} \qquad \Gamma \vdash \pi(k; \overline{t_1})}{\Gamma \vdash \pi(k; \overline{t_2})}\ [\, \pi \in P^f \,]$$

$$(\mathsf{P}^h)\ \frac{\Gamma \vdash k_1 = k_2 \ \ \Gamma \vdash \pi(k_1; \overline{t_1})}{\Gamma \vdash \pi(k_2; \delta_{k_1/k_2}(\overline{t_1}))}\ [\, \pi \in P^f \,] \qquad (\mathsf{Ret}_0)\ \frac{\Gamma \vdash @_k\, \rho}{\Gamma \vdash \rho}\ [\, \rho \text{ is atomic} \,]$$

[a] For brevity, $\Gamma \vdash \overline{k_1} = \overline{k_2}$ stands for $\Gamma \vdash k_{1,i} = k_{2,i}$ for all indexes i of the two tuples.

Fig. 1. Proof rules for atomic sentences

Definition 11 (Congruence). *Let* $\Delta = (\Sigma^n, \Sigma^r \subseteq \Sigma)$ *be a* HDCLS*-signature, and* (W, M) *a Kripke structure for it. A* Δ-congruence *on* (W, M) *is a family* $\equiv\, = \{\equiv_w\}_{w \in |W|}$ *of* Σ*-congruences* $\equiv_w$ *on* M_w*, for each possible world* $w \in |W|$*, such that* $(\equiv_{w_1,s}) = (\equiv_{w_2,s})$ *for all worlds* $w_1, w_2 \in |W|$ *and rigid sorts* $s \in S^r$*.*

The next construction is a straightforward generalization of its first-order counterpart, and has been studied in several other papers in the literature (see, e.g. [8]). For that reason, we include it for further reference without a proof.

Proposition 12 (Quotient model). *Every* Δ*-congruence* $\equiv$ *on* (W, M) *determines a* quotient-model homomorphism $(\_/{\equiv})\colon (W, M) \to (W, M/{\equiv})$ *that acts as an identity on* W*, and for which* $(M/{\equiv})_w$ *is the quotient* Σ*-model* $M_w/{\equiv_w}$*.*

Moreover, $(\_/{\equiv})$ *has the following* universal property*: for any Kripke homomorphism* $h\colon (W, M) \to (W', M')$ *such that* $\equiv\, \subseteq \ker(h)$*,*[8] *there exists a unique homomorphism* $h'\colon (W, M/{\equiv}) \to (W', M')$ *such that* $(\_/{\equiv}) \mathbin{;} h' = h$*.*[9] □

We prove the atomic completeness of HDCLS in two steps: first, for nominal equations only; then, for arbitrary atomic sentences (both nominal and hybrid). According to the lemma below, every set of nominal equations Γ^n admits a 'least' Kripke structure (W^n, M^n) that encapsulates the formal deduction of equations.

[8] This means that $h_{w,s}(a_1) = h_{w,s}(a_2)$ for all $a_1, a_2 \in M_{w,s}$ such that $a_1 \equiv_{w,s} a_2$.

[9] Note that we use the diagrammatic notation for function composition.

Lemma 13 (Least Kripke structure of a set of nominal equations). *For every set Γ^n of* nominal equations *over a signature Δ, there exists a reachable initial model (W^n, M^n) such that $\Gamma^n \vdash \rho$ if and only if $(W^n, M^n) \models \rho$, for all nominal or hybrid equations ρ over the signature Δ.* □

The following proposition shows that a set Γ of (nominal or hybrid) equations generates a congruence on a reachable Kripke structure (W, M) when Γ entails all the equations satisfied by (W, M). In particular, the result holds when Γ includes the set of all equations that are satisfied by (W, M).

Proposition 14 (Congruence generated by a set of equations). *Consider a set Γ of equations over a signature Δ, and a reachable Δ-model (W, M) such that $\Gamma \vdash \rho$ for all equations ρ satisfied by (W, M). For all $w \in |W|$, let $\equiv_w$ be the relation on M_w defined by $\tau_1 \equiv_w \tau_2$ whenever $\Gamma \vdash t_1 =_k t_2$ for some $k \in T_{\Sigma^n}$ and $t_1, t_2 \in T_k^{\Delta}$ such that $w = W_k$, and $\tau_i = M_{w,t_i}$. Then:*

P1. $[t_1] \equiv_{[k]} [t_2]$ *iff* $\Gamma \vdash t_1 =_k t_2$*, for all* $k \in T_{\Sigma^n}$ *and* $t_1, t_2 \in T_k^{\Delta}$*;*
P2. $\equiv$ *is a Δ-congruence on (W, M).* □

Now we can finally prove the completeness result for atomic sentences.

Theorem 15 (Atomic completeness). *Every set Γ of atomic sentences over a signature Δ has a reachable initial model (W^{Γ}, M^{Γ}) such that $\Gamma \vdash \rho$ if and only if $(W^{\Gamma}, M^{\Gamma}) \models \rho$, for all atomic sentences ρ over Δ.*

Proof. Let Γ^n be the subset of nominal equations in Γ. By Lemma 13, there exists a initial model (W^n, M^n) of Γ^n such that $\Gamma^n \vdash \rho$ iff $(W^n, M^n) \models \rho$ for all equations ρ over Δ. Then (W^n, M^n) satisfies the hypotheses of Proposition 14 with respect to the set of all (nominal or hybrid) equations in Γ. It follows that the relation $\equiv$ defined by $[t_1] \equiv_{[k]} [t_2]$ whenever $\Gamma \vdash t_1 =_k t_2$, for all nominals k and all terms $t_1, t_2 \in T_{k,s}^{\Delta}$, is a congruence on (W^n, M^n). We define (W^{Γ}, M^{Γ}) as the model obtained from $(W^n, M^n/{\equiv})$ by interpreting:

- each nominal relation symbol $\lambda \in P^n$ as $W_{\lambda}^{\Gamma} = \{[\bar{k}] \in |W^n| \mid \Gamma \vdash \lambda(\bar{k})\}$;
- each relation symbol $\varpi \in P^r$ as $M_{[k],\varpi}^{\Gamma} = \{[\bar{t}]/{\equiv_{[k]}} \in M_{[k]}^{\Gamma} \mid \Gamma \vdash \varpi(\bar{t})\}$;
- each relation symbol $\pi \in P^f$ as $M_{[k],\pi}^{\Gamma} = \{[\bar{t}]/{\equiv_{[k]}} \in M_{[k]}^{\Gamma} \mid \Gamma \vdash \pi(k; \bar{t})\}$.

Note that the interpretations of $\varpi \in P^r$ and $\pi \in P^f$ are independent of the choice of the nominal k. For example, for flexible relation symbols, if $[k] = [k']$ then $\Gamma \vdash k = k'$; therefore, if $\Gamma \vdash \pi(k; \bar{t})$, we also have $\Gamma \vdash \pi(k'; \bar{t'})$ by (P^h), where $\bar{t'} = \delta_{k/k'}(\bar{t})$ is a tuple of hybrid terms that satisfies $[\bar{t}] \equiv_{[k]} [\bar{t'}]$.

The fact that (W^{Γ}, M^{Γ}) is a reachable model of Γ follows in a straightforward manner by construction. Therefore, we focus on the initiality property. Let (W, M) be a Δ-model that satisfies Γ. In particular, (W, M) satisfies all nominal equations in Γ. By Lemma 13, we deduce that there exists a unique homomorphism $h \colon (W^n, M^n) \to (W, M)$. We also know that (W, M) satisfies all hybrid equations in Γ, which implies that $\equiv \subseteq \ker(h)$. By Proposition 12, this means that there exists a unique Kripke homomorphism $h' \colon (W^n, M^n/{\equiv}) \to (W, M)$ such that $(\_/{\equiv}) \mathbin{;} h' = h$. To finalize this part of the proof, we need to ensure that h' preserves the interpretation of all relation symbols (nominal or hybrid) satisfied by (W^Γ, M^Γ). We only consider the case of flexible relation symbols. Nominal relations and rigid relations can be treated in a similar manner. Suppose $\pi \in P^{f}_{ar}$ and $\bar{\tau} \in M^{\Gamma}_{[k],\pi}$, for an arbitrary but fixed nominal $k \in T_{\Sigma^n}$. Then:

1	$\Gamma \vdash \pi(k; \bar{t})$ for some tuple of terms $\bar{t} \in T^{\Delta}_{k,ar}$ such that $\bar{\tau} = [\bar{t}]/{\equiv_{[k]}}$	by the definition of $M^{\Gamma}_{[k],\pi}$
2	$\Gamma \models \pi(k; \bar{t})$	by Proposition 10
3	$(W, M) \models \pi(k; \bar{t})$	since $(W, M) \models \Gamma$
4	$M_{w,\bar{t}} \in M_{w,\pi}$ for $w = W_k$	by the definition of $\models$
5	$h'(\bar{\tau}) \in M_{w,\pi}$	since $h'(\bar{\tau}) = h'([\bar{t}]/{\equiv_{[k]}}) = M_{w,\bar{t}}$

Lastly, we show that $\Gamma \vdash \rho$ iff $(W^\Gamma, M^\Gamma) \models \rho$, for all atomic sentences ρ. The 'only if' part is straightforward since (W^Γ, M^Γ) is a model of Γ. For the 'if' part, we proceed by case analysis on the structure of ρ. The more interesting cases are those of relational atoms. Suppose, for instance that $(W^\Gamma, M^\Gamma) \models \pi(k; \bar{t})$, where $\pi \in P^{f}_{ar}$, $k \in T_{\Sigma^n}$, and $\bar{t} \in T^{\Delta}_{k,ar}$. If follows that:

1	$[\bar{t}]/{\equiv_{[k]}} \in M^{\Gamma}_{[k],\pi}$	by the definition of $\models$
2	$\Gamma \vdash \pi(k; \bar{t'})$ for some tuple of terms $\bar{t'} \in T^{\Delta}_{k,ar}$ such that $[\bar{t'}] \equiv_{[k]} [\bar{t}]$	by the definition of $M^{\Gamma}_{[k],\pi}$
3	$\Gamma \vdash \bar{t} =_{k,ar} \bar{t'}$	by Proposition 14
4	$\Gamma \vdash \pi(k; \bar{t})$	by the proof rule (P^f) $\square$

5 Quasi-completeness

The main contribution in this section is the construction, for any set of Horn clauses, of an initial model that encapsulates the syntactic deduction of atomic sentences and action relations. An initiality result is obtained in [11] as well, but in that paper it is based on the semantic entailment. In contrast, the present result is based on syntactic deduction, which requires a higher level of complexity, and it is developed in the context of a modular approach to completeness. This means that the present results are applicable to other modal logics, where some of the sentence-building operators considered here may be disregarded.

We focus on entailments of the form $\Gamma \models \rho$, where Γ is an arbitrary set of Horn clauses, and ρ is either an atomic sentence, or an action relation. To that end, let $\vdash$ be the syntactic entailment relation generated by the rules listed in Figures 1, 2 and 3. The soundness and compactness result presented in Section 4 can be generalized with ease for the entailment relation $\vdash$ that we consider here.

Proposition 16. *The entailment relation $\vdash$ is sound and compact.* $\square$

Fact 17 (Retrieve redundancies). For all nominals $k_1, k_2 \in T_{\Sigma^n}$ and all sentences γ over a signature Δ, the sentences $@_{k_1} @_{k_2} \gamma$ and $@_{k_2} \gamma$ are both syntactically and semantically equivalent. Moreover, if ρ is atomic or an action relation, then $@_{k_n} \rho$ is syntactically and semantically equivalent to ρ.

$$(\mathsf{Comp})\ \frac{\Gamma \vdash \mathfrak{a}_1(k_1,k_2) \quad \Gamma \vdash \mathfrak{a}_2(k_2,k_3)}{\Gamma \vdash (\mathfrak{a}_1 \mathbin{;} \mathfrak{a}_2)(k_1,k_3)} \qquad (\mathsf{Union})\ \frac{\Gamma \vdash \mathfrak{a}_i(k_1,k_2)}{\Gamma \vdash (\mathfrak{a}_1 \cup \mathfrak{a}_2)(k_1,k_2)}\ [\ i \in \{1,2\}\ ]$$

$$(\mathsf{Refl})\ \frac{\Gamma \vdash k_1 = k_2}{\Gamma \vdash \mathfrak{a}^*(k_1,k_2)} \qquad (\mathsf{Star})\ \frac{\Gamma \vdash \mathfrak{a}(k_i,k_{i+1}) \text{ for } 0 \leqslant i < n}{\Gamma \vdash \mathfrak{a}^*(k_0,k_n)} \qquad (\mathsf{Ret_a})\ \frac{\Gamma \vdash @_k\, \mathfrak{a}(k_1,k_2)}{\Gamma \vdash \mathfrak{a}(k_1,k_2)}$$

Fig. 2. Proof rules for action relations

$$(\mathsf{Ret_@})\ \frac{\Gamma \vdash @_{k_1} @_{k_2} \gamma}{\Gamma \vdash @_{k_2} \gamma} \qquad (\mathsf{Ret_I})\ \frac{\Gamma \vdash \gamma}{\Gamma \vdash @_k \gamma} \qquad (\mathsf{Imp_E})\ \frac{\Gamma \vdash @_k (\bigwedge H \Rightarrow \gamma)}{\Gamma \cup H \vdash @_k \gamma}$$

$$(\mathsf{Store_E})^a\ \frac{\Gamma \vdash @_k \downarrow z \cdot \gamma}{\Gamma \vdash @_k\, \theta_{z \leftarrow k}(\gamma)} \qquad (\mathsf{Subst_q})^b\ \frac{\Gamma \vdash @_k \forall X \cdot \gamma}{\Gamma \vdash @_k\, \theta(\gamma)}$$

$$(\mathsf{Nec_E})\ \frac{\Gamma \vdash @_{k_1} [\mathfrak{a}]\gamma \quad \Gamma \vdash \mathfrak{a}(k_1,k_2)}{\Gamma \vdash @_{k_2} \gamma} \qquad (\mathsf{Next_E})\ \frac{\Gamma \vdash @_k \langle o \rangle \gamma}{\Gamma \vdash @_{o(k)} \gamma}$$

[a] Recall that $\theta_{z \leftarrow k} \colon \{z\} \to \varnothing$ is the substitution that maps z to the nominal k.
[b] $\theta \colon X \to \varnothing$ is a ground substitution.

Fig. 3. Proof rules for Horn clauses

To prove that $\vdash$ is also complete, we first extend Theorem 15 to entailments $\Gamma \vdash \rho$ for which Γ is a set of atoms and ρ is either atomic or an action relation.

Proposition 18 (Extending atomic completeness). *Let Γ be a set of atomic sentences over a signature Δ, and (W^Γ, M^Γ) a reachable initial model of Γ as in Theorem 15. Then $\Gamma \vdash \rho$ if and only if $(W^\Gamma, M^\Gamma) \models \rho$, for all atomic sentences or action relations ρ over the signature Δ.* □

The result below shows that, in order to obtain an initial model of a set Γ of clauses, it suffices to consider the initial model $(W^{\Gamma_0}, M^{\Gamma_0})$ of the set Γ_0 of atoms entailed by Γ. Moreover, $(W^{\Gamma_0}, M^{\Gamma_0})$ satisfies all clauses entailed by Γ.

Theorem 19 (Initiality preserves formal deductions). *Let Γ be a set of clauses over a signature Δ, $\Gamma_0 = \{\rho \in \mathsf{Sen}^{\mathsf{HDCLS}}(\Delta) \mid \Gamma \vdash \rho \text{ \& } \rho \text{ is atomic}\}$, and $(W^{\Gamma_0}, M^{\Gamma_0})$ a reachable initial model of Γ_0 as in Theorem 15. Then $\Gamma \vdash \gamma$ implies $(W^{\Gamma_0}, M^{\Gamma_0}) \models \gamma$ for all Horn clauses γ over Δ.*

Proof. Since the model $(W^{\Gamma_0}, M^{\Gamma_0})$ is reachable, it suffices to prove that $\Gamma \vdash @_k \gamma$ implies $(W^{\Gamma_0}, M^{\Gamma_0}) \models @_k \gamma$ for all nominals $k \in T_{\Sigma^n}$ and Horn clauses $\gamma \in \mathsf{Sen}^{\mathsf{HDCLS}}(\Delta)$. We proceed by structural induction on γ.

For the base case, assume $\Gamma \vdash @_k \gamma$, where γ is atomic. It follows that:

1	$\Gamma \vdash \gamma$	by $(\mathsf{Ret_0})$ in Figure 1
2	$\gamma \in \Gamma_0$	by the definition of Γ_0
3	$\Gamma_0 \vdash \gamma$	by the monotonicity of $\vdash$
4	$(W^{\Gamma_0}, M^{\Gamma_0}) \models \gamma$	by Theorem 15
5	$(W^{\Gamma_0}, M^{\Gamma_0}) \models @_k \gamma$	by Fact 17

For the induction step, we proceed by case analysis on the topmost sentence-building operator of γ. We only present the case corresponding to the *necessity* operator. Proofs for the remaining cases can be found in Appendix C.

[$\Gamma \vdash @_k [\mathfrak{a}]\gamma$] Let $w = W^{\Gamma_0}_k$. We want to show that $(W^{\Gamma_0}, M^{\Gamma_0}) \models^{w'} \gamma$ for all possible worlds w' such that $(w, w') \in W^{\Gamma_0}_{\mathfrak{a}}$. Given such a possible world, since the model $(W^{\Gamma_0}, M^{\Gamma_0})$ is reachable, we know that there exists a nominal k' such that $w' = W^{\Gamma_0}_{k'}$. It follows that:

1	$(W^{\Gamma_0}, M^{\Gamma_0}) \models \mathfrak{a}(k, k')$	since $(w, w') \in W^{\Gamma_0}_{\mathfrak{a}}$
2	$\Gamma_0 \vdash \mathfrak{a}(k, k')$	by Proposition 18
3	$\Gamma_f \vdash \mathfrak{a}(k_1, k_2)$ for some finite $\Gamma_f \subseteq \Gamma_0$	since $\vdash$ is compact
4	$\Gamma \vdash \mathfrak{a}(k, k')$	since $\Gamma \vdash \Gamma_f$ and $\Gamma_f \vdash \mathfrak{a}(k_1, k_2)$
5	$\Gamma \vdash @_{k'} \gamma$	by $(\mathsf{Nec_E})$
6	$(W^{\Gamma_0}, M^{\Gamma_0}) \models @_{k'} \gamma$	by the induction hypothesis
7	$(W^{\Gamma_0}, M^{\Gamma_0}) \models^{w'} \gamma$	since $w' = W^{\Gamma_0}_{k'}$.

□

We are now finally ready to tackle the quasi-completeness of HDCLS: the initial model of a set of Horn clauses encapsulates the formal deduction of both atomic sentences and action relations. Note that, in general, action relations are not Horn clauses; nonetheless, we discuss their case too because it provides an important technical tool for the final completeness result.

Corollary 20 (Quasi-completeness). *Under the notations and hypotheses of Theorem 19, $(W^{\Gamma_0}, M^{\Gamma_0})$ is also an initial model of Γ. Moreover, for all atomic sentences or action relations ρ, the following statements are equivalent:*

1. $\Gamma \models \rho$ *2.* $(W^{\Gamma_0}, M^{\Gamma_0}) \models \rho$ *3.* $\Gamma \vdash \rho$ □

6 Horn-clause completeness

This final technical section deals with Birkhoff completeness, which corresponds to the existence of a syntactic characterization for the semantic entailment relation of HDCLS. This is practically very useful, because Horn clauses facilitate the development of an operational semantics of formal specifications based on rewriting. For example, action relations can provide logical support for the *rewriting rules* used in Maude [3], or for the *transitions* from CafeOBJ [6].

In order to generalize completeness to arbitrary Horn clauses, we need to consider additional rules, which are particular to different kinds of clauses. We say that a sentence is *action-free* if it contains no occurrences of any of the action-building operators (*composition*, *union*, or *transitive closure*), and that it is *star-free* if it contains no occurrences of the *transitive-closure* operator.

Notation 21. Consider the following fragments of HDFOLS. Each of them is obtained through a specific restriction on sentences:

$\mathsf{HDFOLS}^{(1)}$ – corresponding to action-free Horn clauses;

$$(\mathsf{Ret_E})\ \frac{\Gamma \vdash_{\Delta[z]} @_z\, \gamma}{\Gamma \vdash_{\Delta} \gamma} \qquad (\mathsf{Imp_I})\ \frac{\Gamma \cup H \vdash @_k\, \gamma}{\Gamma \vdash @_k\, (\bigwedge H \Rightarrow \gamma)} \qquad (\mathsf{Store_I})\ \frac{\Gamma \vdash @_k\, \theta_{z \leftarrow k}(\gamma)}{@_k \downarrow z \cdot \gamma}$$

$$(\mathsf{Quant_I})\ \frac{\Gamma \vdash_{\Delta[X]} @_k\, \gamma}{\Gamma \vdash_{\Delta} @_k\, \forall X \cdot \gamma} \qquad (\mathsf{Nec_I})\ \frac{\Gamma \cup \{\mathfrak{a}(k,z)\} \vdash_{\Delta[z]} @_z\, \gamma}{\Gamma \vdash_{\Delta} @_k\, [\mathfrak{a}]\gamma} \qquad (\mathsf{Next_I})\ \frac{\Gamma \vdash @_{o(k)}\, \gamma}{\Gamma \vdash @_k\, (o)\, \gamma}$$

Fig. 4. Additional proof rules for Horn clauses

$$(\mathsf{Comp_I})\ \frac{E \cup \{\mathfrak{a}_1(k_1,z), \mathfrak{a}_2(z,k_2)\} \vdash^{(2)}_{\Delta[z]} e}{E \cup \{(\mathfrak{a}_1 \mathbin{;} \mathfrak{a}_2)(k_1,k_2)\} \vdash^{(2)}_{\Delta} e}\ [\ E \cup \{e\} \subseteq \mathtt{Sen}^{\mathsf{HDFOLS}}(\Delta)\ ]$$

$$(\mathsf{Union_I})\ \frac{E \cup \{\mathfrak{a}_i(k_1,k_2)\} \vdash^{(2)} e \text{ for } i \in \{1,2\}}{E \cup \{(\mathfrak{a}_1 \cup \mathfrak{a}_2)(k_1,k_2)\} \vdash^{(2)} e}\ [\ E \cup \{e\} \subseteq \mathtt{Sen}^{\mathsf{HDFOLS}}(\Delta)\ ]$$

$$(\mathsf{Star_I})^a\ \frac{E \cup \{\mathfrak{a}^n(k_1,k_2)\} \vdash^{(3)} e \text{ for all } n \in \mathbb{N}}{E \cup \{\mathfrak{a}^*(k_1,k_2)\} \vdash^{(3)} e}\ [\ E \cup \{e\} \subseteq \mathtt{Sen}^{\mathsf{HDFOLS}}(\Delta)\ ]$$

[a] Note that this rule is infinitary; we only use it in the final result in Section 6.

Fig. 5. Additional proof rules for action relations

$\mathsf{HDFOLS}^{(2)}$ – corresponding to star-free Horn clauses and action relations;
$\mathsf{HDFOLS}^{(3)}$ – corresponding to Horn clauses and action relations.

Notice that $\mathsf{HDFOLS}^{(3)}$ is the richest fragment, and that γ is a clause in HDFOLS iff it is a Horn clause in $\mathsf{HDFOLS}^{(3)}$. We also define three entailment relations:

1. $\vdash^{(1)}$ is generated by the proof rules in Figures 1–4, but restricts the applications of $(\mathsf{Nec_I})$ to situations where $\mathfrak{a}$ is a modality (i.e., an atomic action);
2. $\vdash^{(2)}$ is generated by the proof rules in Figures 1–5, except $(\mathsf{Star_I})$, and restricted to applications of $(\mathsf{Comp_I})$ and $(\mathsf{Union_I})$ to star-free sentences;
3. $\vdash^{(3)}$ is generated by all proof rules in Figures 1–5.

Notice also that $\vdash^{(3)}$ is the most general one. Given a set of Horn clauses, $\vdash^{(3)}$ can be used to derive arbitrary Horn clauses from it, whereas $\vdash^{(2)}$ can only be used to derive star-free Horn clauses, and $\vdash^{(1)}$ only action-free Horn clauses.

It is easy to check that all these entailment relations are sound – similarly to Propositions 10 and 22, along the lines of [8]. Compactness, however, holds only for the first two. That is because the rule $(\mathsf{Star_I})$ in Figure 5 is infinitary.

Proposition 22 (Soundness & compactness). *The entailment relation $\vdash^{(x)}$ is sound, for all $x \in \{1,2,3\}$. Moreover, $\vdash^{(1)}$ and $\vdash^{(2)}$ are also compact.* □

Our approach to completeness relies on the introduction rules in Figures 4 and 5. These allow us to simplify, for example, the action relations that may appear in the left-hand side of the turnstile symbol during the proof process.

Theorem 23 (Birkhoff completeness). *Let $x \in \{1,2,3\}$. For every set Γ of Horn clauses in* HDFOLS*, and for every clause γ in* $\mathsf{HDFOLS}^{(x)}$*,*

$$\Gamma \models \gamma \qquad \textit{implies} \qquad \Gamma \vdash^{(x)} \gamma.$$

Proof. Notice that $\Gamma \models \gamma$ implies $\Gamma \models @_k \gamma$, for any nominal k. Therefore, given the proof rule $(\mathsf{Ret_E})$, it suffices to prove that $\Gamma \models @_k \gamma$ implies $\Gamma \vdash^{(x)} @_k \gamma$. We proceed by induction on the structure of the sentence γ.

For the base case, where γ is an atomic sentence, the conclusion follows by Fact 17, Corollary 20, and the fact that $\Gamma \vdash \gamma$ implies $\Gamma \vdash^{(x)} \gamma$.

For the induction step, we consider only the case where γ is universally quantified. The remaining cases can be proved in a similar fashion; see Appendix D.

[$\Gamma \models @_k \forall X \cdot \gamma$] Then:

1	$\Gamma \models_{\Delta[X]} @_k \gamma$	by the general properties of $\models$
2	$\Gamma \vdash^{(x)}_{\Delta[X]} @_k \gamma$	by the induction hypothesis
3	$\Gamma \vdash^{(x)}_{\Delta} @_k \forall X \cdot \gamma$	by $(\mathsf{Quant_I})$ $\square$

To come to an end, notice that the entailment relation $\vdash^{(3)}$ is sound (by Proposition 22) and complete (by Theorem 23), but it is not compact, since the rule $(\mathsf{Star_I})$ is infinitary. The next proposition shows this is the best result we can obtain, because the semantic entailment relation in HDCLS is not compact.

Proposition 24 (Lack of compactness). HDCLS *is not compact.*

Proof (sketch). It suffices to consider a signature Δ with two nominals, k and k', and two modalities, λ and α, and the set $\Gamma = \{\lambda^n(k,k') \Rightarrow \alpha(k,k') \mid n \in \mathbb{N}\}$ of Horn clauses over Δ. Then the following properties hold:

1. $\Gamma \models \lambda^*(k,k') \Rightarrow \alpha(k,k')$;
2. There is no finite subset $\Gamma_f \subseteq \Gamma$ such that $\Gamma_f \models \lambda^*(k,k') \Rightarrow \alpha(k,k')$. $\square$

7 Conclusions

The hybrid-dynamic first-order logic that we have studied in this paper is obtained by enriching first-order logic with a unique combination of features that are specific to hybrid and to dynamic logics. This provides a language that is particularly well suited for specifying and reasoning about reconfigurable systems. More precisely, it allows us to capture reconfigurable systems as Kripke structures whose possible worlds (a) have an algebraic structure, which supports operations on configurations, and (b) are labelled with constrained first-order models that capture the local structure of configurations. From a syntactic perspective, we define nominals and hybrid terms to refer to possible worlds and to the elements of the first-order structures associated to those worlds. Terms are then used to form nominal and hybrid equations, as well as relational atoms, from which we build complex sentences using Boolean connectives, quantifiers, hybrid-logic operators such as *retrieve* and *store*, and dynamic-logic operators such as *necessity* over actions, i.e., regular expressions over modalities.

In this context, we have developed a layered approach towards a Birkhoff completeness result for hybrid-dynamic first-order logic. There are three major layers to consider: first, the atomic layer, which deals with entailments where both the premises and the conclusion are atomic sentences; second, a mixed layer, which deals with entailments where the premises are Horn clauses, but the conclusion is only an atomic sentence or an action relation; and third, the general, Horn-clause layer, which deals with entailments where both the premises and the conclusion are Horn clauses. For each of these layers, we have developed sound and complete proof systems. Moreover, for the first two layers, the proof systems considered have also been shown to be compact.

The third layer deserves more attention. In that case, we distinguish between two main proof systems: (a) one that is compact, but complete only for entailments whose conclusion is a star-free clause; and (b) one that is not compact, but it is complete for all entailments. To conclude this line of developments, we have shown that this is the best result one can obtain for hybrid-dynamic logic.

As mentioned already, thanks to its features and expressive power, hybrid-dynamic first-order logic is a promising formalism for reasoning about reconfigurable systems. The work reported in this paper provides a rigorous foundation for that purpose. Therefore, an important task to pursue further is the development of a language, specification methodology, and appropriate tool support (that implements the proof systems presented here) for this new logic.

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A Proofs for results presented in Section 2

Proof (of Lemma 2: The universal property of hybrid term models).

The fact that (W, T^W) is a Δ-model is straightforward. For the 'freeness' part, we define $\{h_w : T^W_w \to M'_{f(w)}\}_{w \in |W|}$ by structural induction:

1. $h_{w,s}(\sigma(\bar{\tau})) = M'_{f(w_0),\sigma}(h_{w_0,ar}(\bar{\tau}))$ for all $\sigma \in F^r_{ar \to s}$, $w_0, w \in |W|$, $\bar{\tau} \in T^W_{w_0,ar}$;
2. $h_{w,s}(\sigma(w_0; \bar{\tau})) = M'_{f(w_0),\sigma}(h_{w_0,ar}(\bar{\tau}))$ for all flexible operations $\sigma \in F^f_{ar \to s}$ such that $s \in S^r$, and all $w_0, w \in |W|$ and $\bar{\tau} \in T^W_{w_0,ar}$;
3. $h_{w,s}(\sigma(w; \bar{\tau})) = M'_{f(w),\sigma}(h_{w,ar}(\bar{\tau}))$ for all flexible operations $\sigma \in F^f_{ar \to s}$ such that $s \in S^f$, and all $w \in |W|$ and $\bar{\tau} \in T^W_{w,ar}$.

It is again straightforward to check that $h = (f, \{h_w\}_{w \in |W|})$ is a Kripke homomorphism $(W, T^W) \to (W', M')$, and that it is unique with this property. □

Proof (of Proposition 4: A term model is reachable by hybrid terms whenever its space state is reachable by nominals).

By hypothesis, $h : T_{\Sigma^n} \to W$ is surjective. Therefore, all we need to prove is that, for every nominal k, the Σ-homomorphism $h_k : T^\Delta_k \to T^W_{h(k)}$ is surjective. We proceed by induction on the structure of the hybrid terms:

1. Assume that $\bar{\tau} \in T^W_{h(k_0),ar}$ and $\sigma \in F^r_{ar \to s}$ (hence, $s \in S^r$). By the induction hypothesis, there exists tuple of hybrid terms $\bar{t} \in T^\Delta_{k_0,ar}$ such that $h_{k_0,ar}(\bar{t}) = \bar{\tau}$. Therefore, $h_{k,s}(\sigma(\bar{t})) = h_{k_0,s}(\sigma(\bar{t})) = \sigma(h_{k_0,ar}(\bar{t})) = \sigma(\bar{\tau})$.
2. Assume that $\bar{\tau} \in T^W_{h(k_0),ar}$, $\sigma \in F^f_{ar \to s}$ and $s \in S^r$. By the induction hypothesis, there exists $\bar{t} \in T^\Delta_{k_0,ar}$ such that $h_{k_0,ar}(\bar{t}) = \bar{\tau}$. Therefore, $h_{k,s}(\sigma(k_0; \bar{t})) = h_{k_0,s}(\sigma(k_0; \bar{t})) = \sigma(h(k_0); h_{k_0,ar}(\bar{t})) = \sigma(h(k_0); \bar{\tau})$.
3. Assume that $\bar{\tau} \in T^W_{h(k),ar}$, $\sigma \in F^f_{ar \to s}$ and $s \in S^f$. By the induction hypothesis, there exists $\bar{t} \in T^\Delta_{k,ar}$ such that $h_{k,ar}(\bar{t}) = \bar{\tau}$. Therefore, $h_{k,s}(\sigma(k; \bar{t})) = \sigma(h(k); h_{k,ar}(\bar{t})) = \sigma(h(k); \bar{\tau})$. □

B Proofs for results presented in Section 4

Many of the results presented in this work (including Lemma 13, which we discuss below) rely on the following general properties of entailment relations:

Monotonicity: $\Gamma' \subseteq \Gamma$ implies $\Gamma \vdash \Gamma'$;
Transitivity: $\Gamma \vdash \Gamma'$ and $\Gamma' \vdash \Gamma''$ imply $\Gamma \vdash \Gamma''$;
Union: $\Gamma \vdash \Gamma'$ and $\Gamma \vdash \Gamma''$ imply $\Gamma \vdash \Gamma' \cup \Gamma''$.

From these, we can derive a form of *cut*: $\Gamma \vdash H$ and $\Gamma \cup H \vdash \gamma$ imply $\Gamma \vdash \gamma$.

Proof (of Lemma 13: Every set of nominal equations admits a reachable initial model that encapsulates the formal deduction of equations from that set).

Let Γ^n be a set of nominal equations over Δ, and $\equiv^n = \{(k_1, k_2) \in T_{\Sigma^n} \times T_{\Sigma^n} \mid \Gamma^n \vdash k_1 = k_2\}$. It is straightforward to check, based on the rules (R^n), (S^n),

(T^n), (F^n), and (P^n) in Figure 1, that $\equiv^n$ is a Σ^n-congruence on T_{Σ^n}. Therefore, we can define W^n as the quotient Σ^n-model $T_{\Sigma^n}/{\equiv^n}$, and M^n as the family of all sets of hybrid terms over W^n. Clearly, (W^n, M^n) is reachable.

To show that it is an initial model of Γ^n, note that, for every $(k_1 = k_2) \in \Gamma^n$:

1	$\Gamma^n \vdash k_1 = k_2$	by the monotonicity of $\vdash$
2	$[k_1] = [k_2]$	by the definition of $\equiv^n$
3	$(W^n, M^n) \models k_1 = k_2$	by the definition of $\models$

Furthermore, for every Δ-model (W, M) that satisfies Γ^n, the first-order model W satisfies Γ^n too, so there exists a unique Σ^n-homomorphism $f \colon W^n \to W$. By Lemma 2, f extents to a unique Δ-homomorphism $h \colon (W^n, M^n) \to (W, M)$.

Now let us show that $\Gamma^n \vdash \rho$ and $(W^n, M^n) \models \rho$ are equivalent. If $\Gamma^n \vdash \rho$, then we know that $\Gamma^n \models \rho$ by Proposition 10. Given that (W^n, M^n) is a model of Γ^n, it follows, by the definition of the semantic entailment, that $(W^n, M^n) \models \rho$.

For the converse, we distinguish two cases: if ρ is a nominal equation, the conclusion follows with ease from the definition of $\equiv^n$; on the other hand, the case where ρ is a hybrid equation deserves more attention.

Let $[\_] \colon (T_{\Sigma^n}, T^{\Delta}) \to (W^n, M^n)$ be the unique arrow from Notation 3. It suffices to show that for all $k_1, k_2 \in T_{\Sigma^n}$, $s \in S$, $t_1 \in T^{\Delta}_{k_1,s}$ and $t_2 \in T^{\Delta}_{k_2,s}$ such that $[k_1] = [k_2]$ and $[t_1] = [t_2]$, we have $\Gamma \vdash t_1 =_{k_1,s} \delta_{k_2/k_1}(t_2)$.

We proceed by structural induction on the term t_1:

[$t_1 = \sigma(\overline{t'_1})$, *where* $\sigma \in F^r_{ar\to s}$ *and* $\overline{t'_1} \in T^{\Delta}_{k_1,ar}$]

1	$[t_1] = \sigma([\overline{t'_1}])$	since $[\_]$ is a homomorphism
2	$t_2 = \sigma(\overline{t'_2})$ for some $\overline{t'_2} \in T^{\Delta}_{k_2,ar} = T^{\Delta}_{k_1,ar}$	by regarding $\sigma([\overline{t'_1}]) = [t_2]$ as an equality of trees
3	$[\overline{t'_1}] = [\overline{t'_2}]$	since $\sigma([\overline{t'_1}]) = \sigma([\overline{t'_2}])$
4	$\Gamma \vdash \overline{t'_1} = \overline{t'_2}$	by the induction hypothesis
5	$\Gamma \vdash \sigma(\overline{t'_1}) = \sigma(\overline{t'_2})$	by (F^r)

[$t_1 = \sigma(k'_1; \overline{t'_1})$ *where* $\sigma \in F^f_{ar\to s}$, $\overline{t'_1} \in T^{\Delta}_{k'_1,ar}$ *and* $s \in S^r$]

1	$[t_1] = \sigma([k'_1]; [\overline{t'_1}])$	since $[\_]$ is a homomorphism
2	$t_2 = \sigma(k'_2; \overline{t'_2})$ for some $k'_2 \in T_{\Sigma^n}$ and $\overline{t'_2} \in T^{\Delta}_{k'_2,ar}$	since $\sigma([k'_1]; [\overline{t'_1}]) = [t_2]$ can be regarded as equality of trees
3	$[k'_1] = [k'_2]$ and $[\overline{t'_1}] = [\overline{t'_2}]$	since $\sigma([k'_1]; [\overline{t'_1}]) = \sigma([k'_2]; [\overline{t'_2}])$
4	$\Gamma \vdash \overline{t'_1} =_{k'_1} \delta_{k'_2/k'_1}(\overline{t'_2})$	by the induction hypothesis
5	$\Gamma \vdash \sigma(k'_1; \overline{t'_1}) = \sigma(k'_1; \delta_{k'_2/k'_1}(\overline{t'_2}))$	by (F^f)
6	$\Gamma \vdash \sigma(k'_1; t'_1) = \delta_{k'_2/k'_1}(\sigma(k'_2; \overline{t'_2}))$	by the definition of $\delta_{k'_2/k'_1}$
7	$\Gamma \vdash \sigma(k'_1; \overline{t'_1}) = \sigma(k'_2; \overline{t'_2})$	by (W^h), since $\Gamma \vdash k'_1 = k'_2$

[$t_1 = \sigma(k_1; \overline{t'_1})$ *where* $\sigma \in F^f_{ar\to s}$, $\overline{t_1} \in T^{\Delta}_{k,ar}$ *and* $s \in S^f$]

1	$[t] = \sigma([k_1]; [\overline{t'_1}])$	since $[\_]$ is a homomorphism
2	$t_2 = \sigma(k_2; \overline{t'_2})$ for some $\overline{t'_2} \in T^{\Delta}_{k_2,ar}$	since $\sigma([k_1]; [\overline{t'_1}]) = [t']$ can be regarded as equality of trees

3	$[\overline{t'_1}] = [\overline{t'_2}]$	since $\sigma([\overline{t'_1}]) = \sigma([\overline{t'_2}])$
4	$\Gamma \vdash \overline{t'_1} =_{k_1} \delta_{k_2/k_1}(\overline{t'_2})$	by the induction hypothesis
5	$\Gamma \vdash \sigma(k_1; \overline{t'_1}) =_{k_1} \sigma(k_1; \delta_{k_2/k_1}(\overline{t'_2}))$	by (F^f)
6	$\Gamma \vdash \sigma(k_1; \overline{t'_1}) =_{k_1} \delta_{k_2/k_1}(\sigma(k_2; \overline{t'_2}))$	by the definition of δ_{k_2/k_1} $\square$

Proof (of Proposition 14: A set of equations generates a congruence on a reachable Kripke model if it entails all the equations satisfied by that model).

In regard to the characterization of $\equiv$ (as per the item P1), the 'if' part follows immediately by the very definition of $\equiv$. Therefore, we focus on the 'only if' part. To that end, suppose $[t_1] \equiv_{[k]} [t_2]$. It follows that:

1	$\Gamma \vdash t'_1 =_{k'} t'_2$ for some nominal k' and terms $t'_1, t'_2 \in T^{\Delta}_{k',s}$ such that $[k] = [k']$ and $[t_i] = [t'_i]$	by the definition of $\equiv$
2	$\Gamma \vdash k = k'$	since $(W, M) \models k = k'$
3	$\Gamma \vdash \delta_{k'/k}(t'_1) =_k \delta_{k'/k}(t'_2)$	by (W^f)
4	$[t_i] = [t'_i] = [\delta_{k'/k}(t'_i)]$	since $[k] = [k']$
5	$\Gamma \vdash t_i =_k \delta_{k'/k}(t'_i)$	$(W, M) \models t_i = \delta_{k'/k(t'_i)}$
6	$\Gamma \vdash t_1 =_k t_2$	from 5 and 3, by (T^h)

Let us now show that $\equiv$ is a Δ-congruence on (W, M). For each nominal k, the reflexivity, symmetry, and transitivity of $\equiv_{[k]}$ are straightforward consequences of the proof rules (R^h), (S^h) and (T^h), of the characterization given at P1, and of the fact that (W, M) is reachable. For instance, in regard to the reflexivity of $\equiv_{[k]}$, for every $\tau \in M_{[k],s}$ we know there exists a term $t \in T^{\Delta}_{k,s}$ such that $\tau = [t]$. By (R^h), we have $\Gamma \vdash t = t$, which implies, by P1, that $\tau \equiv_{[k]} \tau$.

For the compatibility of $\equiv$ with the operations in F, assume that $\sigma \in F_{ar \to s}$ and $\overline{\tau_1}, \overline{\tau_2} \in M_{[k],ar}$ such that $\overline{\tau_1} \equiv_{[k],ar} \overline{\tau_2}$. There is no significant distinction between the case where σ is rigid and the case where σ is flexible. Therefore, we choose to focus on the latter case, corresponding to $\sigma \in F^{\mathsf{f}}_{ar \to s}$. We have:

1	$\Gamma \vdash \overline{t_1} =_{k,ar} \overline{t_2}$ for some tuples of terms $\overline{t_i} \in T^{\Delta}_{k,ar}$ such that $\overline{\tau_i} = [\overline{t_i}]$	by P1
2	$\Gamma \vdash \sigma(k; \overline{t_1}) =_{k,s} \sigma(k; \overline{t_2})$	by the proof rule (F^f)
3	$[\sigma(k; \overline{t_1})] \equiv_{[k],s} [\sigma(k; \overline{t_2})]$	by the definition of $\equiv_{[k]}$
4	$M_{[k],\sigma}(\overline{\tau_1}) \equiv_{[k],s} M_{[k],\sigma}(\overline{\tau_2})$	since $[\sigma(k; \overline{t_1})] = M_{[k],\sigma}(\overline{\tau_1})$

It remains to check that $(\equiv_{[k],s}) = (\equiv_{[k'],s})$ for all nominals $k, k' \in T_{\Sigma^n}$ and all rigid sorts $s \in S^r$. This follows easily from P1 and the proof rule (W^r). $\square$

C Proofs for results presented in Section 5

Proof (of Proposition 18: Atomic completeness extends to action completeness).

Let Γ be a set of atomic sentences over Δ, and (W^{Γ}, M^{Γ}) an initial reachable model of Γ as in Theorem 15. Building on that result, it suffices to prove that

$$\Gamma \vdash \mathfrak{a}(k_1, k_2) \quad \text{iff} \quad (W^{\Gamma}, M^{\Gamma}) \models \mathfrak{a}(k_1, k_2)$$

for all actions $\mathfrak{a} \in A^n$ and all nominals $k_1, k_2 \in T_{\Sigma^n}$. As in the case of atomic completeness, the 'only if' part follows with ease from the soundness of $\vdash$ and the fact that (W^Γ, M^Γ) is a model of Γ. Therefore, we only give a detailed account for the 'if' part of the statement.

We prove the result by induction on the structure of the action $\mathfrak{a}$. For the base case, which corresponds to the fact that $\mathfrak{a}$ is a modality, the conclusion follows by Theorem 15. Hence, it suffices to analyse the induction steps that correspond to the *composition*, *union*, and *transitive-closure* operators.

[$(W^\Gamma, M^\Gamma) \models (\mathfrak{a}_1 \mathbin{;} \mathfrak{a}_2)(k_1, k_2)$] It follows that:

1	$([k_1], [k_2]) \in W^\Gamma_{\mathfrak{a}_1 ; \mathfrak{a}_2}$	by the definition of $\models$
2	$([k_1], [k_2]) \in W^\Gamma_{\mathfrak{a}_1} ; W^\Gamma_{\mathfrak{a}_2}$	since $W^\Gamma_{\mathfrak{a}_1 ; \mathfrak{a}_2} = W^\Gamma_{\mathfrak{a}_1} ; W^\Gamma_{\mathfrak{a}_2}$
3	$([k_1], [k]) \in W^\Gamma_{\mathfrak{a}_1}$ and $([k], [k_2]) \in W^\Gamma_{\mathfrak{a}_2}$ for some nominal $k \in T_{\Sigma^n}$	since (W^Γ, M^Γ) is reachable
4	$(W^\Gamma, M^\Gamma) \models \mathfrak{a}_1(k_1, k), \mathfrak{a}_2(k, k_2)$	by the definition of $\models$
5	$\Gamma \vdash \mathfrak{a}_1(k_1, k)$ and $\Gamma \vdash \mathfrak{a}_2(k, k_2)$	by the ind. hypothesis
6	$\Gamma \vdash (\mathfrak{a}_1 ; \mathfrak{a}_2)(k_1, k_2)$	by (Comp)

[$(W^\Gamma, M^\Gamma) \models (\mathfrak{a}_1 \cup \mathfrak{a}_2)(k_1, k_2)$] It follows that:

1	$([k_1], [k_2]) \in W^\Gamma_{\mathfrak{a}_1 \cup \mathfrak{a}_2}$	by the definition of $\models$
2	$([k_1], [k_2]) \in W^\Gamma_{\mathfrak{a}_i}$ for some $i \in \{1, 2\}$	since $W^\Gamma_{\mathfrak{a}_1 \cup \mathfrak{a}_2} = W^\Gamma_{\mathfrak{a}_1} \cup W^\Gamma_{\mathfrak{a}_2}$
3	$(W^\Gamma, M^\Gamma) \models \mathfrak{a}_i(k_1, k_2)$ for some $i \in \{1, 2\}$	by the definition of $\models$
4	$\Gamma \vdash \mathfrak{a}_i(k_1, k_2)$ for some $i \in \{1, 2\}$	by the ind. hypothesis
5	$\Gamma \vdash (\mathfrak{a}_1 \cup \mathfrak{a}_2)(k_1, k_2)$	by (Union)

[$(W^\Gamma, M^\Gamma) \models \mathfrak{a}^*(k_1, k_2)$] By the definition of the satisfaction of $\mathfrak{a}^*(k_1, k_2)$, it follows that $([k_1], [k_2]) \in (W^\Gamma_{\mathfrak{a}})^n$ for some $n \in \mathbb{N}$. If $n = 0$, then:

1	$[k_1] = [k_2]$	since $(W^\Gamma_{\mathfrak{a}})^0 = id_{\lvert W^\Gamma \rvert}$
2	$(W^\Gamma, M^\Gamma) \models k_1 = k_2$	by the definition of $\models$
3	$\Gamma \vdash k_1 = k_2$	by Theorem 15
4	$\Gamma \vdash \mathfrak{a}^*(k_1, k_2)$	by (Refl)

On the other hand, if $n > 0$, then there exist nominals $\{l_i\}_{0 \leqslant i \leqslant n}$ such that $l_0 = k_1$, $l_n = k_2$, and $([l_i], [l_{i+1}]) \in W^\Gamma_{\mathfrak{a}}$ for all $0 \leqslant i < n$. It follows that:

1	$(W^\Gamma, M^\Gamma) \models \mathfrak{a}(l_i, l_{i+1})$ for all $0 \leqslant i < n$	by the definition of $\models$
2	$\Gamma \vdash \mathfrak{a}(l_i, l_{i+1})$ for all $0 \leqslant i < n$	by the ind. hypothesis
3	$\Gamma \vdash \mathfrak{a}^*(l_0, l_n)$	by (Star)
4	$\Gamma \vdash \mathfrak{a}^*(k_1, k_2)$	since $l_0 = k_1$, $l_n = k_2$ $\square$

Proof (of Theorem 19: Initiality preserves formal deductions).

In what follows, we continue the case analysis on page 13 for the proof of Theorem 19. The case that corresponds to the *necessity* operator is already discussed in Section 5. Therefore, we focus on the remaining five cases.

[$\Gamma \vdash @_k @_{k'} \gamma$] The conclusion follows from the list of inferences below:

1	$\Gamma \vdash @_{k'} \gamma$	by ($\mathsf{Ret}_@$) in Figure 3
2	$(W^{\Gamma_0}, M^{\Gamma_0}) \models @_{k'} \gamma$	by the induction hypothesis
3	$(W^{\Gamma_0}, M^{\Gamma_0}) \models @_k @_{k'} \gamma$	by Fact 17

[$\Gamma \vdash @_k (\bigwedge H \Rightarrow \gamma)$] What we need to show is that $(W^{\Gamma_0}, M^{\Gamma_0}) \models^w H$ implies $(W^{\Gamma_0}, M^{\Gamma_0}) \models^w \gamma$, where $w = W^{\Gamma_0}_k$. So, suppose $(W^{\Gamma_0}, M^{\Gamma_0}) \models^w H$. Then:

1	$(W^{\Gamma_0}, M^{\Gamma_0}) \models H$	by Fact 5
2	$\Gamma_0 \vdash H$	by Proposition 18
3	$\Gamma_f \vdash H$ for some finite subset $\Gamma_f \subseteq \Gamma_0$	since $\vdash$ is compact
4	$\Gamma \vdash \Gamma_f$	by the union property of $\vdash$
5	$\Gamma \vdash H$	from 4 and 3, by transitivity
6	$\Gamma \cup H \vdash @_k \gamma$	by ($\mathsf{Imp_E}$)
7	$\Gamma \vdash @_k \gamma$	from 5 and 6, by *cut*
8	$(W^{\Gamma_0}, M^{\Gamma_0}) \models @_k \gamma$	by the induction hypothesis
9	$(W^{\Gamma_0}, M^{\Gamma_0}) \models^w \gamma$	by the definition of $\models$

[$\Gamma \vdash @_k \downarrow z \cdot \gamma$] We need to show that $(W^{\Gamma_0}, M^{\Gamma_0}) \models^w \downarrow z \cdot \gamma$, where $w = W^{\Gamma_0}_k$. To that end, we proceed as follows:

1	$\Gamma \vdash @_k \theta_{z\leftarrow k}(\gamma)$	by ($\mathsf{Store_E}$)
2	$(W^{\Gamma_0}, M^{\Gamma_0}) \models @_k \theta_{z\leftarrow k}(\gamma)$	by the induction hypothesis
3	$(W^{\Gamma_0}, M^{\Gamma_0}) \models^w \theta_{z\leftarrow k}(\gamma)$	by the definition of $\models$
4	$(W^{\Gamma_0}, M^{\Gamma_0})\restriction_{\theta_{z\leftarrow k}} \models^w \gamma$	by the local sat. cond. for $\theta_{z\leftarrow k}$
5	$(W^{\Gamma_0}, M^{\Gamma_0})^{z\leftarrow w} \models^w \gamma$	by Fact 8
6	$(W^{\Gamma_0}, M^{\Gamma_0}) \models^w \downarrow z \cdot \gamma$	by the definition of $\models$

[$\Gamma \vdash @_k \forall X \cdot \gamma$] Let $w = W^{\Gamma_0}_k$. We want to show that $(W, M) \models^w \gamma$ for any $\Delta[X]$-expansion (W, M) of $(W^{\Gamma_0}, M^{\Gamma_0})$. Therefore, consider one such expansion. Since the model $(W^{\Gamma_0}, M^{\Gamma_0})$ is reachable, by Proposition 9, there exists a substitution $\theta\colon X \to \varnothing$ such that $(W^{\Gamma_0}, M^{\Gamma_0})\restriction_\theta = (W, M)$. It follows that:

1	$\Gamma \vdash @_k \theta(\gamma)$	by ($\mathsf{Subst_q}$)
2	$(W^{\Gamma_0}, M^{\Gamma_0}) \models @_k \theta(\gamma)$	by the induction hypothesis
3	$(W^{\Gamma_0}, M^{\Gamma_0}) \models^w \theta(\gamma)$	by the definition of $\models$
4	$(W^{\Gamma_0}, M^{\Gamma_0})\restriction_\theta \models^w \gamma$	by the local sat. cond. for θ
5	$(W, M) \models^w \gamma$	since $(W^{\Gamma_0}, M^{\Gamma_0})\restriction_\theta = (W, M)$

[$\Gamma \vdash @_k (o)\, \gamma$] It suffices to show that $(W^{\Gamma_0}, M^{\Gamma_0}) \models^w \gamma$, where $w = W^{\Gamma_0}_{o(k)}$.

1	$\Gamma \vdash @_{o(k)} \gamma$	by ($\mathsf{Next_E}$)
2	$(W^{\Gamma_0}, M^{\Gamma_0}) \models @_{o(k)} \gamma$	by the induction hypothesis
3	$(W^{\Gamma_0}, M^{\Gamma_0}) \models^w \gamma$	since $w = W^{\Gamma_0}_{o(k)}$ $\square$

Proof (of Corollary 20: The initial model of a set of Horn clauses encapsulates the formal deduction of both atomic sentences and action relations).

Let Γ be a set of Horn clauses over Δ. *For the first part of the corollary*, recall that, by Theorem 15, $(W^{\Gamma_0}, M^{\Gamma_0})$ is a reachable initial model of Γ_0. By soundness, $\Gamma \models \Gamma_0$. Therefore, for every model (W, M) of Γ, we have $(W, M) \models \Gamma_0$. By the initiality property of $(W^{\Gamma_0}, M^{\Gamma_0})$, we obtain a unique homomorphism $(W^{\Gamma_0}, M^{\Gamma_0}) \to (W, M)$. All this means that, in order to prove that $(W^{\Gamma_0}, M^{\Gamma_0})$ is an initial model of Γ, it suffices so show that $(W^{\Gamma_0}, M^{\Gamma_0}) \models \Gamma$. To that end, let $\gamma \in \Gamma$ and $w \in |W^{\Gamma_0}|$.

1	$w = [k]$ for some nominal $k \in T_{\Sigma^n}$	since $(W^{\Gamma_0}, M^{\Gamma_0})$ is reachable
2	$\Gamma \vdash @_k\, \gamma$	by $(\mathsf{Ret_I})$, since $\Gamma \vdash \gamma$
3	$(W^{\Gamma_0}, M^{\Gamma_0}) \models @_k\, \gamma$	by Theorem 19
4	$(W^{\Gamma_0}, M^{\Gamma_0}) \models^w \gamma$	by the definition of $\models$

For the second part of the corollary, we proceed as follows:

[*1* $\Rightarrow$ *2*] From the definition of semantic entailment, since $(W^{\Gamma_0}, M^{\Gamma_0}) \models \Gamma$.
[*2* $\Rightarrow$ *3*] Assume that $(W^{\Gamma_0}, M^{\Gamma_0})$ satisfies ρ. In that case, we obtain:

1	$\Gamma_0 \vdash \rho$	by Proposition 18
2	$\Gamma_f \vdash \rho$ for some finite $\Gamma_f \subseteq \Gamma_0$	since $\vdash$ is compact
3	$\Gamma \vdash \Gamma_f$	by the union property of $\vdash$
4	$\Gamma \vdash \rho$	from 3 and 2, by transitivity

[*3* $\Rightarrow$ *1*] By Proposition 16. □

D Proofs for results presented in Section 6

The following fact forms a semantic basis for Lemma 26 (also presented below), which shows how the completeness of any of the entailment relations $\vdash^{(x)}$ can be generalized to situations where action relations can be used as premises for formal reasoning. Lemma 26 has an important role in dealing with implications and with the *necessity* operator in the proof of Theorem 23.

Fact 25. In HDFOLS, for each set Γ of sentences and any sentence γ, we have:

1. $\Gamma \cup \{(\mathfrak{a}_1 \mathbin{;} \mathfrak{a}_2)(k_1, k_2)\} \models_\Delta \gamma$ implies $\Gamma \cup \{\mathfrak{a}_1(k_1, z), \mathfrak{a}_2(z, k_2)\} \models_{\Delta[z]} \gamma$;
2. $\Gamma \cup \{(\mathfrak{a}_1 \cup \mathfrak{a}_2)(k_1, k_2)\} \models \gamma$ implies $\Gamma \cup \{\mathfrak{a}_i(k_1, k_2)\} \models \gamma$ for both $i \in \{1, 2\}$;
3. $\Gamma \cup \{\mathfrak{a}^*(k_1, k_2)\} \models \gamma$ implies $\Gamma \cup \{\mathfrak{a}^n(k_1, k_2)\} \models \gamma$ for all $n \in \mathbb{N}$.

Lemma 26. *Let $x \in \{1, 2, 3\}$, and consider a finite set H of action relations and a Horn clause γ in* $\mathsf{HDFOLS}^{(x)}$ *such that $\Gamma \models \gamma$ implies $\Gamma \vdash^{(x)} \gamma$ for all sets Γ of Horn clauses in* HDFOLS.[10] *Then $\Gamma \cup H \models \gamma$ implies $\Gamma \cup H \vdash^{(x)} \gamma$.*

Proof. We prove the statement by well-founded (Noetherian) induction on (m, n), where m is the total number of occurrences of the *transitive-closure* operator in H, and n is the total number of occurrences of the *composition* or *union* operators in H. We also denote these numbers by $ntc(H)$ and $ncu(H)$, respectively.

Notice that:

[10] Note that, if $x = 1$, then H consists only of nominal relations.

- $ntc(H) = ncu(H) = 0$ when H is a set of action relations in $\mathsf{HDFOLS}^{(1)}$;
- $ntc(H) = 0$ when H is a set of action relations in $\mathsf{HDFOLS}^{(2)}$.

For the base case, corresponding to $ntc(H) = ncu(H) = 0$, we have that H consists only of atomic sentences. Therefore, $\Gamma \cup H$ is a set of clauses, which means that we can use the hypothesis on γ to infer $\Gamma \cup H \vdash^{(x)} \gamma$.

For the induction step, suppose that the statement holds for all sets H as above such that $(ntc(H), ncu(H)) < (m, n)$, in lexicographic order; and let H be a finite subset of action relations in $\mathsf{HDFOLS}^{(x)}$ for which $(0, 0) < (ntc(H), ncu(H)) = (m, n)$,[11] Γ' an arbitrary set of Horn clauses in HDFOLS, and γ a Horn clause in $\mathsf{HDFOLS}^{(x)}$ such that:

H1. $\Gamma \models \gamma$ implies $\Gamma \vdash^{(x)} \gamma$ for all sets Γ of Horn clauses, and
H2. $\Gamma' \cup H \models \gamma$.

It follows that $H = H' \cup \{\mathfrak{a}(k_1, k_2)\}$, where $\mathfrak{a}(k_1, k_2)$ is a non-atomic action relation. We proceed by case analysis on the topmost action operator in $\mathfrak{a}$.

[$\mathfrak{a} = \mathfrak{a}_1 \mathbin{;} \mathfrak{a}_2$] In this case, we have:

1	$\Gamma' \cup H' \cup \{(\mathfrak{a}_1 \mathbin{;} \mathfrak{a}_2)(k_1, k_2)\} \models_\Delta \gamma$	by the hypothesis H2
2	$\Gamma' \cup H' \cup \{\mathfrak{a}_1(k_1, z), \mathfrak{a}_2(z, k_2)\} \models_{\Delta[z]} \gamma$	by Fact 25
3	$\Gamma' \cup \underbrace{H' \cup \{\mathfrak{a}_1(k_1, z), \mathfrak{a}_2(z, k_2)\}}_{H''} \vdash^{(x)}_{\Delta[z]} \gamma$	by the ind. hypothesis, since $ntc(H'') = m$, $ncu(H'') < n$
4	$\Gamma' \cup H' \cup \{(\mathfrak{a}_1 \mathbin{;} \mathfrak{a}_2)(k_1, k_2)\} \vdash^{(x)}_\Delta \gamma$	by $(\mathsf{Comp_I})$

[$\mathfrak{a} = \mathfrak{a}_1 \cup \mathfrak{a}_2$] In this case, we have:

1	$\Gamma' \cup H' \cup \{(\mathfrak{a}_1 \cup \mathfrak{a}_2)(k_1, k_2)\} \models \gamma$	by the hypothesis H2
2	$\Gamma' \cup H' \cup \{\mathfrak{a}_i(k_1, k_2)\} \models \gamma$ for both $i \in \{1, 2\}$	by Fact 25
3	$\Gamma' \cup \underbrace{H' \cup \{\mathfrak{a}_i(k_1, k_2)\}}_{H''_i} \vdash^{(x)} \gamma$ for $i \in \{1, 2\}$	by the ind. hypothesis, since $ntc(H''_i) = m$, $ncu(H''_i) < n$
4	$\Gamma' \cup H' \cup \{(\mathfrak{a}_1 \cup \mathfrak{a}_2)(k_1, k_2)\} \vdash^{(x)} \gamma$	by $(\mathsf{Union_I})$

[$\mathfrak{a} = \mathfrak{a}_1^*$] In this case, we have:

1	$\Gamma' \cup H' \cup \{\mathfrak{a}_1^*(k_1, k_2)\} \models \gamma$	by the hypothesis H2
2	$\Gamma' \cup H' \cup \{\mathfrak{a}_1^p(k_1, k_2)\} \models \gamma$ for all $p \in \mathbb{N}$	by Fact 25
3	$\Gamma' \cup \underbrace{H' \cup \{\mathfrak{a}_1^p(k_1, k_2)\}}_{H''} \vdash^{(x)} \gamma$ for all $p \in \mathbb{N}$	by the ind. hypothesis, since $ntc(H'') < m$
4	$\Gamma' \cup H' \cup \{\mathfrak{a}_1^*(k_1, k_2)\} \vdash^{(x)} \gamma$	by $(\mathsf{Star_I})$ □

Proof (proof of Theorem 23: Birkhoff completeness).

In what follows, we continue the case analysis on page 15 for the proof of Theorem 23. The case that corresponds to universally quantified sentences is already discussed in Section 6. Therefore, we focus on the remaining five cases.

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[11] Note that the ind. step is vacuously true for $\vdash^{(1)}$, since no H in $\mathsf{HDFOLS}^{(1)}$ satisfies $(0, 0) < (ntc(H), ncu(H))$. A similar observation holds for $\vdash^{(2)}$ and the case $\mathfrak{a} = \mathfrak{a}_1^*$.

[$\Gamma \models @_k\, @_{k_1}\, \gamma$] Then:

1	$\Gamma \models @_{k_1}\, \gamma$	by Fact 17
2	$\Gamma \vdash^{(x)} @_{k_1}\, \gamma$	by the induction hypothesis
3	$\Gamma \vdash^{(x)} @_k\, @_{k_1}\, \gamma$	by Fact 17

[$\Gamma \models @_k\, (\bigwedge H \Rightarrow \gamma)$] Then:

1	$\Gamma \cup H \models @_k\, \gamma$	by the general properties of $\models$
2	$\Gamma \cup H \vdash^{(x)} @_k\, \gamma$	by ind. hypothesis and Lemma 26
3	$\Gamma \vdash^{(x)} @_k\, (\bigwedge H \Rightarrow \gamma)$	by $(\mathsf{Imp_I})$

[$\Gamma \models @_k \downarrow z \cdot \gamma$] Let $\theta_{z \leftarrow k} \colon \{z\} \to \varnothing$ be the Δ-substitution that maps the nominal variable z to the (ground) term k. Then:

1	$\Gamma \models @_k\, \theta_{z \leftarrow k}(\gamma)$	since $@_k \downarrow z \cdot \gamma \mathrel{\models\!\!\!\mid} @_k\, \theta_{z \leftarrow k}(\gamma)$
2	$\Gamma \vdash^{(x)} @_k\, \theta_{z \leftarrow k}(\gamma)$	by the induction hypothesis
3	$\Gamma \vdash^{(x)} @_k \downarrow z \cdot \gamma$	by $(\mathsf{Store_I})$

[$\Gamma \models @_k\, [\mathfrak{a}]\gamma$] Then:

1	$\Gamma \cup \{\mathfrak{a}(k, z)\} \models_{\Delta[z]} @_z\, \gamma$	by the general properties of $\models$
2	$\Gamma \cup \{\mathfrak{a}(k, z)\} \vdash^{(x)}_{\Delta[z]} @_z\, \gamma$	by ind. hypothesis and Lemma 26
3	$\Gamma \vdash^{(x)}_{\Delta} @_k\, [\mathfrak{a}]\gamma$	by $(\mathsf{Nec_I})$

[$\Gamma \models @_k\, (o)\, \gamma$] Then:

1	$\Gamma \models @_{o(k)}\, \gamma$	since $@_k\, (o)\, \gamma \mathrel{\models\!\!\!\mid} @_{o(k)}\, \gamma$
2	$\Gamma \vdash^{(x)} @_{o(k)}\, \gamma$	by the induction hypothesis
3	$\Gamma \vdash^{(x)} @_k\, (o)\, \gamma$	by $(\mathsf{Next_I})$ ☐

Proof (of Proposition 24: HDCLS *is not compact).*

By the definition of compactness, it suffices to find a signature Δ, a set Γ of Horn clauses over Δ, and a clause γ such that $\Gamma \models \gamma$, but $\Gamma_f \not\models \gamma$ for all finite subsets $\Gamma_f \subseteq \Gamma$. Therefore, let Δ be the signature that consists of only two nominal constants, k and k', and only two modalities, λ and α, and let $\Gamma = \{\lambda^n(k, k') \Rightarrow \alpha(k, k') \mid n \in \mathbb{N}\}$. We show that $\Gamma \models \lambda^*(k, k') \Rightarrow \alpha(k, k')$ and $\Gamma_f \not\models \lambda^*(k, k') \Rightarrow \alpha(k, k')$ for all finite subsets $\Gamma_f \subseteq \Gamma$.

The first part follows easily from the definition of the satisfaction relation. For the second part, let Γ_f be a finite subset of Γ, and $n \in \mathbb{N}$ the largest natural number such that the sentence $\lambda^n(k, k') \Rightarrow \alpha(k, k')$ belongs to Γ_f.

Now consider a Δ-model (W, M) that has $n + 1$ possible worlds – that is, $|W| = \{w_1, \ldots, w_{n+1}\}$ – and interprets k, k', λ and α as follows: $W_k = w_0$, $W_{k'} = w_{n+1}$, $W_\lambda = \{(w_i, w_{i+1}) \mid 1 \leqslant i \leqslant n\}$, and $W_\alpha = \varnothing$.

In other words, we can regard (W, M) as a chain $w_1 \xrightarrow{\lambda} w_2 \xrightarrow{\lambda} w_3 \cdots w_n \xrightarrow{\lambda} w_{n+1}$. It is easy to check that (W, M) is a model of Γ_f (because it does not satisfy $\lambda^i(k, k')$, for any $1 \leqslant i \leqslant n$) and that it does not satisfy $\lambda^*(k, k') \Rightarrow \alpha(k, k')$. ☐