# Bell numbers, partition moves and the eigenvalues of the random-to-top shuffle in Dynkin Types A, B and D

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**Abstract**

Let $B_t(n)$ be the number of set partitions of a set of size $t$ into at most $n$ parts and let $B'_t(n)$ be the number of set partitions of $\{1,\ldots,t\}$ into at most $n$ parts such that no part contains both 1 and $t$ or both $i$ and $i+1$ for any $i \in \{1,\ldots,t-1\}$. We give two new combinatorial interpretations of the numbers $B_t(n)$ and $B'_t(n)$ using sequences of random-to-top shuffles, and sequences of box moves on the Young diagrams of partitions. Using these ideas we obtain a very short proof of a generalization of a result of Phatarfod on the eigenvalues of the random-to-top shuffle. We also prove analogous results for random-to-top shuffles that may flip certain cards. The proofs use the Solomon descent algebras of Types A, B and D. We give generating functions and asymptotic results for all the combinatorial quantities studied in this paper.

*Keywords:* Bell number, Stirling number, symmetric group, partition, Young diagram, random-to-top shuffle, top-to-random shuffle
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## 1. Introduction

For $t, n \in \mathbf{N}_0$, let $B_t(n)$ be the number of set partitions of $\{1,\ldots,t\}$ into at most $n$ parts. If $n \geq t$ then $B_t(n)$ is the Bell number $B_t$; the difference $B_t(n) - B_t(n-1)$ is $\left\{ t \atop n \right\}$, the Stirling number of the second kind. Let $B'_t(n)$

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be the number of set partitions of $\{1, \dots, t\}$ into at most $n$ parts such that no part contains both 1 and $t$ or both $i$ and $i+1$ for any $i \in \{1, \dots, t-1\}$.

The first object of this paper is to give two combinatorial interpretations of the numbers $B_t(n)$ and $B'_t(n)$, one involving certain sequences of random-to-top shuffles, and another involving sequences of box removals and additions on the Young diagrams of partitions. The first of these interpretation is justified by means of an explicit bijection. The second interpretation is considerably deeper, and its justification is less direct: our argument requires the Branching Rule for representations of the symmetric group $\mathrm{Sym}_n$, and a basic result from the theory of Solomon's descent algebra. Using these ideas we obtain a very short proof of a generalization of a result due to Phatarfod [19] on the eigenvalues of the random-to-top shuffle.

We also state and prove analogous results for random-to-top shuffles that may flip the moved card from face-up to face-down, using the descent algebras associated to the Coxeter groups of Type B and D. In doing so, we introduce analogues of the Bell numbers corresponding to these types; these appear not to have been studied previously. We give generating functions, asymptotic formulae and numerical relationships between these numbers, and the associated Stirling numbers, in §6 below.

We now define the quantities which we shall show are equal to either $B_t(n)$ or $B'_t(n)$.

**Definition.** For $m \in \mathbf{N}$, let $\sigma_m$ denote the $m$-cycle $(1, 2, \dots, m)$. A *random-to-top shuffle* of $\{1, \dots, n\}$ is one of the $n$ permutations $\sigma_1, \dots, \sigma_n$. Let $S_t(n)$ be the number of sequences of $t$ random-to-top shuffles whose product is the identity permutation. Define $S'_t(n)$ analogously, excluding the identity permutation $\sigma_1$.

We think of the permutations $\sigma_1, \dots, \sigma_n$ as acting on the $n$ positions in a deck of $n$ cards; thus $\sigma_m$ is the permutation moving the card in position $m$ to position 1 at the top of the deck. If the cards are labelled by a set $C$ and the card in position $m$ is labelled by $c \in C$ then we say that $\sigma_m$ *lifts* card $c$.

We represent partitions by Young diagrams. Motivated by the Branching Rule for irreducible representations of symmetric groups, we say that a box in a Young diagram is *removable* if removing it leaves the Young diagram of a partition; a position to which a box may be added to give a Young diagram of a partition is said to be *addable*.

**Definition.** A *move* on a partition consists of the removal of a removable box and then addition in an addable position of a single box. A move is *exceptional* if it consists of the removal and then addition in the same place of the lowest removable box. Given partitions $\lambda$ and $\mu$ of the same size, let $M_t(\lambda,\mu)$ be the number of sequences of $t$ moves that start at $\lambda$ and finish at $\mu$. Let $M'_t(\lambda,\mu)$ be defined analogously, considering only non-exceptional moves. For $n \in \mathbf{N}_0$, let $M_t(n) = M_t\big((n),(n)\big)$ and let $M'_t(n) = M'_t\big((n),(n)\big)$.

We note that if the Young diagram of $\lambda$ has exactly $r \in \mathbf{N}$ removable boxes then $M_1(\lambda,\lambda) = r$ and $M'_1(\lambda,\lambda) = r-1$. For example, $(2,1)$ has moves to $(2,1)$, in two ways, and also to $(3)$, $(1^3)$.

Our first main result, proved in §2 below, is as follows.

**Theorem 1.1.** *For all $t$, $n \in \mathbf{N}_0$ we have $B_t(n) = S_t(n) = M_t(n)$ and $B'_t(n) = S'_t(n) = M'_t(n)$.*

The first equality in Theorem 1.1 is a special case of [20, Theorem 15]. The authors thank an anonymous referee for this reference.

In §3 we generalize random-to-top shuffles to $k$-shuffles for $k \in \mathbf{N}$ and use our methods to give a short proof of Theorem 4.1 in [8] on the eigenvalues of the associated Markov chain. Theorem 1.1 and this result have analogues for Types B and D. We state and prove these results in §4 and §5, obtaining Theorem 4.3, Proposition 4.9, Theorem 5.1 and Proposition 5.5. In §6 we give generating functions, asymptotic formulae and numerical relationships between the numbers studied in this paper.

The sequences of partition moves counted by $M'_t\big(\lambda,\mu\big)$ are in bijection with the Kronecker tableaux $KT^t_{\lambda,\mu}$ defined in [10, Definition 2]. In §7 we show that, when $n \geq t$, the equality $B'_t(n) = M'_t(n)$ is a corollary of a special case of Lemma 1 in [10]. The proof of this lemma depends on results on the RSK correspondence for oscillating tableaux, as developed in [7] and [22]. Our proof, which goes via the numbers $S'_t(n)$, is different and significantly shorter. In §7 we also make some remarks on the connections with earlier work of Bernhart [1] and Fulman [9] and show an unexpected obstacle to a purely bijective proof of Theorem 1.1: this line of argument is motivated by [9] and [10].

The numbers studied in this paper have been submitted to the Online Encyclopedia of Integer Sequences [17]. We end in §8 by discussing their appearances prior to this submission. With the expected exception of $B_t(n)$ and $\left\{ {t \atop n} \right\}$ these are few, and occur for particular choices of the parameters $t$ and $n$. References are given to the new sequences arising from this work.

## 2. Proof of Theorem 1.1

We prove the first equality in Theorem 1.1 using an explicit bijection. This is a special case of the bijection in the proof of Theorem 15 of [20].

**Lemma 2.1.** *If $t$, $n \in \mathbf{N}_0$ then $B_t(n) = S_t(n)$ and $B'_t(n) = S'_t(n)$.*

*Proof.* Suppose that $\tau_1, \ldots, \tau_t$ is a sequence of top-to-random shuffles such that $\tau_1 \ldots \tau_t = \mathrm{id}_{\mathrm{Sym}_n}$. Take a deck of cards labelled by $\{1, \ldots, n\}$, so that card $c$ starts in position $c$, and apply the shuffles so that at time $s \in \{1, \ldots, t\}$ we permute the positions of the deck by $\tau_s$. For each $c \in \{1, \ldots, n\}$ let $A_c$ be the set of $s \in \{1, \ldots, t\}$ such that $\tau_s$ lifts card $c$. Removing any empty sets from the list $A_1, \ldots, A_n$ we obtain a set partition of $\{1, \ldots, t\}$ into at most $n$ sets.

Conversely, given a set partition of $\{1, \ldots, t\}$ into $m$ parts where $m \leq n$, we claim that there is a unique way to label its parts $A_1, \ldots, A_m$, and a unique sequence of $t$ random-to-top shuffles leaving the deck invariant, such that $A_c$ is the set of times when card $c$ is lifted by the shuffles in this sequence. If such a labelling exists, then for each $c \in \{1, \ldots, m\}$, the set $A_c$ must contain the greatest element of $\{1, \ldots, t\} \backslash (A_1 \cup \cdots \cup A_{c-1})$, since otherwise a card $c'$ with $c' > c$ is lifted after the final time when card $c$ is lifted, and from this time onwards, cards $c$ and $c'$ are in the wrong order. Using this condition to fix $A_1, \ldots, A_m$ determines the card lifted at each time, and so determines a unique sequence of random-to-top shuffles that clearly leaves the deck invariant. It follows that $B_t(n) = S_t(n)$.

A set partition corresponds to a sequence of non-identity shuffles if and only if the same card is never lifted at consecutive times, and the first card lifted is not card 1. Therefore the bijection just defined restricts to give $B'_t(n) = S'_t(n)$. □

The second half of the proof is algebraic. Let $\mathrm{Cl}(\mathrm{Sym}_n)$ denote the ring of class functions of $\mathrm{Sym}_n$. Let $\pi \in \mathrm{Cl}(\mathrm{Sym}_n)$ be the natural permutation character of $\mathrm{Sym}_n$, defined by $\pi(\tau) = |\mathrm{Fix}\ \tau|$ for $\tau \in \mathrm{Sym}_n$. Let $\chi^\lambda$ denote the irreducible character of $\mathrm{Sym}_n$ canonically labelled by the partition $\lambda$ of $n$. Let $\vartheta = \pi - 1_{\mathrm{Sym}_n}$ where $1_{\mathrm{Sym}_n}$ is the trivial character of $\mathrm{Sym}_n$; note that $\vartheta = \chi^{(n-1,1)}$. The main idea in the following lemma is well known: it appears in [2, Lemma 4.1], and the special case for $M'_t\big((n), \mu\big)$ is proved in [10, Proposition 1].

**Lemma 2.2.** *Let $t \in \mathbf{N}_0$. If $\lambda$ and $\mu$ are partitions of $n \in \mathbf{N}$ then $M_t(\lambda, \mu) = \langle \chi^\lambda \pi^t, \chi^\mu \rangle$ and $M'_t(\lambda, \mu) = \langle \chi^\lambda \vartheta^t, \chi^\mu \rangle$.*

*Proof.* We have

$$\chi^\lambda \pi = \chi^\lambda \big( 1 \big\uparrow_{\mathrm{Sym}_{n-1}}^{\mathrm{Sym}_n} \big) = \big( \chi^\lambda \big\downarrow_{\mathrm{Sym}_{n-1}} \big) \big\uparrow^{\mathrm{Sym}_n}. \tag{1}$$

By the Branching Rule for $\mathrm{Sym}_n$ (see [15, Chapter 9]) we have $\chi^\lambda \big\downarrow_{\mathrm{Sym}_{n-1}} = \sum_\nu \chi^\nu$ where the sum is over all partitions $\nu$ whose Young diagram is obtained from $\lambda$ by removing a single box. By Frobenius reciprocity we have

$\langle \chi^\mu, \chi^\nu\uparrow^{\mathrm{Sym}_n}\rangle = \langle \chi^\mu\downarrow_{\mathrm{Sym}_{n-1}}, \chi^\nu\rangle$. Hence $\chi^\nu\uparrow^{\mathrm{Sym}_n} = \sum_\mu \chi^\mu$ where the sum is over all partitions $\mu$ whose Young diagram is obtained from $\nu$ by adding a single box. Therefore the right-hand side of (1) is $\sum_\mu M_1(\lambda,\mu)\chi^\mu$, where the sum is over all partitions $\mu$ of $n$. This proves the case $t = 1$ of the first part of the lemma; the general case follows by induction. The second part is proved similarly, using that $\chi^\lambda\vartheta = \chi^\lambda(\pi - 1_{\mathrm{Sym}_n}) = \big(\chi^\lambda\downarrow_{\mathrm{Sym}_{n-1}}\big)\uparrow^{\mathrm{Sym}_n} - \chi^\lambda$. $\square$

In [21, Theorem 1], Solomon considers an arbitrary Coxeter group $G$ and defines an associated *descent algebra* $\mathrm{Des}(G)$ inside the rational group algebra $\mathbf{Q}G$. In the special case of $\mathrm{Sym}_n$, a simpler definition is possible. (See also [4, page 7] for an equivalent presentation.) Recall that a permutation $\sigma \in \mathrm{Sym}_n$ has a *descent* in position $k$ if $k\sigma > (k+1)\sigma$. For each $I \subseteq \{1,\ldots,n-1\}$, let $\Xi_I = \sum_\sigma \sigma \in \mathbf{Q}\mathrm{Sym}_n$ where the sum is over all $\sigma \in \mathrm{Sym}_n$ such that the descents of $\sigma$ occur in a subset of the positions $I$. Then $\mathrm{Des}(\mathrm{Sym}_n) = \langle \Xi_I : I \subseteq \{1,\ldots,n-1\}\rangle_{\mathbf{Q}}$ is an algebra with unit element $\Xi_\varnothing = \mathrm{id}_{\mathrm{Sym}_n}$. This fact is proved in [21] in the general setting of Coxeter groups; an elegant alternative proof due to Bidigare [3] in the special case of symmetric groups is presented in [4, Appendix B].

Let $I = \{i_1,\ldots,i_\ell\}$ and let $\mathrm{Sym}_I = \mathrm{Sym}_{i_1} \times \mathrm{Sym}_{i_2-i_1} \times \cdots \times \mathrm{Sym}_{i_\ell - i_{\ell-1}} \leq \mathrm{Sym}_n$. By a special case of [21, Theorem 1], there is an algebra epimorphism $\mathrm{Des}(\mathrm{Sym}_n) \to \mathrm{Cl}(\mathrm{Sym}_n)$ defined by $\Xi_I \mapsto \pi_I$, where $\pi_I$ is the permutation character of $\mathrm{Sym}_n$ acting on cosets of $\mathrm{Sym}_I$. We refer to this map as the *canonical epimorphism*. Define a bilinear form on $\mathbf{Q}\mathrm{Sym}_n$ by $(g,h) = 1$ if $g = h^{-1}$ and $(g,h) = 0$ if $g \neq h^{-1}$. By [4, Theorem 1.2], the canonical epimorphism is an isometry with respect to the bilinear form on $\mathrm{Des}(\mathrm{Sym}_n)$ defined by restriction of $(-,-)$ and the usual inner product on $\mathrm{Cl}(\mathrm{Sym}_n)$, defined by $\langle \chi, \phi\rangle = \frac{1}{n!}\sum_{\sigma\in\mathrm{Sym}_n} \chi(\sigma)\overline{\phi(\sigma)}$ for $\chi$, $\phi \in \mathrm{Cl}(\mathrm{Sym}_n)$.

Let $\Xi = \sum_{m=1}^n \sigma_m^{-1}$ and let $\Delta = \Xi - \mathrm{id}_{\mathrm{Sym}_n}$. Note that $\tau \in \mathrm{Sym}_n$ is a summand of $\Delta$ if and only if

$$1\tau > 2\tau < 3\tau < \ldots < n\tau.$$

Thus $\Delta$ is the sum of all $\tau \in \mathrm{Sym}_n$ such that the unique descent of $\tau$ is in position 1. Hence $\Xi = \Xi_{\{1\}}$ and $\Delta = \Xi_{\{1\}} - \mathrm{id}_{\mathrm{Sym}_n}$ in the notation above. Since $\pi = \pi_{\{1\}}$, under the canonical algebra epimorphism $\mathrm{Des}(\mathrm{Sym}_n) \to \mathrm{Cl}(\mathrm{Sym}_n)$, we have $\Xi \mapsto \pi$, $\mathrm{id}_{\mathrm{Sym}_n} \mapsto 1_{\mathrm{Sym}_n}$ and $\Delta \mapsto \vartheta$.

**Lemma 2.3.** *If $t$, $n \in \mathbf{N}_0$ then $S_t(n) = M_t(n)$ and $S'_t(n) = M'_t(n)$.*

*Proof.* If $n = 0$ then clearly $S_t(0) = M_t(0) = S'_t(0) = M'_t(0)$; the common value is 0 if $t \in \mathbf{N}$ and 1 if $t = 0$. Suppose that $n \in \mathbf{N}$. By Lemma 2.2 it is sufficient to prove that $\langle \pi^t, 1_{\mathrm{Sym}_n}\rangle = S_t(n)$ and $\langle \vartheta^t, 1_{\mathrm{Sym}_n}\rangle = S'_t(n)$.

Write $[\mathrm{id}_{\mathrm{Sym}_n}]$ for the coefficient of the identity permutation in an element of $\mathbf{Q}\,\mathrm{Sym}_n$. By the remark before this lemma, if $\Gamma \in \mathrm{Des}(\mathrm{Sym}_n)$ maps to $\phi \in \mathrm{Cl}(\mathrm{Sym}_n)$ under the epimorphism $\mathrm{Des}(\mathrm{Sym}_n) \to \mathrm{Cl}(\mathrm{Sym}_n)$, then

$$[\mathrm{id}_{\mathrm{Sym}_n}]\Gamma = (\Gamma, \mathrm{id}_{\mathrm{Sym}_n}) = \langle \phi, 1_{\mathrm{Sym}_n} \rangle. \tag{2}$$

The two required results now follow from Lemma 2.2 using the obvious equalities $[\mathrm{id}_{\mathrm{Sym}_n}]\Xi^t = S_t(n)$ and $[\mathrm{id}_{\mathrm{Sym}_n}]\Delta^t = S'_t(n)$. $\square$

Theorem 1.1 now follows from Lemmas 2.1 and 2.3.

## 3. Eigenvalues of the $k$-shuffle

Fix $n$, $k \in \mathbf{N}$ with $k \leq n$. A *$k$-shuffle* of a deck of $n$ cards takes any $k$ cards in the deck and moves them to the top of the deck, preserving their relative order. Note that a 1-shuffle is a random-to-top shuffle as already defined and that the inverse of a $k$-shuffle is a riffle shuffle in which the pack is first split into the top $k$ cards and the remaining $n-k$ cards. Let $P(k)$ be the transition matrix of the Markov chain on $\mathrm{Sym}_n$ in which each step is given by multiplication by one of the $\binom{n}{k}$ $k$-shuffles, chosen uniformly at random. Thus for $\sigma$, $\tau \in \mathrm{Sym}_n$ we have

$$P(k)_{\sigma\tau} = \begin{cases} \binom{n}{k}^{-1} & \text{if } \sigma^{-1}\tau \text{ is a } k\text{-shuffle} \\ 0 & \text{otherwise.} \end{cases}$$

It was proved by Phatarfod in [19] that the eigenvalues of $P(1)$ are exactly the numbers $|\mathrm{Fix}\ \tau|/n$ for $\tau \in \mathrm{Sym}_n$. More generally, it follows from a statement by Diaconis, Fill and Pitman [8, (6.1)] that if $\pi_k(\tau)$ is the number of fixed points of $\tau \in \mathrm{Sym}_n$ in its action on $k$-subsets of $\{1, \ldots, n\}$, then the eigenvalues of $P(k)$ are exactly the numbers $\pi_k(\tau)/\binom{n}{k}$ for $\tau \in \mathrm{Sym}_n$. The proof of this statement in [8] refers to unpublished work of Diaconis, Hanlon and Rockmore. We provide a short proof here.

Observe that $\mathrm{Tr}\, P(k)^t/n!$ is the probability that $t$ sequential $k$-shuffles leave the deck invariant. It is easily seen that $\tau^{-1}$ is a $k$-shuffle if and only if $1\tau < \ldots < k\tau$ and $(k+1)\tau < \ldots < n\tau$. Hence the sum of the inverses of the $k$-shuffles is the basis element $\Xi^{(k,n-k)} \in \mathrm{Des}(\mathrm{Sym}_n)$ defined in [4, page 7], which maps to $\pi_k = 1_{\mathrm{Sym}_k \times \mathrm{Sym}_{n-k}} {\uparrow}^{\mathrm{Sym}_n}$ under the canonical epimorphism $\mathrm{Des}(\mathrm{Sym}_n) \to \mathrm{Cl}(\mathrm{Sym}_n)$. Therefore (2) in the proof of Lemma 2.3 implies that

$$\begin{aligned}\frac{\operatorname{Tr} P(k)^t}{n!} &= [\mathrm{id}_{\mathrm{Sym}_n}] \binom{n}{k}^{-t} \Xi^{(k,n-k)^t} \\ &= \binom{n}{k}^{-t} \langle \pi_k^t, 1_{\mathrm{Sym}_n} \rangle \\ &= \binom{n}{k}^{-t} \frac{1}{n!} \sum_{\tau \in \mathrm{Sym}_n} \pi_k(\tau)^t\end{aligned}$$

for all $t \in \mathbf{N}_0$. It follows that if $\varepsilon_1, \ldots, \varepsilon_{n!}$ are the eigenvalues of $P(k)$, then $\sum_{i=1}^{n!} \varepsilon_i^t = \sum_{\tau \in \mathrm{Sym}_n} \pi_k(\tau)^t / \binom{n}{k}^t$ for all $t \in \mathbf{N}_0$. Thus the multisets $\{\varepsilon_i : 1 \leq i \leq n!\}$ and $\{\pi_k(\tau)/\binom{n}{k} : \tau \in \mathrm{Sym}_n\}$ are equal, as required.

The analogous result for the $k$-shuffle with the identity permutation excluded is stated below.

**Proposition 3.1.** *Let $n \geq 2$ and let $P'(k)$ be the transition matrix of the $k$-shuffle, modified so that at each time one of the $\binom{n}{k} - 1$ non-identity $k$-shuffles is chosen uniformly at random. The eigenvalues of $P'(k)$ are $(\pi_k(\tau) - 1)/(\binom{n}{k} - 1)$ for $\tau \in \mathrm{Sym}_n$.*

*Proof.* The proposition may be proved by adapting our proof of the result of Diaconis, Fill and Pitman. Alternatively, it may be obtained as a straightforward corollary to that result, by observing that

$$\left(\binom{n}{k} - 1\right) P'(k) = \binom{n}{k} P(k) - I_n. \qquad \square$$

We note that the eigenvalues of the $k$-top-to-random shuffles considered in Theorem 4.1 and the following Remark 1 in [8] may be determined by a similar short argument using the element $\Xi_{\{1,\ldots,k\}} \in \mathrm{Des}(\mathrm{Sym}_n)$.

## 4. Oriented random-to-top shuffles: Type B

In this section we state and prove the analogue of Theorem 1.1 for the Coxeter group of Type B. Henceforth we shall always use the symbol † to indicate quantities relevant to this type.

### *4.1. Type B set partitions, shuffles and partition moves*

We begin by defining the three quantities that we shall show are equal. Let $t$, $n \in \mathbf{N}_0$.

Let $B_t^\dagger(n)$ be the number of set partitions of $\{1, \ldots, t\}$ into at most $n$ parts with an even number of elements in each part distinguished by *marks*. Let $B_t^{\dagger\prime}(n)$ be defined similarly, counting only those set partitions such that if 1 and $t$ are in the same part then 1 is marked, and if both $i$ and $i+1$ are in the same part then $i+1$ is marked.

We define the Coxeter group $\mathrm{BSym}_n$ to be the maximal subgroup of the symmetric group on the set $\{-1,\ldots,-n\} \cup \{1,\ldots,n\}$ that permutes $\{-1,1\},\ldots,\{-n,n\}$ as blocks for its action. Let $\sigma \in \mathrm{BSym}_n$. We interpret $\sigma$ as the shuffle of a deck of $n$ cards, each oriented either *face-up* or *face-down*, that moves a card in position $i \in \{1,\ldots,n\}$ to position $|i\sigma|$, changing the orientation of the card if and only if $i\sigma \in \{-1,\ldots,-n\}$.

**Definition.** For $m \in \mathbf{N}$ let

$$\begin{aligned}\rho_m &= (-1,\ldots,-m)(1,\ldots,m),\\ \bar{\rho}_m &= (-1,\ldots,-m,1,\ldots,m).\end{aligned}$$

We say that $\rho_m$ and $\bar{\rho}_m$ are *oriented random-to-top shuffles*. Let $S_t^{\dagger}(n)$ be the number of sequences of $t$ oriented random-to-top shuffles whose product is the identity permutation. Define $S_t^{\dagger\prime}(n)$ analogously, excluding the identity permutation $\rho_1$.

Observe that, under our card shuffling interpretation, $\bar{\rho}_m$ lifts the card in position $m$ to the top of the deck and then flips it.

A pair $(\lambda,\lambda^\star)$ of partitions such that the sum of the sizes of $\lambda$ and $\lambda^\star$ is $n \in \mathbf{N}_0$ will be called a *double partition of* $n$. By [16, Theorem 4.3.34] the irreducible characters of $\mathrm{BSym}_n$ are canonically labelled by double partitions of $n$. Since this theorem is considerably more general than we need, we give an independent proof. For $r \in \mathbf{N}$, let $H_r \leq \mathrm{BSym}_r$ be the index 2 subgroup of permutations that, in our card shuffling interpretation, flip evenly many cards. Let $\zeta_r : \mathrm{BSym}_r \to \{1,-1\}$ be the non-trivial linear character of $\mathrm{BSym}_r$ with kernel $H_r$. Given a character $\chi$ of $\mathrm{Sym}_r$, we define the *inflation* of $\chi$ to $\mathrm{BSym}_r$ by $\mathrm{Inf}_{\mathrm{Sym}_r}^{\mathrm{BSym}_r} \chi(\sigma) = \chi(\overline{\sigma})$, where $\overline{\sigma} \in \mathrm{Sym}_r$ is the image of $\sigma \in \mathrm{BSym}_r$ under the canonical quotient map $\mathrm{BSym}_r \to \mathrm{Sym}_r$ with kernel $\langle (1,-1),\ldots,(r,-r)\rangle$. Given a double partition $(\lambda,\lambda^\star)$ of $n$ such that $|\lambda| = \ell$ and $|\lambda^\star| = \ell^\star$, we define

$$\chi^{(\lambda,\lambda^\star)} = \big(\mathrm{Inf}_{\mathrm{Sym}_\ell}^{\mathrm{BSym}_\ell} \chi^\lambda \times \zeta_{\ell^\star} \mathrm{Inf}_{\mathrm{Sym}_{\ell^\star}}^{\mathrm{BSym}_{\ell^\star}} \chi^{\lambda^\star}\big)\big\uparrow^{\mathrm{BSym}_n}.$$

**Lemma 4.1.** *The characters $\chi^{(\lambda,\lambda^\star)}$ for $(\lambda,\lambda^\star)$ a double partition of $n$ are exactly the irreducible characters of* $\mathrm{BSym}_n$.

*Proof.* We outline a proof using the Clifford theory presented in [14, Chapter 6]. Let $\eta : C_2 \to \{\pm 1\}$ be the non-trivial character of $C_2$ and let $B = \langle (1,-1),\ldots,(n,-n)\rangle \trianglelefteq \mathrm{BSym}_n$. Let $\phi^{(\lambda,\lambda^\star)} = \mathrm{Inf}_{\mathrm{Sym}_\ell}^{\mathrm{BSym}_\ell} \chi^\lambda \times \zeta_{\ell^\star} \mathrm{Inf}_{\mathrm{Sym}_{\ell^\star}}^{\mathrm{BSym}_{\ell^\star}} \chi^{\lambda^\star}$.

By the definition of inflation, the kernel of $\mathrm{Inf}_{\mathrm{Sym}_\ell}^{\mathrm{BSym}_\ell}\chi^\lambda$ contains $C_2^\ell$, and the kernel of $\mathrm{Inf}_{\mathrm{Sym}_{\ell^\star}}^{\mathrm{BSym}_{\ell^\star}}\chi^{\lambda^\star}$ contains $C_2^{\ell^\star}$. Hence

$$\phi^{(\lambda,\lambda^\star)}\big\downarrow_B = \phi(1)1_{C_2^\ell}\times\eta_{C_2^{\ell^\star}}.$$

The character $1_{C_2^\ell}\times\eta_{C_2^{\ell^\star}}$ of $B$ has inertial group $\mathrm{BSym}_\ell\times\mathrm{BSym}_{\ell^\star}\le\mathrm{BSym}_n$. Since the $\phi^{(\lambda,\lambda^\star)}$ are distinct irreducible characters of $\mathrm{BSym}_\ell\times\mathrm{BSym}_{\ell^\star}$, it follows from [14, Theorem 6.11] that the characters $\chi^{(\lambda,\lambda^\star)}$ are distinct and irreducible.

The sum of the squares of the degrees of the irreducible characters of a finite group is equal to its order. The degree of $\chi^{(\lambda,\lambda^\star)}$ is $\binom{n}{\ell}\chi^\lambda(1)\chi^{\lambda^\star}(1)$, so

$$\sum_{\substack{\ell+\ell^\star=n\\ \lambda\in\mathrm{Par}(\ell)\\ \lambda^\star\in\mathrm{Par}(\ell^\star)}}\binom{n}{\ell}^2\chi^\lambda(1)^2\chi^{\lambda^\star}(1)^2=\sum_{\ell=0}^n\binom{n}{\ell}^2\ell!(n-\ell)!=n!\sum_{\ell=0}^n\binom{n}{\ell}=n!2^n,$$

which is the order of $\mathrm{BSym}_n$. Therefore we have constructed all irreducible characters of $\mathrm{BSym}_n$. □

For use in the proof of Lemma 4.4, we record a branching rule for $\mathrm{BSym}_n$.

**Lemma 4.2.** *If $(\lambda,\lambda^\star)$ is a double partition of $n$ then $\chi^{(\lambda,\lambda^\star)}\downarrow_{\mathrm{BSym}_{n-1}}=\sum\chi^{(\nu,\nu^\star)}$ where the sum is over all double partitions $(\nu,\nu^\star)$ of $n-1$ obtained by removing a single box from either the Young diagram of $\lambda$ or the Young diagram of $\lambda^\star$.*

*Proof.* Let $\phi=\mathrm{Inf}_{\mathrm{Sym}_\ell}^{\mathrm{BSym}_\ell}\chi^\lambda\times\zeta_{\ell^\star}\mathrm{Inf}_{\mathrm{Sym}_{\ell^\star}}^{\mathrm{BSym}_{\ell^\star}}\chi^{\lambda^\star}$. Let $K=\mathrm{BSym}_\ell\times\mathrm{BSym}_{\ell^\star}$, where the factors act on $\{1,\dots,\ell\}$ and $\{\ell+1,\dots,n\}$, respectively. Identifying $\mathrm{BSym}_n/\mathrm{BSym}_{n-1}$ with $\{-1,\dots,-n\}\cup\{1,\dots,n\}$, we see that $K$ has two orbits on $\mathrm{BSym}_n/\mathrm{BSym}_{n-1}$, with representatives $\ell$ and $n$. The corresponding double cosets are $K\mathrm{BSym}_{n-1}$ and $K(\ell,n)\mathrm{BSym}_{n-1}$. Define

$$\begin{aligned}G&=K\cap\mathrm{BSym}_{n-1}=\mathrm{BSym}_{\{1,\dots,\ell\}}\times\mathrm{BSym}_{\{\ell+1,\dots,n-1\}},\\ H&=K^{(\ell,n)}\cap\mathrm{BSym}_{n-1}=\mathrm{BSym}_{\{1,\dots,\ell-1\}}\times\mathrm{BSym}_{\{\ell,\ell+1,\dots,n-1\}}.\end{aligned}$$

By Mackey's Theorem (see for instance [14, Problem 5.6]), $\chi^{(\lambda,\lambda^\star)}\downarrow_{\mathrm{BSym}_{n-1}}=\phi\uparrow^{\mathrm{BSym}_n}\downarrow_{\mathrm{BSym}_{n-1}}=\phi\downarrow_G\uparrow^{\mathrm{BSym}_{n-1}}+\phi^{(\ell,n)}\downarrow_H\uparrow^{\mathrm{BSym}_{n-1}}$. By the definition of inflation and the Branching Rule for symmetric groups, as stated after (1), we have

$$\begin{aligned}\phi\big\downarrow_G&=\mathrm{Inf}_{\mathrm{Sym}_\ell}^{\mathrm{BSym}_\ell}\chi^\lambda\times\zeta_{\ell^\star-1}\sum_{\nu^\star}\mathrm{Inf}_{\mathrm{Sym}_{\ell^\star-1}}^{\mathrm{BSym}_{\ell^\star-1}}\chi^{\nu^\star},\\ \phi^{(\ell,n)}\big\downarrow_H&=\sum_\nu\mathrm{Inf}_{\mathrm{Sym}_{\ell-1}}^{\mathrm{BSym}_{\ell-1}}\chi^\nu\times\zeta_{\ell^\star}\mathrm{Inf}_{\mathrm{Sym}_{\ell^\star}}^{\mathrm{BSym}_{\ell^\star}}\chi^{\lambda^\star},\end{aligned}$$

where $\nu^\star$ and $\nu$ vary as in the statement of this lemma. The lemma therefore follows from Mackey's Theorem. □

**Definition.** Let $(\lambda,\lambda^\star)$ be a double partition. A *double-move* on $(\lambda,\lambda^\star)$ consists of the removal and then addition of a single box on the corresponding Young diagrams. (The box need not be added to the diagram from which it is removed.) A double-move is *exceptional* if it consists of the removal and then addition in the same place of the lowest removable box in the diagram of $\lambda$, or if $\lambda$ is empty, of the lowest removable box in the diagram of $\lambda^\star$. Let $M^\dagger_t\big((\lambda,\lambda^\star),(\mu,\mu^\star)\big)$ be the number of sequences of $t$ double-moves that start at $(\lambda,\lambda^\star)$ and finish at $(\mu,\mu^\star)$. Let $M^{\dagger\prime}_t\big((\lambda,\lambda^\star),(\mu,\mu^\star)\big)$ be defined analogously, considering only non-exceptional double-moves. Let $M^\dagger_t(n) = M^\dagger_t\big(((n),\varnothing),((n),\varnothing)\big)$ and let $M^{\dagger\prime}_t(n) = M^{\dagger\prime}_t\big(((n),\varnothing),((n),\varnothing)\big)$.

This definition of 'exceptional' is convenient for small examples, but can be replaced with any other that always excludes exactly one double-move from the set counted by $M^\dagger_1\big((\lambda,\lambda^\star),(\lambda,\lambda^\star)\big)$.

We can now state the analogue of Theorem 1.1 for Type B.

**Theorem 4.3.** *For all $t$, $n \in \mathbf{N}_0$ we have $B^\dagger_t(n) = S^\dagger_t(n) = M^\dagger_t(n)$ and $B^{\dagger\prime}_t(n) = S^{\dagger\prime}_t(n) = M^{\dagger\prime}_t(n)$.*

The proof of this theorem follows the same general plan as the proof of Theorem 1.1 so we shall present it quite briefly. The most important difference is that we now make explicit use of the root system.

*4.2. Proof of Theorem 4.3*

To prove Lemma 2.1 we used a bijection between the set partitions of $\{1,\ldots,t\}$ into at most $n$ parts and sequences of $t$ random-to-top shuffles leaving a deck of $n$ cards invariant. This bijection can be modified to give a bijection between the marked set partitions counted by $B^\dagger_t(n)$ and the shuffle sequences counted by $S^\dagger_t(n)$: given a marked set partition $\mathcal{P}$, the corresponding shuffle flips the orientation of the card lifted at time $s \in \{1,\ldots,t\}$ if and only if $s$ is a marked element of $\mathcal{P}$. This map restricts to a bijection between the marked set partitions counted by $B^{\dagger\prime}_t(n)$ and the shuffle sequences counted by $S^{\dagger\prime}_t(n)$. Thus $B^\dagger_t(n) = S^\dagger_t(n)$ and $B^{\dagger\prime}_t(n) = S^{\dagger\prime}_t(n)$ for all $t$, $n \in \mathbf{N}_0$.

Let $\mathrm{BSym}_{n-1} = \{\sigma \in \mathrm{BSym}_n : n\sigma = n\}$ and let $\pi^\dagger = 1_{\mathrm{BSym}_{n-1}}\big\uparrow^{\mathrm{BSym}_n}$. Inducing via the subgroup $\mathrm{BSym}_{n-1} \times \langle(-n,n)\rangle$ of $\mathrm{BSym}_n$ one finds that

$$\pi^\dagger = \chi^{((n),\varnothing)} + \chi^{((n-1,1),\varnothing)} + \chi^{((n-1),(1))}.$$

Let $\vartheta^\dagger = \pi^\dagger - 1_{\mathrm{BSym}_n}$. The analogue of Lemma 2.2 is as follows.

**Lemma 4.4.** *If $(\lambda,\lambda^\star)$ and $(\mu,\mu^\star)$ are double partitions of $n \in \mathbf{N}$ then*

$$M^\dagger_t\big((\lambda,\lambda^\star),(\mu,\mu^\star)\big) = \langle \chi^{(\lambda,\lambda^\star)}\pi^t, \chi^{(\mu,\mu^\star)}\rangle,$$
$$M^{\dagger\prime}_t\big((\lambda,\lambda^\star),(\mu,\mu^\star)\big) = \langle \chi^{(\lambda,\lambda^\star)}\vartheta^t, \chi^{(\mu,\mu^\star)}\rangle.$$

*Proof.* By Lemma 4.2, we have $\chi^{(\lambda,\lambda^\star)}\downarrow_{\mathrm{BSym}_{n-1}} = \sum_{(\nu,\nu^\star)} \chi^{(\nu,\nu^\star)}$ where the sum is over all $(\nu, \nu^\star)$ obtained from $(\lambda, \lambda^\star)$ by removing a single box from the Young diagram of either $\lambda$ or $\lambda^\star$. By Frobenius reciprocity it follows that $\chi^{(\nu,\nu^\star)}\uparrow^{\mathrm{BSym}_n} = \sum_{(\mu,\mu^\star)} \chi^{(\mu,\mu^\star)}$, where the sum is over all $(\mu, \mu^\star)$ obtained from $(\nu, \nu^\star)$ by adding a single box to the Young diagram of either $\nu$ or $\nu^\star$. This proves the case $t = 1$ of the first part of the lemma; the general case then follows by induction on $t$. The second part is proved similarly, again in the same way as Lemma 2.2. □

The group $\mathrm{BSym}_n$ acts on the $n$-dimensional real vector space $E = \langle e_1, \ldots, e_n\rangle$ by $e_i\sigma = \pm e_j$ where $j = |i\sigma|$ and the sign is the sign of $i\sigma$. Let $\Phi^+ = \{e_i - e_j : 1 \le i < j \le n\} \cup \{e_i + e_j : 1 \le i < j \le n\} \cup \{e_k : 1 \le k \le n\}$. Then $\Phi^+$ is the set of positive roots in a root system of type B, having simple roots $\alpha_i = e_i - e_{i+1}$ for $1 \le i < n$ and $\beta = e_n$. Let $\mathrm{Des}(\mathrm{BSym}_n)$ be the descent subalgebra of the rational group algebra $\mathbf{Q}\mathrm{BSym}_n$, as defined in [21, Theorem 1], for this choice of simple roots. Thus $\mathrm{Des}(\mathrm{BSym}_n)$ has as a basis the elements $\Xi_I^\dagger$ defined for each $I \subseteq \{\alpha_1, \ldots, \alpha_{n-1}, \beta\}$ by taking the sum of all the permutations $\sigma \in \mathrm{BSym}_n$ such that

$$\{\alpha \in \Phi^+ : \alpha\sigma \in -\Phi^+\} \subseteq I.$$

By [21, Theorem 1], there is an algebra epimorphism $\mathrm{Des}(\mathrm{BSym}_n) \to \mathrm{Cl}(\mathrm{BSym}_n)$ under which $\Xi_I^\dagger$ maps to the permutation character of $\mathrm{BSym}_n$ acting on the cosets of the parabolic subgroup corresponding to $I$. By [13, Theorem 3.1], this epimorphism is an isometry with respect to the bilinear form on $\mathrm{Des}(\mathrm{BSym}_n)$ defined by restriction of the form $(-,-)$ on $\mathbf{Q}\mathrm{Sym}_{2n}$.

We need the following basic lemma describing descents in Type B for our choice of simple roots.

**Lemma 4.5.** *Let $\sigma \in \mathrm{BSym}_n$. If $\ell$ and $m$ are such that $\alpha_\ell\sigma, \ldots, \alpha_{m-1}\sigma \in \Phi^+$ then either $\ell\sigma, \ldots, m\sigma$ all have the same sign and $\ell\sigma < \ldots < m\sigma$, or there exists a unique $q \in \{\ell, \ldots, m-1\}$ such that $1 \le \ell\sigma < \ldots < q\sigma$ and $(q+1)\sigma < \ldots < m\sigma \le -1$.*

*Proof.* It is routine to check that $\alpha_i\sigma \in \Phi^+$ if and only if one of

(a) $i\sigma \in \{1, \ldots, n\}$, $(i+1)\sigma \in \{1, \ldots, n\}$ and $i\sigma < (i+1)\sigma$;

(b) $i\sigma \in \{1, \ldots, n\}$, $(i+1)\sigma \in \{-1, \ldots, -n\}$;

(c) $i\sigma \in \{-1, \ldots, -n\}$, $(i+1)\sigma \in \{-1, \ldots, -n\}$ and $i\sigma < (i+1)\sigma$.

The lemma now follows easily. □

Thus if $i\sigma$ and $(i+1)\sigma$ have the same sign then $\alpha_i$ is a descent of $\sigma$ (that is, $\alpha_i\sigma \in -\Phi^+$) if and only if $i\sigma < (i+1)\sigma$. However, by (b), a change from positive to negative is *never* a descent, and a change from negative to positive is *always* a descent. We remark that in [5] and [18], an alternative choice of simple roots is used under which 'descent' has its expected numerical meaning in all cases. However, this choice corresponds to random-to-bottom shuffles and so is not convenient for our purposes.

Let $\Xi^\dagger = \sum_{m=1}^n (\rho_m^{-1} + \bar{\rho}_m^{-1})$ and let $\Delta^\dagger = \Xi - \mathrm{id}_{\mathrm{BSym}_n}$.

**Lemma 4.6.** *Let $n \geq 2$. Then $\Xi^\dagger$, $\Delta^\dagger \in \mathrm{Des}(\mathrm{BSym}_n)$ and under the canonical algebra epimorphism $\mathrm{Des}(\mathrm{BSym}_n) \to \mathrm{Cl}(\mathrm{BSym}_n)$ we have $\Xi^\dagger \mapsto \pi^\dagger$ and $\Delta^\dagger \mapsto \pi^\dagger - 1_{\mathrm{Sym}_n}$.*

*Proof.* By Lemma 4.5, if $\alpha_i\sigma \in \Phi^+$ for $2 \leq i < n$ then there exists a unique $p \in \{1,\ldots,n\}$ such that $\{2,\ldots,p\}\sigma \subseteq \{1,\ldots,n\}$ and $\{p+1,\ldots,n\}\sigma \subseteq \{-1,\ldots,-n\}$. If in addition $\beta\sigma \in \Phi^+$, we must have $p = n$. Therefore $\alpha_1$ is the unique simple root $\delta$ such that $\delta\sigma \notin \Phi^+$ if and only if $\{2,\ldots,n\}\sigma \subseteq \{1,\ldots,n\}$ and *either*

(1) $1\sigma \in \{1,\ldots,n\}$ and $1\sigma > 2\sigma < \ldots < n\sigma$, *or*

(2) $1\sigma \in \{-1,\ldots,-n\}$ and $2\sigma < \ldots < n\sigma$.

The permutations in Case (1) are $\{\rho_2^{-1},\ldots,\rho_n^{-1}\}$ and the permutations in Case (2) are $\{\bar{\rho}_1^{-1},\ldots,\bar{\rho}_n^{-1}\}$. Hence $\Xi^\dagger = \Xi^\dagger_{\{\alpha_1\}}$. This proves the first part of the lemma.

The reflections in the simple roots $\alpha_2,\ldots,\alpha_{n-1}$ and $\beta$ are $s_{\alpha_i} = (i, i+1)(-i, -(i+1))$ for $2 \leq i \leq n-1$ and $s_\beta = (-n, n)$. These generate the subgroup $\mathrm{BSym}_{\{2,\ldots,n\}}$ of $\mathrm{BSym}_n$. Hence $\mathrm{BSym}_{\{2,\ldots,n\}}$ is the parabolic subgroup of $\mathrm{BSym}_n$ corresponding to $\{\alpha_1\}$ and, since $\mathrm{BSym}_{\{2,\ldots,n\}}$ is conjugate to $\mathrm{BSym}_{n-1}$, the permutation character of $\mathrm{BSym}_n$ acting on the cosets of $\mathrm{BSym}_{\{2,\ldots,n\}}$ is $\pi^\dagger$. This proves the second part of the lemma. □

We are now ready to prove the second equality in Theorem 4.3.

**Lemma 4.7.** *For $t$, $n \in \mathbf{N}_0$ we have $S_t^\dagger(n) = M_t^\dagger(n)$ and $S_t^{\dagger\prime}(n) = M_t^{\dagger\prime}(n)$.*

*Proof.* If $n = 0$ the result is clear. When $n = 1$ the oriented top-to-random shuffles are the identity and $(-1, 1)$ and it is easily checked that $S_0^\dagger(1) = M_0^\dagger(1) = 1$ and $S_t^\dagger(1) = M_t^\dagger(1) = 2^{t-1}$ for all $t \in \mathbf{N}$. Similarly $S_t^{\dagger\prime}(1) = M_t^{\dagger\prime}(1) = 1$ if $t$ is even and $S_t^{\dagger\prime}(1) = M_t^{\dagger\prime}(1) = 0$ if $t$ is odd. When $n \geq 2$ we follow the proof of Lemma 2.3, using that $[\mathrm{id}_{\mathrm{BSym}_n}]\Xi^{\dagger t} = S_t^\dagger(n)$ and $[\mathrm{id}_{\mathrm{BSym}_n}]\Delta^{\dagger t} = S_t^{\dagger\prime}(n)$. □

This completes the proof of Theorem 4.3.

*4.3. Eigenvalues of the oriented k-shuffle*

The analogue for Type B of the $k$-shuffle is most conveniently defined using our interpretation of elements of $\mathrm{BSym}_n$ as shuffles (with flips) of a deck of $n$ cards.

**Definition.** Let $n$, $k \in \mathbf{N}$ with $k \leq n$. An *oriented $k$-shuffle* is performed as follows. Remove $k$ cards from the deck. Then choose any $j \in \{0, 1, \ldots, k\}$ of the $k$ cards and flip these $j$ cards over as a block. Place the $j$ flipped cards on top of the deck, and then put the $k - j$ unflipped cards on top of them.

Thus there are $2^k\binom{n}{k}$ oriented $k$-shuffles. After an oriented $k$-shuffle that flips $j$ cards, the newly flipped cards occupy positions $k - j + 1, \ldots, k$, and appear in the reverse of their order in the original deck.

For $k < n$, let $\Xi^\dagger_k = \Xi^\dagger_{\{\alpha_k\}} \in \mathrm{Des}(\mathrm{BSym}_n)$ and let $\Xi^\dagger_n = \Xi^\dagger_{\{\beta\}} \in \mathrm{Des}(\mathrm{BSym}_n)$. The following lemma is proved by extending the argument used in the proof of Lemma 4.6.

**Lemma 4.8.** *Let $k \leq n$. Let $\sigma \in \mathrm{BSym}_n$. Then $\sigma$ is in the support of $\Xi^\dagger_k$ if and only if $\sigma^{-1}$ is an oriented $k$-shuffle.*

*Proof.* Let $\sigma$ be in the support of $\Xi^\dagger_k$. Lemma 4.5 implies that either $1 \leq 1\sigma < \ldots < k\sigma$ or there exists a unique $j \in \{1, \ldots, k\}$ such that $1 \leq 1\sigma < \ldots < (j-1)\sigma$ and $j\sigma < \ldots < k\sigma \leq -1$. Moreover, if $k < n$, then since $\beta\sigma \in \Phi^+$, Lemma 4.5 implies that $1 \leq (k+1)\sigma < \ldots < n\sigma$.

Thus in our card shuffling interpretation, $\sigma$ takes the top $k$ cards, flips the bottom $j$ of these cards as a block, for some $j \in \{0, 1, \ldots, k\}$, and then inserts the blocks of $k - j$ unflipped cards and $j$ flipped cards into the deck, preserving the order within each block. These shuffles are exactly the inverses of the oriented $k$-shuffles. $\square$

Under the canonical epimorphism $\mathrm{Des}(\mathrm{BSym}_n) \to \mathrm{Cl}(\mathrm{BSym}_n)$, the image of $\Xi^\dagger_k$ is the permutation character $\pi^\dagger_k$ of $\mathrm{BSym}_n$ acting on the cosets of the parabolic subgroup $\mathrm{Sym}_k \times (C_2 \wr \mathrm{Sym}_{n-k})$ generated by the reflections in the simple roots other than $\alpha_k$ if $k < n$, or other than $\beta$ if $k = n$.

Let $P^\dagger(k)$ be the transition matrix of the Markov chain on $\mathrm{BSym}_n$ in which each step is given by choosing one of the $2^k\binom{n}{k}$ oriented $k$-shuffles uniformly at random. Let $P^{\dagger'}(k)$ be the analogous transition matrix when only non-identity shuffles are chosen. The same argument used in §3 now proves the following proposition.

**Proposition 4.9.** *The eigenvalues of $P^\dagger(k)$ are $\pi^\dagger_k(\tau)/2^k\binom{n}{k}$ and the eigenvalues of $P^{\dagger'}(k)$ are $(\pi^\dagger_k(\tau) - 1)/(2^k\binom{n}{k} - 1)$, both for $\tau \in \mathrm{BSym}_n$.* $\square$

A convenient model for the cosets of $\mathrm{Sym}_k \times (C_2 \wr \mathrm{Sym}_{n-k})$ in $\mathrm{BSym}_n$ is the set $\Omega_k$ of $k$-subsets of the short roots $\{\pm e_1, \ldots, \pm e_n\}$ that each contain at most one element of each pair $\{e_i, -e_i\}$. The action of $\mathrm{BSym}_n$ on $\Omega_k$ is inherited from its action on $\langle e_1, \ldots, e_n \rangle$. Using the notation for elements of $\mathrm{BSym}_n$ introduced immediately before (4.1.12) in [16], it is clear that a $k$-subset $A = \{s_j e_j : j \in J\} \in \Omega_k$ is fixed by $(u_1, \ldots, u_n; \tau) \in \mathrm{BSym}_n$ if and only if (i) $J$ is fixed by $\tau$ and (ii) $u_j s_j = s_{j\tau}$ for each $j \in J$.

Using this we make the two extreme cases in Proposition 4.9 more explicit. It is clear that $\pi_1^\dagger(\sigma) = \pi^\dagger(\sigma)$ for $\sigma \in \mathrm{BSym}_n$; the model $\Omega_1$ shows that $\pi^\dagger\big((u_1, \ldots, u_n; \tau)\big)$ is the number of short roots $e_i$ such that $u_i = 1$ and $i\tau = i$. An $n$-shuffle is the inverse of the shuffle performed by separating the deck into two parts, flipping the part containing the bottom card, and then riffle-shuffling the two parts. By (i) and (ii) we see that $\pi_n^\dagger\big((u_1, \ldots, u_n; \tau)\big)$ is the number of sequences $(s_1, \ldots, s_n)$ such that $s_j \in \{+1, -1\}$ and $u_j s_j = s_{j\tau}$ for each $j$. Hence the corresponding eigenvalue is $2^{d-n}$, where $d$ is the number of orbits of $\tau$ on $\{1, \ldots, n\}$.

## 5. Cheating random-to-top shuffles: Type D

### *5.1. Type D shuffles and partition moves*

For $n \in \mathbf{N}$ we define the Coxeter group $\mathrm{DSym}_n$ of Type D by

$$\mathrm{DSym}_n = \big\{\sigma \in \mathrm{BSym}_n : \big|\{1, \ldots, n\}\sigma \cap \{-1, \ldots, -n\}\big| \text{ is even}\big\}.$$

The shuffles and partition moves relevant to this case are defined as follows. We shall use the superscript $\ddagger$ to denote quantities relevant to this type.

**Definition.** Let $n \in \mathbf{N}$ and let $k \leq n$. Let $\sigma \in \mathrm{DSym}_n$. Then $\sigma$ is a *cheating $k$-shuffle* if and only if either $\sigma$ or $\sigma(-n, n)$ is an oriented $k$-shuffle. A cheating 1-shuffle will be called a *cheating random-to-top shuffle*. Let $S_t^\ddagger(n)$ be the number of sequences of $t$ cheating random-to-top shuffles whose product is $\mathrm{id}_{\mathrm{DSym}_n}$. Define $S_t^{\ddagger\prime}(n)$ analogously, excluding the identity permutation $\rho_1$.

Thus the cheating $k$-shuffles are obtained from the oriented $k$-shuffles by composing with a 'cheating' flip of the bottom card, whenever this is necessary to arrive in the group $\mathrm{DSym}_n$. It easily follows that the cheating random-to-top shuffles are

$$\{\rho_m : 1 \leq m \leq n\} \cup \{\bar{\rho}_m(-n, n) : 1 \leq m \leq n\}$$

where $\rho_m$ and $\bar{\rho}_m$ are as defined in §4.1. Note that a cheating random-to-top shuffle either flips no cards, or flips both the lifted card and the (new, in the case of $\bar{\rho}_n$) bottom card.

**Definition.** Let $t$, $n \in \mathbf{N}_0$. Let $M_t^{\ddagger}(n)$ be the number of sequences of $t$ double-moves that start at $((n), \varnothing)$ and finish at either $((n), \varnothing)$ or $(\varnothing, (n))$. Define $M_t^{\ddagger\prime}(n)$ analogously, considering only non-exceptional double-moves.

As in the case of Type B, the sequences of shuffles counted by $S_t^{\ddagger}(n)$ and $S_t^{\ddagger\prime}(n)$ are in bijection with certain marked set partitions of $\{1, \ldots, t\}$ into at most $n$ parts, but it now seems impossible to give a more direct description of these partitions. Our strongest analogue of Theorem 1.1 is as follows.

**Theorem 5.1.** *For all $t \in \mathbf{N}_0$ and $n \in \mathbf{N}$ such that $n \geq 2$ we have $S_t^{\ddagger}(n) = M_t^{\ddagger}(n)$ and $S_t^{\ddagger\prime}(n) = M_t^{\ddagger\prime}(n)$.*

We remark that since $\mathrm{DSym}_1$ is the trivial group we have $S_t^{\ddagger}(1) = 1$ whereas $M_t^{\ddagger}(1) = 2^t$ for all $t \in \mathbf{N}_0$. Similarly $S_0^{\ddagger\prime}(1) = 1$ and $S_t^{\ddagger\prime}(1) = 0$ for all $t \in \mathbf{N}$, whereas $M_t^{\ddagger\prime}(1) = 1$ for all $t \in \mathbf{N}_0$.

*5.2. Proof of Theorem 5.1*

We begin with the character-theoretic part of the proof. Let $\pi^{\ddagger} = \pi^{\dagger}\!\downarrow_{\mathrm{DSym}_n}$ and let $\vartheta^{\ddagger} = \pi^{\ddagger} - 1_{\mathrm{DSym}_n}$.

**Lemma 5.2.** *Let $t \in \mathbf{N}_0$ and let $n \in \mathbf{N}$. Then $M_t^{\ddagger}(n) = \langle \pi^{\ddagger t}, 1_{\mathrm{DSym}_n} \rangle$ and $M_t^{\ddagger\prime}(n) = \langle \vartheta^{\ddagger t}, 1_{\mathrm{DSym}_n} \rangle$.*

*Proof.* Since $1_{\mathrm{DSym}_n}\!\uparrow^{\mathrm{BSym}_n} = \chi^{((n),\varnothing)} + \chi^{(\varnothing,(n))}$, it follows from Frobenius reciprocity that

$$\langle \pi^{\ddagger t}, 1_{\mathrm{DSym}_n} \rangle = \langle \pi^{\dagger t}\!\downarrow_{\mathrm{DSym}_n}, 1_{\mathrm{DSym}_n} \rangle = \langle \pi^{\dagger t}, \chi^{((n),\varnothing)} + \chi^{(\varnothing,(n))} \rangle.$$

The first part of the lemma now follows from Lemma 4.4. The second part is proved analogously. $\square$

A root system for $\mathrm{DSym}_n$, constructed inside the span of the root system for $\mathrm{BSym}_n$ already defined, has positive roots $\Psi^+ = \{e_i - e_j : 1 \leq i < j \leq n\} \cup \{e_i + e_j : 1 \leq i < j \leq n\}$, and simple roots $\alpha_1, \ldots, \alpha_{n-1}$ and $\gamma$, where $\gamma = e_{n-1} + e_n$. Let $\mathrm{Des}(\mathrm{DSym}_n)$ be the descent subalgebra of the rational group algebra $\mathbf{Q}\mathrm{DSym}_n$, as defined in [21, Theorem 1], for this choice of simple roots. Let $\Xi_I^{\ddagger}$ for $I \subseteq \{\alpha_1, \ldots, \alpha_{n-1}, \gamma\}$ be the canonical basis elements, defined in the same way as for Type B.

Let $\Xi^{\ddagger} = \sum_{m=1}^{n} (\rho_m^{-1} + (-n, n)\bar{\rho}_m^{-1})$ and let $\Delta^{\ddagger} = \Xi^{\ddagger} - \mathrm{id}_{\mathrm{DSym}_n}$. The following lemma is the analogue of Lemma 4.6 for Type D.

**Lemma 5.3.** *Let $n \in \mathbf{N}$ be such that $n \geq 3$. Then* $\Xi^\ddagger$, $\Delta^\ddagger \in \mathrm{Des}(\mathrm{DSym}_n)$ *and under the canonical algebra epimorphism* $\mathrm{Des}(\mathrm{DSym}_n) \to \mathrm{Cl}(\mathrm{DSym}_n)$ *we have* $\Xi^\ddagger \mapsto \pi^\ddagger$ *and* $\Delta^\ddagger \mapsto \pi^\ddagger - 1_{\mathrm{Sym}_n}$.

*Proof.* Let $\sigma \in \mathrm{DSym}_n$. Suppose that $\alpha_1$ is the unique simple root $\delta$ such that $\delta\sigma \notin \Psi^+$. Observe that if $n\sigma \in \{-1, \ldots, -n\}$ then, since $\gamma\sigma \in \Psi^+$, we have $(n-1)\sigma \in \{1, \ldots, n\}$ and $(n-1)\sigma < |n\sigma|$. It therefore follows from Lemma 4.5 that $\{2, \ldots, n-1\}\sigma \subseteq \{1, \ldots, n\}$. Since $\big|\{1, \ldots, n\}\sigma \cap \{-1, \ldots, -n\}\big|$ is even, either $\{1, n\}\sigma \subseteq \{1, \ldots, n\}$ or $\{1, n\}\sigma \subseteq \{-1, \ldots, -n\}$. Hence *either*

(1) $1\sigma, n\sigma \in \{1, \ldots, n\}$ and $1\sigma > 2\sigma < \ldots < n\sigma$, *or*

(2) $1\sigma, n\sigma \in \{-1, \ldots, -n\}$ and $2\sigma < \ldots < (n-1)\sigma < |n\sigma|$.

The permutations $\sigma$ in Case (1) are $\{\rho_m^{-1} : 1 \leq m \leq n\}$. In Case (2) the chain of inequalities implies that $|n\sigma| \geq n-1$ and $(n-1)\sigma \geq n-2$. If $n\sigma = -(n-1)$ then $(n-1)\sigma = n-2$ and the unique permutation is

$$(-n, n)\bar{\rho}_n^{-1} = (n, -(n-1), \ldots -1)(-n, (n-1), \ldots, 1).$$

The permutations such that $n\sigma = -n$ are $(-n, n)\bar{\rho}_m^{-1}$ for $1 \leq m \leq n-1$. Hence $\Xi^\ddagger = \Xi_{\{\alpha_1\}} \in \mathrm{DSym}_n$.

It is easily seen that $\pi^\ddagger = 1_{\mathrm{DSym}_{n-1}}\big\uparrow^{\mathrm{DSym}_n}$, and that $\mathrm{DSym}_{\{2,\ldots,n\}}$ is generated by the reflections in the simple roots $\alpha_i$ for $2 \leq i \leq n-1$ and $\gamma$. Therefore, by [21, Theorem 1], under the canonical algebra epimorphism $\mathrm{Des}(\mathrm{DSym}_n) \to \mathrm{Cl}(\mathrm{DSym}_n)$ we have $\Xi^\ddagger \mapsto \pi^\ddagger$. This completes the proof. $\square$

We are now ready to prove Theorem 5.1. When $n \geq 3$ the theorem follows from Lemma 5.2 and Lemma 5.3 by the argument used to prove Lemma 4.7. In the remaining case $\mathrm{DSym}_2 = \langle (-1,-2)(1,2), (-1,1)(-2,2) \rangle$ is the Klein 4-group and every permutation in $\mathrm{DSym}_2$ is a cheating top-to-random shuffle. Moreover, $\pi$ is the sum of all irreducible characters of $\mathrm{DSym}_2$ and $\vartheta$ is the sum of all non-trivial irreducible characters of $\mathrm{DSym}_2$. Since $\mathrm{DSym}_2$ is abelian, the group algebra $\mathbf{Q}\mathrm{DSym}_2$ is isomorphic to the algebra of class functions $\mathrm{Cl}(\mathrm{DSym}_2)$. Hence the coefficient of $\mathrm{id}_{\mathrm{DSym}_n}$ in $\Xi^{\ddagger t}$ is $\langle \pi^{\ddagger t}, 1_{\mathrm{DSym}_n} \rangle$, and the coefficient of $\mathrm{id}_{\mathrm{DSym}_n}$ in $\Delta^{\ddagger t}$ is $\vartheta^{\ddagger t}$. The theorem now follows from Lemma 5.2.

*5.3. Eigenvalues of the cheating oriented $k$-shuffle*

For $k < n$ let $\Xi^\ddagger_k = \Xi^\ddagger_{\{\alpha_k\}}$ and let $\Xi^\ddagger_n = \Xi^\ddagger_{\{\gamma\}}$. The following lemma is the analogue of Lemma 4.8 and generalizes Lemma 5.3. Let $\mathcal{S}(\Gamma)$ denote the support of $\Gamma \in \mathbf{Q}\mathrm{BSym}_n$.

**Lemma 5.4.** *Let $k \le n-2$. Let $\sigma \in \mathrm{DSym}_n$. Then $\sigma$ is in the support of $\Xi^{\ddagger}_k$ if and only if $\sigma^{-1}$ is a cheating $k$-shuffle.*

*Proof.* Let $\sigma \in \mathrm{BSym}_n$. The sets $\Phi^+$ and $\Psi^+$ agree except with respect to $\beta = \varepsilon_n$ and $\gamma = \varepsilon_{n-1} + \varepsilon_n$. Since $k \le n-2$, it is clear that $\sigma \in \mathcal{S}(\Xi^{\ddagger}_k)$ and $n\sigma \in \{1,\ldots,n\}$ if and only if $\sigma \in \mathcal{S}(\Xi^{\dagger}_k) \cap \mathrm{DSym}_n$. By Lemma 4.8, this is the case if and only if $\sigma^{-1}$ is a cheating $k$-shuffle that leaves the bottom card unflipped.

Suppose that $n\sigma \in \{-1,\ldots,-n\}$. Then $\gamma\sigma \in \Psi^+$ if and only $(n-1)\sigma < |n\sigma|$; in this case, as seen in the proof of Lemma 5.3, we have $\beta(-n,n)\sigma \in \Phi^+$ and $\alpha_{n-1}(-n,n)\sigma = \varepsilon_{(n-1)\sigma} - \varepsilon_{|n\sigma|} \in \Phi^+$. Hence $\sigma \in \mathcal{S}(\Xi^{\ddagger}_k)$ and $n\sigma \in \{-1,\ldots,-n\}$ if and only if $(-n,n)\sigma \in \mathcal{S}(\Xi^{\dagger}_k)$ and $\sigma \in \mathrm{DSym}_n$. By Lemma 4.8, this is the case if and only if $\sigma^{-1}$ is a cheating $k$-shuffle that flips the bottom card. $\square$

We remark that if $\sigma \in \mathrm{DSym}_n$ then $\sigma^{-1}$ is a cheating $(n-1)$-shuffle if and only if $\sigma \in S(\Xi^{\ddagger}_{\{\alpha_{n-1},\gamma\}})$ and $\sigma^{-1}$ is a cheating $n$-shuffle if and only if $\sigma \in S(\Xi^{\ddagger}_{\{\alpha_{n-1},\gamma\}}) \backslash \{\sigma \in \mathrm{DSym}_n : \alpha_{n-1}\sigma \in -\Phi^+ \text{ and } \gamma\sigma \in -\Phi^+\}$. For example $(n-1,-(n-1))(n,-n)$ is a cheating $(n-1)$-shuffle having two descents; this shuffle is obtained from the oriented $(n-1)$-shuffle that flips card $n-1$ (while fixing all other cards) by the cheating flip.

Under the canonical epimorphism $\mathrm{Des}(\mathrm{DSym}_n) \to \mathrm{Cl}(\mathrm{DSym}_n)$, the image of $\Xi^{\ddagger}_k$ is the permutation character $\pi^{\ddagger}_k$ of $\mathrm{DSym}_n$ acting on the cosets of the parabolic subgroup $\big(\mathrm{Sym}_k \times (C_2 \wr \mathrm{Sym}_{n-k})\big) \cap \mathrm{DSym}_n$. Thus $\pi^{\ddagger}_k = \pi^{\dagger}_k{\downarrow_{\mathrm{DSym}_n}}$.

Let $P^{\ddagger}(k)$ be the transition matrix of the Markov chain on $\mathrm{DSym}_n$ in which each step is given by choosing one of the $2^k\binom{n}{k}$ cheating $k$-shuffles uniformly at random. Let $P^{\ddagger'}(k)$ be the analogous chain where only non-identity shuffles are chosen. The same argument used in §3 now proves the following proposition.

**Proposition 5.5.** *Let $k \le n-2$. The eigenvalues of $P^{\ddagger}(k)$ are $\pi^{\ddagger}_k(\tau)/2^k\binom{n}{k}$ and the eigenvalues of $P^{\ddagger'}(k)$ are $(\pi^{\ddagger}_k(\tau)-1)/(2^k\binom{n}{k}-1)$, both for $\tau \in \mathrm{DSym}_n$.* $\square$

In particular, we observe that the eigenvalues of the Type D shuffle are exactly the eigenvalues of the Type B shuffle coming from elements of $\mathrm{DSym}_n \le \mathrm{BSym}_n$.

## 6. Generating functions and asymptotics

For $t \in \mathbf{N}_0$ and $n \in \mathbf{N}$, the Stirling number of the second kind $\left\{ {t \atop n} \right\}$ satisfies $\left\{ {t \atop n} \right\} = B_t(n) - B_t(n-1)$. By analogy we define $\left\{ {t \atop n} \right\}' = B'_t(n) -$

$B_t'(n-1)$, $\left\{ {t \atop n} \right\}^\dagger = B_t^\dagger(n) - B_t^\dagger(n-1)$ and $\left\{ {t \atop n} \right\}^{\dagger\prime} = B_t^{\dagger\prime}(n) - B_t^{\dagger\prime}(n-1)$. When $n = 0$ each generalized Stirling number is defined to be 1 if $t = 0$ and otherwise 0.

Theorem 1.1 and Theorem 4.3 give several combinatorial interpretations of these numbers. In particular, we note that $\left\{ {t \atop n} \right\}$ is the number of sequences of $t$ random-to-top shuffles that leave a deck of $n$ cards invariant while lifting every card at least once, and $\left\{ {t \atop n} \right\}^\dagger$ is the number of such sequences of oriented random-to-top shuffles. Moreover $\left\{ {t \atop n} \right\}'$ and $\left\{ {t \atop n} \right\}^{\dagger\prime}$ have similar interpretations, considering only sequences of non-identity shuffles.

In this section we give generating functions and asymptotic results on the generalized Bell and Stirling Numbers. Along the way we shall see a number of relationships between these numbers. Some of these results are obtained using basic arguments from residue calculus: we refer the reader to [23, Section 5.2] for an account of this method.

### *6.1. Relating $B_t(n)$ to $B_t'(n)$ and the asymptotics of $B_t'(t)$*

The numbers $B_t'(t)$ are considered by Bernhart [1], who describes the associated set partitions as *cyclically spaced*. The following lemma generalizes a result in §3.5 of [1]. The bijective proof given therein also generalizes, but we give instead a short algebraic proof as an application of Theorem 1.1 and Lemma 2.2.

**Lemma 6.1.** *If $t$, $n \in \mathbf{N}$ then $B_t'(n) + B_{t-1}'(n) = B_{t-1}(n-1)$.*

*Proof.* By Theorem 1.1 it is equivalent to prove that $M_t'(n) + M_{t-1}'(n) = M_{t-1}(n-1)$. Let $\pi$ be the natural permutation character of $\mathrm{Sym}_n$ and let $\vartheta = \chi^{(n-1,1)}$, as in Lemma 2.2. Note that $\vartheta\!\downarrow_{\mathrm{Sym}_{n-1}}$ is the natural permutation character of $\mathrm{Sym}_{n-1}$. Since restriction commutes with taking products of characters, it follows from Lemma 2.2 that $M_{t-1}(n-1) = \langle \vartheta^{t-1}\!\downarrow_{\mathrm{Sym}_{n-1}}, 1_{\mathrm{Sym}_{n-1}} \rangle$. By Frobenius reciprocity, $\langle \vartheta^{t-1}\!\downarrow_{\mathrm{Sym}_{n-1}}, 1_{\mathrm{Sym}_{n-1}} \rangle = \langle \vartheta^{t-1}, \pi \rangle$. Now

$$\langle \vartheta^{t-1}, \pi \rangle = \langle \vartheta^{t-1}, \vartheta \rangle + \langle \vartheta^{t-1}, 1_{\mathrm{Sym}_n} \rangle = \langle \vartheta^{t}, 1_{\mathrm{Sym}_n} \rangle + \langle \vartheta^{t-1}, 1_{\mathrm{Sym}_n} \rangle$$

which is equal to $M_t'(n) + M_{t-1}'(n)$, again by Lemma 2.2. $\square$

We note that the quantity $B_t'(n) + B_{t-1}'(n)$ appearing in Lemma 6.1 has a natural interpretation: it counts set partitions which are *spaced*, but not necessarily *cyclically spaced*. To prove this we use the associated Stirling numbers: for $t \in \mathbf{N}$ and $n \in \mathbf{N}_0$ define $\left\{ {t \atop n} \right\}^\star = \left\{ {t \atop n} \right\}' + \left\{ {t-1 \atop n} \right\}'$.

**Proposition 6.2.** *For $t \in \mathbf{N}$ and $n \in \mathbf{N}_0$, $\left\{ {t \atop n} \right\}^\star$ is equal to the number of set partitions of $\{1, \ldots, t\}$ into $n$ parts, such that $i$ and $i+1$ are not in the same part for any $i$.*

*Proof.* The set partitions of $\{1, \ldots, t\}$ into exactly $n$ sets counted by $\left\{ {t \atop n} \right\}^\star$ but not by $\left\{ {t \atop n} \right\}'$ are those with 1 and $t$ in the same part. But in this case $t-1$ cannot also be in the same part as 1, and so deleting $t$ gives a bijection between these extra set partitions and the set partitions counted by $\left\{ {t-1 \atop n} \right\}'$. □

We now use Lemma 6.1 to get the exponential generating function for the $B'_t(t)$.

**Proposition 6.3.**

$$\sum_{t=0}^{\infty} \frac{B'_t(t)}{t!} x^t = \exp\bigl(\exp(x) - 1 - x\bigr).$$

*Proof.* Taking $t \in \mathbf{N}$ and $n \geq t$, Lemma 6.1 gives $B'_t(t) + B'_{t-1}(t-1) = B_{t-1}(t-1)$. It follows that if $F(x) = \sum_{t=0}^{\infty} B'_t(t) x^t/t!$ is the exponential generating function for $B'_t(t)$ then

$$F(x) + F'(x) = \sum_{t=0}^{\infty} \frac{B_t(t)}{t!} x^t = \exp(\exp(x) - 1).$$

Since $\exp(\exp(x) - 1 - x)$ solves this differential equation, and agrees with $F$ when $x = 0$, we have $F(x) = \exp(\exp(x) - 1 - x)$. □

Thus $\sum_{t=0}^{\infty} B'_t(t) x^t/t! = \exp(-x) \sum_{t=0}^{\infty} B_t(t) x^t/t!$. As an immediate corollary we get $B'_t(t) = \sum_{s=0}^{t} \binom{t}{s} (-1)^{t-s} B_s(s)$ and $B_t(t) = \sum_{s=0}^{t} \binom{t}{s} B'_s(s)$; these formulae are related by binomial inversion. Using this formula for $B'_t(t)$ and the standard result

$$B_t(t) = \frac{1}{\mathrm{e}} \sum_{j=0}^{\infty} \frac{j^t}{j!} \tag{3}$$

(see for example [23, Equation (1.41)], or sum Equation (7) below over all $n \in \mathbf{N}_0$) we obtain

$$B'_t(t) = \frac{1}{\mathrm{e}} \sum_{j=0}^{\infty} \frac{(j-1)^t}{j!}. \tag{4}$$

This equation appears to offer the easiest route to the asymptotics of $B'_t(t)$. Let $W(t)$ denote Lambert's $W$ function, defined for $x \in \mathbf{R}^{\geq 0}$ by the equation $W(x)\mathrm{e}^{W(x)} = x$.

**Corollary 6.4.** *We have*

$$\frac{B_t'(t)}{B_t(t)} \sim \frac{W(t)}{t} \quad \textit{as } t \to \infty.$$

*Proof.* Let $m(t) = \lfloor \exp W(t-1/2) \rfloor$. The solution to Exercise 9.46 in [11] can easily be adapted to show that

$$B_t'(t) = \mathrm{e}^{m(t)-t-1/2} m(t)^{t-1} \sqrt{\frac{m(t)}{m(t)+t}} \bigl(1 + \mathrm{O}(t^{-1/2}\log t)\bigr).$$

Comparing with the analogous formula for $B_t(t)$ proved in [11] we obtain $B_t'(t)/B_t(t) \sim 1/m(t)$ as $t \to \infty$. We now use the fact that

$$m(t) \sim \exp W(t-1/2) = \frac{t-1/2}{W(t-1/2)} \sim \frac{t}{W(t)} \quad \text{as } t \to \infty,$$

where the final asymptotic equality follows from the elementary bounds $\log t - \log\log t \le W(t) \le \log t$. □

We also obtain the following result on the number $W_t$ of set partitions of $\{1, \ldots, t\}$ into parts of size at least 2.

**Corollary 6.5.** *If $t \in \mathbf{N}_0$ then $B_t'(t)$ is equal to $W_t$.*

*Proof.* Since $\exp(x) - 1 - x$ is the exponential generating function enumerating sets of size at least two, the corollary follows from Proposition 6.3 using [23, Theorem 3.11]. □

This result is proved in [1, §3.5], as a corollary of two explicit bijections showing that $B_t'(t) + B_{t+1}'(t+1) = B_t(t)$ (as noted earlier, this is a special case of Lemma 6.1) and $W_t + W_{t+1} = B_t(t)$. It would be interesting to have a direct bijective proof of the corollary.

It is worth noting that Corollary 6.5 does not extend to the Stirling numbers $\left\{ {t \atop n} \right\}'$. For example, if $t \ge 2$ then $\left\{ {t \atop 1} \right\}' = 0$, whereas the unique set partition of $\{1, \ldots, t\}$ into a single part obviously has all parts of size at least 2.

*6.2. Generating functions and asymptotics for $\left\{ {t \atop n} \right\}'$ and $B_t'(n)$*

Let $t$, $n \in \mathbf{N}$. By definition we have $\left\{ {t \atop n} \right\}' = B_t'(n) - B_t'(n-1)$. Provided $n \ge 2$, Lemma 6.1 applies to both summands, and we obtain $\left\{ {t \atop n} \right\}' + \left\{ {t-1 \atop n} \right\}' = \left\{ {t-1 \atop n-1} \right\}$. From the ordinary generating function $\sum_{t=0}^\infty \left\{ {t \atop n} \right\} x^t = x^n \prod_{j=1}^n 1/(1-jx)$ (see for example [23, (1.36)]) we now get

$$\sum_{t=0}^{\infty} \left\{ {t \atop n} \right\}' x^t = \frac{x^n}{1+x} \prod_{j=1}^{n-1} \frac{1}{1-jx} \tag{5}$$

for $n \geq 2$. (When $n = 1$ we have $\left\{ {t \atop 1} \right\}' = 0$ for all $t \in \mathbf{N}$, so the generating function is zero.) A simple residue calculation now shows that provided $n \geq 3$, we have

$$\left\{ {t \atop n} \right\}' \sim \frac{(n-1)^t}{n!} \quad \text{as } t \to \infty.$$

(When $n = 2$ we have $\left\{ {t \atop 2} \right\}' = 1$ if $t$ is even, and $\left\{ {t \atop 2} \right\}' = 0$ if $t$ is odd.) It easily follows that same asymptotic relation holds for $B'_t(n)$. Thus if $n \geq 3$ then $\frac{1}{n}\left\{ {t \atop n-1} \right\}$, $\left\{ {t \atop n} \right\}'$, $\frac{1}{n}B_t(n-1)$ and $B'_t(n)$ are all asymptotically equal to $(n-1)^t/n!$ as $t \to \infty$. Moreover, by calculating all residues in (5) we obtain the explicit formula

$$\left\{ {t \atop n} \right\}' = \frac{1}{n!}\sum_{k=0}^{n-1}(-1)^k\binom{n}{k}(n-k-1)^t + \frac{(-1)^{n+t}}{n!} \tag{6}$$

valid for $n \geq 2$. (The summand for $n-1$ is included so that the formula holds also when $t = 0$.) This formula may be compared with the well known identity

$$\left\{ {t \atop n} \right\} = \frac{1}{n!}\sum_{k=0}^{n}(-1)^k\binom{n}{k}(n-k)^t, \tag{7}$$

valid for all $t$, $n \in \mathbf{N}_0$, which has a short direct proof using the Principle of Inclusion and Exclusion. When $n \geq 3$, an easy corollary of (5) is the recurrence $\left\{ {t \atop n} \right\}' = \left\{ {t-1 \atop n-1} \right\}' + (n-1)\left\{ {t-1 \atop n} \right\}'$, analogous to the well known $\left\{ {t \atop n} \right\} = \left\{ {t-1 \atop n-1} \right\} + n\left\{ {t-1 \atop n} \right\}$. However, while the recurrence for $\left\{ {t \atop n} \right\}$ has a very simple bijective proof, the authors know of no such proof for the recurrence for $\left\{ {t \atop n} \right\}'$.

*6.3. Generating function and asymptotics of $B^\dagger_t(t)$*

The exponential generating function enumerating non-empty sets with an even number of their elements marked is $\frac{1}{2}(\exp 2x - 1)$. Hence, by [23, Theorem 3.11], we have

$$\sum_{n=0}^{\infty} \frac{B^\dagger_t(t)}{t!}x^t = \exp\left(\tfrac{1}{2}(\exp 2x - 1)\right).$$

Since there are $2^{t-n}$ ways to mark the elements of a set partition of $\{1, \ldots, t\}$ into $n$ parts so that an even number of elements in each part are marked, we have $\left\{ {t \atop n} \right\}^\dagger = 2^{t-n}\left\{ {t \atop n} \right\}$. Using this, a routine adaption of the proof of [23, Equation 1.41] shows that

$$B^\dagger_t(t) = \frac{1}{\sqrt{e}}\sum_{j=0}^{\infty}\frac{(2j)^t}{2^j j!}. \tag{8}$$

The method used to prove Corollary 6.4 then shows that

$$B_t^\dagger(t) = 2^t \mathrm{e}^{\ell(t)-t}\ell(t)^t \sqrt{\frac{\ell(t)}{\ell(t)+t}}\big(1+\mathrm{O}(t^{-1/2}\log t)\big)$$

where $\ell(t)$ is defined by the equation $(\log \ell(t) + \log 2)\ell(t) = t - 1/2$.

*6.4. Relating $B_t^\dagger(n)$ to $B_t^{\dagger\prime}(n)$ and the asymptotics of $B_t^{\dagger\prime}(n)$*

Lemma 6.1 has the following analogue in Type B.

**Lemma 6.6.** *For $t$, $n \in \mathbf{N}$ we have*

$$B_t^{\dagger\prime}(n) + B_{t-1}^{\dagger\prime}(n) = \sum_{s=0}^{t-1} \binom{t-1}{s} B_s^\dagger(n-1).$$

*Proof.* Let $\pi_{(n)}^\dagger = 1_{\mathrm{BSym}_{n-1}}\big\uparrow^{\mathrm{BSym}_n}$ and $\vartheta_{(n)}^\dagger = \pi_{(n)}^\dagger - 1_{\mathrm{BSym}_n}$. By Lemma 4.4,

$$B_t^{\dagger\prime}(n) + B_{t-1}^{\dagger\prime}(n) = \big\langle \big(\vartheta_{(n)}^\dagger\big)^{t-1}\pi_{(n)}^\dagger, 1_{\mathrm{BSym}_n}\big\rangle.$$

Recall from §4.2 that

$$\pi_{(n)}^\dagger = \chi^{((n),\varnothing)} + \chi^{((n-1,1),\varnothing)} + \chi^{((n-1),(1))}.$$

By the Branching Rule for $\mathrm{BSym}_n$ stated in Lemma 4.4 we have

$$\vartheta_{(n)}^\dagger\big\downarrow_{\mathrm{BSym}_{n-1}} = 2\chi^{((n-1),\varnothing)} + \chi^{((n-2,1),\varnothing)} + \chi^{((n-2),(1))} = \pi_{(n-1)}^\dagger + 1_{\mathrm{BSym}_{n-1}}.$$

A straightforward calculation using Lemma 4.4 now shows that

$$\begin{aligned}
B_t^{\dagger\prime}(n) + B_{t-1}^{\dagger\prime}(n) &= \big\langle (\vartheta_{(n)}^\dagger)^{t-1}\pi_{(n)}, 1_{\mathrm{BSym}_n}\big\rangle \\
&= \big\langle \big(\vartheta_{(n)}^\dagger\big\downarrow_{\mathrm{BSym}_{n-1}}\big)^{t-1}\big\uparrow^{\mathrm{BSym}_n}, 1_{\mathrm{BSym}_n}\big\rangle \\
&= \big\langle \big(\pi_{(n-1)}^\dagger + 1_{\mathrm{BSym}_{n-1}}\big)^{t-1}\big\uparrow^{\mathrm{BSym}_n}, 1_{\mathrm{BSym}_n}\big\rangle \\
&= \big\langle \big(\pi_{(n-1)}^\dagger + 1_{\mathrm{BSym}_{n-1}}\big)^{t-1}, 1_{\mathrm{BSym}_{n-1}}\big\rangle \\
&= \sum_{s=0}^{t-1} \binom{t-1}{s} \big\langle (\pi_{(n-1)}^\dagger)^s, 1_{\mathrm{BSym}_{n-1}}\big\rangle,
\end{aligned}$$

and the result follows. □

In order to state a Type B analogue of Proposition 6.2, we define a further family of Type B Stirling numbers by $\left\{ {t \atop n} \right\}^{\dagger\star} = \left\{ {t \atop n} \right\}^{\dagger\prime} + \left\{ {t-1 \atop n} \right\}^{\dagger\prime}$ for $t \in \mathbf{N}$ and $n \in \mathbf{N}_0$.

**Proposition 6.7.** *For $t \in \mathbf{N}$ and $n \in \mathbf{N}_0$, $\left\{ {t \atop n} \right\}^{\dagger\star}$ is equal to the number of marked set partitions of $\{1, \ldots, t\}$ into $n$ parts, such that an even number of elements of each part are marked, and such that if $i$ and $i+1$ are in the same part then $i+1$ is marked.*

*Proof.* By definition of $\left\{ {t \atop n} \right\}^{\dagger\prime}$, the marked set partitions of $\{1, \ldots, t\}$ into exactly $n$ parts counted by $\left\{ {t \atop n} \right\}^{\dagger\star}$ but not by $\left\{ {t \atop n} \right\}^{\dagger\prime}$ are those with 1 and $t$ in the same part, but with 1 unmarked. The following procedure defines a bijection between these partitions and those counted by $\left\{ {t-1 \atop n} \right\}^{\dagger\prime}$: if $t$ is marked then transfer the mark to 1; then delete $t$. $\square$

Continuing the analogy with Type A, we now use Lemma 6.6 to get the generating function for $B_t^{\dagger\prime}(t)$.

**Proposition 6.8.**

$$\sum_{t=0}^{\infty} \frac{B_t^{\dagger\prime}(t)}{t!} x^t = \exp\bigl(\tfrac{1}{2}(\exp 2x - 1 - 2x)\bigr).$$

*Proof.* Taking $t \in \mathbf{N}$ and $n \geq t$, Lemma 6.6 gives $B_t^{\dagger\prime}(t) + B_{t-1}^{\dagger\prime}(t-1) = \sum_{s=0}^{t-1} B_s^{\dagger}(s)\binom{t-1}{s}$. Observe that if $G(x) = \sum_{t=0}^{\infty} a_t x^t/t!$ then $x^k G(x)/k! = \sum_{t=0}^{\infty} a_t x^{t+k}\binom{t+k}{k}/(t+k)!$. It follows that if $H(x) = \sum_{t=0}^{\infty} B_t^{\dagger\prime}(t)x^t/t!$ then

$$H(x) + H'(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} \sum_{t=0}^{\infty} \frac{B_t^{\dagger}(t)}{t!} x^t = \exp\bigl(x + \tfrac{1}{2}(\exp 2x - 1)\bigr).$$

Since $\exp\bigl(\frac{1}{2}(\exp 2x - 1 - 2x)\bigr)$ solves this differential equation, and agrees with $H$ when $x = 0$, we have $H(x) = \exp\bigl(\frac{1}{2}(\exp 2x - 1 - 2x)\bigr)$. $\square$

Thus $\sum_{t=0}^{\infty} B_t^{\dagger\prime}(t)x^t/t! = \exp(-x)\sum_{t=0}^{\infty} B_t^{\dagger}(t)x^t/t!$. As an immediate corollary we get $B_t^{\dagger\prime}(t) = \sum_{s=0}^{t}\binom{t}{s}(-1)^{t-s}B_s^{\dagger}(s)$ and $B_t^{\dagger}(t) = \sum_{s=0}^{t}\binom{t}{s}B_s^{\dagger\prime}(s)$; again these formulae are related by binomial inversion. Using this formula for $B_t^{\dagger\prime}(t)$ and (8), we obtain

$$B_t^{\dagger\prime}(t) = \frac{1}{\sqrt{e}} \sum_{j=0}^{\infty} \frac{(2j-1)^t}{2^j j!}. \tag{9}$$

This equation has the same relationship to (8) as (4) has to (3), and the same method used to prove Corollary 6.4 gives

$$B_t^{\dagger\prime}(t) = 2^{t-1/2} \mathrm{e}^{\ell(t)-t} \ell(t)^{t-1/2} \sqrt{\frac{\ell(t)}{\ell(t)+t}} \bigl(1 + \mathrm{O}(t^{-1/2}\log t)\bigr)$$

and

$$\frac{B_t^{\dagger\prime}(t)}{B_t^{\dagger}(t)} \sim \frac{1}{\sqrt{2\ell(t)}} \text{ as } t \to \infty$$

where $\ell(t)$ is as defined in §6.3.

We also obtain the expected analogue of Corollary 6.5. The example of $\left\{ {t \atop 1} \right\}^{\dagger\prime}$ shows that, as before, this corollary does not extend to the corresponding Stirling numbers.

**Corollary 6.9.** *If $t \in \mathbf{N}_0$ then $B_t^{\dagger\prime}(t)$ is the number of set partitions of $\{1, \ldots, t\}$ into parts of size at least two with an even number of their elements marked.* □

*Proof.* As in the Type A case, this follows from [23, Theorem 3.11] since $\frac{1}{2}\bigl(\exp(2x) - 1 - 2x\bigr)$ is the exponential generating function enumerating sets of size at least two with an even number of their elements marked. □

*6.5. Relating $B_t^{\dagger}(n)$ to $B_t^{\dagger\prime}(n)$ and the asymptotics of $\left\{ {t \atop n} \right\}^{\dagger}$, $\left\{ {t \atop n} \right\}^{\dagger\prime}$, $B_t'(n)$ and $B_t^{\dagger\prime}(n)$*

Let $t$, $n \in \mathbf{N}$. Since $\left\{ {t \atop n} \right\}^{\dagger} = 2^{t-n}\left\{ {t \atop n} \right\}$, Equation (7) implies that

$$\left\{ {t \atop n} \right\}^{\dagger} = \frac{1}{2^n n!} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \bigl(2(n-k)\bigr)^t. \tag{10}$$

Moreover, the ordinary generating function for $\left\{ {t \atop n} \right\}^{\dagger}$ is

$$\sum_{t=0}^{\infty} \left\{ {t \atop n} \right\}^{\dagger} x^t = x^n \prod_{j=1}^{n} \frac{1}{1-2jx} \tag{11}$$

and so $\left\{ {t \atop n} \right\}^{\dagger}$ and $B_t^{\dagger}(n)$ are both asymptotically equal to $(2n)^t/2^n n!$ as $t \to \infty$. Lemma 6.6 implies that $\left\{ {t \atop n} \right\}^{\dagger\prime} + \left\{ {t-1 \atop n} \right\}^{\dagger\prime} = \sum_{s=0}^{t-1} \binom{t-1}{s} \left\{ {s \atop n-1} \right\}^{\dagger}$ provided $n \geq 2$. The generating function for the right-hand side is

$$\begin{aligned}
\sum_{t=0}^{\infty}\sum_{s=0}^{t-1} \binom{t-1}{s} \left\{ {s \atop n-1} \right\}^{\dagger} y^t &= \sum_{s=0}^{\infty} \left\{ {s \atop n-1} \right\}^{\dagger} \frac{y^s}{(1-y)^{s+1}} \\
&= \frac{y^n}{1-y} \prod_{j=1}^{n-1} \frac{1}{1-(2j+1)y}
\end{aligned}$$

where the second line follows by substituting $y/(1-y)$ for $x$ in (11), with $n$ replaced with $n-1$. Hence

$$\sum_{t=0}^{\infty} \left\{ {t \atop n} \right\}^{\dagger\prime} x^t = \frac{x^n}{1+x} \prod_{j=1}^{n} \frac{1}{1-(2j-1)x} \tag{12}$$

provided $n \geq 2$. A residue calculation then shows that

$$\left\{ {t \atop n} \right\}^{\dagger\prime} \sim \frac{(2n-1)^t}{2^n n!} \quad \text{as } t \to \infty.$$

provided $n \geq 2$. (When $n = 1$ we have $\left\{ {t \atop 1} \right\}^{\dagger\prime} = 1$ if $t$ is even and $\left\{ {t \atop 0} \right\}^{\dagger\prime} = 0$ if $t$ is odd.) It easily follows that if $n \geq 2$ then $\left\{ {t \atop n} \right\}^{\dagger\prime}$ and $B_t^{\dagger\prime}(n)$ are both asymptotically equal to $(2n-1)^t/2^n n!$ as $t \to \infty$. Moreover, by calculating all residues we obtain the explicit formula

$$\left\{ {t \atop n} \right\}^{\dagger\prime} = \frac{1}{2^n n!} \sum_{k=0}^{n-1} (-1)^k \binom{n}{k} \bigl(2(n-k)-1\bigr)^t + \frac{(-1)^{n+t}}{2^n n!} \tag{13}$$

valid for $n \geq 2$. This formula has a striking similarity to (6). We remark that an alternative derivation of (9) is given by summing this formula over all $n \in \mathbf{N}_0$, using the identity

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} \binom{n}{j} = \frac{1}{\sqrt{e}} \frac{(-1)^j}{2^j j!},$$

which can easily be proved by comparing coefficients of $x^j$ in

$$\sum_{j=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} \binom{n}{j} x^j = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} (1+x)^n = \exp\bigl(-\tfrac{1}{2}(1+x)\bigr) = \frac{\exp(-\frac{1}{2}x)}{\sqrt{e}}.$$

The recurrences for $\left\{ {t \atop n} \right\}^{\dagger}$ and $\left\{ {t \atop n} \right\}^{\dagger\prime}$ obtained from (11) and (12) are $\left\{ {t \atop n} \right\}^{\dagger} = \left\{ {t-1 \atop n-1} \right\}^{\dagger} + 2n \left\{ {t-1 \atop n} \right\}^{\dagger}$ for $n \geq 1$ and $\left\{ {t \atop n} \right\}^{\dagger\prime} = \left\{ {t-1 \atop n-1} \right\}^{\dagger\prime} + (2n-1) \left\{ {t-1 \atop n} \right\}^{\dagger\prime}$ for $n \geq 3$.

*6.6. Stirling numbers in Type D*

It is possible to define Stirling numbers for Type D, by $\left\{ {t \atop n} \right\}^{\ddagger} = M_t^{\ddagger}(n) - M_t^{\ddagger}(n-1)$ and $\left\{ {t \atop n} \right\}^{\ddagger\prime} = M_t^{\ddagger\prime}(n) - M_t^{\ddagger\prime}(n-1)$. However, just as the set partitions corresponding to Type D shuffles have no apparent natural description, these Type D Stirling numbers (which take negative values in many cases) have no obvious combinatorial interpretation. We shall say nothing more about these quantities here. We note however that there is an analogue of Lemmas 6.1 and 6.6 in Type D, namely

$$M_t^{\ddagger\prime}(n) + M_{t-1}^{\ddagger\prime}(n) = \sum_{s=0}^{t-1} \binom{t-1}{s} M_t^{\ddagger}(n)$$

for $t$, $n \in \mathbf{N}$. This has a similar proof to Lemma 6.6, replacing Lemma 4.4 with Lemma 5.2. We also note that $S_t^{\dagger}(n) = S_t^{\ddagger}(n)$ if $t < n$, since no sequence of $t$ oriented random-to-top shuffles leaving the deck fixed can lift the bottom card, and evenly many card flips occur in any such sequence. Moreover $S_t^\dagger(t) + 1 = S_t^\ddagger(t)$; the exceptional sequence lifts the bottom card to the top $t$ times, flipping it every time.

## 7. Earlier work on shuffles and Kronecker powers, and an obstruction to an explicit bijection between the sequences counted by $B_t(n)$ and $M_t(n)$

The connection between the random-to-top shuffle (or its inverse, the top-to-random shuffle) and Solomon's descent algebra is well known. See for example the remark on Corollary 5.1 in [8].

In [10] the authors study the powers of the irreducible character $\chi^{(n-1,1)}$. Suppose that $t \leq n$. The special case $\lambda = \varnothing$ of Lemma 2 of [10] gives a bijection between the sequences of partition moves counted by $M'_t(n)$ and permutations $\pi$ of $\{1, \ldots, t\}$ such that every cycle of $\pi$ is decreasing, and $\pi$ has no fixed points. Such permutations are in bijection with set partitions of $\{1, \ldots, t\}$ into non-singleton parts, and so, using the result of Bernhart mentioned earlier, we obtain a bijective proof that $M'_t(t) = M'_t(n) = B'_t(n) = B'_t(t)$. The restriction $t \leq n$, which allows the use of the Bernhart result, arises from the use of the RSK correspondence for oscillating tableaux, and appears essential to the proof in [10].

In [9] Fulman defines, for each finite group $G$ and each subgroup $H$ of $G$, a Markov chain on the irreducible representations of $G$. In the special case when $G = \mathrm{Sym}_n$ and $H = \mathrm{Sym}_{n-1}$ this gives a Markov chain $J$ on the set of partitions of $n$ in which the transition probability from the partition $\lambda$ to the partition $\mu$ is

$$\frac{M_1(\lambda, \mu)}{n} \frac{\chi^\mu(1)}{\chi^\lambda(1)}.$$

The relevant special case of his Theorem 3.1 is as follows.

**Proposition 7.1** (Fulman)**.** *Let $\lambda$ be a partition of $n$. Starting at $(n)$, the probability that $J$ is at $\lambda$ after $t$ steps is equal to the probability that a product of $t$ random-to-top shuffles is sent to a pair of tableaux of shape $\lambda$ by the RSK correspondence.*

Fulman's theorem relates partitions and shuffle sequences, but it is qualitatively different to Lemma 2.3. In particular, we note that Lemma 2.3 has no very natural probabilistic interpretation, since choosing individual moves uniformly at random fails to give a uniform distribution on the set of sequences of all partition moves from $(n)$ to $(n)$.

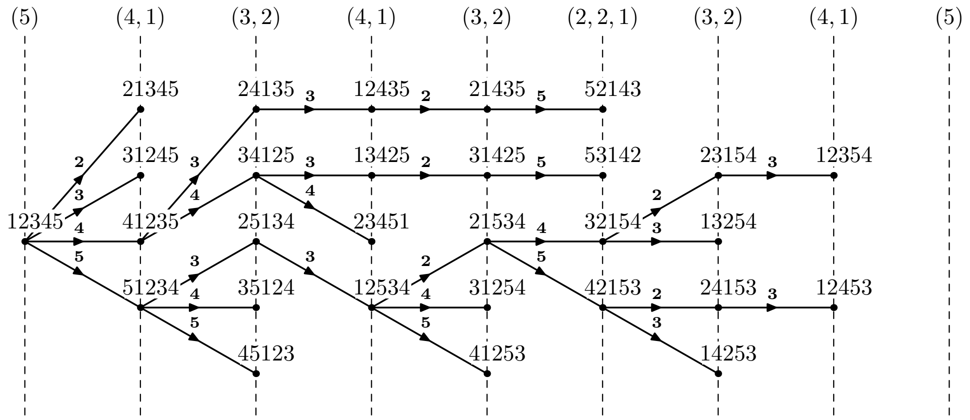


Figure 1: All sequences $(\tau_1, \tau_2, \ldots, \tau_r)$ of random-to-top shuffles such that $\mathrm{sh}(\tau_1, \ldots \tau_i) = \lambda^{(i)}$ for each $i \in \{0, 1, \ldots, r\}$, where $\lambda^{(1)}, \ldots, \lambda^{(8)}$ are as defined in Proposition 7.2. Labels on vertices show the inverses of the permutations $\tau_1 \ldots \tau_i$ and so give the order of the deck; an arrow labelled $m$ corresponds to the random-to-top shuffle $\sigma_m$.

Despite this, it is very natural to ask whether the RSK correspondence can be used to give a bijective proof of Lemma 2.3, particularly in view of the related RSK correspondence used in [10], and the fact that if $\tau \in \mathrm{Sym}_n$ then the RSK shape of $\tau\sigma_m$ differs from the RSK shape of $\tau$ by a move. (This follows easily from Greene's characterization of the RSK shape of a permutation by increasing subsequences: see [12].) The following proposition shows that, perhaps surprisingly, the answer to this question is negative. The proof is a brute-force verification, which to make this paper self-contained we present in Figure 1 above.

**Proposition 7.2.** *Let* $\mathrm{sh}(\tau)$ *denote the RSK shape of the permutation* $\tau$ *and let*

$$(\lambda^{(0)}, \ldots, \lambda^{(8)}) = \big((5), (4,1), (3,2), (4,1), (3,2), (2,2,1), (3,2), (4,1), (5)\big).$$

*There does not exist a sequence* $(\tau_1, \ldots, \tau_8)$ *of random-to-top shuffles such that* $\mathrm{sh}(\tau_1 \ldots \tau_i) = \lambda^{(i)}$ *for all* $i \in \{0, 1, \ldots, 8\}$. $\square$

Computer calculation show that if $t \leq 7$ then taking shapes of RSK correspondents gives a bijection proving Theorem 1.1 whenever $n \leq 12$. When $t = 8$, similar examples to the one above exists for $5 \leq n \leq 8$. (These claims may be verified using the Magma [6] code available from the second author's website[1].) Since a shuffle sequence counted by $S_t(n)$ never

[1] See `www.ma.rhul.ac.uk/~uvah099/`.

moves a card that starts in position $t+1$ or lower, taking shapes of RSK correspondents fails to give a bijective proof of Theorem 1.1 whenever $t = 8$ and $n \geq 5$. It is therefore an open problem to find a bijective proof of Theorem 1.1 that deals with the case $t > n$ not covered by Lemma 2 of [10]. Finding bijective proofs of Theorems 4.3 and 5.1 are also open problems.

## 8. Occurrences of Type A and B Bell and Stirling numbers in the On-Line Encyclopedia of Integer Sequences

The numbers studied in this paper have been submitted to the Online Encyclopedia of Integer Sequences (OEIS) [17]: $\left\{ {t \atop n} \right\}'$, $B_t'(n)$, $B_t^\dagger(n)$, $\left\{ {t \atop n} \right\}^{\dagger'}$, $B_t^{\dagger'}(n)$ for $t \in \mathbf{N}$ and $n \leq t$ are now present as the 'triangular' sequences A261139, A261137, A261275, A261318 and A261319, respectively. Here we discuss the appearances prior to this submission.

The numbers $B_t'(t)$ appear as sequence A000296; the interpretations at the start of §6.1 and in Corollary 6.5 are both given. We have $B_t'(2) = 1$ if $t$ is even and $B_t'(2) = 0$ if $t$ is odd; this is sequence A000035. By (6) we have $\left\{ {t \atop 3} \right\}' = \frac{1}{6}(2^t - 3 - (-1)^t)$; this is sequence A000975. It follows that $B_t'(3) = \frac{1}{6}(2^t + 2(-1)^t)$; this is sequence A001045. The sequence $\left\{ {t \atop 4} \right\}'$ appears as A243869, counting the number of set partitions of $\{1, \ldots, t\}$ into four parts, satisfying our condition.

The Type B Stirling numbers $\left\{ {t \atop n} \right\}^\dagger$ appear as sequence A075497; they are defined there by their characterization as ordinary Stirling numbers scaled by particular powers of 2. They do not appear to have been connected before with Type B Coxeter groups or descent algebras.

The other statistics introduced here for types B and D have not hitherto appeared in OEIS in any generality, and appear not to have been defined prior to this investigation. They appear in OEIS only for certain very particular choices of parameters, usually those for which the numbers have particularly simple expressions. For instance, we see from (10) that $\left\{ {t \atop 1} \right\}^\dagger = 2^{t-1}$, and that $\left\{ {t \atop 2} \right\}^\dagger = 2^{t-2}(2^{t-1} + 1)$, giving sequences A000079 and A007582 respectively .

The sequence $B_t^\dagger(3)$ also appears, as sequence A233162. Let $C$ be a set of $2n$ unlabelled colours arranged into $n$ pairs. Let $A$ be an $s \times t$ array of boxes. Define $Q_{s,t}(n)$ to be the number of ways of colouring $A$ with colours of $C$, so that no two horizontally or vertically adjacent boxes receive colours from the same pair. (Since the colours are unlabelled, two colourings of $A$ that differ by a permutation of the $n$ pairs, or by swapping colours within a pair, are regarded as the same.) The sequence A233162 gives the numbers

$Q_{1,t}(4)$. However the coincidence with the Type B Bell numbers extends to any number of colours, as follows.

**Proposition 8.1.** *$Q_{1,t}(n) = B^{\dagger}_{t-1}(n-1)$ for all $t \geq 1$.*

*Proof.* Let the boxes be labelled $0, \ldots, t-1$ from left to right. We shall show that the number of colourings using exactly $n$ colour pairs (though not necessarily using both colours in each pair) is $\left\{ {t-1 \atop n-1} \right\}^{\dagger}$; this is the number of marked partitions of a set of size $t-1$ into exactly $n-1$ parts. Since $B^{\dagger}_{t-1}(n-1) = \sum_{k=0}^{n-1} \left\{ {t-1 \atop k} \right\}^{\dagger}$, this is enough to prove the proposition.

Suppose we are given a colouring of the boxes using exactly $n$ colour pairs, such that consecutive boxes receive colours from distinct pairs. We label the colour pairs with the numbers $1, \ldots, n$ in order of their first appearance in the colouring, reading from left to right. Let $c_i$ be the label of the colour of box $i$. We define a function $f$ on $\{1, \ldots, t-1\}$ by

$$f(i) = \begin{cases} c_i & \text{if } c_i < c_{i-1}, \\ c_i - 1 & \text{if } c_i > c_{i-1}. \end{cases}$$

(By assumption, the case $c_i = c_{i-1}$ does not occur.) We notice that if the first occurrence of the colour pair $k$ occurs in box $i$, where $i > 0$, then $c_j < k$ for all $j < i$. It follows that $f(i) = k - 1$, and hence that the image of $f$ is $\{1, \ldots, n-1\}$. Now the kernel congruence of $f$ defines a set partition of $\{1, \ldots, t-1\}$ into exactly $n-1$ parts; it remains to determine the marks.

Within each colour pair, we now distinguish a marked and an unmarked colour: the marked colour is the one which first appears in our colouring. (It is not necessary that the unmarked colour should appear at all.) Let $X$ be a part of the partition just described, and let $x$ be the least element of $X$. We add marks to the elements of $X$ as follows: for each element $y \in X$ other than $x$, we add a mark to $y$ if box $y$ receives a marked colour. Then we add a mark to $x$ if necessary to make the total number of marks in $X$ even.

Given a marked partition of $\{1, \ldots, t-1\}$ into exactly $n-1$ parts, it is easy to reverse the procedure just described in order to reconstruct the unique colouring of $t$ boxes with $n$ unlabelled colour pairs with which it is associated. So we have a bijective correspondence, and the proposition is proved. $\square$

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