SYMMETRIC SEMINORMS AND THE LEIBNIZ PROPERTY

ZOLTÁN LÉKA

ABSTRACT. We show that certain symmetric seminorms on $\mathbb{R}^n$ satisfy the Leibniz inequality. As an application, we obtain that L^p norms of centered bounded real functions, defined on probability spaces, have the same property. Even though this is well-known for the standard deviation it seems that the complete result has never been established. In addition, we shall connect the results with the differential calculus introduced by Cipriani and Sauvageot and Rieffel's non-commutative Riemann metric.

1. INTRODUCTION

Let $(S, \mathcal{F}, \mu)$ denote a probability space and let $1 \leq p < \infty$. The seminorm given by the pth absolute central moment of a random variable $f \colon S \to \mathbb{R}$ is

$$\sigma_p(f; \mu) = \|f - \mathbb{E}f\|_p = \left(\int_S \left| f - \int_S f \, d\mu \right|^p d\mu \right)^{1/p}.$$

One of the most used quantity in probability theory and statistics is the standard deviation (when $p = 2$). Recently M.A. Rieffel observed that the standard deviation in ordinary and non-commutative probability spaces satisfies the strong Leibniz inequality and even matricial seminorms have the same property [23]. To be more precise, we say that a seminorm L on a unital normed algebra $(\mathcal{A}, \|\cdot\|)$ is strongly Leibniz if (i) $L(1_\mathcal{A}) = 0$, (ii) the Leibniz property

$$L(ab) \leq \|a\|L(b) + \|b\|L(a)$$

holds for all $a, b \in \mathcal{A}$ and, furthermore, (iii) for all invertible a, $L(a^{-1}) \leq \|a^{-1}\|^2 L(a)$ follows. Given an ordinary probability space $(S, \mathcal{F}, \mu)$, this means that for all f and $g \in L^\infty(S, \mu)$, we have the inequalities

$$\|fg - \mathbb{E}(fg)\|_2 \leq \|g\|_\infty \|f - \mathbb{E}f\|_2 + \|f\|_\infty \|g - \mathbb{E}g\|_2$$

and

$$\|f^{-1} - \mathbb{E}(f^{-1})\|_2 \leq \|f^{-1}\|_\infty^2 \|f - \mathbb{E}f\|_2 \quad \text{if } f^{-1} \in L^\infty(S, \mu).$$

The study of the strong Leibniz property associated with non-commutative metrics on quantum metric spaces was initiated by M. Rieffel in his seminal papers [20], [22], [21]. This property played a crucial role in the development of a quantum theory for the Gromov–Hausdorff distance between classical metric spaces. A quantized version of the non-commutative theory was established in the recent papers by Li and Kerr [11], Wei Wu [28]. A novel approach to the long-standing problem of finding a proper metric between compact C^*-metric spaces has been offered by F. Latrémolière in [12], [13]. For a thorough survey on these types of results, we refer the reader to [14].

Very natural sources of strongly Leibniz seminorms are normed first-order differential calculi. We now recall that a normed first order differential calculus is a couple (Ω, ∂), where Ω is a normed bimodule over a unital normed algebra $\mathcal{A}$ such that

$$\|a\omega b\|_\Omega \leq \|a\|\|\omega\|_\Omega\|b\| \quad \text{for all } a, b \in \mathcal{A} \text{ and all } \omega \in \Omega,$$

and $\partial\colon \mathcal{A} \to \Omega$ is a derivation which satisfies the Leibniz rule $\partial(ab) = \partial(a)b + a\partial(b)$. Readily,

$$L(a) = \|\partial a\|_\Omega$$

is a (strongly) Leibniz seminorm on $\mathcal{A}$ (see [23, Proposition 1.1]).

A prototype of Leibniz seminorms is the Lipschitz number

$$L_\rho(f) = \sup\{|f(x) - f(y)|/\rho(x,y)\colon x \neq y\}$$

of complex-valued continuous functions defined on any compact metric space (X, ρ) ([27, Proposition 1.5.3]). It might happen that $L_\rho(f) = +\infty$ but L_ρ is well-defined on a dense $*$-subalgebra of $C(X)$. Interestingly, one can obtain L_ρ by means of a normed first order differential calculus ([26, Proposition 8], [22, Example 11.5]). We direct the interested reader to [6], [19], [26] and [27] for a comprehensive study of general Lipschitz seminorms, Lip-norms, and the associated Lipschitz algebras. Although, we are unaware of any characterization of the Leibniz property [19, Question 6.3], the lattice inequality $L(f \vee g) \leq L(f) \vee L(g)$, for all real f and g, is sufficient to conclude that a Lip-norm L is Leibniz ([19, Theorem 8.1]).

On the other hand, it is important to notice that any symmetric Dirichlet form $(D(\mathscr{E}), \mathscr{E})$ defined on a dense domain $D(\mathscr{E})$ of the real Hilbert space $L^2(S, \mu)$ satisfies the Leibniz inequality, see e.g. [8, Theorem 1.4.2], [3, Corollary 3.3.2]. See [16] and [15] for Dirichlet forms on finite sets, graphs and fractals. Furthermore, F. Cipriani and J.–L. Sauvageot [7] showed that every regular C^*-Dirichlet form can be represented as a quadratic form associated with a derivation taking its values in a Hilbert module, which is a direct link to the Leibniz rule.

Back to the standard deviation, one can present a direct simple proof of its strong Leibniz property (see [22], [1]). More interestingly, the (quantum) standard deviation completely fits into the aforementioned machinery of differential 1-forms. This idea was exploited independently from [7] in some depth in Rieffel's papers [22], [23] and based on the observation that the variance of any random variable $f \in L^2(S, \mu)$ can be written as

$$\|f - \mathbb{E}f\|_2^2 = \frac{1}{2} \iint_{S \times S} |f(x) - f(y)|^2 \, d\mu(x) \, d\mu(y)$$

(that is, it is a Dirichlet form on $L^2(S, \mu)$). His differential calculus is defined through the concept of spectral triples, introduced by Alain Connes, and what he calls non-commutative Riemann metric, see [6] and [22]. Hence one can say that the standard deviation, commutative or not, shares a flavor of Connes' noncommutative geometry.

The main goal of this paper is to show that the Leibniz inequality

$$\|fg - \mathbb{E}(fg)\|_p \leq \|g\|_\infty \|f - \mathbb{E}f\|_p + \|f\|_\infty \|g - \mathbb{E}g\|_p$$

is satisfied for all $1 \leq p \neq 2 < \infty$ and real $f, g \in L^\infty(S, \mu)$, which does not seem to have been noticed previously. We remark that the end-point case $p = \infty$ has already been settled in the recent paper [1]. First, we prove the result for the finite state space $S_n = \{1, \ldots, n\}$ endowed with the uniform probability measure. To get a friendly approach to the subject, we shall replace the ℓ^p norms with symmetric norms on $\mathbb{R}^n$. It should be stressed here that the essential part of the paper works with symmetric norms and the uniform case. In Section 3 we shall investigate the results in terms of different differential calculi, including

the Cipriani–Sauvageot algebraic construction of differential 1-forms, and a very brief connection with Rieffel's non-commutative Riemann metric. In Section 4, the failure of the normed bimodule property will lead us to a finite dimensional example of a Leibniz seminorm that is not strongly Leibniz. Lastly, in Section 5, we shall derive the Leibniz inequality for arbitrary probability measures by applying our earlier results on Leibniz seminorms in probability spaces [1]. Our paper is in part an attempt to reveal a possible link between normed differential calculi and absolute central moments of bounded functions (random variables). Our future plan is to study the corresponding results in non-commutative matrix and C^*-algebras as well as the case of complex-valued functions.

2. LEIBNIZ INEQUALITY FOR SYMMETRIC SEMINORMS

At first, we collect a few notations we require in order to prove the main results.

2.1. **Symmetric norms.** From now on let x_i denote the i-th component of any vector $x \in \mathbb{R}^n$. Let $|x|$ be the vector obtained by applying the absolute value to each component of x. We say that a norm $\|\cdot\|$ on $\mathbb{R}^n$ is symmetric if it is invariant under sign-changes and permutations of the components. Symmetric norms are monotone which means that

$$\|x\| \leq \|y\| \quad \text{if} \quad |x|^\downarrow \leq |y|^\downarrow,$$

where $|x|^\downarrow$ denotes the usual non-increasing rearrangement of the vector $|x|$ and the order between two vectors is the product order. Furthermore, the norm $\|\cdot\|$ is absolute so

$$\|x\| = \||x|\|$$

for all $x \in \mathbb{R}^n$ (see [2, Section 2]).

The vector k-norms (or Ky Fan k-norms) are special examples of symmetric norms. Indeed, the vector k-norm of x is defined by

$$\|x\|_{(k)} = \sum_{i=1}^{k} |x|_i^\downarrow.$$

In cases of $k = n$ and $k = 1$, we obtain the ℓ^1 and ℓ^∞ norms on $\mathbb{R}^n$, denoted by $\|\cdot\|_1$ and $\|\cdot\|_\infty$, respectively. We now recall that the dual norm of any symmetric norm is symmetric as well. This follows easily from the duality relation $\|x\|_* = \max\{\langle x, y\rangle \colon \|y\| \leq 1\}$, where $\langle \cdot, \cdot \rangle$ denotes the usual inner product on $\mathbb{R}^n$.

A celebrated theorem of Ky Fan says that, for any $x, y \in \mathbb{R}_+^n$, the inequalities

$$\|x\|_{(k)} \leq \|y\|_{(k)}$$

hold for all $1 \leq k \leq n$, that is, x is weakly majorized by y, if and only if

$$\|x\| \leq \|y\|$$

for all symmetric norms $\|\cdot\|$ on $\mathbb{R}^n$ (see [2] or [25, Chapter 15]). Hence one can look upon the vector k-norms as the cornerstones of symmetric norms.

Following Barry Simon's terminology in [25, p. 248], we introduce a class of real matrices.

Definition. We say that a matrix $A \in M_n(\mathbb{R})$ is real substochastic if

$$\sum_{i=1}^{n} |a_{ij}| \leq 1, \quad j = 1, \ldots, n,$$

$$\sum_{j=1}^{n} |a_{ij}| \leq 1, \quad i = 1, \ldots, n.$$

It is simple to see that A is real substochastic if and only if A is a contraction on $\mathbb{R}^n$ endowed with the ℓ^1 norm and the ℓ^∞ norm; i.e. $\|Ay\|_1 \le \|y\|_1$ and $\|Ay\|_\infty \le \|y\|_\infty$ for all $y \in \mathbb{R}^n$. Recall that the Calderón–Mityagin theorem (see [5], [17] or [25, Theorem 15.17]) says that if A is a real substochastic matrix then

$$\|Ay\| \le \|y\|$$

follows for all $y \in \mathbb{R}^n$ and symmetric norms $\|\cdot\|$.

Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Let us introduce the symmetric matrix Θ_x with zero row and column sums defined by

$$(\Theta_x)_{ij} = \begin{cases} \frac{1}{2n}(x_i + x_j) & \text{if } i \ne j \\ -\sum_{k:k \ne i}(\Theta_x)_{ik} & \text{if } i = j. \end{cases}$$

Let us define the matrix

$$I_x = I_n + \Theta_x,$$

where I_n denote the $n \times n$ identity matrix. Throughout this section we shall use the notation $\mathbb{E}f = \frac{1}{n}\sum_{i=1}^n f_i \mathbf{1}$, where $f = (f_1, \ldots, f_n) \in \mathbb{R}^n$ and $\mathbf{1}$ stands for the constant 1 vector. Moreover, we shall consistently use f and g for vectors in $\mathbb{R}^n$ and fg for their pointwise product.

Our first proposition links the product of two vectors $f, g \in \mathbb{R}^n$ with the matrices $I_{f+\mathbf{1}}, I_{g+\mathbf{1}}$.

Proposition 2.1. *For any $f, g \in \mathbb{R}^n$,*

$$I_{f+\mathbf{1}}(g - \mathbb{E}g) + I_{g+\mathbf{1}}(f - \mathbb{E}f) = \mathbb{E}(fg) - fg.$$

Proof. Clearly, it is enough to show that

$$I_f(g - \mathbb{E}g) + I_g(f - \mathbb{E}f) = \mathbb{E}((f - \mathbf{1})(g - \mathbf{1})) - (f - \mathbf{1})(g - \mathbf{1})$$

holds. A straightforward calculation gives for every index $1 \leq m \leq n$ that

$$n(I_f(g - \mathbb{E}g) + I_g(f - \mathbb{E}f))_m$$

$$= \frac{1}{2n} \sum_{1 \leq i \neq m \leq n} \sum_{1 \leq j \leq n} (f_i + f_m)(g_i - g_j)$$

$$+ \left(1 - \frac{1}{2n} \sum_{1 \leq i \neq m \leq n} (f_m + f_i)\right) \sum_{1 \leq j \leq n} (g_m - g_j)$$

$$+ \frac{1}{2n} \sum_{1 \leq i \neq m \leq n} \sum_{1 \leq j \leq n} (g_i + g_m)(f_i - f_j)$$

$$+ \left(1 - \frac{1}{2n} \sum_{1 \leq i \neq m \leq n} (g_m + g_i)\right) \sum_{1 \leq j \leq n} (f_m - f_j)$$

$$= \frac{1}{2n} \sum_{1 \leq i \neq m \leq n} (f_i + f_m)\left(\sum_{1 \leq j \leq n} (g_i - g_j) - \sum_{1 \leq j \leq n} (g_m - g_j)\right)$$

$$+ \frac{1}{2n} \sum_{1 \leq i \neq m \leq n} (g_i + g_m)\left(\sum_{1 \leq j \leq n} (f_i - f_j) - \sum_{1 \leq j \leq n} (f_m - f_j)\right)$$

$$+ \sum_{1 \leq i \leq n} (g_m - g_i) + \sum_{1 \leq i \leq n} (f_m - f_i)$$

$$= \frac{1}{2} \sum_{1 \leq i \leq n} ((f_i + f_m)(g_i - g_m) + (g_i + g_m)(f_i - f_m))$$

$$+ \sum_{1 \leq i \leq n} (g_m - g_i + f_m - f_i)$$

$$= \sum_{1 \leq i \leq n} (f_i g_i - f_m g_m + g_m - g_i + f_m - f_i)$$

$$= \left(\sum_{1 \leq i \leq n} (f_i - 1)(g_i - 1)\right) - n(f_m - 1)(g_m - 1)$$

$$= n(\mathbb{E}((f - \mathbf{1})(g - \mathbf{1})) - (f - \mathbf{1})(g - \mathbf{1}))_m,$$

which is what we intended to have. $\qquad\square$

Let us remember that the dual norm of the vector k-norm is

$$\|x\|_{(k)^*} = \max\left(\|x\|_\infty, \frac{\|x\|_1}{k}\right) \qquad x \in \mathbb{R}^n$$

(e.g. [2, Ex. IV.2.12]).

Let $\mathfrak{B}_{(k)^*} = \{x \in \mathbb{R}^n \colon \|x\|_{(k)^*} \leq 1\}$ denote the closed unit ball of the dual space $(\mathbb{R}^n, \|\cdot\|_{(k)})^*$. Then the set of extreme points of $\mathfrak{B}_{(k)^*}$ can be readily described. The result is well-known, however, we were unable to find any reference of it hence we sketch a short proof for the sake of completeness.

Lemma 2.2.

$$\operatorname{ext} \mathfrak{B}_{(k)^*} = \left\{\sum_{i \in S} \pm e_i \colon S \subseteq \{1, \ldots, n\} \text{ and } |S| = k\right\},$$

where e_i-s denote the standard basis elements of $\mathbb{R}^n$.

Proof. Let $\mathfrak{K}_0$ be the subset of the n-cube $[-1, 1]^n$ whose elements have at most k non-zero coordinates. It is not difficult to see that

$$\operatorname{conv} \mathfrak{K}_0 = \mathfrak{B}_{(k)^*}.$$

In fact, pick a point v in $\mathfrak{B}_{(k)^*}$ which has at most $k+1$ non-zero coordinates. Denote v_i a coordinate of v which has the smallest non-zero modulus. Obviously, $|v_i| \leq 1$. Now choose a vector $c \in \{-1, 0, 1\}^n$ such that the support of c has cardinality k, $i \in \operatorname{supp} c$ and $\operatorname{sign} c_j = \operatorname{sign} v_j$ for every $j \in \operatorname{supp} c$. Then it is simple to see that

$$\frac{v - |v_i|c}{1 - |v|_i} \in \mathfrak{B}_{(k)^*}.$$

Iterating the previous process, we arrive a point which has at most k non-zero coordinates. This point is the convex combination of vertices of a proper k-cube in $[-1, 1]^n$.   □

Now we are ready to prove the following proposition.

Proposition 2.3. *For any $f \in [-1, 1]^n$ and $1 \leq k \leq n$, the operator*

$$I_{f+\mathbf{1}}^* \colon (\mathbb{R}^n, \| \cdot \|_{(k)^*}) \to (\mathbb{R}^n, \| \cdot \|_{(k)^*})/\mathbb{R}, \quad x \mapsto I_{f+\mathbf{1}}x + \mathbb{R}$$

is a contraction.

Proof. First, to get an upper bound on the norm of $I_{f+\mathbf{1}}^*$, it is enough to calculate the norm of the class $I_{f+\mathbf{1}}v$ for all extreme points v of the unit ball $(\mathbb{R}^n, \| \cdot \|_{(k)^*})$. From Lemma 2.2, we can assume that

$$v = \sum_{i \in S_+} e_i - \sum_{i \in S_-} e_i$$

for some disjoint sets $S_+, S_- \subseteq \mathbb{Z}_n$ such that $|S_-| + |S_+| = k$. For any $x, y \in \mathbb{R}^n$ and $0 \leq s \leq 1$, we have $I_{sx+(1-s)y}^* = sI_x^* + (1-s)I_y^*$. Furthermore, since the quotient norm is convex, one has

$$\begin{aligned}
\|I_{f+\mathbf{1}}v\|_{(k)^*} &= \min_{\lambda \in \mathbb{R}} \|I_{f+\mathbf{1}}v - \lambda\mathbf{1}\|_{(k)^*} \\
&\leq \max_{x \in [0,2]^n} \min_{\lambda \in \mathbb{R}} \|I_x v - \lambda\mathbf{1}\|_{(k)^*} \\
&= \max_{x \in \{0,2\}^n} \min_{\lambda \in \mathbb{R}} \|I_x v - \lambda\mathbf{1}\|_{(k)^*}.
\end{aligned}$$

Next, pick an $x \in \{0, 2\}^n$. Set

$$r_v = \frac{1}{n}\langle x, v \rangle.$$

In order to prove that $I_x v$ is in the unit ball of the quotient space, we check that

$$\|I_x v - r_v 1\|_{(k)^*} \leq 1.$$

In fact,

$$\begin{aligned}
\|I_x v - r_v 1\|_\infty &= \max_{1 \leq i \leq n} \left| \langle I_x e_i - n^{-1}x, v \rangle \right| \\
&\leq \max_{1 \leq i \leq n} \left\| (I_x - n^{-1}x \otimes 1)e_i \right\|_{(k)} \|v\|_{(k)^*} \\
&\leq \max_{1 \leq i \leq n} \left\| (I_x - n^{-1}x \otimes 1)e_i \right\|_1.
\end{aligned}$$

Let $s = \operatorname{card}\{i : x_i = 2\}$. For any $1 \leq i \leq n$, note that

$$\begin{aligned}
\left\| (I_x - n^{-1}x \otimes 1)e_i \right\|_1 &= \left| 1 - \frac{1}{2n}\sum_{j=1}^n (x_i + x_j) \right| + \frac{1}{2n}\sum_{j=1}^n |x_i - x_j| \\
&= \begin{cases} \frac{s}{n} + \frac{n-s}{n} & \text{if } x_i = 2, \\ \left(1 - \frac{s}{n}\right) + \frac{s}{n} & \text{if } x_i = 0 \end{cases} \\
&= 1.
\end{aligned}$$

Thus
$$\|I_x v - r_v 1\|_\infty \le 1.$$

Now, let P_S denote the projection $\sum_{i=1}^n x_i e_i \mapsto \sum_{i \in S} x_i e_i$ on $\mathbb{R}^n$, where $S = S_- \cup S_+$ is the support of v. Then

$$
\begin{aligned}
\|I_x v - r_v 1\|_1 &= \sum_{i=1}^n \left| \left\langle P_S \left(I_x e_i - \frac{1}{n} x \right), v \right\rangle \right| \\
&\le \sum_{i=1}^n \left\| P_S \left(I_x e_i - \frac{1}{n} x \right) \right\|_{(k)} \|v\|_{(k)^*} \\
&\le \sum_{i=1}^n \left\| P_S \left(I_x e_i - \frac{1}{n} x \right) \right\|_1 \\
&= \sum_{i \in S} \left(\left| 1 - \frac{1}{2n} \sum_{j=1}^n (x_i + x_j) \right| + \frac{1}{2n} \sum_{j \in S} |x_i - x_j| \right) \\
&\quad + \sum_{i \notin S} \frac{1}{2n} \sum_{j \in S} |x_i - x_j| \\
&= \sum_{i \in S} \left(\left| 1 - \frac{1}{2n} \sum_{j=1}^n (x_i + x_j) \right| + \frac{1}{2n} \sum_{j=1}^n |x_i - x_j| \right),
\end{aligned}
$$

that is,

$$
\begin{aligned}
\|I_x v - r_v 1\|_1 &\le \sum_{i \in S} \left\| (I_x - n^{-1} x \otimes 1) e_i \right\|_1 \\
&= |S|.
\end{aligned}
$$

Hence
$$\|I_x v - r_v 1\|_{(k)^*} \le 1,$$

and the proof is complete. $\square$

Let us define the hyperplane

$$\mathfrak{X}_0 := \{ x \in \mathbb{R}^n : \mathbb{E}x = \sum_{i=1}^n x_i = 0 \} \subseteq \mathbb{R}^n.$$

Obviously, the dual of the Banach space $(\mathfrak{X}_0, \|\cdot\|_{(k)})$ is the quotient space $(\mathbb{R}^n, \|\cdot\|_{(k)^*})/\mathbb{R}$. In fact, $\mathfrak{X}_0$ is a one co-dimensional subspace of $\mathbb{R}$, whilst $\langle y, x - \mathbb{E}x \rangle = 0$ holds for all $y \in \mathbb{R}1$. Clearly, $I_{f+1} 1 = 1$. Hence the adjoint of $I_{f+1} \colon (\mathfrak{X}_0, \|\cdot\|_{(k)}) \to (\mathbb{R}^n, \|\cdot\|_{(k)})$ is the operator

$$I_{f+1}^* \colon (\mathbb{R}^n, \|\cdot\|_{(k)^*}) \to (\mathbb{R}^n, \|\cdot\|_{(k)^*})/\mathbb{R}, \quad x \mapsto I_{f+1} x + \mathbb{R},$$

of Proposition 2.3. Since $\|I_{f+1}|\mathfrak{X}_0\| = \|(I_{f+1}|\mathfrak{X}_0)^*\|$ (see e.g. [18, Proposition 2.3.10]), a straightforward corollary is the following statement.

Proposition 2.4. *Let* $f \in [-1, 1]^n$. *The operator* I_{f+1} *is a contraction on the normed space* $(\mathfrak{X}_0, \|\cdot\|_{(k)})$.

Furthermore, this leads us to the next proposition.

Proposition 2.5. *For any symmetric* $\|\cdot\|$ *on* $\mathbb{R}^n$ *and* $f \in [-1, 1]^n$, I_{f+1} *is a contraction on* $(\mathfrak{X}_0, \|\cdot\|)$.

Proof. For all $x \in \mathfrak{X}_0$ and $1 \le k \le n$, Proposition 2.4 says that

$$\sum_{i=1}^{k} |I_{f+\mathbf{1}}x|_i^\downarrow \le \sum_{i=1}^{k} |x|_i^\downarrow.$$

Thus the vector $|I_{f+\mathbf{1}}x|$ is weakly majorized by $|x|$. Now the absolute property of $\|\cdot\|$ and Ky Fan's theorem for symmetric norms give that

$$\|I_{f+\mathbf{1}}x\| = \||I_{f+\mathbf{1}}x|\| \le \||x|\| = \|x\|,$$

which is what we intended to have. $\hfill\square$

Now one can readily prove the following Leibniz inequality for symmetric norms.

Theorem 2.6. *Let $\|\cdot\|$ be a symmetric norm on $\mathbb{R}^n$. For all $f, g \in \mathbb{R}^n$, we have*

$$\|fg - \mathbb{E}(fg)\| \le \|g\|_\infty \|f - \mathbb{E}f\| + \|f\|_\infty \|g - \mathbb{E}g\|.$$

Proof. Without loss of generality, we can assume that $\|f\|_\infty = \|g\|_\infty = 1$. Applying Proposition 2.1 and Proposition 2.5, it follows that

$$\begin{aligned}
\|fg - \mathbb{E}(fg)\| &= \|I_{f+\mathbf{1}}(g - \mathbb{E}g) + I_{g+\mathbf{1}}(f - \mathbb{E}f)\| \\
&\le \|I_{f+\mathbf{1}}|\mathfrak{X}_0\| \|g - \mathbb{E}g\| + \|I_{g+\mathbf{1}}|\mathfrak{X}_0\| \|f - \mathbb{E}f\| \\
&= \|g - \mathbb{E}g\| + \|f - \mathbb{E}f\|,
\end{aligned}$$

and the proof is complete. $\hfill\square$

Remark. One can give a direct proof of Proposition 2.5 via the Calderón–Mityagin interpolation result as we briefly indicate. For any $x \in [0, 2]^n$, let us consider the matrix

$$L_x = I_x - \frac{1}{n} x \otimes \mathbf{1}.$$

We note that the off-diagonal part of L_x is skew-symmetric: $(L_x)_{i,j} = -(L_x)_{j,i}$ for every $i \ne j$, hence $\|L_x^T\|_{1\to 1} = \|L_x^T\|_{\infty\to\infty}$. From the proof of Proposition 2.3, it follows that

$$\|L_x^T\|_{1\to 1} \le 1 \quad \text{and} \quad \|L_x^T\|_{\infty\to\infty} \le 1.$$

Moreover, for any symmetric norm $\|\cdot\|$, the adjoint of $I_x \colon (\mathfrak{X}_0, \|\cdot\|) \to (\mathbb{R}^n, \|\cdot\|)$, $v \mapsto I_x v$, is the operator

$$I_x^* \colon (\mathbb{R}^n, \|\cdot\|_*) \to (\mathbb{R}^n, \|\cdot\|_*)/\mathbb{R},$$

where

$$I_x^* v = I_x v + \mathbb{R}$$

and $\|\cdot\|_*$ denotes the dual norm. Again, for any $v \in \mathbb{R}^n$, let $r_v = \frac{1}{n}\langle x, v\rangle$. Then

$$\begin{aligned}
\|I_x v - r_v \mathbf{1}\|_* &= \|I_x v - \frac{1}{n}\langle x, v\rangle\|_* \\
&= \|\langle (I_x - \frac{1}{n} x \otimes 1)e_i, v\rangle_i\|_* \\
&= \|L_x^T v\|_*.
\end{aligned}$$

Since the dual norm $\|\cdot\|_*$ is symmetric, the Calderón–Mityagin theorem says that

$$\min_{\lambda \in \mathbb{R}} \|I_x v - \lambda \mathbf{1}\|_* \le \|L_x^T v\|_* \le \|v\|_*.$$

That is,

$$\|I_x^*\| \le 1,$$

and the operator I_x is a contraction on $(\mathfrak{X}_0, \|\cdot\|)$.

Remark. Perhaps it is appropriate to note that if $x \in [0,1]^n$ then I_x is doubly stochastic. Hence, the Birkhoff–von Neumann theorem gives that $\|I_x\|_{\|\cdot\| \to \|\cdot\|} \leq 1$ for any permutation invariant norm $\|\cdot\|$ on $\mathbb{R}^n$. Now assume that f, g are nonnegative and $\|f\|_\infty = \|g\|_\infty = 1$. Then

$$I_{-f+\mathbf{1}}(\mathbb{E}g - g) + I_{-g+\mathbf{1}}(\mathbb{E}f - f) = \mathbb{E}(fg) - fg,$$

and the matrices $I_{-f+\mathbf{1}}$, $I_{-g+\mathbf{1}}$ are doubly stochastic as well. A simple corollary is the following statement.

Theorem 2.7. *Let $\|\cdot\|$ be a permutation invariant norm on $\mathbb{R}^n$. For any nonnegative vectors f and g in $\mathbb{R}^n_+$, we have*

$$\|fg - \mathbb{E}(fg)\| \leq \|g\|_\infty \|f - \mathbb{E}f\| + \|f\|_\infty \|g - \mathbb{E}g\|.$$

3. Derivations and the Leibniz inequality

To have a description of the Leibniz inequality in terms of derivations, we shall need to introduce the fundamental concepts of Laplacians and related Dirichlet forms on finite sets [16].

3.1. Laplacians and Dirichlet forms.
We recall that a Laplacian matrix Δ is a non-positive definite matrix such that its kernel is the subspace $\mathbb{R}\mathbf{1}$ and all of its off-diagonals are non-negative. Let us remember that every Laplacian Δ determines a Dirichlet form $\mathscr{E}_\Delta(u, v) = -\langle u, \Delta v \rangle$ on $\mathbb{R}^n \times \mathbb{R}^n$. To be precise, for any $f \in \mathbb{R}^n$ let us define the vector

$$\overline{f}_i = \begin{cases} 0 & \text{if } f_i \leq 0, \\ f_i & \text{if } 0 < f_i < 1, \\ 1 & \text{if } 1 \leq f_i. \end{cases}$$

A symmetric bilinear form $\mathscr{E}$ on $\mathbb{R}^n \times \mathbb{R}^n$ is a Dirichlet form if it satisfies the following properties:

(i) $\mathscr{E}(f, f) \geq 0$,
(ii) $\mathscr{E}(f, f) = 0$ if and only if $f \in \mathbb{R}\mathbf{1}$,
(iii) $\mathscr{E}(\overline{f}, \overline{f}) \leq \mathscr{E}(f, f)$ (Markovian property).

Actually, there is a one-to-one correspondence between the Laplacians and the Dirichlet forms on finite dimensional space, see [16, Proposition 2.1.3].

On the other hand, it is simple to see that

$$|f_i g_i - f_j g_j| = |f_i g_i - f_j g_i + f_j g_i - f_j g_j|$$
$$\leq \|g\|_\infty |f_i - f_j| + \|f\|_\infty |g_i - g_j|,$$

hence the Leibniz inequality

$$\mathscr{E}^{1/2}(fg, fg) \leq \|g\|_\infty \mathscr{E}^{1/2}(f, f) + \|f\|_\infty \mathscr{E}^{1/2}(g, g)$$

follows immediately (see [15, p. 281]).

Now let $f \in \mathbb{R}^n$ and P denote the orthogonal projection $f \mapsto \frac{1}{n} \sum_{i=1}^n f_i \mathbf{1}$ with respect to the usual inner product on $\mathbb{R}^n$. Then the matrix

$$\Delta_u = P - I_n = \frac{1}{n} \begin{pmatrix} 1-n & 1 & \cdots & 1 \\ 1 & 1-n & & 1 \\ \vdots & & \ddots & \vdots \\ 1 & \cdots & 1 & 1-n \end{pmatrix}$$

is a Laplacian and

$$\mathscr{E}_{\Delta_u}(f, f) = \|f - \mathbb{E}f\|_2^2,$$

where $\|\cdot\|_2$ denotes the ℓ_n^2-norm.

3.2. **Derivations and the Leibniz inequality.** Let μ be in general a probability measure on the set S_n. Then the variance of any vector f can be written as a Dirichlet form

$$\sigma_2^2(f;\mu) = -\langle f, \Delta_\mu f \rangle = \frac{1}{2} \sum_{x,y \in S_n} (f(x) - f(y))^2 \mu(x)\mu(y),$$

where the off-diagonal part of the Laplacian Δ_μ is $(\Delta_\mu)_{i,j} = \mu(i)\mu(j)$, $1 \le i \ne j \le n$. Now the deviation $\sigma_2(f;\mu)$ can be represented as the L^2-norm of the gradient vector $\partial_u f$, where ∂_u is the universal derivation $\partial_u f = f \otimes 1 - 1 \otimes f$, in the Hilbert space $L^2(S_n \times S_n, \mu \otimes \mu)$.

To obtain σ_p as a norm of a derivation we need a refined approach. Let us consider the matrix algebra $M_n(\mathbb{R}) = \ell_n^\infty \otimes \ell_n^\infty$ endowed with the Hilbert–Schmidt inner product as a bimodule over the finite dimensional algebra ℓ_n^∞, where the left and right actions are defined by linearity from

$$a(b \otimes c)d = ab \otimes cd.$$

Define the derivation $\partial \colon \ell_n^\infty \to M_n(\mathbb{R})$ by

$$(3.1) \qquad \partial f = \frac{1}{\sqrt{2n}}(f \otimes 1 - 1 \otimes f),$$

which satisfies the Leibniz equality, i.e. $\partial(fg) = \partial f \cdot g + f \cdot \partial g$. The adjoint operator ∂^*, defined by $\mathrm{Tr}(A^T \partial f) = \langle \partial^* A, f \rangle$ for any $A \in M_n(\mathbb{R})$, is the operator

$$(3.2) \qquad (\partial^* A)_i = -\frac{1}{\sqrt{2n}}(A(1 \otimes 1) - (1 \otimes 1)A)_{ii}.$$

Indeed, let ι denote the canonical embedding of the algebra ℓ_n^∞ into $M_n(\mathbb{R})$ as the diagonal algebra. Then one has that

$$\partial f = \frac{1}{\sqrt{2n}}((\iota f)1 \otimes 1 - 1 \otimes 1(\iota f)).$$

Since the extended derivation $d \colon A \mapsto A(1 \otimes 1) - (1 \otimes 1)A$ is a skew adjoint map on $M_n(\mathbb{R})$ with respect to the Hilbert–Schmidt inner product, we get $\partial^* = -\iota^* d$.

An elementary calculation implies the following lemma, whence we omit its proof.

Lemma 3.1. *One has the decomposition*

$$-\Delta_u = \partial^* \partial.$$

Then the following definition is quite natural.

Definition. Fix a symmetric norm $\|\cdot\|$ on $\mathbb{R}^n$ and let $\|\cdot\|_*$ denote its dual norm. We define a seminorm on the matrix algebra $M_n(\mathbb{R})$ by

$$\|A\|_\partial = \max \{ \mathrm{Tr}(A^T \partial f) \colon \|f\|_* \le 1\}.$$

The next proposition links the differential calculus $(M_n(\mathbb{R}), \partial)$ over ℓ_n^∞ with the norms of centered vectors.

Proposition 3.2. *Let $f = (f_1, \ldots, f_n) \in \ell_n^\infty$ and $\|\cdot\|$ be a symmetric norm on $\mathbb{R}^n$. Then the equality*

$$\left\| f - \frac{1}{n} \sum_{i=1}^n f_i \mathbf{1} \right\| = \|\partial f\|_\partial$$

holds.

Proof. From Lemma 3.1 and duality

$$\|\Delta_u f\| = \|\partial^* \partial f\| = \max\left\{\langle \partial^* \partial f, g \rangle \colon \|g\|_* \le 1\right\}$$
$$= \max\left\{\operatorname{Tr}(\partial f^T \partial g) : \|g\|_* \le 1\right\}$$
$$= \|\partial f\|_\partial.$$

$\square$

The next theorem shows a certain module property of the seminorm $\|\cdot\|_\partial$.

Theorem 3.3. *For any f and $g \in \mathbb{R}^n$,*

$$\|\partial f \cdot g\|_\partial \le \|g\|_\infty \|\partial f\|_\partial,$$

$$\|g \cdot \partial f\|_\partial \le \|g\|_\infty \|\partial f\|_\partial.$$

Proof. First, we have

$$(\partial f \cdot g)_{ij} = g_j (\partial f)_{ij} \quad \text{and} \quad (g \cdot \partial f)_{ij} = g_i (\partial f)_{ij}.$$

For any $a \in \mathbb{R}^n$, note that $I_g + a \otimes \mathbf{1} = I_g$ holds on the subspace $\mathfrak{X}_0 = (I_n - P)\mathbb{R}^n$. Thus

$$I_{g+\mathbf{1}} = I_{g+\mathbf{1}} - \frac{\mathbf{1}}{n} \otimes 1 = \Theta_g \quad \text{on} \quad \mathfrak{X}_0.$$

From Proposition 2.5

$$\|\Theta_g h\| \le \|g\|_\infty \|h\| \quad \text{for any } h \in \mathfrak{X}_0.$$

On the other hand, a direct calculation shows that

$$\langle h, \Theta_g f \rangle = \frac{1}{2n} \sum_{i,j=1}^n (f_i - f_j)(g_i + g_j) h_i = \frac{1}{2n} \sum_{i,j=1}^n (f_i - f_j) g_i (h_i - h_j)$$
$$= \operatorname{Tr}((g \cdot \partial f)^T \partial h).$$

Thus

$$\|g \cdot \partial f\|_\partial = \max\{\langle h, \Theta_g f \rangle \colon \|h\|_* \le 1\}$$
$$= \|\Theta_g f\|$$
$$= \|\Theta_g((I_n - P)f \oplus Pf)\|$$
$$= \|\Theta_g(I_n - P)f\|$$
$$\le \|g\|_\infty \|(I_n - P)f\|$$
$$= \|g\|_\infty \|\partial f\|_\partial.$$

The same argument gives that $\|\partial f \cdot g\|_\partial \le \|g\|_\infty \|\partial f\|_\partial$ holds, hence the proof is complete. $\square$

We saw in Theorem 2.6 that the Leibniz inequality is true with symmetric norms. Now one can provide a transparent reformulation of its proof relying upon the previous results.

Proof of Theorem 2.6.

$$\left\| fg - \frac{1}{n}\sum_{i=1}^n f_i g_i \mathbf{1} \right\| = \|\partial^* \partial(fg)\| = \|\partial^*(\partial f \cdot g) + \partial^*(f \cdot \partial g)\|$$
$$\le \|\partial f \cdot g\|_\partial + \|f \cdot \partial g\|_\partial$$
$$\le \|g\|_\infty \|\partial f\|_\partial + \|f\|_\infty \|\partial g\|_\partial$$
$$= \|g\|_\infty \left\| f - \frac{1}{n}\sum_{i=1}^n f_i \mathbf{1} \right\| + \|f\|_\infty \left\| g - \frac{1}{n}\sum_{i=1}^n g_i \mathbf{1} \right\|.$$

$\square$

At this point one can ask if the inequality

$$(3.3) \qquad \|f(\partial g)h\|_\partial \leq \|f\|_\infty \|h\|_\infty \|\partial g\|_\partial$$

holds for all f, g and $h \in \mathbb{R}^n$ and a normed bimodule structure might appear on a certain subspace of $M_n(\mathbb{R})$. Then the strong Leibniz inequality would be an immediate corollary of (3.3) due to the derivation rule $\partial f^{-1} = -f^{-1}(\partial f)f^{-1}$. Unfortunately, this is rarely the case as we can now see. Let us consider the seminorm $\|\cdot\|_{1,\partial} \equiv \|\partial^* \cdot\|_1$ on $M_n(\mathbb{R})$. Then, with choice of the vectors $f = h = (1, -1, 1, 1, 1)$ and $g = (1, -1, 0, 0, 0)$ in $\mathbb{R}^5$, one has

$$\|\partial g\|_{1,\partial} < \|f(\partial g)h\|_{1,\partial}.$$

However, in the case of $\|\cdot\|_{2,\partial}$ seminorm one can apply the differential calculus invented by Cipriani and Sauvageot [7] in order to prove (3.3) (see Proposition (3.6) below).

The crucial point in the previous proof of Theorem 2.6 and in the failure of (3.3) is the decomposition $-\Delta_u = \partial^*\partial$ which depends heavily on the Hilbert–Schmidt inner product. Another decomposition would emerge from the aforementioned Hilbert bimodule structure on $M_n(\mathbb{R})$ used by Cipriani and Sauvageot. This choice will be studied in the next section.

3.2.1. *Cipriani–Sauvageot differential calculus.* Here we shall briefly describe the Hilbert bimodule structure introduced in [7] (the interested reader might see [24] as well). The motivation there was to prove that any regular C^*-Dirichlet form can be represented as a quadratic form associated with a closable derivation. To have a natural connection with the previous parts of the paper, we shall only describe their algebraic construction in the real finite dimensional case.

Let us consider the left and right actions of ℓ_n^∞ on $\ell_n^\infty \otimes \ell_n^\infty = M_n(\mathbb{R})$ by linearity from

$$a(b \otimes c) = ab \otimes c - a \otimes bc$$
$$(b \otimes c)d = b \otimes cd.$$

Let $\mathscr{E}$ be a Dirichlet form on the set $S_n = \{1, \dots, n\}$. Then a positive bilinear form on $M_n(\mathbb{R})$ is given by

$$(c \otimes d, a \otimes b)_{\mathcal{H}} = \frac{1}{2}(\mathscr{E}(c, abd) + \mathscr{E}(cdb, a) - \mathscr{E}(db, ca)).$$

The Hilbert space $\mathcal{H}$ is obtained by taking the factor space by the zero-norm subspace. As a result, we get $\mathcal{H}$ is a Hilbert bimodule over ℓ_n^∞ [7, Theorem 3.7]. Furthermore, the map $\partial_0 \colon \ell_n^\infty \to \mathcal{H}$ defined by

$$\partial_0 f = f \otimes 1$$

is a derivation on ℓ_n^∞. Indeed, one can easily see that the Leibniz equality $\partial_0(fg) = \partial_0 f \cdot g + f \cdot \partial_0 g$ is satisfied. Interestingly, from [7, Theorem 4.7] one has the equality

$$(3.4) \qquad \mathscr{E}(f, f) = \|\partial_0 f\|_{\mathcal{H}}^2.$$

We know that there is a one-to-one correspondence between Dirichlet forms on finite sets and Laplace matrices, hence every Laplace matrix Δ can be decomposed as

$$(3.5) \qquad -\Delta = \partial_0^*\partial_0,$$

where the adjoint ∂_0^* is given by the formula $(\partial_0 f, a \otimes b)_{\mathcal{H}} = \langle f, \partial_0^*(a \otimes b)\rangle$ (see [7, Theorem 8.2] for the general case).

Now let us consider the Dirichlet form $\mathscr{E}_{\Delta_u}$ determined by $-\Delta_u = I - P$. First, let us calculate ∂_0^* with respect to the inner product defined by $\mathscr{E}_{\Delta_u}$.

Lemma 3.4. *The adjoint of the derivation* $\partial_0 \colon \ell_n^\infty \to \mathcal{H}$ *is the linear map*

$$(\partial_0^*(a \otimes b))_i = \frac{1}{2n} \sum_{j=1}^n (a_i - a_j)(b_i + b_j), \qquad 1 \leq i \leq n.$$

Proof. A little computation shows that

$$\langle \partial_0^*(a \otimes b), c \rangle = (a \otimes b, c \otimes 1)_{\mathcal{H}}$$

$$= \frac{1}{2}(\mathscr{E}_{\Delta_u}(c, ab) + \mathscr{E}_{\Delta_u}(a, bc) - \mathscr{E}_{\Delta_u}(b, ac))$$

$$= \frac{1}{2n} \sum_{i,j=1}^n (a_i b_i(c_i - c_j) + b_i c_i(a_i - a_j) - a_i c_i(b_i - b_j))$$

$$= \frac{1}{2n} \sum_{i=1}^n c_i \left(\sum_{j=1}^n (a_i b_i - b_i a_j + a_i b_j - a_j b_j) \right)$$

hence the proof is complete. □

Notice that we have $\partial_0^*(a \otimes b) = \Theta_b a$ with the notations of Section 2.

Lemma 3.5. *For any f, g and $h \in \ell_n^\infty$,*

$$(\partial_0^*(f(\partial_0 g)h))_i = \frac{1}{2n} \sum_{j=1}^n (g_i - g_j)(f_i h_j + f_j h_i).$$

Proof. From Lemma 3.4 notice that

$$\partial_0^*(f(\partial_0 g)h) = \partial_0^*(fg \otimes h - f \otimes gh) = \partial_0^*(f_i h_j(g_i - g_j))_{i,j}$$

$$= \left(\frac{1}{2n} \sum_{j=1}^n (f_i h_j + f_j h_i)(g_i - g_j) \right)_{1 \leq i \leq n},$$

which is what we intended to have. □

The previous lemmas show that the adjoints ∂_0^* and $(2n)^{-1/2}\partial^*$ in (3.2) are the same on the subspace of matrices with zero diagonal and

(3.6) $$\partial_0^*(f(\partial_0 g)h) = \partial^*(f(\partial g)h)$$

(the actions on both sides depend on the derivations ∂ in (3.1) and ∂_0). This also implies that no matter the decomposition $-\Delta_u = \partial^*\partial$ or $-\Delta_u = \partial_0^*\partial_0$ is considered, we need to overcome exactly the same inequalities to obtain the (strong) Leibniz property.

Now we can prove the following bimodule property of the norm $\| \cdot \|_{2,\partial}$ of the previous section.

Proposition 3.6. *For any f, g and $h \in \ell_n^\infty$,*

$$\|\partial^*(f(\partial g)h)\|_2 \leq \|f\|_\infty \|h\|_\infty \|\partial^*\partial g\|_2.$$

Proof. Notice that (3.6) guarantees that it is enough to prove the inequality

$$\|\partial_0^*(f(\partial_0 g)h)\|_2 \leq \|f\|_\infty \|h\|_\infty \|\partial_0^*\partial_0 g\|_2.$$

Since Δ_u is an orthogonal projection in ℓ_n^2, for any $x \in \mathbb{R}^n$,

$$\|\Delta_u x\|_2 = \langle -\Delta_u x, x \rangle^{1/2} = \|\partial_0 x\|_{\mathcal{H}},$$

where we used formula (3.4).

We observe that $\|\partial_0^* A\|_2 \leq \|A\|_{\mathcal{H}}$ for all $A \in M_n(\mathbb{R})$. Indeed, for any $x \in \mathbb{R}^n$, the Cauchy–Schwarz inequality and the previous equality imply

$$\langle \partial_0^* A, \partial_0^* A \rangle = (A, \partial_0 \partial_0^* A)_{\mathcal{H}} \leq \|A\|_{\mathcal{H}} \|\partial_0 \partial_0^* A\|_{\mathcal{H}} = \|A\|_{\mathcal{H}} \|\Delta_u \partial_0^* A\|_2 \leq \|A\|_{\mathcal{H}} \|\partial_0^* A\|_2.$$

Furthermore, [7, Theorem 3.7] gives the inequality

$$\|f(\partial_0 g)h\|_{\mathcal{H}} \le \|f\|_\infty \|h\|_\infty \|\partial_0 g\|_{\mathcal{H}}$$

for all f, g and $h \in \ell_n^\infty$.

Combining these observations, we get

$$
\begin{aligned}
\|\partial_0^*(f(\partial_0 g)h)\|_2 &\le \|f(\partial_0 g)h\|_{\mathcal{H}} \\
&\le \|f\|_\infty \|h\|_\infty \|\partial_0 g\|_{\mathcal{H}} \\
&= \|f\|_\infty \|h\|_\infty \|\partial_0^* \partial_0 g\|_2,
\end{aligned}
$$

so the proof is complete. $\qquad\square$

3.2.2. *Rieffel's non-commutative Riemann metric.* In [22] Rieffel introduced the concept of the non-commutative Riemann metric that turns out to be a rich source of strongly Leibniz seminorms in finite dimensional C^*-algebras. They provide us with an alternative way to obtain the Laplacian Δ_u as the divergence of a derivation. In short, a non-commutative Riemann metric is a normed first order differential calculus (Ω, ∂) over a C^*-algebra $\mathcal{A}$ with an $\mathcal{A}$-valued correspondence $(\cdot, \cdot)_{\mathcal{A}}$ defined on Ω, see [22, Section 3]. In particular, let us define the ℓ_n^∞-valued pre-inner product on the algebraic tensor product $\ell_n^\infty \otimes \ell_n^\infty$ by

$$(c \otimes d, a \otimes b)_{\ell_n^\infty} = bd\Gamma_{\Delta_u}(a, c),$$

where Γ_{Δ_u} is the carré-du-champ operator

$$\Gamma_{\Delta_u}(a, c) = -\frac{1}{2}(a\Delta_u c + c\Delta_u a - \Delta_u(ac)).$$

Then

$$\frac{1}{n}\mathscr{E}_{\Delta_u}(f, g) = \mathbb{E}(\Gamma_{\Delta_u}(f, g)) = \mathrm{Cov}_u(f, g),$$

where $\mathbb{E}$ and Cov_u denotes the expected value and the covariance with respect to the uniform probability measure. A positive bilinear form on $\ell_n^\infty \otimes \ell_n^\infty$ is given by

$$(c \otimes d, a \otimes b)_{\mathcal{H}} = \mathbb{E}(bd\Gamma_{\Delta_u}(a, c)).$$

It is simple to see that

$$
\begin{aligned}
(c \otimes d, a \otimes b)_{\mathcal{H}} &= -\frac{1}{2}\mathbb{E}(bcd\Delta_u a + abd\Delta_u c - bd\Delta_u(ac)) \\
&= \frac{1}{2n}(\mathscr{E}_{\Delta_u}(bcd, a) + \mathscr{E}_{\Delta_u}(c, abd) - \mathscr{E}_{\Delta_u}(bd, ac)).
\end{aligned}
$$

which essentially agrees with the bilinear form used by Cipriani and Sauvageot and leads to (3.5).

4. AN EXAMPLE

The failure of the bimodule inequality (3.3) in the previous section suggests the following real finite dimensional example of a Leibniz seminorm that is not strongly Leibniz. Such an example seems to have been unnoticed so far (see [23, p. 54]).

Let us define the Laplace matrix

$$
\Delta_3 = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}.
$$

Then the seminorm

$$L(f) = \|\Delta_3 f\|_\infty$$

defined on $\mathbb{R}^3$ is a Leibniz seminorm that is not strongly Leibniz.

In fact, let us choose the vector $f = (-0.1, 0.1, -0.2)^T$. A direct calculation gives that the inequality $L(1/f) \leq \|1/f\|_\infty^2 L(f)$ does not hold. For the Leibniz rule, let us consider the decomposition

$$\Delta_3(fg) = \Pi(f)g + \Pi(g)f,$$

where

$$\Pi(x) = \frac{1}{2} \begin{bmatrix} -(2x_1 + x_2 + x_3) & x_1 + x_2 & x_1 + x_3 \\ x_1 + x_2 & -(x_1 + x_2) & 0 \\ x_1 + x_3 & 0 & -(x_1 + x_3) \end{bmatrix}.$$

Then

$$\|\Pi(f)g\|_\infty \leq \|f\|_\infty \|\Delta_3 g\|_\infty \quad \text{and} \quad \|\Pi(g)f\|_\infty \leq \|g\|_\infty \|\Delta_3 f\|_\infty.$$

Indeed, without loss of generality, we can assume that $\|f\|_\infty = 1$. The function $f \mapsto \|\Pi(f)g\|_\infty$ is convex on the cube $[-1,1]^3$, hence it attains its maximum if f is in the vertex set $\{-1,1\}^3$. In addition,

$$\|\Pi(f)g\|_\infty = \max(|\varepsilon_{12}(g_1 - g_2) + \varepsilon_{13}(g_1 - g_3)|, |\varepsilon_{12}(g_1 - g_2)|, |\varepsilon_{13}(g_1 - g_3)|),$$

where $\varepsilon_{ij} = \dfrac{f_i + f_j}{2} \in \{-1, 0, 1\}$. Then it is straightforward to see that

$$\|\Pi(f)g\|_\infty \leq \|\Delta_3 g\|_\infty = \max(|g_1 - g_2 + g_1 - g_3|, |g_1 - g_2|, |g_1 - g_3|).$$

We can derive similarly the rest of the statement, which gives the requested result.

It would be interesting to know if the Leibniz inequality

$$\|\Delta(fg)\| \leq \|f\|_\infty \|\Delta g\| + \|g\|_\infty \|\Delta f\|$$

holds for all $n \times n$ Laplacians Δ and symmetric norms $\|\cdot\|$ on $\mathbb{R}^n$. This would be a particular discrete version of the Kato–Ponce inequality studied intensively in PDEs, see e.g [10], [9] and [4].

5. An application : the continuous case

In probability theory and statistics central moments and absolute central moments are primary objects which usually appear in estimates of probability distribution and their characteristic functions. M. Rieffel proved that the standard deviation is a strongly Leibniz seminorm in commutative and non-commutative probability spaces. He even extended these results to the case of matricial seminorms on a unital C^*-algebra [23].

We are now in a position to prove the Leibniz inequality for higher order moments of bounded real-valued random variables.

Here is one of the main results of the paper.

Theorem 5.1. *Let $(S, \mathcal{F}, \mu)$ be a probability space and $1 \leq p < \infty$. For any real f and $g \in L^\infty(S, \mu)$, we have*

$$\|fg - \mathbb{E}(fg)\|_p \leq \|g\|_\infty \|f - \mathbb{E}f\|_p + \|f\|_\infty \|g - \mathbb{E}g\|_p.$$

Proof. The statement is a corollary of Theorem 2.6 and the equivalence of Proposition 2.1 proved in [1]. $\square$

It would be interesting to have similar estimates in rearrangement invariant Banach function spaces.

Funding

ACKNOWLEDGEMENT

REFERENCES

[1] Á. Besenyei and Z. Léka, Leibniz seminorms in probability spaces, *J. Math. Anal. Appl.*, **429** (2015), 1178–1189.

[2] R. Bhatia, *Matrix analysis*, Springer–Verlag New York, 1997.

[3] N. Bouleau and F. Hirsch, *Dirichlet Forms and Analysis on Wiener space*, de Gruyter Studies in Mathematics, De Gruyter, Berlin, 1991.

[4] J. Bourgain and D. Li, On an endpoint Kato–Ponce inequality, *Differential Integral Equations*, **27** (2014), 1037–1072.

[5] A.-P. Calderón, Spaces between L^1 and L^∞ and the theorem of Marcinkiewicz, *Stud. Math.*, **26** (1966), 273–299.

[6] A. Connes, Compact metric spaces, Fredholm modules and hyperfiniteness, *Ergodic Theory and Dynamical Systems*, **9** (1989), 207–220.

[7] F. Cipriani and J.-L. Sauvageot, Derivations as square roots of Dirichlet forms, *J. Func. Anal.*, **201** (2003), 78–120.

[8] M. Fukushima, *Dirichlet Forms and Markov Processes*, North Holland Mathematical Library, 1980.

[9] L. Grafakos and S. Oh, The Kato–Ponce inequality, *Comm. Partial Differential Equations*, **39** (2014), 1128–1157.

[10] T. Kato and G. Ponce, Commutator estimates and the Euler and Navier-Stokes equations, *Comm. Pure Appl. Math.*, **41** (1988), 891–907.

[11] D. Kerr and H. Li, On Gromov-Hausdorff convergence for operator metric spaces, *J. Operator Theory*, **62** (2009), 83–109.

[12] F. Latrémolière, The dual Gromov–Hausdorff propinquity, *Journal de Mathématiques Pures et Appliquées*, **103** (2015), 303–351.

[13] F. Latrémolière, The quantum Gromov–Hausdorff propinquity, *Trans. Amer. Math. Soc.*, **368** (2016), 365–411.

[14] F. Latrémolière, Quantum metric spaces and the Gromov–Hausdorff propinquity, preprint, http://arxiv.org/pdf/1506.04341.

[15] R. Lyons and Y. Peres, *Probability on trees and networks*, preprint.

[16] J. Kigami, *Analysis on Fractals*, Cambridge University Press, Cambridge, 2001.

[17] B.S. Mityagin, An interpolation theorem for modular spaces (Russian), *Mat. Sb. (N.S.)*, **66** (1965), 473–482.

[18] G.K. Pedersen, *Analysis Now*, Springer–Verlag, 1989.

[19] M.A. Rieffel, Metrics on state spaces, *Doc. Math.*, **4** (1999), 559–600.

[20] M.A. Rieffel, Matrix algebras converge to the sphere for quantum Gromov–Hausdorff distance, Memoirs Amer. Math. Soc, 2004.

[21] M.A. Rieffel, Leibniz seminorms for "matrix algebras converge to the sphere", *Quanta of Maths* **11**, Amer. Math. Soc., Providence, RI, 2010, 543–578.

[22] M.A. Rieffel, Non-commutative resistance networks, *SIGMA Symmetry Integrability Geom. Methods Appl.*, **10** (2014), 2259–2274.

[23] M.A. Rieffel, Standard deviation is a strongly Leibniz seminorm, *New York J. Math.*, **20** (2014), 35–56.

[24] J.–L. Sauvageot, Quantum Dirichlet forms, differential calculus and semigroups, Quantum probability and applications, V (Heidelberg, 1988), 334–346, Lecture Notes in Math., 1442, Springer, Berlin, 1990.

[25] B. Simon, *Convexity: An Analytic Viewpoint*, Cambridge University Press, 2011.

[26] N. Weaver, Lipschitz algebras and derivations of von Neumann algebras, *J. Funct. Anal.*, 139 (1996), 261–300.

[27] N. Weaver, *Lipschitz algebras*, World Scientific Publishing Co., Inc., River Edge, NJ, 1999.

[28] W. Wu, Quantized Gromov-Hausdorff distance, *J. Funct. Anal.*, 238 (2006), 58–98.

ROYAL HOLLOWAY, UNIVERSITY OF LONDON, EGHAM HILL, EGHAM, SURREY, TW20 0EX, UNITED KINGDOM

E-mail address: zoltan.leka@rhul.ac.uk