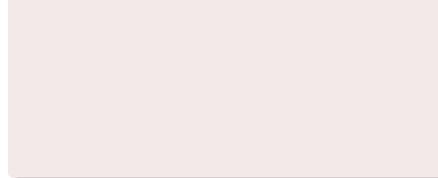
The Davenport constant of finite abelian groups of rank three

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Definitions





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- A sequence S over G is called zero-sum if $|S| = 0_G$.
- The Davenport constant D(G) of G is the smallest $m \in \mathbb{N}$ such that every sequence S over G with $I(S) \ge m$ has a non-empty zero sum subsequence.

Question (Rogers, 1963, and Davenport, 1966)

What is the value of D(G) for an arbitrary finite abelian group G?



Structure theorem

For any non-trivial finite abelian group G there exists unique parameters $1 < d_1 | \cdots | d_r \in \mathbb{N}$ such that $G \cong \mathbb{Z}_{d_1} \oplus \cdots \oplus \mathbb{Z}_{d_r}$.



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- Applications in graph theory: N Alon, S Friedland, and G Kalai. "Regular subgraphs of almost regular graphs". In: *Journal of Combinatorial Theory*, *Series B* 37 (1984), pp. 79–91

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Which groups G satisfy $D(G) = 1 + d^*(G)$ and which satisfy $D(G) > 1 + d^*(G)$?



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<i>p</i> -groups	$\mathbb{Z}_2^{4k}\oplus\mathbb{Z}_{4k+2}$, $k\in\mathbb{N}$
groups of rank ≤ 2	$\mathbb{Z}_m \oplus \mathbb{Z}_n^2 \oplus \mathbb{Z}_{2n}$, <i>m</i> , $n \ge 3$ are odd and $m n $
$\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{3d}, \ d \geq 1$	$\mathbb{Z}_n^n\oplus\mathbb{Z}_{nm}$, $m,\ n\geq 2$ and $m n-1$
many others	some others

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Question

Does the equality $D(G) = 1 + d^*(G)$ hold for all finite abelian groups G of rank three?



$$D(\mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_{10})$$

Theorem (A.S. 2015)

The equality $D(G)=1+d^*(G)$ holds for any finite abelian group $G\cong\mathbb{Z}_5\oplus\mathbb{Z}_5\oplus\mathbb{Z}_{10}$



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- Given an arbitrary prime p, does the equality D(G) = 1 + d*(G) hold for G ≅ Z_p ⊕ Z_p ⊕ Z_{2p}?
- Does the equality $D(G) = 1 + d^*(G)$ hold for $G \cong \mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_{15}$?

Thank you for listening!



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