

Statistical diagonalization of a random biased Hamiltonian: the case of the eigenvectors

Grégoire Ithier and Saeed Ascroft

Department of Physics, Royal Holloway, University of London, United Kingdom

We present a non perturbative calculation technique providing the mixed moments of the Green functions of a random biased Hamiltonian $\hat{H} = \hat{H}_0 + \hat{W}$, where \hat{H}_0 is fixed, chosen arbitrarily and \hat{W} is random unbiased. These moments give access to the statistics of the overlaps between eigenvectors of \hat{H}_0 and eigenvectors of \hat{H} . In the particular case of a Gaussian \hat{W} , we apply this method to calculate the mean Green functions and recover the associated second order moments of the overlaps or Local Density of States. Then we calculate the correlations between Green functions and the associated fourth order moments of the eigenvectors overlaps in the same setting. Such quantities are crucial for understanding the local out of equilibrium dynamics as well as the local stationary states of a large composite quantum system. In this case, \hat{H}_0 is the sum of the Hamiltonians of the system subparts and \hat{W} is an interaction term. We test our predictions with numerical simulations.

I. INTRODUCTION

How are the eigenvalues and eigenvectors of an hermitian matrix or Hamiltonian \hat{H}_0 modified by the addition of another hermitian matrix \hat{W} ? This question is central in many areas of science: e.g. in physics for the quantum many body problem[1–3], quantum chaos[4, 5] and thermalisation[6–8], the Anderson localization problem[9–11], in signal processing for telecommunications and time series analysis[12–14], to name a few. Perturbation theory provides a *deterministic* answer, i.e. for given \hat{H}_0 and \hat{W} , for both the spectrum and the eigenvectors, and provided the typical strength of \hat{W} is much smaller than the minimum level spacing of \hat{H}_0 . This approach has been the main focus in physics for a long time, however many interesting phenomena are non perturbative in nature, e.g. superconductivity or the fractional quantum Hall effect. On the non perturbative side, the Bethe Ansatz[15] provides some exact diagonalization results but only for specific classes of Hamiltonians, see e.g., [16–18]. If one focuses on the particular problem of finding only the spectrum of $\hat{H}_0 + \hat{W}$ for *arbitrary* given matrices \hat{H}_0 and \hat{W} , this is a very difficult problem [19–22]. However, getting a *probabilistic* answer for some classes of random matrices (e.g. for W) is possible and not less satisfactory for the physicist looking for typical properties. For instance, Dyson’s brownian motion[23] provides the spectral properties of $\hat{H}_0 + \hat{W}$ but only when the matrix \hat{W} is random with identically distributed Gaussian entries (up to the hermitian or real symmetry). More recently, first order “free” probability theory has also provided a rigorous probabilistic answer regarding the global spectral properties of $\hat{H}_0 + \hat{W}$ and for larger classes of matrices[24–26], namely matrices in generic position with one another[27]. Some rigorous results on the *local* spectral properties of $\hat{H}_0 + \hat{W}$ given by the second order statistics (i.e. the correlation functions) have been obtained using Random Matrix Theory tools [28] and the concept of second order “freeness” [29–31] also seems promising to deeply understand how correlation functions combine together when summing large matrices. On the question of the eigenvectors, much less work has been done (see however [12–14, 32–35]), and the natural question is: what is the statistics of the overlaps (or scalar products) between eigenvectors of \hat{H}_0 and eigenvectors of $\hat{H}_0 + \hat{W}$.

In this article, we present a non perturbative method for calculating the mixed moments of these overlaps under generic assumptions on a deterministic \hat{H}_0 and a random \hat{W} . This method is inspired from[36] where the authors considered moments of traces of the resolvent operator without source term (i.e. $\hat{H}_0 = 0$) and was also used in [37] for investigating the behavior of eigenvalues under additive matrix deformation. The method is approximate and reminiscent of the loop equations (or Schwinger-Dyson equations) of the diagram technique in Quantum Field Theory [38–40]. It consists in finding a self consistent approximate solution to a set of algebraic equation verified by the mixed moments of the Green functions[41]. Compared to other methods used for quantitatively describing the behavior of eigenvectors under matrix addition, e.g. replica trick[13, 14], supersymmetric formalism[42–45], or flow equation method[12, 34, 46], the method is technically less involved. It relies on the fact that some quantities involved in the calculation are the subject of “measure concentration” (see, e.g., Ref.[47–49]), in other words they are self-averaging which allows to spot easily the main contribution in order to derive and solve the approximate self-consistent equations. In addition, this technique does not require rotational invariance of the probability distribution of the additive term and could be used to investigate the effect of correlation between entries, of a non Gaussian statistics or of a band structure. Theses cases are relevant in many contexts, e.g. for the empirical covariance estimation problem[12] or for a two body interaction in physical systems[50, 51]. As far as the second and fourth order moments of the overlaps in the Gaussian \hat{W} case are concerned, the main application we will have in mind regards the dynamics of quantum

systems, and in particular the process of thermalisation[8, 52]. However, our method is general and could be used to study different statistics on \hat{W} relevant for modelling disordered and interacting quantum systems.

This paper is organized as follows: In Sec.II, we present our calculation framework and illustrate our method with the second order moment calculation by recovering the results of the GOE and GUE cases for \hat{W} with an arbitrary \hat{H}_0 . Then in Sec.III, we focus on the fourth order statistics, i.e. the most important quantity for understanding the out of equilibrium dynamics of embedded quantum systems (as we will explain later), and expose the main result of this paper in the case of a Gaussian unitary distributed \hat{W} (GUE) with arbitrary \hat{H}_0 .

II. ANALYTICAL FRAMEWORK FOR CALCULATING THE MOMENTS OF THE OVERLAP COEFFICIENTS

In this section, we introduce our framework and apply it to the calculation of the second order statistics of the overlaps coefficients. We recall several tools useful for our calculation: the link between the overlap coefficients and the resolvent of the total Hamiltonian, the expansion of a resolvent matrix entry, i.e a Green function (GF), as a function of the interaction Hamiltonian (in Sec.II B 3) and a so called "decoupling" formula (in Sec.II B 4). Using these tools we recover the well known result for the first order statistics of the GFs in the case of a Gaussian interaction. We infer the second order statistics of the overlaps in Sec.II B 7 and compare the analytical prediction to the results of numerical simulations on Fig.1.

A. Hypotheses

1. Decomposing \hat{H} in two parts: \hat{H}_0 and \hat{W}

Following the same notations as in [8, 52], we consider a "bare" Hermitian matrix \hat{H}_0 with eigenvectors $\{|\phi_1\rangle, \dots, |\phi_N\rangle\}$ and respective eigenvalues $\epsilon_1, \dots, \epsilon_N$, and a "dressed" matrix $\hat{H} = \hat{H}_0 + \hat{W}$ (with \hat{W} Hermitian) with eigenvectors $\{|\psi_1\rangle, \dots, |\psi_N\rangle\}$ and eigenvalues $\{\lambda_1, \dots, \lambda_N\}$. Such a separation of the total matrix \hat{H} in two parts is natural (but not unique) in several contexts, e.g. when modelling disordered quantum systems[11, 53] like metals or many body interacting quantum systems like heavy nuclei[54, 55] or atoms[56, 57]. An important focus in mesoscopic physics today is to incorporate *both* disorder and interactions and try to understand their interplay, e.g. when studying the many body localization problem[58–61]. Such separation is also relevant for modeling the effect of noise when considering the empirical estimation of a matrixial quantity, e.g. some covariance between time series, with a relatively small data set[12–14]. As we mainly have physical applications in mind, we will call in the following the matrices \hat{H}_0, \hat{H} the bare and dressed Hamiltonians and \hat{W} the interaction.

2. Introducing randomness

The quantities of interest regarding how the eigenvectors of \hat{H}_0 are modified by the addition of the extra term \hat{W} are the overlap coefficients or scalar products: $\langle \phi_n | \psi_i \rangle$. These coefficients define the transition matrix from the bare basis to the dressed one which physically speaking tells how a bare eigenvector $|\phi_n\rangle$ is *hybridized* with the dressed eigenvectors. Outside the perturbative limit, analytical calculation of the quantities $\langle \phi_n | \psi_i \rangle$ have proved to be difficult for a *deterministic* \hat{W} . Now therefore if physical Hamiltonians should be considered a priori like fully deterministic matrices, it can be interesting to introduce some level of randomness in the modeling. This was Wigner's original idea when considering the nucleus Hamiltonian like a random matrix. Proceeding this way, he was making a crucial step from the usual statistical physics approach where randomness is introduced on the state of a system[62] towards a new statistical physics where randomness is now introduced on the *nature* of the system itself. This radical change of point of view was originally justified on heuristic grounds: in practice, physically relevant quantities, i.e. accessible by experiments, do not depend much on the details of the realization of the disorder associated to the randomness. These physical observables seem to take *typical* values dependent only on some conditions constraining randomness and summarizing its macroscopic properties, e.g., symmetry class: hermitian or real symmetric, spectral variance, the possibility of a block diagonal structure. Recently, the typicality of the quantum dynamics[52] provided a rigorous ground for justifying such introduction of a controlled amount of randomness in the modeling of a quantum system, in this case in the interaction Hamiltonian \hat{W} between a system S (Hamiltonian \hat{H}_s) and its environment (Hamiltonian \hat{H}_e). This typicality property states that the reduced density matrix of S , i.e. the state of S , considered as a function $f(\hat{W})$ of a random interaction \hat{W} (either Wigner band random matrix or a Randomly Rotated Matrix, see [52]), for

all other parameters fixed (initial state $\varrho(0)$, densities of states of \hat{H}_0 and \hat{W}), exhibits a generalized central limit theorem phenomenon known as the ‘‘concentration of measure’’ [47, 48]:

$$f(\hat{W}) = \varrho_s(t) = \text{Tr}_e(\varrho(t)) = \text{Tr}_e\left(U_t \varrho(0) U_t^\dagger\right) \text{ where } U_t = e^{-iHt} \text{ and } \hat{H} = \hat{H}_s + \hat{H}_e + \hat{W},$$

is such that the variance of f with respect to the probability measure on \hat{W} verifies

$$\sigma_f^2 \leq \frac{4\sigma_w^2 t^2}{\hbar^2} \frac{1}{\dim \mathcal{H}_e},$$

where $\sigma_w^2 = \text{Tr}(\hat{W} \cdot \hat{W}^\dagger)/N$, the spectral variance of \hat{H} is assumed to be fixed, independent of N . As a consequence, when $\dim \mathcal{H}_e \rightarrow \infty$, $\sigma_f^2 \rightarrow 0$ and the embedded system S follows a typical dynamics given by $\mathbb{E}[\varrho_s(t)]$ [52].

To illustrate simply this phenomenon, one can consider some experiment whose measurement outcome X is blurred by a random noise. It is then common practice to repeat the measurement n times in stationary conditions and get n outcomes X_1, X_2, \dots, X_n . Calculating the empirical average $\frac{1}{n} \sum_{k=1}^n X_k$ of these n measurements will provide a result with a signal over noise ratio improved by a factor \sqrt{n} compared to a single measurement. This is the simplest illustration of the central limit theorem. Similarly, when considering a thermodynamical system away from criticality, any *extensive* observable on a global scale will be the sum of uncorrelated or weakly correlated mesoscopic contributions and thus will undergo a central limit theorem: it will be self-averaging in the thermodynamical limit. The typicality of the reduced density matrix $\rho_s(t) \approx \mathbb{E}[\rho_s(t)]$ involves a generalization of this phenomenon to highly non linear functions like $f(\hat{W})$.

This typicality has several interesting consequences. First, it provides some explanation for the lack of sensitivity to microscopic details of \hat{W} of processes like for instance thermalisation. Second, numerical simulations using a single realization of the random disorder on \hat{W} provide in a single run the typical dynamics, there is no need to average numerically over several realizations of \hat{W} which is of course computationally interesting (see [52]). Finally, the last consequence is very practical: it allows to calculate an approximation of $\varrho_s(t)$ simply by averaging over the interaction \hat{W} : $\text{Tr}_e(\varrho(t)) \approx \mathbb{E}[\text{Tr}_e(\varrho(t))] = \text{Tr}_e(\mathbb{E}[\varrho(t)])$. Such calculation requires the fourth moments of the overlap coefficients as we will see now.

3. Motivation for calculating the eigenvectors overlaps moments

Our main motivation in the overlap moments calculation concerns the time evolution of a quantum system coupled to a large environment and the so called ‘‘thermalisation’’ problem. It is usually argued in the literature on this problem that the second order moments give complete information about the dynamics. We argue that it is actually the fourth order moments, as far as the state of the embedded system is concerned. Indeed, as the embedded system is open, its state is given by partial tracing the total density matrix of the closed composite system $\varrho_{s+e}(t)$. Expanding the global initial state $\varrho_{s+e}(0)$ on the bare eigenbasis $\varrho_{s+e}(0) = \sum_{m,p} c_{m,p} |\phi_m\rangle\langle\phi_p|$ and the evolution operator \hat{U}_t on the dressed eigenbasis: $\hat{U}_t = \sum_i |\psi_i\rangle\langle\psi_i| e^{-i\lambda_i t}$, one can see easily that the matrix elements of the total density matrix $\langle\phi_n|\varrho_{s+e}(t)|\phi_q\rangle$ involves the quantum channel S given as a tensor of order 4:

$$S_{n,m,p,q}(t) = \sum_{i,j} e^{-i(\lambda_i - \lambda_j)t} \langle\phi_n|\psi_i\rangle\langle\psi_i|\phi_m\rangle\langle\phi_p|\psi_j\rangle\langle\psi_j|\phi_q\rangle.$$

A scattering matrix provides the relations between input and output amplitudes of probabilities and is usually defined on the Hilbert space of open channels, i.e. propagating states, e.g. when studying nuclear reactions. One can think of $S(t)$ as a *scattering tensor*, i.e. an object providing the transfer between state occupations (i.e. the diagonal terms of the initial and final density matrices) and also between coherences (i.e. the outerdiagonal elements). After averaging and taking the large dimension limit in order to consider a continuous approximation, we are left with, on one hand, the two point density function of the dressed spectrum $p(\lambda, \lambda')$ and, on the other hand, the fourth order moments of the overlap coefficients: $\mathbb{E}[\langle\phi_n|\psi_i\rangle\langle\psi_i|\phi_m\rangle\langle\phi_p|\psi_j\rangle\langle\psi_j|\phi_q\rangle]$. These quantities allow to calculate $\mathbb{E}[\varrho_{s+e}(t)]$ and subsequently $\varrho_s(t) \approx \text{Tr}_e(\mathbb{E}[\varrho_{s+e}(t)])$. This motivates our interest in the calculation of the moments of the overlaps, and in particular the fourth order ones.

A similar quantity to S (i.e. a Fourier transform of a product of four overlaps) appears in mesoscopic physics and the Anderson localization problem when considering the probability of quantum diffusion $P(\vec{r}, \vec{r}', t)$ of a particle (e.g. an electron) from point \vec{r} at time $t = 0$ to point \vec{r}' at time t . In this case, the bare basis is the one of spatial position $|\vec{r}\rangle$ (see e.g. the review in [63]). Besides, characterizing the multifractality of the dressed eigenfunctions involves considering moments of the overlaps at *all* orders, and has been under recent renewed attention in the context of generalized Rosenzweig-Porter random matrix ensembles (see e.g. [64, 65]).

B. Calculation tools

1. Link between the eigenvectors overlaps and the GFs

We first remind the well known relations between the overlaps and the matrix elements of the resolvent operator of the dressed Hamiltonian $G_{\hat{H}}(z) = (\hat{H} - z\mathbb{1})^{-1}$ in the bare eigenbasis $\{|\phi_1\rangle, \dots, |\phi_N\rangle\}$: $G_{n,m}(z_1) = \langle\phi_n|G(z_1)|\phi_m\rangle$. These $G_{n,m}(z)$ are similar to the familiar GFs or propagators of quantum field theory[38] which are the matrix elements of the resolvent on the real space $|\vec{r}\rangle$ basis. In our case, in order to stay as general as possible we consider an abstract Hilbert space and the matrix elements of the resolvent are considered on the bare basis $\{|\phi_1\rangle, \dots, |\phi_n\rangle\}$. Using the closure relation verified by the dressed eigenbasis: $G_{n,m}(z) = \sum_j \langle\phi_n|\psi_j\rangle\langle\psi_j|\phi_m\rangle \frac{1}{\lambda_j - z}$, we see that the overlap $\langle\phi_n|\psi_j\rangle\langle\psi_j|\phi_m\rangle$ is the residue of the complex function $z \mapsto G_{n,m}(z)$ at the pole λ_j . Defining the retarded GFs as $G_{n,m}^R(\lambda) = \lim_{\eta \rightarrow 0^+} G_{n,m}(\lambda + i\eta)$, expanding the fraction with $z = \lambda + i\eta$ for $\eta \rightarrow 0^+$, and taking the imaginary part, we get the quantity $\frac{1}{\pi} \text{Im} G_{n,m}^R(\lambda) = \sum_j \langle\phi_n|\psi_j\rangle\langle\psi_j|\phi_m\rangle \delta(\lambda - \lambda_j)$, which, in the case $n = m$, coincides with the Local Density of States (LDoS) also called Strength Function in nuclear physics or condensed matter. This function can be seen as a non perturbative extension of the so-called spectral function and can be probed experimentally, e.g. in neutron scattering experiment for the nuclear LDOS or angle resolved photoelectric emission "ARPES" for the electronic LDOS[66]. In order to introduce the various tools needed for calculating the moments of the overlaps, we focus first on the second order ones: $\mathbb{E}[\langle\phi_n|\psi_j\rangle\langle\psi_j|\phi_m\rangle]$.

Note that in this article, we will not worry about the precise shape of the probability distribution of each overlap $\langle\phi_n|\psi_i\rangle$ and if they may obey some kind of generalized Porter-Thomas distribution (i.e. Gaussian distribution for the overlaps). We refer the reader to [67–70] for experimental evidence, [6, 7] for some insight on this problem, [12] for a full derivation when \hat{W} is Gaussian and [51] for the binary approximation which relies on a Gaussian statistics assumption for the overlaps.

We will assume in the following to be in the non perturbative regime, i.e. the typical matrix element of \hat{W} of order σ_w/\sqrt{N} (σ_w is defined as the spectral variance of the interaction) is large compared to the mean level spacing of the bare Hamiltonian $D \approx \sigma_0/N$, with σ_0 the spectral variance of \hat{H}_0 . As a consequence, after averaging, the GFs no longer have isolated poles but a branch cut along the support of the dressed spectrum. Each second order moment of the overlap is sampling the step height of this branch cut which can be related to the imaginary part of the retarded GF averaged around an average dressed eigenvalue:

$$\mathbb{E}[\langle\phi_n|\psi_j\rangle\langle\psi_j|\phi_m\rangle] \approx \frac{1}{\pi N \rho} \mathbb{E}[\text{Im}(G_{n,m}^R(\bar{\lambda}_j))] \quad (1)$$

where $\bar{\lambda}_j$ is the average j^{th} dressed eigenvalue. We thus need to calculate the averaged GFs.

2. Identifying the zero mean Green functions

This is the first step in the calculation: identifying the zero terms. We use here the same method as in[71] which relies on a large $|z|$ expansion of the resolvent operator:

$$G_H(z) = -\frac{1}{z} \sum_{k=0}^{\infty} \frac{H^k}{z^k}.$$

Assuming \hat{W} to be Gaussian (either GOE or GUE) and using the Wick theorem, one can show easily that $\mathbb{E}[H^k]$ is diagonal $\forall k \in \mathbb{N}^+$ in the bare eigenbasis (see Supp. Mat. of [8]). This implies that all extra diagonal mean GFs are zero: $\mathbb{E}[G_{n,m}(z)] = 0 \forall z$ for $n \neq m$ when \hat{W} is Gaussian. Note that finding the value of the diagonal terms is very difficult using this $1/z$ expansion and involves advanced combinatoric reasoning. We prefer to use the following much simpler loop equation technique, which requires two sets of preliminary formulas: the expansion of the GFs and a so-called "decoupling" formula.

3. Expansion of a Green function $G_{n,m}(z)$ with respect to the interaction \hat{W} .

For the sake of completeness we remind here some well known properties. We consider the expansion of a GF as function of the interaction \hat{W} . Our starting point is the identity involving the resolvent of the sum of two matrices $\hat{H} = \hat{H}_0 + \hat{W}$: $G_{\hat{H}}(z) = G_{\hat{H}_0}(z) - G_{\hat{H}_0}(z)\hat{W}G_{\hat{H}}(z) = G_{\hat{H}_0}(z) - G_{\hat{H}}(z)\hat{W}G_{\hat{H}_0}(z)$ which follows trivially from the

resolvent definition $G_{\hat{H}}(z)(\hat{H} - z\mathbb{1}) = \mathbb{1}$ and is a propagator version of the Lippmann-Schwinger equation. This equation provides several useful well known formulas:

$$G_{n,m}(z) = \frac{1}{\epsilon_n - z} \left(\delta_{n,m} - \sum_{k=1}^N W_{n,k} G_{k,m}(z) \right) = \frac{1}{\epsilon_m - z} \left(\delta_{n,m} - \sum_{k=1}^N G_{n,k}(z) W_{k,m} \right) \quad (2)$$

$$\frac{\partial G_{n,p}(z)}{\partial W_{k,l}} = -G_{n,k}(z) G_{l,p}(z) \quad (3)$$

$$\frac{G_H(z_1) - G_H(z_2)}{z_1 - z_2} = -G_H(z_1) G_H(z_2) \quad \text{which gives} \quad \frac{\partial G_{n,p}(z)}{\partial z} = -\langle \phi_n | G(z)^2 | \phi_p \rangle \quad (4)$$

where $\delta_{n,m}$ is the Kronecker symbol. One should note that the expansions in Eq.(2) are equalities and not approximations, i.e. they are not Taylor expansions. These equations can be considered as equations of motion. One can also note that a diagram perturbation calculation at order k would mean to iterate the expansion process k times and neglect the residual. Here, we need only a *single* such expansion.

4. “Decoupling” formula.

This is the core tool of our method for averaging GFs and their products: a “decoupling” formula, which was previously used in[36] for calculating the covariance between *traces* of the resolvent of random matrices without source term, i.e. $\hat{H}_0 = 0$. This formula consists in a cumulant expansion approach based on the following simple idea: if ξ is a real random variable, and f a complex value function defined on \mathbb{R} , then $\mathbb{E}[\xi f(\xi)]$ can be written as an expansion over the cumulants $\kappa_n(\xi)$ of ξ :

$$\mathbb{E}[\xi f(\xi)] = \sum_{n=0}^{\infty} \frac{\kappa_{n+1}(\xi)}{n!} \mathbb{E}[f^{(n)}(\xi)] \quad (5)$$

This formula follows easily from integration by parts. We will consider here a generalization of this formula to the multivariate case with $\vec{\xi} = (\xi_1, \xi_2, \dots, \xi_N)$:

$$\mathbb{E}[\xi_1 f(\vec{\xi})] = \kappa_1(\xi_1) \mathbb{E}[f] + \sum_j \frac{\kappa_2(\xi_1, \xi_j)}{1!} \mathbb{E} \left[\frac{\partial f}{\partial \xi_j} \right] + \sum_{j,k} \frac{\kappa_3(\xi_1, \xi_j, \xi_k)}{2!} \mathbb{E} \left[\frac{\partial^2 f}{\partial \xi_j \partial \xi_k} \right] + \dots \quad (6)$$

where the $\kappa_r(\xi_1, \xi_{i_2}, \dots, \xi_{i_r})$ are the mixed cumulants of order r . The “decoupling” effect is now clear: this formula allows to relate the covariance between the input and the output of the function f to the statistics of the input and the statistics of the derivatives of f . This formula simplifies when the random variables $\{\xi_1, \xi_2, \dots, \xi_N\}$ form a centered Gaussian family. Only the second order remains, since all higher order mixed cumulants are zero:

$$\mathbb{E}[\xi_1 f(\vec{\xi})] = \sum_k \kappa_2(\xi_1, \xi_k) \mathbb{E} \left[\frac{\partial f}{\partial \xi_k} \right] \quad \text{if the } \{\xi_1, \dots, \xi_N\} \text{ are real centered Gaussian variables.} \quad (7)$$

$\kappa_2(\xi_i, \xi_j)$ is the covariance matrix of the $\{\xi_i\}$ family. In this article, we will consider such a truncation of the decoupling formula in Eq.(6) at order 2, which means that we will take into account only the Gaussian behavior in the statistics of \hat{W} . Such simplification provides a first path for capturing all the phenomenon important we are interested in (in particular thermalisation[8]) and is sufficient for our purpose. Calculation with higher order cumulants or correlations between entries are more involved and will be investigated in a further publication. Note that thanks to the decoupling formula the familiar power series of perturbation theory has been changed for a cumulant series, where the terms of order higher than 2 are *exactly* zero in the Gaussian interaction case.

5. Covariance between an entry of \hat{W} and a Green function: Gaussian case

Using the decoupling formula from Eq.(7) as well as Eq.(3), we calculate the covariance between $W_{n,k}$ and $G_{k,m}(z)$, a quantity required in the next sections:

$$\mathbb{E}[W_{n,k} G_{k,m}(z)] = - \sum_{p,q} \kappa_{n,k,p,q} \mathbb{E}[G_{k,p}(z) G_{q,m}(z)] \quad (8)$$

where the $\kappa_{n,k,p,q} = \text{cov}(W_{n,k}, W_{p,q})$ is the covariance between entries of the interaction. For Gaussian random matrices, this covariance tensor is $\frac{\sigma_w^2}{N} \delta_{n,q} \delta_{k,p}$ for W in the GUE and $\frac{\sigma_w^2}{N} (\delta_{n,q} \delta_{k,p} + \delta_{n,p} \delta_{k,q})$ for W in the GOE, with σ_w defined as the standard deviation of the spectrum of \hat{W} : $\sigma_w^2 = \text{tr}(\hat{W}^2)$, $\text{tr} = \text{Tr}/N$ being the normalized trace.

6. Loop equations for the mean GFs

We can now start the calculation of the mean of a GF $G_{n,m}(z)$. The method consists in the following steps:

- Expand $G_{n,m}(z)$ using Eq.(2) and average over the interaction to get:

$$(\epsilon_n - z) \mathbb{E}[G_{n,m}(z)] = \delta_{n,m} - \sum_{k=1}^N \mathbb{E}[W_{n,k} G_{k,m}(z)]$$

- Insert Eq.(8) in this last equation to obtain:

$$(\epsilon_n - z) \mathbb{E}[G_{n,m}(z)] = \delta_{n,m} - \sum_{k,p,q} \kappa_{n,k,p,q} \mathbb{E}[G_{k,p} G_{q,m}(z)]$$

Defining the matrix $\Sigma(z)$ of general term $\Sigma_{n,q}(z) = \sum_{k,p} \kappa_{n,k,p,q} G_{k,p}(z)$, the last equation can be rewritten in matricial form like: $(\hat{H}_0 - z\mathbb{1}) \mathbb{E}[G_H(z)] = \mathbb{1} - \mathbb{E}[\Sigma(z) G_H(z)]$. This quantity $\Sigma(z)$ coincides with the well known self-energy in the perturbative limit and extends this concept to the *non perturbative* regime. Besides, it is important to stress the decoupling formula considered in the Gaussian case (Eq.8) is an exact expansion over the second order cumulants of the interaction: the summation in the self-energy is not selective or partial but complete in this case. It is interesting to consider the particular cases:

- $W \in \text{GUE}$: $\kappa_2(W_{n,k}; W_{p,q}) = \delta_{n,q} \delta_{k,p}$ which implies that the self energy is a scalar, $\sigma_w^2 m_{\hat{H}}(z)$ times the identity matrix: $\Sigma(z) = \sigma_w^2 m_{\hat{H}}(z) \mathbb{1}$, where $m_{\hat{H}}(z) = \text{tr}(G_{\hat{H}}(z))$ is the Stieltjes transform of \hat{H} .
- $W \in \text{GOE}$: the self-energy is slightly more complicated: $\Sigma(z) = \frac{\sigma_w^2}{N} {}^t G(z) + \sigma_w^2 m_{\hat{H}}(z) \mathbb{1}$, where ${}^t G(z)$ is the matrix transpose of the resolvent.
- The core hypothesis of our technique is now to assume that $\Sigma(z)$ is self-averaging, i.e. each of its matrix element is concentrated around a mean value. For instance, in the GUE case, the Stieltjes transform of \hat{H} : $m_{\hat{H}}(z)$ is indeed self-averaging for $z = \epsilon + i\eta$ not too close from the support of the spectrum, i.e. $\eta \gg D$ the mean level spacing (see for instance Corollary 4.4.30 in [72]), since it is a sum of a large number of weakly correlated terms. Neglecting the fluctuations provides the approximation $\mathbb{E}[\Sigma(z) G_{\hat{H}}(z)] \approx \mathbb{E}[\Sigma(z)] \mathbb{E}[G_{\hat{H}}(z)]$ and more generally the matricial identity $\mathbb{E}[\hat{W} G_{\hat{H}}(z)] \approx \mathbb{E}[\Sigma(z)] \mathbb{E}[G_{\hat{H}}(z)]$. The self-consistent equation verified by the mean GF follows:

$$\mathbb{E}[G(z)] \approx \frac{1}{\hat{H}_0 - z\mathbb{1} - \Sigma(z)} \quad (9)$$

where $\Sigma(z)$ is identified with its mean value. This equation, also called a loop equation or Dyson equation (see [73] and also Chap. 6 in [74]), is self-consistent since the mean GFs (i.e. $\mathbb{E}[G_{n,n}(z)]$) appear both on the left and the right hand sides.

7. Second order moments of the overlaps and Local Density of States (LDOS)

Combining Eq.(1) and Eq.(9), we get the second order statistics of the overlaps in the case of a Gaussian \hat{W} (either GOE or GUE):

$$\mathbb{E}[\langle \phi_n | \psi_i \rangle \langle \psi_i | \phi_m \rangle] \approx \frac{\delta_{n,m}}{N \rho(\bar{\lambda}_i)} l_{\epsilon_n}(\bar{\lambda}_i) \quad \text{with the LDOS:} \quad l_{\epsilon}(\lambda) = \frac{1}{\pi} \frac{s_{\lambda}}{(\epsilon - \lambda - \tilde{s}_{\lambda})^2 + s_{\lambda}^2}, \quad (10)$$

and where $\tilde{s}_{\lambda} = \lim_{\eta \rightarrow 0^+} \text{Re}(\Sigma(\lambda + i\eta)) = \sigma_w^2 H_{\rho}(\lambda)$ is an energy shift (analog to a Lamb shift) proportional to $H_{\rho}(\lambda)$ the Hilbert transform of the probability density of the spectrum (i.e. $N\rho$ is the dressed Density of States DoS) and

$s_\lambda = \lim_{\eta \rightarrow 0^+} \text{Im}(\Sigma(\lambda + i\eta)) = \pi\sigma_w^2\rho(\lambda)$ is a decay rate. The apparent Lorentzian shape in this formula is reminiscent from the Breit-Wigner law obtained in the context of the so-called “standard model” in nuclear physics[75, 76] and has proved to be ubiquitous in many other fields: in molecular physics with the pre-dissociation of diatomic molecules and its effect on rotational absorption lines[77, 78], atomic physics e.g. with the eigenstates properties of the Ce atom[79], quantum chaos[80], thermalisation [6, 46], financial data analysis with empirical estimation of covariance matrices[12–14, 81], pure mathematics with free probability[26, 82, 83]. However, it is important to note that the LDOS is no longer Lorentzian when the interaction is strong enough that the self-consistent character of Eq.(10) has to be taken into account: the DOS ($N\rho$) depends on λ and can vary significantly on the scale of the width s_λ distorting the LDOS shape. The energy shift \tilde{s}_λ dependence on λ might also alter the LDOS shape. We postpone a systematic study of these effects to further publications. Furthermore, it is also interesting to note that, despite the regime we consider is non perturbative, the width $\Gamma = s_\lambda/\pi = \sigma_w^2\rho$ still have a Fermi Golden rule form (see also [84]). In addition, in the large energy difference limit the Lorentzian shape reproduces the first order perturbative prediction: $\mathbb{E}[|\langle\phi_n|\psi_i\rangle|^2] \approx \sigma_w^2/(\epsilon_n - \lambda_i)^2$. In some sense, Eq.(10) extends the well known perturbative results valid for $|\epsilon_n - \lambda_i|/\sigma'_w \gg 1$ (with $\sigma'_w = \sigma_w/\sqrt{N}$ the typical standard deviation of one entry of W) to any value of this ratio. The predictions obtained are tested numerically on Fig.1 and a satisfactory agreement is found.

8. Generalizations to other statistics on \hat{W} : rotationally invariant ensembles

There is a well known generalization of the Gaussian results to Randomly Rotated Matrices i.e. of the form $\hat{W} = \hat{P}\hat{D}\hat{P}^\dagger$ with \hat{D} diagonal real fixed and \hat{P} unitary or orthogonal Haar distributed. In this case \hat{H}_0 and \hat{W} are said to be in generic position with one another, in the sense that the eigenvectors of \hat{W} are distributed isotropically in the bare eigenbasis. In the limit of infinite dimension, this case provides the framework of *free* probability theory[24, 26, 85–87]. The self-energy $\Sigma(z)$ is related to the analog of a cumulant expansion in a free probability context: the R-transform of \hat{W} [29, 30], $R_{\hat{W}}(z) = \sum_{n \geq 1} \tilde{\kappa}_n(\hat{W})z^{n-1}$ ($\tilde{\kappa}_n(\hat{W})$ being the free-cumulants) by $\Sigma(z) = R_{\hat{H}}(m_{\hat{H}}(z))$. The classical cumulants κ_n have the property that $\kappa_n(X + Y) = \kappa_n(X) + \kappa_n(Y)$ for two commutative independent random variables X, Y . On the free probability side, the free cumulants $\tilde{\kappa}_n$ are such that $\tilde{\kappa}_n(\hat{H}_0 + \hat{W}) = \tilde{\kappa}_n(\hat{H}_0) + \tilde{\kappa}_n(\hat{W})$ for two *non* commutative random “free” variables. In some sense, freeness is the equivalent of independence for non commutative random variables. The Gaussian results correspond to a truncation of the R-transform at second order $\Sigma(z) = \sigma_w^2 m_H(z)$. We note in passing the striking resemblance of these results obtained in the framework of free probability theory with the one obtained with Dynamical Mean Field Theory (DMFT) in condensed matter[88].

III. SECOND ORDER STATISTICS OF THE RESOLVENT AND FOURTH ORDER MOMENTS OF THE OVERLAPS.

In this section, we apply the decoupling technique to the calculation of the fourth order moments of the overlaps for a GUE interaction, which provides the main result of this paper. We compare our analytical formulas with numerical simulations on Fig.(2) and find a satisfactory agreement. Again, we use the fact that the overlap coefficient $\langle\phi_n|\psi_i\rangle\langle\psi_i|\phi_m\rangle$ is (up to a factor 2π) the residue of the meromorphic function $G_{n,m}(z) = \sum_j \frac{\langle\phi_n|\psi_j\rangle\langle\psi_j|\phi_m\rangle}{\lambda_j - z}$ at the pole λ_i . However here, the second order statistics of the resolvent is more complicated, as we will see in the following, after averaging, the function $(z_1, z_2) \mapsto G_{n,m}(z_1)G_{m,n}(z_2)$ has branch cuts and a continuum of singularities (for $z_1 = z_2$ in the support of the dressed spectrum). With a similar procedure as previously done for the first moment of $G_H(z)$, the fourth order moments of the overlap coefficients can be related to the covariance between GFs by

$$\mathbb{E}[\langle\phi_n|\psi_i\rangle\langle\psi_i|\phi_m\rangle\langle\phi_p|\psi_j\rangle\langle\psi_j|\phi_q\rangle] = \lim_{\eta, \eta' \rightarrow 0^+} - \frac{\mathbb{E}[(G_{n,m}(\bar{\lambda}_i + i\eta) - G_{n,m}(\bar{\lambda}_i - i\eta)) (G_{p,q}(\bar{\lambda}_j + i\eta') - G_{p,q}(\bar{\lambda}_j - i\eta'))]}{4\pi^2\rho(\bar{\lambda}_i)\rho(\bar{\lambda}_j)} \quad (11)$$

for $i \neq j$. We are lead to calculate the average of a product of two GFs: $\mathbb{E}[G_{n,m}(z_1)G_{p,q}(z_2)]$.

A. Second order mixed moments of the Green functions

These second order mixed moments of GFs are similar to two particle GFs, i.e. the quantity involved in the calculation of conductivity and susceptibility in condensed matter, and also called “diffusion propagator” in the context of the Anderson localization[11]. We will consider here a Gaussian W , however one body Gaussian interaction,

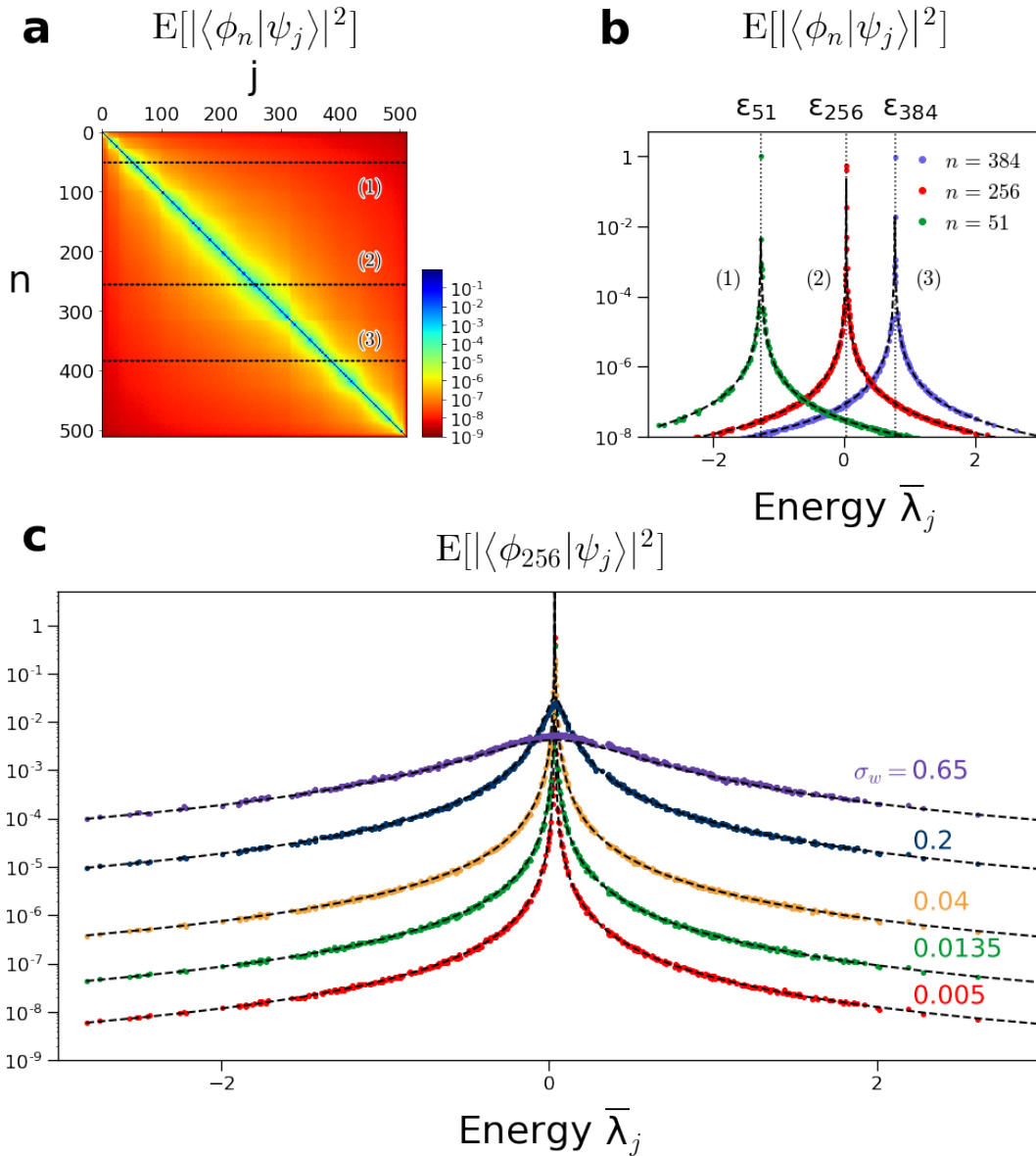


FIG. 1. **Second order moment of the eigenvectors overlaps:** $\mathbb{E}[|\langle \phi_n | \psi_j \rangle|^2]$. Numerical simulations are performed with 512×512 matrices, where \hat{H}_0 is diagonal with Gaussian distributed eigenvalues of standard deviation $\sigma_0 = 1$ and zero mean. The interaction Hamiltonian \hat{W} is taken from the GOE ensemble with spectral variance $\sigma_w^2 = \text{Tr}(W \cdot W^\dagger)/N$. The empirical average is performed over 10^3 realizations of the random \hat{W} . Note that we obtain similar results for \hat{W} in the GUE ensemble. **a**) Color plot of the matrix $C_{n,j} = \mathbb{E}[|\langle \phi_n | \psi_j \rangle|^2]$ in the case $\sigma_w = 0.01$. This overlap matrix quantifies how much each eigenvector of the “bare” Hamiltonian \hat{H}_0 is delocalized in the eigenbasis of the dressed Hamiltonian $\hat{H}_0 + \hat{W}$. The eigenvectors $|\phi_n\rangle$ and $|\psi_i\rangle$ are sorted by order of decreasing eigenvalues. **b**) $\mathbb{E}[|\langle \phi_n | \psi_j \rangle|^2]$ is plotted as a function of the mean dressed eigenvalue $\bar{\lambda}_j$ of the eigenvector $|\psi_j\rangle$, for an interaction strength $\sigma_w = 0.08$ and several values of $n = 51, 256, 384$ (i.e. one quarter, one half and three quarter of the spectrum respectively). The theoretical prediction is provided by Eq.(10) and plotted in dashed line. **c**) $\mathbb{E}[|\langle \phi_{256} | \psi_j \rangle|^2]$ is plotted as a function of $\bar{\lambda}_j$ and for several values of the coupling $\sigma_w = 0.005, 0.0135, 0.04, 0.2, 0.65$ (from red to purple). The theoretical prediction also from Eq.(10) is plotted in dash. For $\sigma_w = 0.005, 0.0135$ the regime is clearly perturbative and one has $\mathbb{E}[|\langle \phi_{256} | \psi_{256} \rangle|^2] \approx 1$. For $\sigma_w = 0.2$ the regime is intermediate and $\Gamma/D = N\pi^2\sigma_w^2\rho^2 \approx 31$, meaning that the eigenvector $|\psi_{256}\rangle$ is delocalized over roughly ≈ 30 bare eigenvectors. For $\sigma_w = 0.65$, the bare eigenvector $|\phi_{256}\rangle$ is strongly delocalized.

suitable for modelling on site disorder, can be treated by the method and will be considered in a further publication. We apply the same method as in Sec.IIB 2 (for spotting the zero cases), and Sec.IIB 6 (for calculating the non zero ones).

1. Zero cases

A large $|z_1|, |z_2|$ series expansion of both matrix elements $G_{n,m}(z_1)$ and $G_{p,q}(z_2)$ is made, and then the Wick theorem is used (the Weingarten calculus can be used for orthogonal or unitary Haar distributed interactions, see Supp. Mat. of [8]). $\mathbb{E}[G_{n,m}(z_1)G_{p,q}(z_2)]$ is zero $\forall z_1, z_2$, except if

- ($n = m$ and $p = q$): this case corresponds to the correlations between diagonal GFs $\mathbb{E}[G_{n,n}(z_1)G_{p,p}(z_2)]$ and is involved in the time evolution of the coherence terms of the total density matrix $\varrho(t)$ under the Hamiltonian \hat{H} (i.e. the *extra* diagonal terms of $\varrho(t)$),
- ($n = q$ and $m = p$): correlations between the two extra diagonal entries $\mathbb{E}[G_{n,m}(z_1)G_{m,n}(z_2)]$. These correlations are involved in the time evolution of the *diagonal* terms of $\varrho(t)$.

In the context of decoherence and thermalisation, the zero cases for the covariance between GFs provide crucial information on the behavior of the *quantum channel* $S(t)$ associated to the time evolution and set in Eq.(1). The preceding results tell us that an initial diagonal state $|\phi_n\rangle\langle\phi_n|$ is only coupled to the diagonal terms $|\phi_m\rangle\langle\phi_m|$. Each initial *outer* diagonal term $|\phi_n\rangle\langle\phi_m|$ ($n \neq m$) is only coupled to itself: it contributes only to the weight of $|\phi_n\rangle\langle\phi_m|$ in the state at time t .

2. Calculation method for the non zero cases

In the following, we focus on a unitary Gaussian interaction W (GUE) and apply the following recipe:

- expand both GFs on the right and on the left using Eq.(2),
- average over the statistics of W and use the decoupling formula from Eq.(8),
- use the relation $G_H(z_1)G_H(z_2) = -(G_H(z_1) - G_H(z_2))/(z_1 - z_2)$,
- get a first set of Dyson equations linking the covariance between extra diagonal GFs and the covariance between diagonal GFs: Eq.(12),
- neglect the correlation between diagonal GFs: $\mathbb{E}[G_{n,n}(z_1)G_{m,m}(z_2)] \approx \mathbb{E}[G_{n,n}(z_1)]\mathbb{E}[G_{m,m}(z_2)]$ and neglect the fluctuations of the Stieltjes transform of the spectrum; $m_{\hat{H}}(z) \approx \mathbb{E}[m_{\hat{H}}(z)]$. This provides a approximate Dyson equations linking the covariance between extra diagonal GFs to products of mean diagonal GFs: Eq.(13) and Eqs.(23), and the covariance between diagonal GFs: Eq.(15).

This procedure, described in detail in the Appendix, provides the following results.

3. Results: covariance between Green functions (GUE case)

- Relation between the covariance of diagonal GFs and the covariance of extra diagonal GFs:

$$(z_1 - z_2)(\epsilon_n - \epsilon_m) \mathbb{E}[G_{n,m}(z_1)G_{m,n}(z_2)] = \frac{\sigma_w^2}{N} (\mathbb{E}[G_{n,n}(z_1)G_{m,m}(z_2)] - \mathbb{E}[G_{n,n}(z_2)G_{m,m}(z_1)]). \quad (12)$$

This equation is remarkable because it is *exact*.

- The covariance between extra diagonal GFs is given by the approximate relations:

$$\mathbb{E}[G_{n,m}(z_1)G_{m,n}(z_2)] \approx \frac{\sigma_w^2}{N} (1 + \Sigma_2(z_1, z_2)) F_{n,m}(z_1, z_2) \quad (13)$$

$$\text{with } \begin{cases} F_{n,m}(z_1, z_2) = \mathbb{E}[G_{m,m}(z_1)] \mathbb{E}[G_{m,m}(z_2)] \mathbb{E}[G_{n,n}(z_1)] \mathbb{E}[G_{n,n}(z_2)] \\ \Sigma_2(z_1, z_2) = \sigma_w^2 \frac{m_H(z_1) - m_H(z_2)}{z_1 - z_2} = \sigma_w^2 \text{tr}(\mathbb{E}[G_H(z_1)G_H(z_2)]). \end{cases} \quad (14)$$

These equations provide a direct link between the first and second order statistics of the GFs. By analogy to the self-energy involved in the first order statistics of the resolvent of $H + W$ (i.e. $\Sigma(z) = \sigma_w^2 \text{tr}(\mathbb{E}[G_H(z)])$ when W is unitary Gaussian), the function $\Sigma_2(z_1, z_2)$ involved in Eq.(13) can be named a ‘‘second order’’ self-energy. This function has a continuum of singularities for $z_1 = z_2$ in the dressed spectrum.

- The covariance between diagonal entries is given by

$$\mathbb{E}[\delta G_{n,n}(z_1)\delta G_{p,p}(z_2)] \approx -\frac{\sigma_w^2}{N} \frac{1}{z_2 - z_1} \frac{\mathbb{E}[(G_{p,n}(z_2) - G_{p,n}(z_1))(G_{n,p}(z_2) - G_{n,p}(z_1))]}{\epsilon_n - \epsilon_p + z_2 - z_1 + \sigma_w^2(m_H(z_2) - m_H(z_1))}. \quad (15)$$

where $\delta G_{n,n}(z) = G_{n,n}(z) - \mathbb{E}[G_{n,n}(z)]$, meaning that the covariance between diagonal terms is smaller by a factor σ_w^2/N compared to the covariance between extra diagonal terms.

From these results on the GFs, one can derive the fourth order moments of the eigenvector overlaps.

B. Fourth order statistics of the overlaps

$$1. \quad \text{Case } n = q, m = p: \mathbb{E}[\langle \phi_n | \psi_i \rangle \langle \psi_i | \phi_m \rangle \langle \phi_m | \psi_j \rangle \langle \psi_j | \phi_n \rangle]$$

Combining Eq.11 and Eq.13, we get the main result of this article, the fourth order moment of the overlaps for $i \neq j$ and $n \neq m$:

$$\mathbb{E}[\langle \phi_n | \psi_i \rangle \langle \psi_i | \phi_m \rangle \langle \phi_m | \psi_j \rangle \langle \psi_j | \phi_n \rangle] \approx -\frac{\sigma_w^2}{N} \frac{1}{\rho(\bar{\lambda}_i)\rho(\bar{\lambda}_j)} \frac{l_{\epsilon_n}(\bar{\lambda}_i)l_{\epsilon_m}(\bar{\lambda}_j) - l_{\epsilon_n}(\bar{\lambda}_j)l_{\epsilon_m}(\bar{\lambda}_i)}{(\bar{\lambda}_i - \bar{\lambda}_j)(\epsilon_n - \epsilon_m)} \quad (16)$$

where $l_\epsilon(\lambda)$ is the Lorentzian function we introduced in Eq.(10). If This formula is tested numerically on Fig.2 and we find a satisfactory agreement. The transfer probability between state $|\phi_m\rangle\langle\phi_m|$ at time $t = 0$ to state $|\phi_n\rangle\langle\phi_n|$ at time t : $p_{m \rightarrow n}(t)$ follows easily by Fourier transformation according to Eq.(1). We will use this formula when studying the out of equilibrium dynamics of an embedded quantum system[89].

$$2. \quad \text{Case } n = m \text{ and } p = q: \mathbb{E}[|\langle \phi_n | \psi_i \rangle|^2 |\langle \psi_j | \phi_p \rangle|^2]$$

Form Eq.(15), we know that two distinct diagonal entries of the resolvent are weakly correlated. As a result the fourth order moment of the overlap follows easily:

$$\mathbb{E}[(G_{n,n}(z_1) - G_{n,n}(z_1^*)) (G_{p,p}(z_2) - G_{p,p}(z_2^*))] = -4 \mathbb{E}[\text{Im}(G_{n,n}(z_1)) \text{Im}(G_{p,p}(z_2))] \approx -4 \text{Im}(\mathbb{E}[G_{n,n}(z_1)]) \text{Im}(\mathbb{E}[G_{p,p}(z_2)])$$

Using Eq.(11), we get the overlap moment:

$$\mathbb{E}[|\langle \phi_n | \psi_i \rangle|^2 |\langle \phi_p | \psi_j \rangle|^2] \approx \mathbb{E}[|\langle \phi_n | \psi_i \rangle|^2] \mathbb{E}[|\langle \phi_p | \psi_j \rangle|^2] \approx \frac{1}{\rho(\bar{\lambda}_i)\rho(\bar{\lambda}_j)} l_{\epsilon_n}(\bar{\lambda}_i)l_{\epsilon_p}(\bar{\lambda}_j). \quad (17)$$

This formula works for all cases, except for the case ($n = p$ and $i = j$).

IV. CONCLUSION

We presented a new method for calculating the mixed moments of the Green functions of a random biased Hamiltonian. These quantities provide access to the mixed moments of the overlaps between eigenvectors of two large Hamiltonians or matrices: \hat{H}_0 deterministic arbitrarily chosen and \hat{W} random. We applied this method to calculate the second and fourth order moments of the overlaps in the Gaussian case for \hat{W} . These quantities are crucial for understanding the out of equilibrium dynamics of an embedded quantum system, as well as its stationary states. They

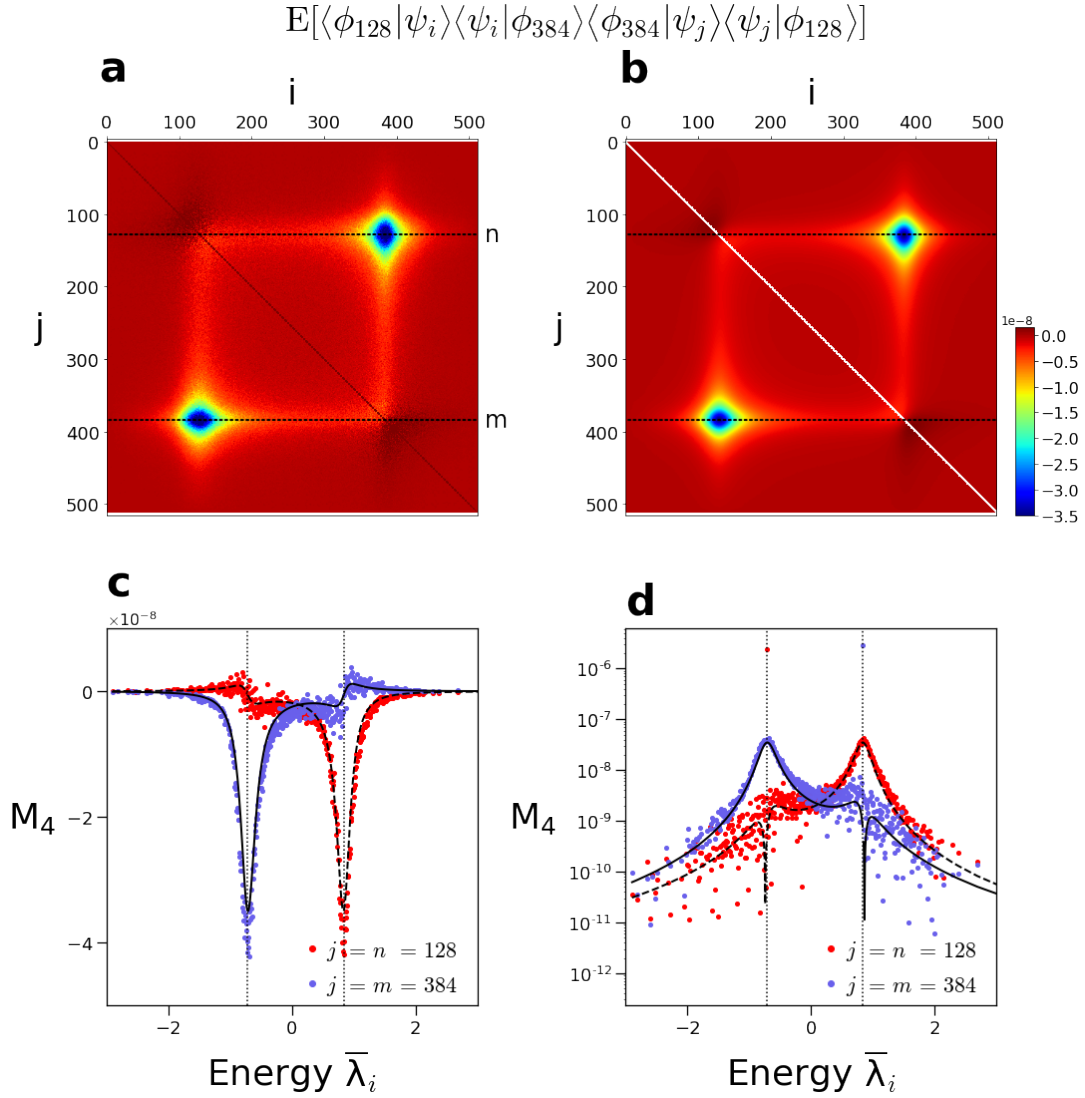


FIG. 2. **Fourth order moments of the eigenvectors overlaps:** $C_{i,j} = E[\langle \phi_n | \psi_i \rangle \langle \psi_i | \phi_m \rangle \langle \phi_m | \psi_j \rangle \langle \psi_j | \phi_n \rangle]$. These moments are related to the correlation between Green functions $E[G_{n,m}(z_1)G_{m,n}(z_2)]$ through Eq.(11). Here \hat{H}_0 is a $N \times N$ ($N = 512$) diagonal matrix with a centered gaussian DOS with variance 1 and \hat{W} is in the GUE with $\sigma_w = \sqrt{\text{Tr}(W^2)/N} = 0.4$. **a)** Numerical results: color plot of the matrix $C_{i,j}$ for fixed values of $n = 128 = N/4$ and $m = 384 = 3N/4$. **b)** Theoretical prediction from Eq.(16). After Fourier transformation, these fourth order moments give the diagonal part of the average total density matrix, i.e. they provide the out of equilibrium dynamics of the probabilities of occupation of a quantum system. **c)** $C_{i,j}$ is plotted as a function of $\bar{\lambda}_i$ for $j = n = 128$ (red) and for $j = m = 384$ (blue) with linear scale. **d)** Same plot in log scale ($|C_{i,j}|$). The numerical averaging is performed over 8.10^5 realizations of \hat{W} .

provide statistical solutions to the many body quantum problem. The formulas we obtain were tested numerically and a satisfactory agreement was found. The method presented in the article is versatile and will be used for investigating other statistics for \hat{W} : Wigner Random Band Matrices, matrices with correlated entries, and Embedded ensembles in further publications.

Acknowledgements: we would like to thank F. Benaych-Georges for insightful discussions and spotting to us useful references.

Appendix

1. Covariance between Green functions

We focus here on the non zero cases: ($n = m$ and $p = q$) and ($n = q$ and $m = p$), and obtain the Dyson equations linking the covariance coefficients between Green functions together by applying the following recipe:

- We expand $G_{n,m}(z_1)$ on the left side using Eq.(2) and get $(\epsilon_n - z_1) \mathbb{E}[G_{n,m}(z_1)G_{p,q}(z_2)] = \delta_{n,m} \mathbb{E}[G_{p,q}(z_2)] - \sum_j \mathbb{E}[W_{n,j}G_{j,m}(z_1)G_{p,q}(z_2)]$.
- We apply the decoupling formula from Eq.(8) and obtain

$$(\epsilon_n - z_1) \mathbb{E}[G_{n,m}(z_1)G_{p,q}(z_2)] = \delta_{n,m} \mathbb{E}[G_{p,q}(z_2)] + \frac{\sigma_w^2}{N} \sum_j (\mathbb{E}[G_{j,j}(z_1)G_{n,m}(z_1)G_{p,q}(z_2)] + \mathbb{E}[G_{j,m}(z_1)G_{p,j}(z_2)G_{n,q}(z_2)])$$

where σ_w^2 is the variance of the spectrum of \hat{W} : $\sigma_w^2 = \text{tr}(\hat{W}^2)$.

- We use the identity $G_H(z_1) - G_H(z_2) = (z_2 - z_1)G_H(z_1)G_H(z_2)$ giving $\sum_j G_{p,j}(z_2)G_{j,m}(z_1) = -(G_{p,m}(z_2) - G_{p,m}(z_1))/(z_2 - z_1)$
- We assume that the Stieltjes transform of the spectrum of H is concentrated, i.e. $\mathbb{E}[\text{tr}(G_H(z_1))G_{n,m}(z_1)G_{p,q}(z_2)] \approx m_H(z_1) \mathbb{E}[G_{n,m}(z_1)G_{p,q}(z_2)]$.

This finally gives us the approximate equation

$$(\epsilon_n - z_1 - \sigma_w^2 m_H(z_1)) \mathbb{E}[G_{n,m}(z_1)G_{p,q}(z_2)] \approx \delta_{n,m} \mathbb{E}[G_{p,q}(z_2)] - \frac{\sigma_w^2}{N} \frac{1}{z_2 - z_1} \mathbb{E}[(G_{p,m}(z_2) - G_{p,m}(z_1))G_{n,q}(z_2)] \quad (18)$$

where the error made is due to the covariance between the Stieltjes transform and the Green functions: $\sigma_w^2 \mathbb{E}[\delta \text{tr}(G_H(z_1))G_{n,m}(z_1)G_{p,q}(z_2)]$. Redoing this very procedure by expanding $G_{n,m}(z_1)$ on the right, then $G_{p,q}(z_2)$ on the left and on the right, we get the following three extra equations verified by the covariance between Green functions:

$$(\epsilon_m - z_1 - \sigma_w^2 m_H(z_1)) \mathbb{E}[G_{n,m}(z_1)G_{p,q}(z_2)] \approx \delta_{n,m} \mathbb{E}[G_{p,q}(z_2)] - \frac{\sigma_w^2}{N} \frac{1}{z_1 - z_2} \mathbb{E}[(G_{n,q}(z_1) - G_{n,q}(z_2))G_{p,m}(z_2)] \quad (19)$$

$$(\epsilon_p - z_2 - \sigma_w^2 m_H(z_2)) \mathbb{E}[G_{n,m}(z_1)G_{p,q}(z_2)] \approx \delta_{p,q} \mathbb{E}[G_{n,m}(z_1)] - \frac{\sigma_w^2}{N} \frac{1}{z_1 - z_2} \mathbb{E}[(G_{n,q}(z_1) - G_{n,q}(z_2))G_{p,m}(z_1)] \quad (20)$$

$$(\epsilon_q - z_2 - \sigma_w^2 m_H(z_2)) \mathbb{E}[G_{n,m}(z_1)G_{p,q}(z_2)] \approx \delta_{p,q} \mathbb{E}[G_{n,m}(z_1)] - \frac{\sigma_w^2}{N} \frac{1}{z_2 - z_1} \mathbb{E}[(G_{p,m}(z_2) - G_{p,m}(z_1))G_{n,q}(z_1)] \quad (21)$$

We consider the following manipulations of the previous equations:

- Eq. 18 – Eq.21 :

$$\begin{aligned} (\epsilon_n - \epsilon_q + z_2 - z_1 + \sigma_w^2(m_H(z_2) - m_H(z_1))) \mathbb{E}[G_{n,m}(z_1)G_{p,q}(z_2)] &\approx \delta_{n,m} \mathbb{E}[G_{p,q}(z_2)] - \delta_{p,q} \mathbb{E}[G_{n,m}(z_1)] \\ &- \frac{\sigma_w^2}{N} \frac{1}{z_2 - z_1} \mathbb{E}[(G_{p,m}(z_2) - G_{p,m}(z_1))(G_{n,q}(z_2) - G_{n,q}(z_1))] \end{aligned} \quad (22)$$

where the error is $\sigma_w^2 \mathbb{E}[\delta \text{tr}(G_H(z_1)) - \delta \text{tr}(G_H(z_2))] G_{n,m}(z_1)G_{p,q}(z_2)$. Let us apply Eq.(22) to the case ($n = q$ and $p = m$ and $n \neq m$), we get the covariance between extra diagonal terms:

$$\mathbb{E}[G_{n,m}(z_1)G_{m,n}(z_2)] \approx -\frac{\sigma_w^2}{N} \frac{1}{z_2 - z_1} \frac{\mathbb{E}[(G_{m,m}(z_2) - G_{m,m}(z_1))(G_{n,n}(z_2) - G_{n,n}(z_1))]}{z_2 - z_1 + \sigma_w^2(m_H(z_2) - m_H(z_1))}$$

Neglecting the correlation between diagonal terms and using the results from the first order statistics of the resolvent, $\mathbb{E}[G_{n,n}(z)] \approx \frac{1}{\epsilon_n - z - \sigma_w^2 m_H(z)}$ we get the covariance between extra diagonal terms of the resolvent:

$$\mathbb{E}[G_{n,m}(z_1)G_{m,n}(z_2)] \approx \delta_{n,m} \mathbb{E}[G_{n,n}(z_1)] \mathbb{E}[G_{n,n}(z_2)] + \frac{\sigma_w^2}{N} \left(1 + \frac{\sigma_w^2}{N} \frac{m_H(z_1) - m_H(z_2)}{z_1 - z_2} \right) F_{n,m}(z_1, z_2) \quad (23)$$

where F is defined by $F_{n,m}(z_1, z_2) = \mathbb{E}[G_{n,n}(z_1)] \mathbb{E}[G_{n,n}(z_2)] \mathbb{E}[G_{m,m}(z_1)] \mathbb{E}[G_{m,m}(z_2)]$. Applying Eq.(22) to the case ($n = m$ and $p = q$), and using the relation

$$\mathbb{E}[G_{n,n}(z_1)] \mathbb{E}[G_{p,p}(z_2)] \approx (\mathbb{E}[G_{p,p}(z_2)] - \mathbb{E}[G_{n,n}(z_1)]) / (\epsilon_n - \epsilon_p + z_2 - z_1 + \sigma_w^2(m_H(z_2) - m_H(z_1))),$$

one can get the covariance between diagonal terms of the resolvent:

$$\boxed{\mathbb{E}[\delta G_{n,n}(z_1) \delta G_{p,p}(z_2)] \approx -\frac{\sigma_w^2}{N} \frac{1}{z_2 - z_1} \frac{\mathbb{E}[(G_{p,n}(z_2) - G_{p,n}(z_1))(G_{n,p}(z_2) - G_{n,p}(z_1))]}{\epsilon_n - \epsilon_p + z_2 - z_1 + \sigma_w^2(m_H(z_2) - m_H(z_1))}} \quad (24)$$

We also see that the covariance between diagonal terms is anti-symmetric under $z_1 \leftrightarrow z_2$ swap and it is smaller by a factor σ_w^2/N compared to the covariance between extra diagonal terms.

- Eq.18 – Eq.19 :

$$(z_1 - z_2)(\epsilon_n - \epsilon_m) \mathbb{E}[G_{n,m}(z_1) G_{p,q}(z_2)] = \frac{\sigma_w^2}{N} (\mathbb{E}[G_{n,q}(z_1) G_{p,m}(z_2)] - \mathbb{E}[G_{n,q}(z_2) G_{p,m}(z_1)]) \quad (25)$$

Note that this equations is exact (there is no error term contrary to Eq.(13) and (15)). Setting $n = q$ and $p = m$, one gets

$$(z_1 - z_2)(\epsilon_n - \epsilon_m) \mathbb{E}[G_{n,m}(z_1) G_{m,n}(z_2)] = \frac{\sigma_w^2}{N} (\mathbb{E}[G_{n,n}(z_1) G_{m,m}(z_2)] - \mathbb{E}[G_{n,n}(z_2) G_{m,m}(z_1)]) \quad (26)$$

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- [1] D. J. Thouless, *The quantum mechanics of many-body systems*. (New York: Academic Press., 1972).
- [2] R. D. Mattuck, *A guide to Feynman diagrams in the many-body problem* (New York: McGraw-Hill, 1976).
- [3] P. Nozières, *Theory of Interacting Fermi Systems* (Addison-Wesley, 1997).
- [4] F. M. Izrailev, *Physics Reports* **196**, 299 (1990).
- [5] F. Haake, *Quantum Signatures of Chaos* (Springer-Verlag, New York, 2nd ed., (2001)).
- [6] J. M. Deutsch, *Physical Review A* **43**, 2046 (1991).
- [7] D. Deutsch, *A closed quantum system giving ergodicity (Notes)* (2009).
- [8] G. Ithier, S. Ascroft, and F. Benaych-Georges, *Phys. Rev. E* **96**, 060102 (2017).
- [9] P. W. Anderson, *Physical Review* **109**, 1492 (1958).
- [10] P. A. Lee and T. V. Ramakrishnan, *Reviews of Modern Physics* **57**, 287 (1985).
- [11] F. Evers and A. D. Mirlin, *Reviews of Modern Physics* **80**, 1355 (2008).
- [12] R. Allez, J. Bun, and J.-P. Bouchaud, (2014), arXiv: 1412.7108.
- [13] J. Bun, R. Allez, J. Bouchaud, and M. Potters, *IEEE Transactions on Information Theory* **62**, 7475 (2016).
- [14] J. Bun, J.-P. Bouchaud, and M. Potters, arXiv:1603.04364 (2016).
- [15] H. Bethe, *Zeitschrift für Physik* **71**, 205 (1931).
- [16] P. L. Krapivsky, J. M. Luck, and K. Mallick, *Journal of Physics A: Mathematical and Theoretical* **48**, 475301 (2015).
- [17] J. Luck, *Journal of Physics A: Mathematical and Theoretical* **49**, 115303 (2016).
- [18] J. M. Luck, *Journal of Physics A: Mathematical and Theoretical* **50**, 355301 (2017).
- [19] H. Weyl, *Mathematische Annalen* **71**, 441 (1912).
- [20] A. A. Klyachko, *Selecta Mathematica* **4**, 419 (1998).
- [21] W. Fulton, *Séminaire Bourbaki*, Tome 40, (1997-1998), Exposé No. 845, p.255-269 (1998).
- [22] A. Knutson and T. Tao, *Notices Amer. Math. Soc.* **48**, 175 (2001).
- [23] F. J. Dyson, *Journal of Mathematical Physics* **3**, 1191 (1962).
- [24] D. Voiculescu, *J. Funct. Anal.* **66** (1986).
- [25] D. Voiculescu, *Communications in Mathematical Physics* **155**, 71 (1993).
- [26] P. Biane, arXiv e-prints (1998), math/9809193.
- [27] By generic position, we mean that the eigenvectors of one matrix are distributed isotropically in the eigenbasis of the other.
- [28] S. O'Rourke and V. Vu, *Random Matrices: Theory App.* **3** (2014).
- [29] J. A. Mingo and R. Speicher, *Journal of Functional Analysis* **235**, 226 (2006).
- [30] J. A. Mingo, P. Sniady, and R. Speicher, *Advances in Mathematics* **209**, 212 (2007).
- [31] B. Collins, J. Mingo, P. A., Sniady, and R. Speicher, *Documenta Mathematica* **12**, 1 (2007).
- [32] M. Wilkinson and P. N. Walker, *Journal of Physics A: Mathematical and General* **28**, 6143 (1995).
- [33] R. Allez and J.-P. Bouchaud, *Physical Review E* **86**, 046202 (2012).
- [34] R. Allez and J.-P. Bouchaud, *Random Matrices: Theory Appl.* **03** (2014).
- [35] K. Truong and A. Ossipov, *EPL (Europhysics Letters)* **116**, 37002 (2016).
- [36] A. M. Khorunzhy, B. A. Khoruzhenko, and L. A. Pastur, *Journal of Mathematical Physics* **37**, 5033 (1996).
- [37] M. Capitaine, Donati-Martin, F. C., D., and M. Février, **16**, 1750 (2011).
- [38] A. A. Abrikosov, L. P. Gorkov, and I. E. Dzyaloshinsky, *Methods of quantum field theory in statistical physics* (Dover, New-York, 1963).
- [39] Y. Makeenko, *Modern Physics Letters A* **06**, 1901 (1991).
- [40] Y. Makeenko, arXiv:hep-th/9112058 (1991).
- [41] We call Green functions the matrix elements of the dressed resolvent operator in the bare basis.
- [42] A. D. Mirlin and Y. V. Fyodorov, *Journal of Physics A: Mathematical and General* **24**, 2273 (1991).
- [43] Y. V. Fyodorov and A. D. Mirlin, *Physical Review Letters* **67**, 2049 (1991).
- [44] Y. V. Fyodorov and A. D. Mirlin, *Physical Review Letters* **69**, 1093 (1992).
- [45] A. D. Mirlin and Y. V. Fyodorov, *Journal of Physics A: Mathematical and General* **26**, L551 (1993).
- [46] S. Genway, A. F. Ho, and D. K. K. Lee, *Physical Review Letters* **111**, 130408 (2013).
- [47] M. Talagrand, *Ann. Probab.* **24**, 1 (1996).
- [48] M. Ledoux, *The concentration of measure phenomenon*, Mathematical surveys and monographs (American Mathematical Society, Providence (R.I.), 2001).
- [49] S. Chatterjee, *Probab. Theory Related Fields* **143**, 1 (2009).
- [50] V. V. Flambaum, G. F. Gribakin, and F. M. Izrailev, *Physical Review E* **53**, 5729 (1996).
- [51] V. K. B. Kota, *Embedded Random Matrix Ensembles in Quantum Physics* (Springer, 2014).
- [52] G. Ithier and F. Benaych-Georges, *Physical Review A* **96**, 012108 (2017).
- [53] P. W. Anderson, *Physical Review* **109**, 1492 (1958).
- [54] E. P. Wigner, *Annals of Mathematics* **62**, 548 (1955).
- [55] E. P. Wigner, *Annals of Mathematics* **65**, 203 (1957).
- [56] A. A. Gribakina, V. V. Flambaum, and G. F. Gribakin, *Physical Review E* **52**, 5667 (1995).
- [57] G. Gribakin, A. A. Gribakina, and V. V. Flambaum, *Australian Journal of Physics* **52**, 443.
- [58] I. V. Gornyi, A. D. Mirlin, and D. G. Polyakov, *Physical Review Letters* **95**, 206603 (2005).
- [59] D. M. Basko, I. L. Aleiner, and B. L. Altshuler, *Annals of Physics* **321**, 1126 (2006).

- [60] D. A. Huse, R. Nandkishore, V. Oganesyan, A. Pal, and S. L. Sondhi, *Physical Review B* **88**, 014206 (2013).
- [61] M. Schreiber, S. S. Hodgman, P. Bordia, H. P. Lüschen, M. H. Fischer, R. Vosk, E. Altman, U. Schneider, and I. Bloch, *Science* **349**, 842 (2015).
- [62] For instance, the microcanonical ensemble assumes all accessible states to be equiprobable.
- [63] E. Akkermans and G. Montambaux, *Mesoscopic Physics of Electrons and Photons*, edited by C. U. Press (2007).
- [64] V. E. Kravtsov, I. M. Khaymovich, E. Cuevas, and M. Amini, *New Journal of Physics* **17**, 122002 (2015).
- [65] C. Monthus, *Journal of Physics A: Mathematical and Theoretical* **50**, 115002 (2017).
- [66] R. Claessen, R. O. Anderson, J. W. Allen, C. G. Olson, C. Janowitz, W. P. Ellis, S. Harm, M. Kalning, R. Manzke, and M. Skibowski, *Physical Review Letters* **69**, 808 (1992).
- [67] C. E. Porter and R. G. Thomas, *Physical Review* **104**, 483 (1956).
- [68] J. L. Rosen, J. S. Desjardins, J. Rainwater, and W. W. Havens, *Physical Review* **118**, 687 (1960).
- [69] J. S. Desjardins, J. L. Rosen, W. W. Havens, and J. Rainwater, *Physical Review* **120**, 2214 (1960).
- [70] J. B. Garg, J. Rainwater, J. S. Petersen, and W. W. Havens, *Physical Review* **134**, B985 (1964).
- [71] V. Kargin, *Probability Theory and Related Fields* **154**, 677 (2012).
- [72] O. Z. G. Anderson, A. Guionnet, *An Introduction to Random Matrices*, Vol. 118 (Cambridge studies in advanced mathematics, 2009).
- [73] A. Guionnet, *Large Random Matrices. Lectures on Macroscopic Asymptotics*, Vol. 118 (Ecole d'Été de Saint-Flour XXXVI, 2006).
- [74] B. Eynard, *Random Matrices. Cours de Physique Théorique de Saclay*.
- [75] B. A. and M. B., *Nuclear structure*, edited by N. Y. Benjamin, Vol. 1 (1969).
- [76] N. Frazier, B. A. Brown, and V. Zelevinsky, *Physical Review C* **54**, 1665 (1996).
- [77] W. G. Brown and G. E. Gibson, *Physical Review* **40**, 529 (1932).
- [78] O. K. Rice, *The Journal of Chemical Physics* **1**, 375 (1933).
- [79] V. V. Flambaum, A. A. Gribakina, G. F. Gribakin, and M. G. Kozlov, *Physical Review A* **50**, 267 (1994).
- [80] Y. V. Fyodorov and A. D. Mirlin, in *On Three Levels: Micro-, Meso-, and Macro-Approaches in Physics*, edited by M. Fannes, C. Maes, and A. Verbeure (Springer US, Boston, MA, 1994) pp. 289–293.
- [81] O. Ledoit and S. Péché, *Probability Theory and Related Fields* **151**, 233 (2011).
- [82] P. Biane, *Mathematische Zeitschrift* **227**, 143 (1998).
- [83] V. Kargin, *Ann. Probab.* **43**, 2119 (2015).
- [84] F. Borgonovi, F. Izrailev, L. Santos, and V. Zelevinsky, *Physics Reports* **626**, 1 (2016).
- [85] D. Voiculescu, *Inventiones mathematicae* **104**, 201 (1991).
- [86] D. Voiculescu, *International Mathematics Research Notices* **1998**, 41 (1998).
- [87] P. Biane, *Indiana University Mathematics Journal* **46(3)** (1997).
- [88] A. Georges, G. Kotliar, W. Krauth, and M. J. Rozenberg, *Reviews of Modern Physics* **68**, 13 (1996).
- [89] G. Ithier and S. Ascroft, In preparation.