A CONTRACTION THEOREM FOR THE LARGEST EIGENVALUE OF A MULTIGRAPH

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ABSTRACT. Let G be a multigraph with loops, and let e be an edge in G. Let H be the multigraph obtained by contracting along the edge e. Let λ_G and λ_H be the largest eigenvalues of G and H respectively. A characterisation theorem is given of precisely when $\lambda_H < \lambda_G$, $\lambda_H = \lambda_G$, or $\lambda_H > \lambda_G$. In the case where H happens to be a simple graph, then so is G, and the theorem subsumes those of Hoffman-Smith and Gumbrell for subdivision of edges or splitting of vertices of a graph.

1. INTRODUCTION

Let H be a graph, and let e be an edge on an internal path (definitions will come later). Hoffman and Smith [3] showed that if one subdivides e to produce a graph G with an extra vertex, splitting e into two edges, then the largest eigenvalue goes down, unless the largest eigenvalue of H was equal to 2, in which case the largest eigenvalue of G is also 2. By contrast, if one subdivides an edge that is not on an internal path, then the largest eigenvalue goes up. Gumbrell extended this to the splitting of a vertex not on an internal path [2].

Reversing the subdivision process in either case, one moves from G to H by contracting along an edge. If one considers contracting along an arbitrary edge of a graph G, one might produce a multigraph H rather than a simple graph: multiple edges may appear. If one considers contracting along an edge of a multigraph, one might meet loops. Thus an attempt to unify and extend the work of Hoffman-Smith and Gumbrell by reversing their subdivision process naturally leads to working with multigraphs with loops. In this setting, we prove a general contraction theorem, Theorem 5, which describes precisely when the largest eigenvalue increases, decreases, or stays the same when an edge is contracted. This theorem subsumes the theorems of Hoffman-Smith and Gumbrell, and generalises them. Even in the setting of graphs it covers some cases not included in their theorems (although these extra graph cases are all trivial consequences of Perron-Frobenius theory).

The plan of the paper is as follows. First we make precise what we mean by a multigraph, define the operations of coalescing vertices and contracting along an edge, and recall some background Perron-Frobenius theory. Then we extend Smith's classification [6] of connected graphs that have largest eigenvalue at most 2 to the realm of multigraphs. We then state the main contraction theorem, Theorem 5, and derive some easy consequences: the theorems of Hoffman-Smith and Gumbrell, and (after proving a general coalescing theorem, Theorem 6) Simić's vertex-splitting theorem [5]. The final section of the paper gives the proof of the main theorem.

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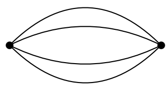
2. MULTIGRAPHS

Multigraphs (sometimes called pseudographs) mean different things to different authors. Let $A = (a_{ij})$ be an $n \times n$ symmetric matrix whose entries are nonnegative integers. We associate to this matrix a *multigraph* G as follows. Let $V = \{v_1, \dots, v_n\}$ be a set containing n distinct elements, called the *vertices* of G. If $i \neq j$ and $a_{ij} \neq 0$, then we say that the vertices v_i and v_j are *adjacent*, and write $v_i \sim v_j$. When $v_i \sim v_j$, the number a_{ij} indicates the number of *edges* between v_i and v_j (there can be more than one). When drawing pictures of graphs, we generally use small discs to indicate the vertices, and draw lines between the discs to indicate edges between those vertices. We should draw a_{ij} distinct lines between vertices v_i and v_j, but it may be more convenient (especially if a_{ij} is large!) to draw a single line labelled by the weight a_{ij} to capture this information. Thus



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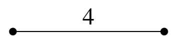
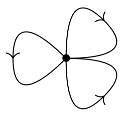


represent the same multigraph, with four edges between the two vertices.

Diagonal entries correspond to loops, and in order to allow odd integers on the diagonal our loops will be directed. If $a_{ii} > 0$, then there are a_{ii} directed loops on the vertex v_i. When drawing pictures, a loop will have an arrowhead on it to remind us that it is directed. Again it is sometimes convenient to draw a single directed loop labelled by the weight a_{ii} rather than drawing a_{ii} separate loops. We may also replace two directed loops by a single undirected loop, drawn without the arrowhead. Thus



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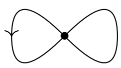

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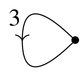


represent the same multigraph: a single vertex with three directed loops on it.

Let σ be any permutation of the set $\{1, \dots, n\}$. We view the matrices $A = (a_{ij})$ and $B = (a_{\sigma(i)\sigma(j)})$ as being *equivalent*. In terms of the associated multigraphs, equivalence of matrices merely corresponds to a relabelling of the elements of the vertex set; in terms of the matrices it involves conjugating by a permutation matrix, that is, applying a permutation to the rows and the same permutation to the columns. Equivalent matrices share the same eigenvalues, and we may refer to these either as the eigenvalues of the multigraph or of the matrix. If we start with a multigraph G described in terms of vertices, edges, and directed loops, then we can define its *adjacency matrix* A_G in the obvious way: label the vertices $v_1, \dots, v_n$, and define $A_G = (a_{ij})$ to be the $n \times n$ matrix where a_{ii} is the number of directed loops on the vertex v_i, and a_{ij} (for $i \neq j$) is the number of edges between v_i and v_j. A different labelling of the vertices would lead to an equivalent adjacency matrix, having the same eigenvalues. Multigraphs with equivalent adjacency matrices are regarded as being identical. Note that all eigenvalues are real, as our matrices are all symmetric real matrices.

The *degree* of a vertex v in a multigraph is the sum of the entries in the corresponding row of the adjacency matrix. It equals the number of edges between v and adjacent vertices, plus the number of directed loops on v.

A sequence of vertices $v_{i_1}, \dots, v_{i_r}$ is called a *pendant path* if:

- the vertices $v_{i_1}, \dots, v_{i_r}$ are distinct;
- $r \geq 2$;
- v_{i_1} has degree 1;

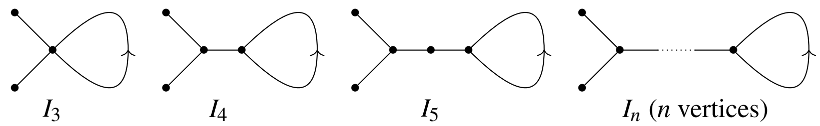
- $v_{i_2}, \ldots, v_{i_{r-1}}$ all have degree 2;
- $v_{i_j} \sim v_{i_{j+1}}$ for $1 \le j < r$.

A sequence of vertices $v_{i_1}, \ldots, v_{i_r}$ is called an *internal path* if:

- the vertices $v_{i_1}, \ldots, v_{i_r}$ are distinct except that we allow the possibility $v_{i_1} = v_{i_r}$;
- $r \ge 2$;
- each of v_{i_1} and v_{i_r} has degree > 2 or a loop;
- $v_{i_2}, \ldots, v_{i_{r-1}}$ all have degree 2;
- $v_{i_j} \sim v_{i_{j+1}}$ for $1 \le j < r$.

For example, the multigraph I_n shown in Figure 1 has one internal path when $n \ge 4$ and two pendant paths for all $n \ge 3$.

FIGURE 1. The multigraphs I_n (n vertices, $n \ge 3$)



3. COALESCING AND CONTRACTING

Let G be a multigraph and let u and v be distinct vertices of G, which might be adjacent or not. We can *coalesce* u and v to form a new multigraph H, which has a single vertex $\widehat{uv}$ which takes over all the edges and loops of u and v. More precisely:

- the vertices of H are those of G without u and v, and with a new vertex $\widehat{uv}$;
- if $z \notin \{u, v\}$ and there is an edge between u and z in G, then there is a corresponding edge between $\widehat{uv}$ and z in H;
- if $z \notin \{u, v\}$ and there is an edge between v and z in G, then there is a corresponding edge between $\widehat{uv}$ and z in H;
- any edge between u and v in G is replaced by two directed loops (equivalent to a single undirected loop) on $\widehat{uv}$ in H;
- any loops on u or v become loops on $\widehat{uv}$.

The example in Figure 2 below should help to make this process clear. Note that the degree of $\widehat{uv}$ in H is the sum of the degrees of u and v in G.

In terms of the adjacency matrices, coalescing u and v has the following effect. In the adjacency matrix of G, let row i and row j correspond to the vertices u and v respectively. The adjacency matrix of H can be formed from that of G in three steps:

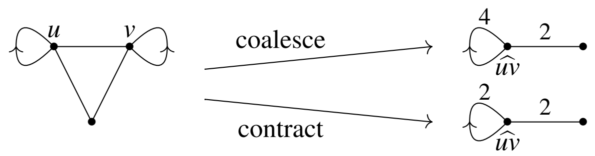
(i) add row i to row j;
(ii) add column i to column j;
(iii) delete row i and column i.

If there is an edge e between vertices u and v in a multigraph G, then we can *contract* along that edge to form a new multigraph H as follows. Intuitively, we shrink the edge to nothing, fusing the end vertices together. Formally, we delete the edge e, then coalesce u and v. The degree of the new vertex $\widehat{uv}$ in H is two less than the sum of the degrees of u and v in G. See Figure 2 below for an example.

In terms of the adjacency matrices, contracting along an edge between u and v has the following effect. In the adjacency matrix of G, let row i and row j correspond to the vertices u and v respectively. The adjacency matrix of H can be formed from that of G in four steps:

(i) subtract 1 from the (i,j) and (j,i) entries (which must have been strictly positive);
(ii) add row i to row j;
(iii) add column i to column j;
(iv) delete row i and column i.

FIGURE 2. Coalescing and contracting



4. RESULTS FROM PERRON-FROBENIUS THEORY

A multigraph G is *connected* if given any distinct vertices u and v in G, there is a sequence of vertices $v_{i_1}, \dots, v_{i_r}$ in G with $v_{i_1} = u$, $v_{i_r} = v$, and $v_{i_j} \sim v_{i_{j+1}}$ for $1 \le j < r$. We shall need the following three results from Perron-Frobenius theory, all contained in [1, Theorem 2.2.1] when applied to the language of multigraphs. For vectors $\mathbf{x} = (x_i)$ and $\mathbf{y} = (y_i)$ of the same dimension, we write $\mathbf{x} \ge \mathbf{y}$ to indicate $x_i \ge y_i$ for all i.

Lemma 1. *Let G be a connected multigraph with adjacency matrix A_G and largest eigenvalue λ_G, and let $\mathbf{v}$ be an eigenvector corresponding to the eigenvalue λ_G. Then $\mathbf{v}$ either has all entries strictly positive or all entries strictly negative. Any eigenvector with all entries strictly positive has eigenvalue λ_G, and is a scalar multiple of $\mathbf{v}$.*

Lemma 2. *Let G be a connected multigraph, and let H be a multigraph obtained by deleting any edge of G (there is no requirement that H be connected). Then the largest eigenvalue of G is strictly greater than that of H. If H is obtained by deleting any vertex of G, along with all incident edges or loops, then again the largest eigenvalue of H is strictly smaller than that of G.*

Lemma 3. *Let A be the adjacency matrix of a connected multigraph H. Suppose that there is nonzero vector $\mathbf{x}$ with nonnegative entries, and a real number t such that $A\mathbf{x} \ge t\mathbf{x}$, and $A\mathbf{x} \ne t\mathbf{x}$. Then the largest eigenvalue of H is strictly greater than t.*

5. CONNECTED MULTIGRAPHS WITH LARGEST EIGENVALUE AT MOST 2

Graphs that have largest eigenvalue at most 2 were classified by Smith [6]. We shall need to extend this classification to multigraphs. The result here can be gleaned from Lemma 5 and Theorem 7 of [4] by restricting to matrices that have nonnegative entries, but we give a direct reduction to Smith's result. By a submultigraph of a multigraph we mean one that is obtained by deleting some number (perhaps none) of the vertices along with all incident edges or loops, and further deleting some number (perhaps none) of the edges or loops.

Proposition 4. *Let G be a connected multigraph with largest eigenvalue at most 2. Then G is a submultigraph of one of those shown in Figure 1 or Figures 3–6: $\tilde{A}_n$ ($n+1$ vertices, $n \geq 0$), $\tilde{D}_n$ ($n+1$ vertices, $n \geq 4$), I_n (n vertices, $n \geq 3$), J_n (n vertices, $n \geq 2$, as $J_1 = \tilde{A}_0$), $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$.*

FIGURE 3. the multigraphs $\tilde{A}_n$ ($n+1$ vertices, $n \geq 0$)

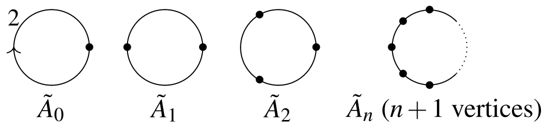
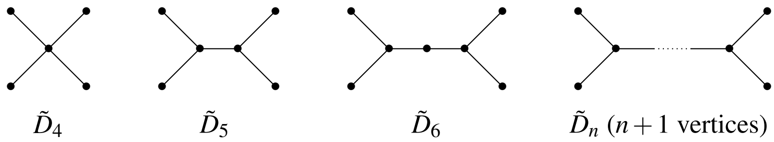


FIGURE 4. the multigraphs $\tilde{D}_n$ ($n+1$ vertices, $n \geq 4$)



Before embarking on the proof, note that Lemma 2 implies that any proper submultigraph of a connected multigraph has strictly smaller largest eigenvalue. Hence if H is a submultigraph of G and G has largest eigenvalue at most 2 and H has largest eigenvalue 2, then $G = H$. We also note that all the multigraphs in Figure 1 or Figures 3–6 have largest eigenvalue exactly 2. For $\tilde{A}_n$ and J_n, the all-ones vector is an eigenvector with eigenvalue 2, and by Perron-Frobenius theory (Lemma 1) this is the largest. For $\tilde{D}_n$ and I_n, one gets an eigenvector with (maximal by Lemma 1) eigenvalue 2 by assigning 1 to the entries corresponding to the degree-1 vertices, and 2 to all other entries. For the $\tilde{E}_n$, the vertices in Figure 6 are labelled with the entries of a Perron-Frobenius eigenvector with eigenvalue 2.

Proof. Suppose that G is a connected multigraph with largest eigenvalue at most 2. If G contains a multiple edge, then since $\tilde{A}_1$ has largest eigenvalue 2 we must have $G = \tilde{A}_1$ (Lemma 2). If G contains more than one loop on any vertex, then since $\tilde{A}_0$ has largest eigenvalue 2 we must have $G = \tilde{A}_0$ (Lemma 2 again). We are thus reduced to multigraphs with no multiple edges and at most one directed loop on each vertex.

If G has a vertex v with a single loop, then either G is a submultigraph of I_3 or $v = v_1$ has degree 2 with a single neighbour, say v_2. Then either G is a submultigraph of one of

FIGURE 5. the multigraphs J_n (n vertices, $n \geq 2$)

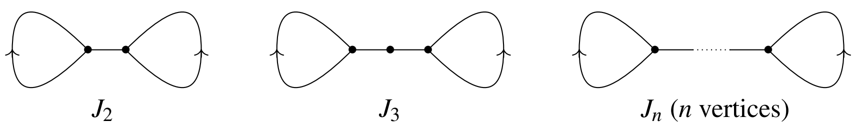
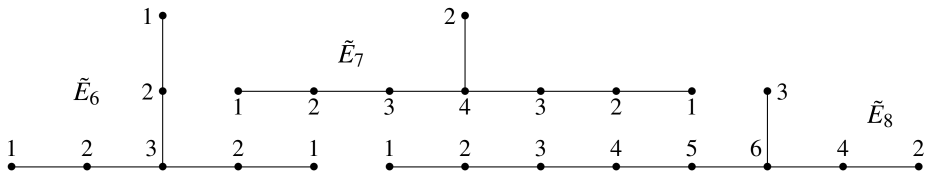


FIGURE 6. the multigraphs $\tilde{E}_n$ ($n+1$ vertices, $n = 6, 7, 8$; the vertex labels indicate Perron-Frobenius eigenvectors)



I_4 or J_2, or v_2 has degree 2 with neighbours v_1 and v_3, say. And so on. We conclude that G is a submultigraph of some I_n or J_n.

This reduces us to the cases where G is a simple graph, for which we can appeal to [6]. $\square$

6. THE CONTRACTION THEOREM

We are now ready to state the main theorem. Note that if G is a connected multigraph with largest eigenvalue strictly less than 2, then it is a connected proper submultigraph of one of the maximal examples of Proposition 4, and we observe that all paths are then pendant.

Theorem 5. *Let G be a connected multigraph with at least one edge, and with largest eigenvalue λ_G. Let e be an edge of G. Let H be the multigraph obtained by contracting along e, and let λ_H be the largest eigenvalue of H. Then:*

(i) *If e is on a pendant path, then $\lambda_H < \lambda_G$.*
(ii) *If e is not on a pendant path, and G is one of $\tilde{A}_n$ ($n \geq 1$), $\tilde{D}_n$ ($n \geq 5$), I_n ($n \geq 4$), or J_n ($n \geq 2$), then $\lambda_H = \lambda_G$.*
(iii) *If e is not on a pendant path, and G is not one of the multigraphs listed in (ii), then $\lambda_H > \lambda_G$.*

Note that the distinction between cases (ii) and (iii) can be expressed more simply in terms of the largest eigenvalue λ_G (given that e is not on a pendant path): if $\lambda_G = 2$, then $\lambda_H = \lambda_G = 2$, whilst if $\lambda_G > 2$ then $\lambda_H > \lambda_G$.

7. CONSEQUENCES OF THEOREM 5

Suppose that H, the contraction of G along an edge $\{u,v\}$, happens to be a graph. Then G itself is a graph.

- If either u or v has degree 2, then G is obtained from H by subdividing an appropriate edge, and Theorem 5 is seen to imply the Hoffman-Smith theorem [3] (which states that if we subdivide an edge in a graph H to produce a graph G, then $\lambda_G < \lambda_H$ unless the subdivided edge is on a pendant path (in which case $\lambda_G > \lambda_H$) or H is one of $\tilde{A}_n$ ($n \geq 3$) or $\tilde{D}_n$ ($n \geq 5$) (in which case $\lambda_G = \lambda_H$)).
- If u and v both have degree at least 3, then G is formed from H by a Gumbrell splitting of $\widehat{uv}$, and Theorem 5 is seen to imply the Gumbrell vertex-splitting theorem [2] (which states that if G is formed from H by splitting a vertex in a manner most simply described by saying that H is formed from G by contracting along an edge, then $\lambda_G < \lambda_H$, unless $H = \tilde{D}_4$ (in which case $\lambda_G = \lambda_H$)).

- If one of u and v has degree 1 and the other does not have degree 2, then producing G from H is neither a subdivision of an internal path nor a Gumbrell vertex-splitting. In this special case we are contracting along the only edge on a very short pendant path: we are in case (i) of Theorem 5, trivially implied by Perron-Frobenius theory.

Thus Theorem 5 is seen to subsume the graph theorems of Hoffman-Smith and Gumbrell, also covering some (trivial) additional cases. Working with contractions rather than subdivisions or splittings brings one naturally into the world of multigraphs, and Theorem 5 works in this more general settting.

If we coalesce two vertices, the picture is much simpler.

Theorem 6. *Let G be any connected multigraph with at least two vertices, and largest eigenvalue λ_G. Let u and v be distinct vertices of G, which might or might not be adjacent. Let H be formed by coalescing u and v, and let λ_H be the largest eigenvalue of H. Then $\lambda_H > \lambda_G$.*

Proof. Form G' by adding an(other) edge between u and v, and let $\lambda_{G'}$ be the largest eigenvalue of G'. By Lemma 2, $\lambda_{G'} > \lambda_G$. We can produce H from G' by contracting along this new edge. The new edge is not on a pendant path, as deleting it does not disconnect G', so we can appeal to one of cases (ii) or (iii) of Theorem 5 to conclude that $\lambda_H \geq \lambda_{G'} > \lambda_G$. □

If the multigraph H in Theorem 6 happens to be a graph, then so is G. Theorem 6 is then seen to imply the vertex-splitting theorem of Simić [5]. (Simić starts with a connected graph H, and splits a vertex to produce a graph G in a manner that is the reverse of coalescing; he shows that $\lambda_G < \lambda_H$. If G is connected, then this a special case of Theorem 6; if G is not connected, then it is a direct consequence of Perron-Frobenius theory.)

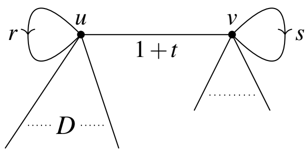
8. Proof of Theorem 5

Case (i) is an immediate consequence of Lemma 2: if e is on a pendant path then deleting the degree-1 vertex from the end of this path gives a multigraph equivalent to H.

Case (ii) follows from inspection of the multigraphs that have largest eigenvalue 2 and a path that is not pendant. Contracting an edge of $\tilde{A}_n$ $(n \geq 1)$ gives $\tilde{A}_{n-1}$; contracting an edge of I_n not on a pendant path $(n \geq 4)$ gives I_{n-1}; contracting an edge of J_n $(n \geq 2)$ gives J_{n-1} (with $J_1 = \tilde{A}_0$); contracting an edge of $\tilde{D}_n$ not on a pendant path $(n \geq 5)$ gives $\tilde{D}_{n-1}$.

The work is in case (iii). Here we have $\lambda_G > 2$ and $e = \{u, v\}$ not on a pendant path. Let A_G, A_H be adjacency matrices for G and H respectively. Let $\mathbf{x}$ be an eigenvector for A_G with eigenvalue λ_G with all entries strictly positive (Lemma 1). We shall construct a nonzero vector $\mathbf{y}$ with nonnegative entries indexed by the rows of A_H (i.e., by the vertices of H) satisfying $A_H\mathbf{y} \geq \lambda_G\mathbf{y}$ but with $A_H\mathbf{y} \neq \lambda_G\mathbf{y}$. By Lemma 3 this will imply that $\lambda_H > \lambda_G$.

For each vertex z of G, let x_z be the entry of $\mathbf{x}$ corresponding to the vertex z. Let $a_{z_1z_2}$ be the entry of A corresponding to any edges from vertex z_1 to vertex z_2 (or loops if $z_1 = z_2$). Swapping the labels of u and v if necessary, we can suppose that $x_u \leq x_v$, and then we can scale $\mathbf{x}$ so that $x_u = 1$ (and $x_v \geq 1$). Suppose that in addition to edge e there are a further $t \geq 0$ edges between u and v, that there are $r \geq 0$ directed loops on u, and $s \geq 0$ directed loops on v. Let N_u be the set of vertices in G adjacent to u other than v, and let N_v be the set of those adjacent to v other than u (of course N_u and N_v might well have non-empty intersection). Suppose that there are $D \geq 0$ edges leading from u to vertices in N_u. Some of this information is captured in the following picture.



Considering the u and v components of the equation $A_G\mathbf{x} = \lambda_G\mathbf{x}$ gives (with $x_u = 1$)

$$(1+t)x_v + r + \sum_{z\in N_u} a_{uz}x_z = \lambda_G\,, \tag{1}$$

and

$$1+t+sx_v + \sum_{z\in N_v} a_{vz}x_z = \lambda_G x_v\,. \tag{2}$$

We define $\mathbf{y}$, with entry y_z corresponding to the vertex z in H, as follows:

$$y_z = \begin{cases} x_v & z = \widehat{uv}\,,\\ x_z + (x_v-1)/\lambda_G & z\in N_u\,,\\ x_z & \text{otherwise.}\end{cases} \tag{3}$$

It would be more precise to use $y_z = x_z + a_{uz}(x_v-1)/\lambda_G$ for $z\in N_u$, but the value given in (3) works, and keeps ensuing formulas simpler. The point of this expression is that the component of $A_H\mathbf{y}$ corresponding to $z\in N_u$ is that of $A_G\mathbf{x}$ plus $a_{uz}(x_v-1)$, so is at least $\lambda_G x_z + (x_v-1)$, and so is at least $\lambda_G y_z$ for the given value of y_z. For vertices z in H other than $\widehat{uv}$ and other than those in N_u, and for the value of y_z in (3), the corresponding entry in $A_H\mathbf{y}$ is at least that in $A_G\mathbf{x}$, so is at least $\lambda_G y_z$. Thus we will be done by Lemma 3 if only we can establish the following strict inequality when we compute the component of $A_H\mathbf{y}$ at $\widehat{uv}$:

$$(r+s+2t)x_v + \sum_{z\in N_u} a_{uz}\Big(x_z + \frac{x_v-1}{\lambda_G}\Big) + \sum_{z\in N_v} a_{vz}x_z + \sum_{z\in N_u\cap N_v} a_{vz}\frac{x_v-1}{\lambda_G} \overset{?}{>} \lambda_G x_v\,. \tag{4}$$

Using (1) and (2) with $D = \sum_{z\in N_u} a_{uz}$, this simplifies to

$$(r+t-1)(x_v-1) + (\lambda_G-2) + \frac{x_v-1}{\lambda_G}\left(D + \sum_{z\in N_u\cap N_v} a_{vz}\right) \overset{?}{>} 0\,. \tag{5}$$

If $r+t\geq 1$, then (5) is trivial, since $\lambda_G>2$ and $x_v\geq 1$.

So we may suppose henceforth that $r=t=0$. Now (1) implies $x_v<\lambda_G$, and hence $x_v(\lambda_G-2) < \lambda_G(\lambda_G-2) = \lambda_G^2-\lambda_G-2+(2-\lambda_G) < \lambda_G^2-\lambda_G-2$, giving

$$\lambda_G^2 - \lambda_G x_v - \lambda_G + 2(x_v-1) > 0\,. \tag{6}$$

Suppose that $D\geq 2$. Then (6) gives

$$\lambda_G - x_v - 1 + D\frac{x_v-1}{\lambda_G} \geq (\lambda_G^2-\lambda_G x_v-\lambda_G+2(x_v-1))/\lambda_G > 0\,,$$

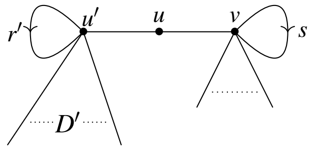
and (5) holds (when $r=t=0$).

We cannot have $D=0$ (given $r=t=0$), else the edge e would be on a pendant path. There remains the awkward case $D=1$. Then u has a unique neighbour u' other than v, with a single edge between u and u', and (5) is implied by

$$x_{u'} - 1 + (x_v-1)/\lambda_G \overset{?}{>} 0$$

(using (1) to give $x_{u'} = \lambda_G - x_v$). If $x_{u'}>1$, then we are done.

There remains the case $x_{u'} \leq 1$. Now we observe that the multigraph obtained by contracting along the edge between u' and u is the same as that obtained by contracting along e (see the following picture, in which u' and v might be adjacent).



We repeat the whole argument with (u,v) replaced by (u',u). With obvious notation (r', D', $t'=0$ for u' corresponding to r, D, t for u), we conclude $\lambda_H > \lambda_G$ unless we are the awkward case $r'=0$, $D'=1$. In that case, u' has a unique neighbour other than u, say u''. Again we conclude $\lambda_H > \lambda_G$ unless $x_{u''} \leq x_{u'}$, in which case we consider contracting along the edge between u'' and u'. And so on. Since our graph is finite, and we cannot loop around in a cycle of degree-2 vertices ($\lambda_G > 2$), this process must terminate with the desired conclusion: $\lambda_H > \lambda_G$.

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