Odd Properly Colored Cycles in Edge-Colored Graphs

Gregory Gutin[1], Bin Sheng[1], and Magnus Wahlström[1]

[1]Royal Holloway, University of London, TW20 0EX, Egham, Surrey, UK

Abstract

It is well-known that an undirected graph has no odd cycle if and only if it is bipartite. A less obvious, but similar result holds for directed graphs: a strongly connected digraph has no odd cycle if and only if it is bipartite. Can this result be further generalized to more general graphs such as edge-colored graphs? In this paper, we study this problem and show how to decide if there exists an odd properly colored cycle in a given edge-colored graph. As a by-product, we show how to detect if there is a perfect matching in a graph with even (or odd) number of edges in a given edge set.

1 Introduction

A graph G is *edge-colored* if each edge of G is assigned a color. (Edge-colorings can be arbitrary, not necessarily proper.) A cycle C in an edge-colored graph is *properly colored (PC)* if no pair of adjacent edges of C have the same color. It is not hard to see that PC cycles in edge-colored graphs generalize directed cycles (dicycles) in digraphs: in a digraph D replace every arc uv by an undirected path $ux_{uv}v$, where x_{uv} is a new vertex, and edges ux_{uv} and $x_{uv}v$ are of colors 1 and 2, respectively. Then PC cycles in the resulting graph correspond exactly to dicycles in D. Similarly, PC cycles generalize cycles in undirected graphs, e.g., by assigning every edge a distinct color.

One of the central topics in graph theory is the existence of certain kinds of cycles in graphs. In digraphs, it is not hard to decide the existence of any dicycle by simply checking whether a given digraph is acyclic [3]. The problem of existence of PC cycles in edge-colored graphs is less trivial. To solve the problem, we may use Yeo's theorem [20]: if an edge-colored graph G has no PC cycle then G contains a vertex z such that no connected component of $G - z$ is joined to z with edges of more than one color. Thus, we can recursively

find such vertices z and delete them from G; if we end up with a trivial graph (containing just one vertex) then G has no PC cycles; otherwise G has a PC cycle. Clearly, the recursive algorithm runs in polynomial time.

One of the next natural questions is to decide whether a digraph has an odd (even, respectively) dicycle, i.e. a dicycle of odd (even) length, respectively. For odd dicycles we can employ the following well-known result (see, e.g., [3, 8]): A strongly connected digraph is bipartite if and only if it has no odd dicycle. (Note that this result does not hold for non-strongly connected digraphs.) Thus, to decide whether a digraph D has an odd cycle, we can find strongly connected components of D and check whether all components are bipartite. This leads to a simple polynomial-time algorithm. The question of whether we can decide in polynomial time whether a digraph has an even dicycle, is much harder and was an open problem for quite some time till it was solved, in affirmative, independently by McCuaig, and Robertson, Seymour and Thomas (see [14]) who found highly non-trivial proofs.

In this paper we consider the problem of deciding the existence of an odd PC cycle (and of finding one, if it exists) in an edge-colored graph in polynomial time. We show that while a natural extension of the odd dicycle solution does not work, an algebraic approach using Tutte matrices and the Schwartz-Zippel lemma allows us to prove that there is a randomized polynomial-time algorithm for solving the problem. The existence of a deterministic polynomial-time algorithm for the odd PC cycle problem remains an open question, as does the existence of a polynomial-time algorithm for the even PC cycle problem.

In this paper, we allow our graphs to have multiple edges (but no loops) and call them, for clarity, *multigraphs*. In edge-colored multigraphs, we allow parallel edges of different colors (there is no need to consider parallel edges of the same color). For an edge xy and a vertex v, we use $\chi(xy)$ and $\chi(v)$ to denote the color of xy and the set of colors of edges incident to v, respectively. All our graphs are undirected; as discussed above, edge-colored multigraphs properly generalize digraphs, even though they are undirected. For any other terminology and notation not provided here, we refer the readers to [3]. There is an extensive literature on PC paths and cycles: for a detailed survey of pre-2009 publications, see Chapter 16 of [3]; more recent papers include [1, 5, 9, 10, 11].

2   Graph-Theoretical Approaches

In this section, we consider some graph-theoretical approaches that have been successfully used for detecting odd cycles in (non-colored) directed and undirected graphs, and find that they are unlikely to work for detecting odd PC
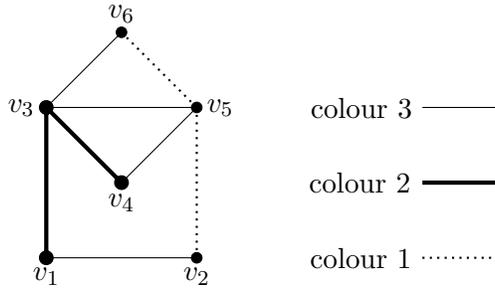
Figure 1: Non-bipartite cyclic connected graph with no odd PC cycle

cycles in edge-colored graphs.

Recall that to solve the odd dicycle problem, in the previous section, we used the following result: A strongly connected digraph is bipartite if and only if it has no odd dicycle. It is not straightforward to generalize strong connectivity to edge-colored multigraphs. Indeed, color-connectivity[1], introduced by Saad [15] under another name, does not appear to be useful to us as, in general, it does not partition vertices into components. Thus, we will use cyclic connectivity introduced by Bang-Jensen and Gutin [2] as follows. Let $P = \{H_1, \ldots, H_p\}$ be a set of subgraphs of an edge colored multigraph G. The *intersection graph* $\Omega(P)$ of P has the vertex set P and the edge set $\{H_i H_j : V(H_i) \cap V(H_j) \neq \emptyset, 1 \leq i < j \leq p\}$. A pair x, y of vertices in an edge-colored multigraph H is *cyclic connected* if H has a collection of PC cycles $P = \{C_1, \ldots, C_p\}$ such that x and y each belong to some cycles in P and $\Omega(P)$ is a connected graph. A maximal cyclic connected induced subgraph of G is called a *cyclic connected component* of G. Note that cyclic connected components partition the vertices of G. Also note that cyclic connectivity for digraphs, where dicycles are considered instead of PC cycles, coincides with strong connectivity. One could wonder whether every non-bipartite cyclic connected edge-colored graph has an odd PC cycle. Unfortunately, it is not true, see a graph H in Fig. 1. It is not hard to check that H is not bipartite and cyclic connected. It has even PC cycles, such as $v_1 v_2 v_5 v_3 v_1$, but no odd PC cycles.

Another natural idea is to find some odd PC closed walk first, and hope to find an odd PC cycle in it. Unfortunately, we cannot generate all possible PC closed walks in polynomial time, and moreover a PC closed walk does not necessarily contain an odd PC cycle, see the graph in Figure 2. It contains an odd PC closed walk, but not an odd PC cycle.

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[1]We will not define color-connectivity; an interested reader can find its definition in Sec. 16.6 of [3].
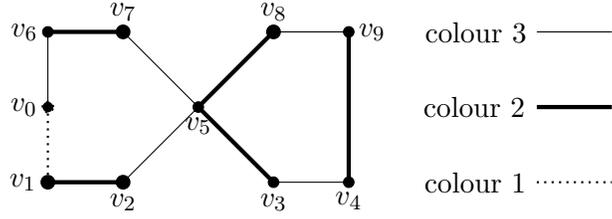
Figure 2: An odd PC closed walk
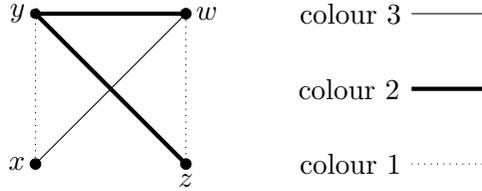
Figure 3: Graph G for which $G' = G$

3 Algebraic Approach

In an edge-colored multigraph G, a vertex v is *monochromatic* if $|\chi(v)| = 1$. Let G' be the multigraph obtained from G by recursively deleting monochromatic vertices such that G' has no monochromatic vertex (for example, see Fig. 3). Following Szeider [16], for every $x \in V(G')$, let G_x denote a graph with vertex set

$$V(G_x) = \{x_i, x_i' :\ i \in \chi(x)\} \cup \{x_a'', x_b''\}$$

and edge set

$$E(G_x) = \{x_a'' x_b'', x_i' x_a'', x_i' x_b'' :\ i \in \chi(x)\} \cup \{x_i x_i' :\ i \in \chi(x)\}.$$

Let G^* denote a graph with vertex set $\bigcup_{x \in V(G')} V(G_x)$ and edge set $E_1 \cup E_2$, where $E_1 = \bigcup_{x \in V(G')} E(G_x)$ and

$$E_2 = \{y_q z_q :\ y_q \in V(G_y), z_q \in V(G_z), yz \in E(G'), \chi(yz) = q\}.$$

For example, see Fig. 4. Let $c = \max\{\chi(x) :\ x \in V(G)\}$. Note that

$$|V(G^*)| = O(c|V(G)|). \tag{1}$$

Observe that each G_x has even number of vertices, thus G^* has even number of vertices.

A subgraph of an edge-colored multigraph is called a *PC cycle subgraph* if it consists of several vertex-disjoint PC cycles. We will use the following result of [7].

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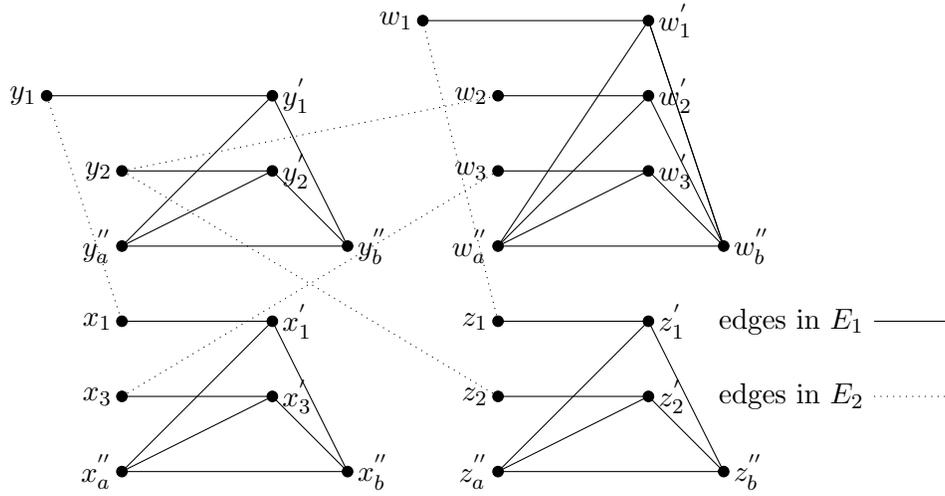
Figure 4: G^* for the graph of Fig. 3

Theorem 1. *Let G be a connected edge-colored multigraph such that G' is non-empty, G^* constructed as above, and $r \geq 2$. Then G has a PC cycle subgraph with r edges if and only if G^* has a perfect matching with exactly r edges in E_2.*

Using Theorem 1, the problem of deciding if there exists an odd PC cycle in G reduces to that of deciding if there is a perfect matching with an odd number of edges from E_2 in the graph G^* (indeed, G^* has an odd PC cycle subgraph if and only if it has an odd PC cycle).

We use the properties of Tutte matrices to solve the reduced problem. For a graph $G = (V, E)$ with $V = \{v_1, v_2, ..., v_n\}$, the *Tutte matrix* A_G is the $n \times n$ multivariate polynomial matrix with entries

$$A_G(i, j) = \begin{cases} x_{ij} & \text{if } v_i v_j \in E \text{ and } i < j \\ -x_{ji} & \text{if } v_i v_j \in E \text{ and } i > j \\ 0 & \text{otherwise,} \end{cases} \tag{2}$$

where x_{ij} are distinct variables. Tutte [18] proved that a graph G has a perfect matching if and only if $\det A_G$ is not identically 0.

We say that a matrix A is *skew symmetric* if $A + A^T = 0$. Note that the Tutte matrix is skew symmetric. In our argument, we will use the notion of Pfaffian of a skew symmetric matrix. Let $A = [a_{ij}]$ be a $2n \times 2n$ skew symmetric matrix. The *Pfaffian* of A is defined as follows.

$$\text{pf } A = \sum_{\sigma} sgn(\sigma) \prod_{i=1}^{n} a_{\sigma(2i-1), \sigma(2i)}, \tag{3}$$

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where $sgn(\sigma) = (-1)^{N(\sigma)}$ is the signature of σ, i.e. $N(\sigma)$ is the number of inversions in σ. The summation is over all permutations σ such that $\sigma(2i-1) < \sigma(2i)$ for each $1 \leq i \leq n$ and $\sigma(2i) < \sigma(2i+2)$ for each $1 \leq i < n$ (i.e., each partition of the set $\{1, \ldots, 2n\}$ into pairs is included in the sum exactly once). Note in particular that for a graph G, this formula for pf A_G enumerates every perfect matching of G exactly once.

Observe that, if we regard a_{ij} as indeterminate, pf A is a multi-linear polynomial. For an odd skew symmetric matrix A, the Pfaffian is defined to be zero. We will use the following well-known relation between Pfaffian and determinants of skew symmetric matrices (see, e.g., [12]).

Theorem 2. *If A is a skew symmetric matrix, then* $\det A = (\text{pf } A)^2$.

Given a graph $G = (V, E)$ and a subset of edges $E_0 \subseteq E(G)$, we now define another skew symmetric matrix A_{G,E_0} whose entries are

$$A_{G,E_0}(i,j) = \begin{cases} -A_G(i,j) & \text{if } v_i v_j \in E_0 \\ A_G(i,j) & \text{if } v_i v_j \notin E_0 \end{cases} \tag{4}$$

It is easy to see that A_{G,E_0} is also a skew symmetric matrix, thus by Theorem 2, $\det A_{G,E_0} = (\text{pf } A_{G,E_0})^2$. Note that A_G and A_{G,E_0} only differ at entries corresponding to edges in E_0. We call a perfect matching M in a graph G E_0-*odd* (E_0-*even*, respectively) if $|M \cap E_0|$ is odd (even, respectively). The E_0-*parity* of M is odd if M is E_0-odd, and even if M is E_0-even.

Here is our key result.

Lemma 1. *Given a graph G with even number of vertices and an edge subset $E_0 \subseteq E(G)$, let A_G and A_{G,E_0} be defined as in (2) and (4). Then $\det A_{G,E_0} = \det A_G$ if and only if all the perfect matchings of G are of same E_0-parity.*

Proof. As both A_G and A_{G,E_0} are skew symmetric, by Theorem 2,

$$\det A_G = (\text{pf } A_G)^2 \text{ and } \det A_{G,E_0} = (\text{pf } A_{G,E_0})^2.$$

Thus $\det A_{G,E_0} = \det A_G$ if and only if pf $A_{G,E_0} = $ pf A_G or pf $A_{G,E_0} = -$ pf A_G. By (3), pf A_G and pf A_{G,E_0} both enumerate perfect matchings of G, and by the definitions of A_G and A_{G,E_0}, for each such matching M, its contributions to pf A_G and pf A_{G,E_0} differ (by a sign term) if and only if M is E_0-odd. Hence pf $A_{G,E_0} = $ pf A_G (pf $A_{G,E_0} = -$ pf A_G, respectively) if and only if each perfect matching in G is E_0-even (E_0-odd, respectively). $\square$

For $G = G^*$ and $E_0 = E_2 \subseteq E(G^*)$, by Lemma 1, if $\det A_{G^*,E_2} \neq \det A_{G^*}$, then G^* has a E_2-odd perfect matching and a E_2-even perfect matching. If $\det A_{G^*,E_2} = \det A_{G^*}$, then every perfect matching of G^* is either E_0-even or E_0-odd. In such a case, we can find a perfect matching M of the graph G^* in

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polynomial time, and decide the parity of $M \cap E_2$. So we have an algorithm for deciding if G^* has a perfect matching with even or odd number of edges in E_2. Unfortunately, we do not know whether this algorithm is polynomial or not as there is no polynomial algorithm to decide whether a multivariate polynomial is identically zero. Fortunately, we can have a polynomial randomized algorithm due to the following well-known lemma, called the Schwartz-Zippel lemma.[2]

Lemma 2. *Let $P(x_1, x_2, \ldots, x_n)$ be a multivariate polynomial of total degree at most d over a field F, and assume that P is not identically zero. Pick $r_1, r_2, \ldots, r_n$ uniformly at random from a finite set S of values where $S \subset F$. Then the probability $\mathbb{P}(P(r_1, r_2, \ldots, r_n) = 0) \leq \frac{d}{|S|}$.*

Now it is not hard to prove the following:

Theorem 3. *Let G be an edge-colored multigraph with n vertices and let $c = \max\{\chi(v): v \in V(G)\}$. There is a randomized algorithm running in time $O((cn)^\omega)$, where $\omega < 2.3729$, that decides if there is an odd PC cycle in G, with false negative probability less than $1/4$.*

Proof. As $f(\mathbf{x}) = \det A_{G^*, E_2} - \det A_{G^*}$ is a multivariate polynomial of degree at most n, we may choose a set S of real values, such that $|S| > 4n$, and use Lemma 2 to decide if $\det A_{G^*, E_2} \neq \det A_{G^*}$ in polynomial time, with false negative less than $1/4$. Note that by repeating the process and taking the majority of the results, we can boost success probability to $1 - \varepsilon$ for any constant $\varepsilon > 0$ without affecting the stated running time. To see that the running time is $O((cn)^\omega)$, recall (1) and observe that computing the determinants of A_{G^*, E_2} and A_{G^*} will take time $O((cn)^\omega)$, where $\omega < 2.3729$, by the algorithm in [19]. Finally, for the case that $f(\mathbf{x}) \equiv 0$, we can use the algorithm of Mucha and Sankowski [13] to find a perfect matching in time $O((cn)^\omega)$, and then decide its E_0-parity. $\square$

4 Open Problems

We have proved that the odd PC cycle problem can be solved in randomized polynomial time. A natural question is whether this problem can be solved in (deterministic) polynomial time. It was proved in [6] that if an edge-colored graph G has no PC closed walk then G has a monochromatic vertex. This can be viewed as a characterization of edge-colored graphs with no PC closed walk and it implies that deciding whether G has a PC closed walk is polynomial-time solvable. We leave open the question of characterizing edge-colored graphs

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[2]It was independently discovered by several authors: Schwartz [17], Zippel [21], DeMillo and Lipton [4].

with no odd PC closed walk, as this differs from graphs with no odd PC cycle as we saw in Section 3.

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