Purely pathwise probability-free Itô integral*

Vladimir Vovk
v.vovk@rhul.ac.uk

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Abstract

This paper gives a simple construction of the pathwise Itô integral $\int_0^t \phi \, d\omega$ for an integrand ϕ and an integrator ω satisfying various topological and analytical conditions. The definition is purely pathwise in that neither ϕ nor ω are assumed to be paths of processes, and the Itô integral exists almost surely in a non-probabilistic finance-theoretic sense. For example, one of the results shows the existence of $\int_0^t \phi \, d\omega$ for a càdlàg integrand ϕ and a càdlàg integrator ω with jumps bounded in a predictable manner.

1 Introduction

The structure of this paper is as follows. To set the scene, Section 2 briefly describes papers and results that I am aware of related to the area of probability-free pathwise Itô integration. Section 3 defines the meaning of the phrase "a property holds almost surely" in a probability-free manner; however, to make our results stronger we will use a stronger condition that the property hold "quasi-always", which is also defined in that section. The main result of Section 3 is the existence of the pathwise Itô integral $\int_0^t \phi \, d\omega$ quasi-always. This result assumes the possibility of trading in ω (interpreted as the price path of a financial security) and the continuity of ϕ and ω (Theorem 1); it is "purely pathwise" in that neither ω nor ϕ are assumed to be paths of processes, and they can be chosen separately. Theorem 1 is proved in the following section, Section 4; the proof relies on a primitive "self-normalized game-theoretic supermartingale" introduced in Appendix A and a game-theoretic version of a classical martingale introduced in Appendix B. The proof can also be extracted from [20] (which, however, does not state Theorem 1 explicitly). Section 5 shows that continuous price paths possess quadratic variation quasi-always; in principle, this is a known result ([26], Theorem 5.1(a)), but we prove it in a slightly different setting (the one required for our Theorem 1). Once we have

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the quadratic variation, we can state a simple version of Itô's formula (Theorem 2) and show the coincidence of our integral with Föllmer's [10] in Section 6. Section 7 gives a definition of the Itô integral $\int_0^t \phi \, \mathrm{d}\omega$ in the case of càdlàg ϕ and ω. Theorem 3, asserting the existence of Itô integral in this case, is proved similarly to Theorem 1. The reader will notice that the setting of the former theorem is more complicated, and so we cannot say that it contains the latter as a special case. We do not compare the definition of Section 7 with Föllmer's since the latter assumes càglàd, rather than càdlàg, integrands. Section 8 makes a first step in defining purely pathwise Itô integrals $\int_0^t \phi \, \mathrm{d}\omega$ for non-regulated (in particular, non-càdlàg and non-càglàd) ϕ assuming, for simplicity, that ω is continuous. Finally, Section 9 concludes by listing some directions of further research.

2 Related literature

The first paper to give a probability-free definition of Itô integral was Föllmer's [10], who defined the integral $\int_0^t \phi \, \mathrm{d}\omega$ in the case where ϕ is obtained by composing a regular function f (namely, $f = F'$ for a C^2 function F) with ω (for simplicity, let us assume that ω is continuous in this introductory section). Föllmer's definition is pathwise in ω but not purely pathwise, as ϕ is a function of ω. Cont and Fournié [5] extend Föllmer's results by replacing the composition of f and ω by applying a non-anticipative functional f (also of the form F' where F is a non-anticipative functional of a class denoted $\mathbb{C}^{1,2}$ and the prime stands for "vertical derivative"). Cont and Fournié's definition is not quite pathwise in ω, but this is repaired by Ananova and Cont in [1] (for the price of additional restrictions on the non-anticipative functional F). Other papers (such as Perkowski and Prömel [19] and Davis et al. [7]) extend Föllmer's results by relaxing the regularity assumptions about f, which requires inclusion of local time. All these papers assume that ω possesses quadratic variation (defined in a pathwise manner), and this assumption is satisfied when ω is a typical price path (see, e.g., [22]; the existence of quadratic variation for such ω was established in, e.g., [26] and [25]; precise definitions will be given below). The existence of local times for typical continuous price paths follows from the main result of [26] (as explained in [19], p. 13) and was explicitly demonstrated, together with its several nice properties, in [19] (Theorem 3.5).

Another definition of pathwise Itô integral is given in the paper [17], but it is not completely probability-free. Besides, it depends on additional axioms of set theory (adding the continuum hypothesis is sufficient), and as the author points out, his "'construction' of the stochastic integral is not 'constructive' in the proper sense; it merely yields an existence result". This paper's construction is explicit.

Another paper on this topic is [25], but the construction used in it is Föllmer's, and the only novelty in [25] is that it shows the existence of quadratic variation for typical càdlàg price paths (under a condition bounding jumps).

To clarify the relation between the usual notion of "pathwise" and what we call "purely pathwise", let us consider two examples in which pathwise definitions are in fact purely pathwise but very restrictive.

Example 1 (Glenn Shafer)**.** Consider the Föllmer-type definition of the Itô integral $\int_0^t f(\omega(s), s)\,\mathrm{d}\omega(s)$ for a time-dependent function f ([24], Corollary 2.3.6; this definition is implicit in [10]). If f does not depend on its first argument, $f(\cdot, s) = \phi(s)$, we obtain a purely pathwise definition of $\int_0^t \phi\,\mathrm{d}\omega$. The problem is that the function f has to be very regular (of class $C^{2,1}$), and so this construction works only for very regular ϕ (such as C^1).

Example 2. The second example is provided by Föllmer's definition of the Itô integral $\int_0^t \nabla F(X(s))\,\mathrm{d}X(s)$ for a function $X : [0, \infty) \to \mathbb{R}^d$ having pathwise quadratic variation (as defined by Föllmer); this definition is given in, e.g., [10], pp. 147–148, and [24], Theorem 2.3.4. Let us take $d = 2$ and denote the components of X as ϕ and ω: $X(t) = (\phi(t), \omega(t))$ for all $t \in [0, \infty)$. For the existence of pathwise quadratic variation, it suffices to assume that ϕ and ω are the price paths of different securities in an idealized financial market (see, e.g., [25], Section 5). Taking $F(\phi, \omega) := \phi\omega$, we obtain the definition of the sum of purely pathwise Itô integrals $\int_0^t \phi\,\mathrm{d}\omega$ and $\int_0^t \omega\,\mathrm{d}\phi$. In this special case the integrand is no longer a function of the integrator, but even if we ignore the fact that $\int_0^t \phi\,\mathrm{d}\omega$ and $\int_0^t \omega\,\mathrm{d}\phi$ are still not defined separately, the fact that ϕ and ω are co-traded in the same market introduces a lot of logical dependence between them; e.g., in the case where $\phi(t) = \omega(t - \epsilon)$ for some $\epsilon > 0$ and for all $t \geq \epsilon$ we would expect the integral $\int_0^t \phi\,\mathrm{d}\omega$ to be well-defined but a market in which such ϕ and ω are traded becomes a money machine (unless ϕ and ω are degenerate, such as constant). Even if ϕ is not a price path of a traded security, the existence of its quadratic variation is a strong and unnecessary assumption. This paper completely decouples ϕ and ω (at least in the càdlàg case), and ϕ is never assumed to be a price path.

This paper is inspired by Rafał Łochowski's recent paper [14], which introduces the Itô integral $\int_0^t \phi\,\mathrm{d}\omega$ for a wide class of trading strategies ϕ as integrands in a probability-free setting similar to that [26] and [20]; the main advance of [14] as compared with [20] is its treatment of càdlàg price processes. The main observation leading to this paper is that $\int_0^t \phi\,\mathrm{d}\omega$ can be defined without assuming that ϕ is the realized path of a given strategy.

Papers that give purely pathwise definitions of Itô integral include [3] (Theorem 7.14) and [11], but the existence results in those papers are not probability-free.

Finally, on the face of it, the paper [20] by Perkowski and Prömel does not give a purely pathwise definition (namely, they assume the integrand to be a process rather than a path). Perkowski and Prömel consider two approaches to defining Itô integral. A disadvantage of their second approach is that it "restricts the set of integrands to those 'locally looking like'" ω ([20], the beginning of Section 4). Their first approach (culminating in their Theorem 3.5) constructs $\int_0^t \phi\,\mathrm{d}\omega$ in the case where ϕ is a path of a process on the sample space of continuous paths in $\mathbb{R}^d$, making ϕ a non-anticipative function of ω. It can, however, be applied to ω consisting of two components that can be used as the integrand and the integrator (as in Example 2 above) and, crucially, the proof of their Theorem 3.5 (see also Corollary 3.6) shows [18] that trading in the integrand is not needed; therefore, it also proves our Theorem 1.

After this paper had been submitted for publication, the technical report [15] extended some results of [20] to càdlàg price paths. The integrands in [15] are processes, but it might still be possible to extract from it purely pathwise results. An important topic of [20] and [15] is the continuity of Itô integration.

3 Definition of Itô integral in the continuous case

In our terminology and definitions we will follow mainly Section 2 of the technical report [26]. We consider a game between Reality (a financial market) and Sceptic (a trader) in continuous time: the time interval is $[0, \infty)$. First Sceptic chooses his trading strategy (to be defined momentarily) and then Reality chooses continuous functions ω and ϕ mapping $[0, \infty)$ to $\mathbb{R}$; ω is interpreted as the price path of a financial security (not required to be nonnegative), and ϕ is simply the function that we wish to integrate by ω. To formalize this picture we will be using Galmarino-style definitions, which are more intuitive than the standard ones (used in the journal version of [26]); see, e.g., [6].

Let
$$\Omega := C[0, \infty)^2 \tag{1}$$
be the set of all possible pairs (ω, ϕ); it is our *sample space.* We equip Ω with the σ-algebra $\mathcal{F}$ generated by the functions $(\omega, \phi) \in \Omega \mapsto (\omega(t), \phi(t))$, $t \in [0, \infty)$ (i.e., the smallest σ-algebra making them measurable). We often consider subsets of Ω and functions on Ω that are measurable with respect to $\mathcal{F}$. As shown in [27], the requirement of measurability is essential: without it, the theory degenerates.

As usual, an *event* is an $\mathcal{F}$-measurable set in Ω, a *random variable* is an $\mathcal{F}$-measurable function of the type $\Omega \to \mathbb{R}$, and an *extended random variable* is an $\mathcal{F}$-measurable function of the type $\Omega \to [-\infty, \infty]$. Each $o = (\omega, \phi) \in \Omega$ is identified with the function $o : [0, \infty) \to \mathbb{R}^2$ defined by
$$o(t) := (\omega(t), \phi(t)), \quad t \in [0, \infty).$$
A *stopping time* is an extended random variable $\tau : \Omega \to [0, \infty]$ such that, for all o and o' in Ω,
$$\left(o|_{[0,\tau(o)]} = o'|_{[0,\tau(o)]}\right) \Longrightarrow \tau(o) = \tau(o'), \tag{2}$$
where $f|_A$ stands for the restriction of f to the intersection of A and f's domain. A random variable X is said to be *determined by time* τ, where τ is a stopping time, if, for all o and o' in Ω,
$$\left(o|_{[0,\tau(o)]} = o'|_{[0,\tau(o)]}\right) \Longrightarrow X(o) = X(o').$$

As customary in probability theory, we will often omit explicit mention of $o \in \Omega$ when it is clear from the context.

A *simple trading strategy* G is defined to be a pair $((\tau_1, \tau_2, \ldots), (h_1, h_2, \ldots))$, where:

- $\tau_1 \le \tau_2 \le \cdots$ is a nondecreasing sequence of stopping times such that, for each $o \in \Omega$, $\lim_{n\to\infty} \tau_n(o) = \infty$;
- for each $n = 1, 2, \ldots$, h_n is a bounded random variable determined by time τ_n.

The *simple capital process* corresponding to a simple trading strategy G and an *initial capital* $c \in \mathbb{R}$ is defined, for $o = (\omega, \phi)$, by

$$\mathcal{K}_t^{G,c}(o) := c + \sum_{n=1}^{\infty} h_n(o)\bigl(\omega(\tau_{n+1} \wedge t) - \omega(\tau_n \wedge t)\bigr), \quad t \in [0, \infty),$$

where the zero terms in the sum are ignored (which makes the sum finite for each t). The value $h_n(o)$ is Sceptic's *bet* at time $\tau_n = \tau_n(o)$, and $\mathcal{K}_t^{G,c}(o)$ is Sceptic's *capital* at time t. The intuition behind this definition is that Sceptic is allowed to bet only on ω, but the current and past values of both ω and ϕ can be used for choosing the bets.

A *nonnegative capital process* is any function $\mathfrak{S}$ that can be represented in the form

$$\mathfrak{S}_t := \sum_{n=1}^{\infty} \mathcal{K}_t^{G_n, c_n}, \tag{3}$$

where the simple capital processes $\mathcal{K}^{G_n, c_n}$ are required to be nonnegative (i.e., $\mathcal{K}_t^{G_n, c_n}(o) \ge 0$ for all t and $o \in \Omega$), and the nonnegative series $\sum_{n=1}^{\infty} c_n$ is required to converge. The sum (3) can take value ∞. Since $\mathcal{K}_0^{G_n, c_n}(o) = c_n$ does not depend on o, $\mathfrak{S}_0(o)$ does not depend on o either and will sometimes be abbreviated to $\mathfrak{S}_0$.

The *outer measure* of a set $E \subseteq \Omega$ (not necessarily $E \in \mathcal{F}$) is defined as

$$\overline{\mathbb{P}}(E) := \inf\bigl\{\mathfrak{S}_0 \bigm| \forall o \in \Omega : \liminf_{t\to\infty} \mathfrak{S}_t(o) \ge \mathbf{1}_E(o)\bigr\}, \tag{4}$$

where $\mathfrak{S}$ ranges over the nonnegative capital processes and $\mathbf{1}_E$ stands for the indicator function of E. The set E is *null* if $\overline{\mathbb{P}}(E) = 0$. This condition is equivalent to the existence of a nonnegative capital process $\mathfrak{S}$ such that $\mathfrak{S}_0 = 1$ and, on the event E, $\lim_{t\to\infty} \mathfrak{S}_t = \infty$ (see, e.g., [26], Section 2). A property of $o \in \Omega$ will be said to hold *almost surely* if the set of o where it fails is null.

Remark 1. The definition (4) is less popular than its modification proposed in [20] (the latter has been also used in, e.g., [19], [2], [13], and [14]). Our rationale for the choice of the original definition (4) is that it is more conservative and, therefore, makes our results stronger. Its financial interpretation is that E is null if Sceptic can become infinitely rich splitting an initial capital of only one monetary unit into a countable number of accounts and running a simple trading strategy on each account making sure that no account ever goes into debt.

The intuition behind an event $E \subseteq \Omega$ holding almost surely is supposed to be that we do not expect it to happen in a market that is efficient to the weakest possible degree: indeed, there is a trading strategy that makes Sceptic starting with one monetary unit infinitely rich whenever the event fails to happen. However, the weakness of this interpretation is that becoming infinitely rich at the infinite time (cf. the lim inf in (4)) is not so surprising. Let us say that a property $E \subseteq [0, \infty) \times \Omega$ of time $t \in [0, \infty)$ and $o \in \Omega$ holds *quasi-always* (*q.a.*) if there exists a nonnegative capital process $\mathfrak{S}$ such that $\mathfrak{S}_0 = 1$ and, for all $t \in [0, \infty)$ and $o \in \Omega$,

$$(\exists s \in [0, t) : (s, o) \notin E) \Longrightarrow \mathfrak{S}_t(o) = \infty.$$

Intuitively, we require that Sceptic become infinitely rich immediately after the property becomes violated.

A *process* is a real-valued function X on the Cartesian product $[0, \infty) \times \Omega$; we will use $X_t(o)$ as the notation for the value of X at (t, o). A sequence of processes X^n converges to a process X *uniformly on compacts quasi-always* (*ucqa*) if the property

$$\lim_{n\to\infty} \sup_{s\in[0,t]} |X_s^n(o) - X_s(o)| = 0$$

of t and o holds quasi-always. A process X is *continuous* if its every path $t \in [0, \infty) \mapsto X_t(o)$ is. Notice that an ucqa limit of continuous processes has continuous paths almost surely.

Now we have all we need to define the Itô integral $\int_0^t \phi \, \mathrm{d}\omega$. First we define a sequence of stopping times T_k^n, $k = 0, 1, 2, \ldots$, inductively by $T_0^n(o) := 0$, where $o = (\omega, \phi)$, and

$$T_k^n(o) := \inf\Big\{ t > T_{k-1}^n(o) \mid \big|\omega(t) - \omega(T_{k-1}^n)\big| \vee \big|\phi(t) - \phi(T_{k-1}^n)\big| = 2^{-n} \Big\} \quad (5)$$

for $k = 1, 2, \ldots$ (as usual, $\inf \emptyset := \infty$); we do this for each $n = 1, 2, \ldots$. We let $T^n(o)$ stand for the *nth partition*, i.e., the set

$$T^n(o) := \{T_k^n(o) \mid k = 0, 1, \ldots\}.$$

Notice that the nestedness of the partitions, $T^1 \subseteq T^2 \subseteq \cdots$, is neither required nor implied by our definition.

Remark 2. The definition of the sequence (5) is different from the one in [26], Section 5, in that it uses not only the values of ω but also those of ϕ. In this respect it is reminiscent of the definitions in [3] (Theorem 7.14) and [11], where similar sequences of stopping times depend only on the values of ϕ.

For all $t \in [0, \infty)$, $\phi \in C[0, \infty)$, and $\omega \in C[0, \infty)$, define

$$(\phi \cdot \omega)_t^n := \sum_{k=1}^{\infty} \phi(T_{k-1}^n \wedge t)\Big(\omega(T_k^n \wedge t) - \omega(T_{k-1}^n \wedge t)\Big), \quad n = 1, 2, \ldots . \quad (6)$$

Theorem 1. *The processes $(\phi \cdot \omega)^n$ converge ucqa as $n \to \infty$.*

The limit whose existence is asserted in Theorem 1 will be denoted $\phi \cdot \omega$ and called the Itô integral of ϕ by ω. Its value at time t will be denoted $(\phi \cdot \omega)_t$ or $\int_0^t \phi \, \mathrm{d}\omega$. Since the convergence is uniform over $s \in [0, t]$ for each t, $(\phi \cdot \omega)_s$ is a continuous function of $s \in [0, t]$ quasi-always (and a continuous function of $s \in [0, \infty)$ almost sure).

4 Proof of Theorem 1

Let us first check the following basic property of the stopping times T_k^n (which will allow us to use these stopping times as components of simple trading strategies).

Lemma 1. *For each n, $T_k^n \to \infty$ as $k \to \infty$.*

Proof. Let us fix n and t and show that $T_k^n > t$ for some k. Each $s \in [0, t]$ has a neighbourhood in which ω and ϕ change by less than 2^{-n}. By the compactness of the interval $[0, t]$ we can choose a finite cover of this interval consisting of such neighbourhoods, and each such neighbourhood contains at most one T_k^n. □

We will often use the following technical lemma.

Lemma 2. *For any sequence $\mathcal{K}^n$, $n = 1, 2, \ldots$, of continuous nonnegative capital processes satisfying $\mathcal{K}_0^n \le 1$, we have $\sup_{s \in [0,t]} \mathcal{K}_s^n = O(n^2)$ as $n \to \infty$ q.a.*

Proof. Fix such a sequence of nonnegative capital processes $\mathcal{K}^n$. It suffices to show that $\sup_{s \in [0,t]} \mathcal{K}_s^n \le n^2$ from some n on q.a. Let $\tilde{\mathcal{K}}^n$ be the nonnegative capital process $\mathcal{K}^n$ stopped at the moment when it reaches level n^2: $\tilde{\mathcal{K}}_t^n := \mathcal{K}_{t \wedge \tau}^n$, where $\tau := \inf\{t \mid \mathcal{K}_t^n = n^2\}$ (it is here that we use the continuity of $\mathcal{K}^n$). Set $\tilde{\mathcal{K}} := \sum_n n^{-2} \tilde{\mathcal{K}}^n$. It remains to notice that $\tilde{\mathcal{K}}_0 < \infty$ and $\tilde{\mathcal{K}}_t = \infty$ whenever $\sup_{s \in [0,t]} \mathcal{K}_s^n > n^2$ for infinitely many n. □

The value of t will be fixed throughout the rest of this section. It suffices to prove that the sequence of functions $(\phi \cdot \omega)_s^n$ on the interval $s \in [0, t]$ is Cauchy (in the uniform metric) quasi-always.

Let us arrange the stopping times $T_0^n, T_1^n, T_2^n, \ldots$ and $T_0^{n-1}, T_1^{n-1}, T_2^{n-1}, \ldots$ into one strictly increasing sequence (removing duplicates if needed) a_k, $k = 0, 1, \ldots$: $0 = a_0 < a_1 < a_2 < \cdots$, each a_k is equal to one of the T_k^n or one of the T_k^{n-1}, each T_k^n is among the a_k, and each T_k^{n-1} is among the a_k. Let us apply the strategy leading to the supermartingale (30) (eventually we will be interested in (31)) to

$$\begin{aligned} x_k &:= b_n \left(\left((\phi \cdot \omega)_{a_k}^n - (\phi \cdot \omega)_{a_{k-1}}^n \right) - \left((\phi \cdot \omega)_{a_k}^{n-1} - (\phi \cdot \omega)_{a_{k-1}}^{n-1} \right) \right) \\ &= b_n \Big(\phi(a'_{k-1}) \left(\omega(a_k) - \omega(a_{k-1}) \right) - \phi(a''_{k-1}) \left(\omega(a_k) - \omega(a_{k-1}) \right) \Big) \\ &= b_n \left(\phi(a'_{k-1}) - \phi(a''_{k-1}) \right) \left(\omega(a_k) - \omega(a_{k-1}) \right), \end{aligned} \tag{7}$$

where $b_n := n^2$ (the rationale for this choice will become clear later), $a'_{k-1} := T^n_{k'}$ with k' being the largest integer such that $T^n_{k'} \le a_{k-1}$, and $a''_{k-1} := T^{n-1}_{k''}$ with k'' being the largest integer such that $T^{n-1}_{k''} \le a_{k-1}$. (Notice that either $a'_{k-1} = a_{k-1}$ or $a''_{k-1} = a_{k-1}$.) Informally, we consider the simple capital process $\mathcal{K}^n$ with starting capital 1 corresponding to betting $\mathcal{K}^n_{a_{k-1}}$ on x_k at each time a_{k-1}, $k = 1, 2, \dots$. Formally, the bet (on ω) at time a_{k-1} is $\mathcal{K}^n_{a_{k-1}} b_n \left(\phi(a'_{k-1}) - \phi(a''_{k-1})\right)$.

We often do not reflect n in our notation (such as a_k and x_k), but this should not lead to ambiguities.

The condition of Lemma 10 is satisfied as

$$|x_k| \le b_n 2^{-n+1} 2^{-n} \le 0.5, \tag{8}$$

where the last inequality (ensuring that (30) and (31) are really supermartingales) is true for all $n \ge 1$. By Lemma 10, we will have

$$\mathcal{K}^n_{a_K} \ge \prod_{k=1}^{K} \exp(x_k - x_k^2), \quad K = 0, 1, \dots .$$

Lemma 10 also shows that, in addition,

$$\mathcal{K}^n_s \ge \mathcal{K}^n_{a_{k-1}} \exp(x_{k,s} - x_{k,s}^2), \quad k = 1, 2, \dots, \quad s \in [a_{k-1}, a_k],$$

where

$$\begin{aligned} x_{k,s} &:= b_n \left(\left((\phi\cdot\omega)^n_s - (\phi\cdot\omega)^n_{a_{k-1}} \right) - \left((\phi\cdot\omega)^{n-1}_s - (\phi\cdot\omega)^{n-1}_{a_{k-1}} \right) \right) \\ &= b_n \left(\phi(a'_{k-1}) - \phi(a''_{k-1})\right) \left(\omega(s) - \omega(a_{k-1})\right) \end{aligned} \tag{9}$$

(cf. (7); notice that (8) remains true for $x_{k,s}$ in place of x_k). This simple capital process $\mathcal{K}^n$ is obviously nonnegative.

To cover both (7) and (9), we modify (9) to

$$x_{k,s} := b_n \left(\phi(a'_{k-1}) - \phi(a''_{k-1})\right) \left(\omega(a_k \wedge s) - \omega(a_{k-1} \wedge s)\right). \tag{10}$$

We have a nonnegative capital process $\mathcal{K}^n$ that starts from 1 and whose value at time s is at least

$$\exp\left(b_n \left((\phi\cdot\omega)^n_s - (\phi\cdot\omega)^{n-1}_s \right) - \sum_{k=1}^{\infty} x_{k,s}^2 \right). \tag{11}$$

Let us show that

$$\sup_{s\in[0,t]} \sum_{k=1}^{\infty} x_{k,s}^2 = o(1) \tag{12}$$

as $n \to \infty$ quasi-always. It suffices to show that

$$\sup_{s\in[0,t]} \sum_{k=1}^{\infty} \left(n^2 2^{-n+1} \left(\omega(a_k \wedge s) - \omega(a_{k-1} \wedge s)\right) \right)^2 = o(1) \qquad \text{q.a.} \tag{13}$$

Using the trading strategy leading to the K29 martingale (32), we obtain the simple capital process

$$
\begin{aligned}
\tilde{\mathcal{K}}_s^n &= n^{-3} + \sum_{k=1}^{\infty} \left(n^2 2^{-n+1} \left(\omega(a_k \wedge s) - \omega(a_{k-1} \wedge s) \right) \right)^2 \\
&\quad - \left(\sum_{k=1}^{\infty} n^2 2^{-n+1} \left(\omega(a_k \wedge s) - \omega(a_{k-1} \wedge s) \right) \right)^2 \\
&= n^{-3} + \sum_{k=1}^{\infty} n^4 2^{-2n+2} \left(\omega(a_k \wedge s) - \omega(a_{k-1} \wedge s) \right)^2 \\
&\quad - n^4 2^{-2n+2} \left(\omega(s) - \omega(0) \right)^2 .
\end{aligned}
\tag{14}
$$

Formally, this simple capital process corresponds to the initial capital $\tilde{\mathcal{K}}_0^n = n^{-3}$ and betting $-2n^4 2^{-2n+2}(\omega(a_{k-1}) - \omega(0))$ at time a_{k-1}, $k = 1, 2, \ldots$ (cf. (33) on p. 25). Let us make this simple capital process nonnegative by stopping trading at the first moment s when $n^4 2^{-2n+2} \left(\omega(s) - \omega(0) \right)^2$ reaches n^{-3} (which will not happen before time t for sufficiently large n); notice that this will make $\tilde{\mathcal{K}}^n$ nonnegative even if the addend $\sum_{k=1}^{\infty} \cdots (\cdots)^2$ in (14) is ignored. Since $\tilde{\mathcal{K}}^n$ is a continuous nonnegative capital process with initial value n^{-3}, applying Lemma 2 to $n^3 \tilde{\mathcal{K}}^n$ gives $\sup_{s \le t} \tilde{\mathcal{K}}_s^n = O(n^{-1}) = o(1)$ q.a. Therefore, the sum $\sum_{k=1}^{\infty} \cdots (\cdots)^2$ in (14) is $o(1)$ uniformly over $s \in [0, t]$ q.a., which completes the proof of (12).

In combination with (12), (11) implies

$$
\mathcal{K}_s^n \ge \exp\Big(b_n \left((\phi \cdot \omega)_s^n - (\phi \cdot \omega)_s^{n-1} \right) - 1 \Big)
$$

for all $s \le t$ from some n on quasi-always. Applying the strategy leading to the supermartingale (30) to $-x_{k,s}$ in place of $x_{k,s}$ and averaging the resulting simple capital processes (as in (31)), we obtain a simple capital process $\bar{\mathcal{K}}^n$ satisfying $\bar{\mathcal{K}}_0^n = 1$ and

$$
\bar{\mathcal{K}}_s^n \ge \frac{1}{2} \exp\Big(b_n \left| (\phi \cdot \omega)_s^n - (\phi \cdot \omega)_s^{n-1} \right| - 1 \Big) \tag{15}
$$

for all $s \le t$ from some n on quasi-always.

By the definition of $\bar{\mathcal{K}}^n$ and Lemma 2, we obtain that

$$
\sup_{s \in [0,t]} \frac{1}{2} \exp \left(n^2 \left| (\phi \cdot \omega)_s^n - (\phi \cdot \omega)_s^{n-1} \right| - 1 \right) = O(n^2) \qquad \text{q.a.}
$$

The last equality implies

$$
\sup_{s \in [0,t]} \left| (\phi \cdot \omega)_s^n - (\phi \cdot \omega)_s^{n-1} \right| = O\left(\frac{\log n}{n^2} \right) \qquad \text{q.a.}
$$

Since the series $\sum_n (\log n) n^{-2}$ converges, we have the ucqa convergence of $(\phi \cdot \omega)^n$ as $n \to \infty$.

5 Quadratic variation

In this section we will show that the quadratic variation of ω along T^n_k exists quasi-always. This was shown in, e.g., [26] and [25], but with "a.s." in place of "q.a." and for a different sequence of partitions (in fact, these are minor differences).

Define (essentially following [26], Section 5)

$$A^n_t(o) := \sum_{k=1}^{\infty} \left(\omega(T^n_k \wedge t) - \omega(T^n_{k-1} \wedge t)\right)^2, \quad n = 1, 2, \ldots,$$

for $o = (\omega, \phi)$.

Lemma 3. *The sequence of processes $A^n : (t, o) \mapsto A^n_t(o)$ converges ucqa as $n \to \infty$.*

We will use the notation $A_t(o)$ for the limit (when it exists) of $A^n_t(o)$ and will call it the *quadratic variation of ω at t*. We will also use the notation $A(o)$ for the *quadratic variation* $t \geq 0 \mapsto A_t(o)$ of the price path ω.

Proof of Lemma 3. The proof will be modelled on that of Theorem 1 in Section 4 (but will be simpler); we start from fixing the value of t. Let us check that the sequence $A^n|_{[0,t]}$ is Cauchy in the uniform metric quasi-always.

Let us apply the supermartingale (30) to

$$\begin{aligned}
x_k &:= b_n \left(\left(A^n_{a_k}(o) - A^n_{a_{k-1}}(o) \right) - \left(A^{n-1}_{a_k}(o) - A^{n-1}_{a_{k-1}}(o) \right) \right) \\
&= b_n \Big(\left(\omega(a_k) - \omega(a'_{k-1})\right)^2 - \left(\omega(a_{k-1}) - \omega(a'_{k-1})\right)^2 \\
&\qquad - \left(\omega(a_k) - \omega(a''_{k-1})\right)^2 + \left(\omega(a_{k-1}) - \omega(a''_{k-1})\right)^2 \Big) \\
&= b_n \Big(-2\omega(a_k)\omega(a'_{k-1}) + 2\omega(a_{k-1})\omega(a'_{k-1}) \\
&\qquad + 2\omega(a_k)\omega(a''_{k-1}) - 2\omega(a_{k-1})\omega(a''_{k-1}) \Big) \\
&= 2b_n \left(\omega(a''_{k-1}) - \omega(a'_{k-1})\right) \left(\omega(a_k) - \omega(a_{k-1})\right)
\end{aligned}$$

and to $-x_k$, where a'_{k-1}, a''_{k-1}, and b_n are defined as before and we are interested only in $n \geq 4$. Instead of the bound (8) we now have

$$|x_k| \leq 2b_n 2^{-n+1} 2^{-n} = b_n 2^{-2n+2} \leq 0.5$$

(the last inequality depending on our assumption $n \geq 4$). The analogue of (15) is

$$\bar{\mathcal{K}}^n_s \geq \frac{1}{2} \exp\Big(b_n \left| A^n_s(o) - A^{n-1}_s(o) \right| - 1 \Big),$$

and so we have

$$\sup_{s \in [0,t]} \frac{1}{2} \exp\left(n^2 \left| A^n_s - A^{n-1}_s \right| - 1 \right) = O(n^2) \qquad \text{q.a.}$$

This implies

$$\sup_{s\in[0,t]}\left|A^n_s - A^{n-1}_s\right| = O\left(\frac{\log n}{n^2}\right) \qquad \text{q.a.}$$

and thus the uniform convergence of A^n_s over $s \in [0,t]$ quasi-always as $n \to \infty$. ☐

6 Itô's formula

In this section we state a version of Itô's formula which shows that our definition of Itô integral agrees with that of Föllmer [10] (when the latter is specialized to the continuous case and our sequence of partitions).

Theorem 2. *Let $F : \mathbb{R} \to \mathbb{R}$ be a function of class C^2. Then*

$$F(\omega(t)) = F(\omega(0)) + \int_0^t F'(\omega)\,\mathrm{d}\omega + \frac{1}{2}\int_0^t F''(\omega)\,\mathrm{d}A(\omega, F'(\omega)) \qquad q.a. \tag{16}$$

The notation $F'(\omega)$ and $F''(\omega)$ in (16) stands for compositions: e.g., $F'(\omega)(s) := F'(\omega(s))$ for $s \geq 0$. The integral $\int_0^t F''(\omega)\,\mathrm{d}A(\omega, F'(\omega))$ can be understood in the Stiltjes sense (either Riemann–Stiltjes or Lebesgue–Stiltjes, since the integrand is continuous), and A is the quadratic variation of ω. The arguments "$(\omega, F'(\omega))$" of A refer to the sequence of partitions ((5) with $\phi := F'(\omega)$) used when defining A.

Proof. By Taylor's formula,

$$\begin{aligned}F(\omega(T^n_k)) - F(\omega(T^n_{k-1})) &= F'(\omega(T^n_{k-1}))\Big(\omega(T^n_k) - \omega(T^n_{k-1})\Big)\\ &\quad + \frac{1}{2}F''(\xi_k)\Big(\omega(T^n_k) - \omega(T^n_{k-1})\Big)^2,\end{aligned}$$

where ξ_k is between $\omega(T^n_{k-1})$ and $\omega(T^n_k)$. It remains to sum this equality over $k = 1,\dots,K$, where K is the largest k such that $T^n_k \leq t$, and to pass to the limit as $n \to \infty$. ☐

Since Itô's formula (16) holds for Föllmer's [10] integral $\int_0^t F'(\omega)\,\mathrm{d}\omega$ as well (see the theorem in [10]), Föllmer's integral (defined only in the context of $\int F'(\omega)\,\mathrm{d}\omega$) coincides with ours quasi-always. This is true for the sequence of partitions (5) with $\phi := F'(\omega)$, provided it is dense (as required in Föllmer's definitions, which in this case are equivalent to ours: cf. [25], Proposition 4 of the journal version).

7 The case of càdlàg integrand and integrator

In this section we allow ω and ϕ to be càdlàg functions, and this requires adding further components to Reality's move, càdlàg functions ω_* and ω^* that control the jumps of ω in a predictable manner. The sample space Ω (the set of all possible moves by Reality) now becomes

$$\Omega := \left\{(\omega, \omega_*, \omega^*, \phi) \in D[0,\infty)^4 \mid \forall t \in (0,\infty) : \omega_*(t-) \leq \omega(t) \leq \omega^*(t-)\right\}, \quad (17)$$

where $D[0,\infty)$ is the Skorokhod space of all càdlàg real-valued functions on $[0,\infty)$, and $f(t-)$ stands for the left limit $\lim_{s\uparrow t} f(s)$ of f at $t > 0$.

The Ω of the previous section, (1), embeds into the Ω of this section, (17), by setting $\omega_* := \omega$ and $\omega^* := \omega$.

Remark 3. The condition on the jumps of ω given in (17) is similar to the condition given in [25], which assumes that ω_* and ω^* are functions of ω (i.e., that there are functions f_* and f^* such that $\omega_*(t) = f_*(\omega(t))$ and $\omega^*(t) = f^*(\omega(t))$ for all $t \in [0,\infty)$) and that $\omega = (\omega_* + \omega^*)/2$. It covers two important special cases:

- the jumps $\Delta\omega(t) := \omega(t) - \omega(t-)$ of ω, where $\Delta\omega(0) := 0$, are bounded by a known constant C in absolute value; this corresponds to $\omega_* := \omega - C$ and $\omega^* := \omega + C$;

- ω is known to be nonnegative (as price paths in real-world markets often are) and the relative jumps $\Delta\omega(t)/\omega(t-)$ (with $0/0 := 0$) are bounded above by a known constant C; this corresponds to $\omega_* := 0$ and $\omega^* := (1 + C)\omega$.

Each $o = (\omega, \omega_*, \omega^*, \phi) \in \Omega$ is identified with the function $o : [0,\infty) \to \mathbb{R}^4$ defined by

$$o(t) := (\omega(t), \omega_*(t), \omega^*(t), \phi(t)), \quad t \in [0,\infty).$$

The sample space Ω is equipped with the universal completion $\mathcal{F}$ of the σ-algebra generated by the functions $o \in \Omega \mapsto o(t)$, $t \in [0,\infty)$. After this change, the definitions of events, random variables, stopping times τ, and random variables determined by time τ remain as before (but with the new sample space Ω and new σ-algebra $\mathcal{F}$).

We need universal completion in the definition of $\mathcal{F}$ to have the following lemma.

Lemma 4. *If $A \subseteq \mathbb{R}$ is a closed set, its entry time by ω,*

$$\tau(o) := \min\{t \in [0,\infty) : \omega(t) \in A\},$$

o standing for $(\omega, \omega_*, \omega^*, \phi)$, is a stopping time.

Proof. See, e.g., the third example in [8] (combined with the universal measurability of analytic sets, Theorem III.33 in [9]). For completeness, however, we will spell out the simple argument. The condition (2) is obvious (as $\omega(\tau) \in A$), so we only need to check that τ is universally measurable. Fix $t \in [0,\infty)$; we will see that $\{\tau \leq t\}$ is universally measurable and even analytic. Let $\mathcal{B}_t$ be for the Borel σ-algebra on $[0,t]$ and $\mathcal{F}_t$ be the σ-algebra generated by the functions

$o \in \Omega \mapsto o(s)$, $s \in [0,t]$. Since A is closed, $\{\tau \le t\}$ is the projection onto Ω of the set $\{(s,o) \in [0,t] \times \Omega \mid \omega(s) \in A\}$. In combination with the progressive measurability of càdlàg processes (such as $\mathfrak{S}_s(o) := \omega(s)$) this implies that, since $\{(s,o) \in [0,t] \times \Omega \mid \omega(s) \in A\}$ is in the product σ-algebra $\mathcal{B}_t \times \mathcal{F}_t$, the set $\{\tau \le t\}$ is analytic. □

Remark 4. The analogues of Lemma 4 also hold for ϕ, ω_*, and ω^* in place of ω (as the same argument shows).

The definitions of a simple trading strategy, a simple capital process, a nonnegative capital process, and the outer measure stay the same as in Section 3 apart from replacing the argument "$o = (\omega, \phi)$" by "$o = (\omega, \omega_*, \omega^*, \phi)$"; "almost sure" and "quasi-always" are also defined as before.

The definition (5) of T_k^n is modified by replacing the equality with an inequality: $T_0^n(o) := 0$ and

$$T_k^n(o) := \inf\Big\{t > T_{k-1}^n(o) \mid \big|\omega(t) - \omega(T_{k-1}^n)\big| \vee \big|\phi(t) - \phi(T_{k-1}^n)\big| \ge 2^{-n}\Big\}, \qquad k = 1, 2, \dots . \tag{18}$$

After this change is made, the definition of $(\phi \cdot \omega)^n$ stays as before, (6). The analogue of Lemma 1 still holds:

Lemma 5. *For each n, $T_k^n \to \infty$ as $k \to \infty$.*

Proof. The proof is analogous to the proof of Lemma 1, except that now we choose a neighbourhood of each $s \in [0,t]$ in which ω changes by less than $|\Delta\omega(s)| + 2^{-n}$ and ϕ changes by less than $|\Delta\phi(s)| + 2^{-n}$. In each such neighbourhood there are fewer than 10 values of T_k^n (for a fixed n). □

The following theorem asserts the existence of Itô integral quasi-always in our current context.

Theorem 3. *The processes $(\phi \cdot \omega)^n$ converge ucqa as $n \to \infty$.*

Proof. Fix $t > 0$ and let E be the event that $(\phi \cdot \omega)_s^n$ fails to converge uniformly over $s \in [0,t]$ as $n \to \infty$. It suffices to prove that E is t-*null*, by which we mean the existence of a nonnegative capital process $\mathfrak{S}$ such that $\mathfrak{S}_0 = 1$ and, on the event E, $\mathfrak{S}_t = \infty$; we will say that such $\mathfrak{S}$ *witnesses* that E is t-null.

Assume, without loss of generality, that $\omega(0) = 0$ (this can be done as (6) is invariant with respect to adding a constant to ω).

First we notice (as in the proof of Theorem 1 of [25]) that it suffices to consider the modified game in which Reality does not output ω_* and ω^* but instead is constrained to producing $\omega \in D[0,\infty)$ satisfying $\sup_{s \in [0,\infty)} |\omega(s)| \le c$ for a given constant $c > 0$. Indeed, suppose that the statement in the first paragraph of the proof (for the given t) holds in the modified game for any c, and let $\mathfrak{S}^c$ be a nonnegative capital process witnessing that the analogue of the event E in the modified game is t-null. A nonnegative capital process $\mathfrak{S}$ witnessing that E is t-null in the original game can be defined as

$$\mathfrak{S}_s := \sum_{L=1}^{\infty} 2^{-L} \mathfrak{S}^{2^L}_{s\wedge\sigma_L} \tag{19}$$

where σ_L is the stopping time

$$\sigma_L := \inf\left\{s \mid \omega^*(s) \vee (-\omega_*(s)) \ge 2^L\right\}$$

(intuitively σ_L is the first moment when we can no longer guarantee that ω will not jump to or above 2^L in absolute value straight away; this is a stopping time by Lemma 4 and Remark 4). Let us check that each addend in (19) is nonnegative not only in the modified but also in the original game. Indeed, if $\mathfrak{S}^{2^L}_s < 0$ for some $s \le \sigma_L$, the nonnegativity of $\mathfrak{S}^{2^L}$ in the modified game (with $c = 2^L$) implies that, for some $s' \in [0, s]$, $|\omega(s')| > 2^L$. By (17), the last inequality implies $\omega^*(s'-) > 2^L$ or $\omega_*(s'-) < -2^L$. Therefore, $\omega^*(s'') > 2^L$ or $\omega_*(s'') < -2^L$ for some $s'' < s' \le s \le \sigma_L$, which contradicts the definition of σ_L. Let us now check that $\mathfrak{S}$ (which we already know to be nonnegative in the original game) witnesses that E is t-null. If $(\omega, \omega_*, \omega^*, \phi) \in E$, there is a constant c bounding $-\omega_*|_{[0,t]}$ and $\omega^*|_{[0,t]}$ from above. Any addend in (19) for which $2^L > c$ will be infinite at time t.

In the rest of this proof Reality is constrained to $\sup_s |\omega(s)| \le c$. Without loss of generality, set $c := 0.5$. We follow the same scheme as for Theorem 1, again defining x_k by (7) and $x_{k,s}$ by (10), with the same b_n. Notice that, for $n \ge 2$, we always have

$$\left|\phi(a'_{k-1}) - \phi(a''_{k-1})\right| \le 2^{-n+1} \tag{20}$$

in (7) and (10); therefore, we can replace (8) by

$$|x_k| \le b_n 2^{-n+1} \le 0.5 \tag{21}$$

(with the analogous inequality for $x_{k,s}$), where the last inequality is true $n \ge 8$, which we assume from now on in this proof.

Essentially the same argument as in Section 4 shows that (12) still holds quasi-always. Indeed, it suffices to check (13). The nonnegativity of the process $\tilde{\mathcal{K}}^n$ follows, for sufficiently large n, from $\left|\omega|_{[0,t]}\right| \le 0.5$; namely, when $n^4 2^{-2n+2} 0.25 \le n^{-3}$, $\tilde{\mathcal{K}}^n$ will be nonnegative even when the addend $\sum_{k=1}^{\infty} \cdots (\cdots)^2$ in (14) is ignored. Applying Lemma 2 now again gives (12).

The proof is now completed in the same way as the proof of Theorem 1. $\square$

8 The case of a nearly càdlàg integrand and a continuous integrator

In this section we will consider non-càdlàg integrands ϕ, motivated by, first of all, Tanaka's formulas, which involve integrands such as $\phi(t) := \mathbf{1}_{\{\omega(t)>a\}}$ (lower semicontinuous for continuous ω), $\phi(t) := \mathbf{1}_{\{\omega(t) \ge a\}}$ (upper semicontinuous for continuous ω), or $\phi(t) := \operatorname{sign}(\omega(t) - a)$ (in general neither). Such functions are not even regulated: they have *essential discontinuities* (i.e., points t such that at least one of the limits $\phi(t-)$ or $\phi(t+)$ does not exist). We will define the Itô integral $\int \phi \, d\omega$ for such ϕ in the case where there is some kind of synergy between ϕ and ω: very roughly, we will require that ω does not change much around the essential discontinuities of ϕ (which will cover the examples given at the beginning of this paragraph). The results of this section are very preliminary; in particular, in this version of the paper we only consider the case of continuous ω.

Intuitively, we will require that the integrand ϕ be non-càdlàg in a controllable and predictable manner. Let $\mathbb{R}^{[0,\infty)}$ be the set of all real-valued functions on $[0,\infty)$ and $2^{[0,\infty)}$ be the family of all subsets of $[0,\infty)$. Formally, we now define the sample space Ω to be the set

$$\Omega := \bigl\{(\omega, \phi, D) \in C[0,\infty) \times \mathbb{R}^{[0,\infty)} \times 2^{[0,\infty)} \mid \\ D \text{ is closed, } \phi \text{ is bounded, and } \phi|_{[0,\infty)\setminus D} \text{ is càdlàg}\bigr\} \qquad (22)$$

consisting of triples

$$o = (\omega, \phi, D) \qquad (23)$$

such that ϕ has left limits and is right-continuous at each point of the open set $[0,\infty) \setminus D$ in $[0,\infty)$ (in the relative topology).

Let us equip the sample space Ω with the universal completion $\mathcal{F}$ of the σ-algebra generated by the functions $o \in \Omega \mapsto (\phi(t), \omega(t),\ t \in [0,\infty)$, and by the events $E \cap (t_1, t_2) \ne \emptyset$, $0 \le t_1 < t_2 < \infty$, where $o = (\omega, \phi, E) \in \Omega$. The definitions of events, random variables, stopping times τ, random variables determined by time τ, simple trading strategies, etc., carry over to this case as well.

The Ω of Section 3, (1), embeds into the Ω of this section, (22), by setting $D := \emptyset$. The Ω of Section 7, (17), embeds in the same way under the restriction that ω is continuous.

One of the conditions of the main result (Theorem 4) of this section will involve a slight modification of the standard notion of box-counting dimension (analogous to the modification of Riemann integrals to Riemann–Stiltjes integrals). For an interval I of the real line and a real-valued function f defined on I, the *oscillation* of f over I is

$$\operatorname{osc}_I(f) := \sup_{t_1, t_2 \in I} |f(t_1) - f(t_2)| = \sup_{t \in I} f(t) - \inf_{t \in I} f(t).$$

Let ω be a real-valued function defined on $[0,\infty)$, E be a subset of $[0,\infty)$, and $\epsilon > 0$. Set

$$M_\omega(E, \epsilon) := \min \left\{ k \ge 1 \mid \exists I_1 \cdots \exists I_k : E \subseteq \bigcup_{i=1}^{k} I_i \ \ \& \ \max_{i=1,\dots,k} \operatorname{osc}_{I_i}(\omega) \le \epsilon \right\},$$

where $I_1, I_2, \dots$ range over the intervals of $[0,\infty)$, and set

$$\dim_\omega(E) := \limsup_{\epsilon \downarrow 0} \frac{\log M_\omega(E, \epsilon)}{\log(1/\epsilon)}.$$

For the identity function $\omega(t) := t$, $\forall t \in [0,\infty)$, this becomes the usual definition of upper box-counting dimension (also known as Minkowski dimension, although it was first given in this form only by Pontryagin and Shnirel'man [21]).

Let us say that (23) is *tame* at time t, $\mathrm{tame}_t(o)$ in symbols, if $\dim_\omega(D \cap [0,t]) < 2$.

Now the definition (18) of T_k^n is modified by setting $T_0^n(o) := 0$ and, for $k \geq 1$:

- if $T_{k-1}^n \notin D$,

$$T_k^n(o) := \inf\Big\{t > T_{k-1}^n(o) \mid \left|\omega(t) - \omega(T_{k-1}^n)\right| \vee \left|\phi(t) - \phi(T_{k-1}^n)\right| \geq 2^{-n} \text{ or } t \in D\Big\}; \quad (24)$$

- if $T_{k-1}^n \in D$,

$$T_k^n(o) := \inf\Big\{t > T_{k-1}^n(o) \mid \left|\omega(t) - \omega(T_{k-1}^n)\right| \geq 2^{-n}\Big\}. \quad (25)$$

With this change, the definition of $(\phi \cdot \omega)^n$ is (6).

Theorem 4. *The processes $(\phi\cdot\omega)^n$ converge ucqa as $n \to \infty$ when the property* tame *is satisfied.*

Before we prove Theorem 4, we briefly discuss its statement, especially the condition that the property tame be satisfied. A more detailed statement of Theorem 4 would be: for each $t > 0$, quasi-always,

- either the sequence of functions $s \in [0,t] \mapsto (\phi \cdot \omega)_s^n$ converges in the uniform metric on $[0,t]$ as $n \to \infty$
- or $\mathrm{tame}_t(o)$ is false.

The following lemma shows that the condition tame is mild in a certain sense; it says that $\dim_\omega([0,t]) \leq 2$ q.a. (more precisely, $\dim_\omega([0,t]) \in \{0,2\}$ q.a.).

Lemma 6. *Quasi-always, ω is constant over $[0,t]$ or $\dim_\omega([0,t]) = 2$.*

Proof. Let us use Theorem 3.1 in [26] (a probability-free version of the Dubins–Schwarz result). The quadratic variation of ω was defined in Section 5, but here we can also use the definition given in [26] (and not involving ϕ). It suffices to prove that $\dim_\omega([0,c]) = 2$ for a typical path ω of standard Brownian motion and a given constant $c > 0$. If we divide $[0,c]$ into n equal parts of length c/n, Lévy's modulus of continuity theorem (see, e.g., [16], Theorem 1.14) shows that the oscillation of ω on each part is equivalent to $\sqrt{2(c/n)\ln n}$ or less as $n \to \infty$. For $\epsilon := \sqrt{2(c/n)\ln n}$ we get

$$M_\omega([0,c],\epsilon) \leq n = O\left(\frac{1}{\epsilon^2}\log\frac{1}{\epsilon}\right),$$

and so letting $\epsilon \downarrow 0$ gives $\dim_\omega([0,c]) \leq 2$.

To see that $\dim_\omega([0,c]) \geq 2$, for a given $\epsilon > 0$ (sufficiently small), choose $n \in \mathbb{N}$ such that $\sqrt{c/n} \in [10\epsilon, 11\epsilon]$. Divide the interval $[0,c]$ into n equal disjoint subintervals of length c/n. Over each of those subintervals, the oscillation of ω exceeds ϵ with probability at least 0.3. Therefore, for sufficiently small ϵ (i.e., sufficiently large n), the fraction of the subintervals on which the oscillation of ω exceeds ϵ is more than 0.2. Let J_j, $j = 1, \ldots$, be an enumeration of such subintervals. It suffices to check that $M_\omega([0,c],\epsilon) \geq 0.1n$ for such ϵ. Arguing indirectly, suppose $k := M_\omega([0,c],\epsilon) < 0.1n$ for such an ϵ, and choose a cover $I_1, \ldots, I_k$ of $[0,c]$ such that $\operatorname{osc}_{I_i}(\omega) \leq \epsilon$ for all $i \in \{1,\ldots,k\}$. For each interval J_j fix an interval I_i such that $I_i \cap J_j \neq \emptyset$. At most two different J_j will be mapped into the same I_i (if three different J_j were mapped into the same I_i, one of those J_j would be a subset of that I_i, which would contradict $\operatorname{osc}_{I_i}(\omega) \leq \epsilon < \operatorname{osc}_{J_j}(\omega)$). Therefore, the number of J_j is at most twice the number k of I_i, which implies that it is at most $0.2n$. This contradiction completes the proof. $\square$

Lemma 6 can be interpreted to say that the condition $\dim_\omega(D \cap [0,t]) < 2$ implicit in Theorem 4 means that $D \cap [0,t]$ is only slightly less massive than the whole of $[0,t]$. On the other hand, the next lemma shows that the sets $\{\omega = a\}$ of essential discontinuities in Tanaka's formulas are typically much less massive.

Lemma 7. *Let $a \in \mathbb{R}$. Quasi-always,*

$$\dim_\omega(\{\omega = a\} \cap [0,t]) \leq 1. \tag{26}$$

Proof. Similarly to the proof of Lemma 6, it suffices to prove (26) for a typical path ω of standard Brownian motion. And in this case (26) follows immediately from the downcrossing representation of the local time at zero (or at a): see, e.g., [16], Theorem 6.1. $\square$

Lemma 7 shows that Theorem 4 is applicable to Itô integration in the context of Tanaka's formulas. Let us consider, e.g., $\int_0^t \mathbf{1}_{\{\omega(s)>a\}} \, \mathrm{d}\omega(s)$. In this case ϕ and D are defined by

$$\begin{aligned} \phi(t) &:= \mathbf{1}_{\{\omega(t)>a\}}, \\ D &:= \{t \in [0,\infty) \mid \omega(t) = 0\}. \end{aligned}$$

The conditions in (22) are satisfied: D is closed, $\phi \in \{0,1\}$ is bounded, and $\phi|_{[0,\infty)\setminus D}$ is càdlàg (and even constant on the connected components of its domain). Therefore, $\int_0^t \phi \, \mathrm{d}\omega$ exists q.a.

Before proving Theorem 4, let us check that the analogue of Lemma 1 still holds and that T_k^n are stopping times.

Lemma 8. *For each n, $T_k^n \to \infty$ as $k \to \infty$.*

Proof. The proof is a modification of the proof of Lemma 5. As before, fix $t > 0$ and choose a neighbourhood of each $s \in [0,t]$ which is a closed interval and in which ω changes by less than 2^{-n} and ϕ changes by less than $|\Delta\phi(s)| + 2^{-n}$.

Let us check that, for a fixed n, in each such neighbourhood there are at most 10 values of T_k^n. (This is sufficient since there exists a finite subcover.) If the neighbourhood does not contain s such that $\phi_*(s) < \phi^*(s)$, this is weaker than the claim (of fewer than 10 values) that we made in the proof of Lemma 5; therefore, let us suppose that such s exist. Let s_0 be the left-most such s in the neighbourhood. To the right of s_0 we do not have any T_k^n in the neighbourhood. To the left, we have fewer than 10 values of T_k^n in the neighbourhood. Therefore, we have at most 10 values overall. □

Lemma 9. *Each T_k^n is a stopping time.*

Proof. As in Section 7, we will only consider the case $k = 1$; our argument will be very similar to that in the proof of Lemma 4. Fix n and $t \in [0, \infty)$; we will see that the set $\{T_1^n \le t\}$ is analytic. It suffices to consider the case $0 \notin D$. In this case T_1^n is defined by (24), and the inequality concerning ω can be ignored (the corresponding hitting time is obviously a stopping time). The infimum in (24) is obviously attained, and so the set $\{T_1^n \le t\}$ is the projection of the set

$$\{(s, o) \in [0, t] \times \Omega \mid |\phi(t) - \phi(0)| \ge \text{ or } s \in D\} \tag{27}$$

onto Ω. The standard proof of the progressive measurability of right-continuous adapted processes (see, e.g., [4], Theorem 3.2.27) shows that there exists a measurable (w.r. to the product σ-algebra on $\mathcal{B}_t \times \mathcal{F}_t$) function Φ on $[0, t] \times \Omega$ that coincides with ϕ at each point (t, o) such that $t \notin D$ (in the usual notation of (23)). It remains to notice that we can replace ϕ by Φ in (27). □

Proof of Theorem 4. Fix $t > 0$. As in the proof of Theorem 3, we will assume, without loss of generality, that $|\omega| \le 0.5$; this will ensure $|x_{k,s}| \le b_n 2^{-n+1} \le 0.5$, assuming $n \ge 8$ (cf. (21)). Finally, since the intersection of countably many properties that hold quasi-always holds quasi-always, there is no loss of generality in replacing the condition $\dim_\omega(D \cap [0, t]) < 2$ by $\dim_\omega(D \cap [0, t]) < 2 - \delta$ for a given $\delta > 0$. Therefore, we assume that, as $\epsilon \downarrow 0$, $D \cap [0, t]$ can be covered by $O(\epsilon^{\delta-2})$ intervals I such that $\operatorname{osc}_I(\omega) \le \epsilon$. This implies that the number of k such that $T_k^n \in D \cap [0, t]$ is

$$O(2^{(2-\delta)n}). \tag{28}$$

It suffices to show that (12) still holds. Without loss of generality we assume that the sum is over the k such that $a_{k-1} < t$. We divide such k into four kinds (and in each of the four corresponding items below, the default is that $\sum_k$ stands for the sum over the k of the kind considered in that item):

1. The first kind of k are those satisfying $a_{k-1} \in D$ (remember that we are only interested in k such that $a_{k-1} < t$). Since we always have $|\omega(a_k \wedge s) - \omega(a_{k-1} \wedge s)| \le 2^{-n}$,

$$\sup_{s \in [0,t]} \sum_k x_{k,s}^2 = O\left(2^{(2-\delta)n} b_n^2 2^{-2n}\right) = o(1).$$

(We have used the fact that both the number of k such that $T_k^n \in D \cap [0,t]$ is (28), and the number of k such that $T_k^{n-1} \in D \cap [0,t]$ is (28).)

2. The second kind of k are those satisfying $a_{k-1} \notin D$ and $a''_{k-1} \in D$. Such k satisfy $a''_{k-1} < a_{k-1}$ and $a'_{k-1} = a_{k-1}$. We cannot claim that the number of such k is (28) since different k may lead to the same value of a''_{k-1}, so we will need a more complicated argument making use of the K29 martingale (32). First we notice that

$$n^{-10} + \sum_k (\omega(a_k \wedge s) - \omega(a_{k-1} \wedge s))^2 - \sum_{\substack{k=1,2,\ldots:T_{k-1}^{n-1}<t,\\ \phi_*(T_{k-1}^{n-1})<\phi^*(T_{k-1}^{n-1})}} \left(\omega(T_k^{n-1} \wedge s) - \omega(T_{k-1}^{n-1} \wedge s)\right)^2 \quad (29)$$

is the value of a simple capital process at time $s \le t$. The subtrahend in (29) is

$$O\left(2^{(2-\delta)n}2^{-2n}\right) = o(n^{-10}),$$

and so the martingale value (29) is nonnegative from some n on, even if we ignore its second addend $\sum_k (\cdots)^2$. Let us make the simple capital process nonnegative by stopping trading when it becomes zero; this will not affect the process at all for large enough n. By Lemma 2, (29) is $O(n^{-8})$ uniformly in $s \in [0,t]$ quasi-always, and so the second addend of (29) is $O(n^{-8})$, q.a. Therefore,

$$\sup_{s\in[0,t]} \sum_k x_{k,s}^2 = \sup_{s\in[0,t]} \sum_k b_n^2 \left(\phi(a_{k-1}) - \phi(a''_{k-1})\right)^2 \times (\omega(a_k) \wedge s - \omega(a_{k-1}) \wedge s)^2 = O\left(n^4 n^{-8}\right) = o(1) \qquad \text{q.a.}$$

3. The third kind of k are those satisfying $a_{k-1} \notin D$ and $a'_{k-1} \in D$. Such k are treated in the same way as the k of the second kind.

4. The last kind of k are those for which all of a_{k-1}, a'_{k-1}, and a''_{k-1} are outside D. Such k satisfy (20), and we again have

$$\begin{aligned}\sup_{s\in[0,t]} \sum_k x_{k,s}^2 &= \sup_{s\in[0,t]} \sum_k b_n 2^{-n+1} (\omega(a_k \wedge s) - \omega(a_{k-1} \wedge s))^2 \\ &\le \sup_{s\in[0,t]} \sum_{k=1}^{\infty} b_n 2^{-n+1} (\omega(a_k \wedge s) - \omega(a_{k-1} \wedge s))^2 \\ &= o(1) \qquad \text{q.a.}\end{aligned}$$

(for the last equality, see (13)).

□

9 Conclusion

The most obvious directions of further research are:

- to explore the dependence of $\int \phi \, \mathrm{d}\omega$ on the choice of the partitions T^n_k (see [28], Section 4, for a proposal about making the definition of $\int \phi \, \mathrm{d}\omega$ invariant with respect to the choice of the partitions);
- to extend Theorem 2 to convex functions F;
- and to relax the conditions of Theorem 4.

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Appendix A: Useful discrete-time supermartingales

Our proofs of Theorems 1, 3, and 4 are based on a simple large-deviation-type supermartingale, which will be defined in this appendix, and on a classical martingale going back to [12], to be defined in Appendix B below.

We consider the case of discrete time, namely, the following perfect-information protocol:

Betting on bounded below variables

Players: Sceptic and Reality

Protocol:

Sceptic announces $\mathcal{K}_0 \in \mathbb{R}$.
FOR $k = 1, 2, \dots$:
  Sceptic announces $M_k \in \mathbb{R}$.
  Reality announces $x_k \in [-0.5, \infty)$.
  Sceptic announces $\mathcal{K}_k \le \mathcal{K}_{k-1} + M_k x_k$.

We interpret $\mathcal{K}_k$ as Sceptic's capital at the end of round k. Notice that Sceptic is allowed to choose his initial capital $\mathcal{K}_0$ and to throw away part of his money at the end of each round.

A *process* is a real-valued function defined on all finite sequences $(x_1, \dots, x_K)$, $K = 0, 1, \dots$, of Reality's moves. If we fix a strategy for Sceptic, his capital $\mathcal{K}_K$, $K = 0, 1, \dots$, will become a process. Such processes are called *supermartingales*.

Lemma 10. *The process*

$$\mathcal{K}_K := \prod_{k=1}^{K} \exp\left(x_k - x_k^2\right) \tag{30}$$

is a supermartingale.

We do not require the measurability of supermartingales *a priori*, but (30) is, of course, measurable. The corresponding strategy for Sceptic used in the proof will be $M_k := \mathcal{K}_{k-1}$, and so will also be measurable. The lemma will still be true if the interval $[-0.5, \infty)$ in the protocol is replaced by $[-0.683, \infty)$ (but will no longer be true for $[-0.684, \infty)$).

Proof. It suffices to prove that on round k Sceptic can turn a capital of $\mathcal{K} > 0$ into a capital of at least

$$\mathcal{K} \exp\left(x_k - x_k^2\right);$$

in other words, that he can obtain a payoff $M_k x_k$ of at least

$$\exp\left(x_k - x_k^2\right) - 1.$$

This will follow from the inequality

$$\exp\left(x_k - x_k^2\right) - 1 \le x_k.$$

Setting $x := x_k$, moving the 1 to the right-hand side, and taking logs of both sides, we rewrite this inequality as

$$x - x^2 \le \ln(1 + x),$$

where $x \in [-0.5, \infty)$. Since we have an equality for $x = 0$, it remains to notice that the derivative of the left-hand side of the last inequality never exceeds the derivative of its right-hand side for $x > 0$, and that the opposite relation holds for $x < 0$. $\square$

Another useful process is

$$\frac{1}{2}\left(\prod_{k=1}^{K}\exp\left(x_k - x_k^2\right) + \prod_{k=1}^{K}\exp\left(-x_k - x_k^2\right)\right), \tag{31}$$

which is a supermartingale in the protocol of betting on bounded variables, where Reality is required to announce $x_k \in [-0.5, 0.5]$. (It suffices to apply Lemma 10 to x_k and $-x_k$ and to average the resulting supermartingales.)

Remark 5. In this appendix we used the method described in [23], Section 2; in fact, it is shown (using slightly different terminology) in [23] that

$$\prod_{k=1}^{K}\exp\left(x_k - \frac{x_k^2}{2} - |x_k|^3\right)$$

is a supermartingale in the protocol of betting on bounded variables, $|x_k| \le \delta$ for a small enough $\delta > 0$ (it is sufficient to assume $\delta \le 0.8$). This supermartingale can be regarded as a discrete-time version of the Doléans exponential.

Appendix B: Another useful discrete-time supermartingale

In this appendix we will define another process used in the proofs of the main results of this paper (in principle, we could have also used this process to replace in those proofs the process defined in Appendix A).

We still consider the case of discrete time. The perfect-information protocol of this appendix is:

Betting on arbitrary variables

Players: Sceptic and Reality

Protocol:

Sceptic announces $\mathcal{K}_0 \in \mathbb{R}$.
FOR $k = 1, 2, \dots$:
  Sceptic announces $M_k \in \mathbb{R}$.
  Reality announces $x_k \in \mathbb{R}$.
  $\mathcal{K}_k := \mathcal{K}_{k-1} + M_k x_k$.

Sceptic's capital $\mathcal{K}_K$ as function of Reality's moves $x_1, \dots, x_K$ for a given strategy for Sceptic is a process called a *martingale* (this term is natural as our new protocol does not allow Sceptic to throw money away).

Lemma 11. *The process*

$$\mathcal{K}_K := \sum_{k=1}^{K} x_k^2 - \left(\sum_{k=1}^{K} x_k\right)^2 \tag{32}$$

is a martingale.

We will refer to (32) as the *K29 martingale*.

Proof. The increment of (32) on round K is

$$x_K^2 - \left(\sum_{k=1}^{K} x_k\right)^2 + \left(\sum_{k=1}^{K-1} x_k\right)^2 = -2\left(\sum_{k=1}^{K-1} x_k\right) x_K \qquad (33)$$

and, therefore, is indeed of the form $M_K x_K$. $\square$