PH4100: The Ability of $SO(10)$ Grand Unified Theories to Unify the Fundamental Forces of Nature at High Energy

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Abstract

In this paper, the possibilities for unification of the fundamental forces via $SO(10)$ Grand Unified Theories (GUTs) will be investigated. Particle states appearing at energy scales of $Q \geq 1\,\text{TeV}$ will be theorised, and introduced into an extension of the Standard Model of particle physics. Such states have the ability to change the running of the strong, weak and electromagnetic couplings with energy, and are capable of unifying these couplings into a single coupling at high energy. Once the potential for unification has been ascertained, successful particle state structures can then be embedded into representations of $SO(10)$, or combinations thereof, such that $SO(10)$ GUT models can be generated.

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1 Introduction

The Standard Model (SM) of particle physics, first proposed in the 1960s, is remarkably successful in its ability to describe almost all natural phenomena currently encountered. Almost every one of its theoretical predictions has been verified experimentally, most of which to an extremely high precision. It took over fifty years from the initial proposition of the SM for humanity to develop technology capable of testing the most recently confirmed prediction, the existence of the Higgs boson, discovered in 2012 at the Large Hadron Collider. In spite of its successes, the SM falls short of explaining the fundamental force of gravitation (in light of general relativity), providing a viable dark matter candidate, accounting for the accelerating Universe, incorporating neutrino oscillations (for which the 2015 Nobel Prize in Physics was awarded), or circumventing the gauge hierarchy problem [1].

Grand Unified Theories (GUTs) are a type of Beyond the Standard Model (BSM) theory, which, as well as providing solutions for many of the problems outlined above, also reduce the unfavourably high number of SM free parameters. GUTs postulate that the strong, weak, and electromagnetic forces of the SM are actually low-energy facets of a single fundamental force, such that over the course of the evolution of the SM gauge couplings (parameters describing the strength of the force exerted by a particular interaction) with energy, which is known as "running", these couplings merge into a single unified coupling, once some very large energy scale, M_{GUT} (typically $\sim 10^{16}\,\mathrm{GeV}$), is reached [2].

In this paper, particle states of masses $\geq 1\,\mathrm{TeV}$ will be theorised, and added to the SM to influence the running of the couplings, in an attempt to unify three of the four fundamental SM interactions within an $SO(10)$ GUT, $SO(10)$ being a large gauge group with the ability to contain the smaller SM gauge group, $\mathcal{G}_{\mathrm{SM}}$. In this way, the SM can be contained within some broader structure, reducing the number of free parameters, whilst also potentially realising a Grand Unification of the couplings. The additional theoretical states will be representations of $SU(5)$ or $SO(10)$, gauge groups whose representations will readily embed into an $SO(10)$ GUT.

In Section 2 of the paper, gauge theories, which have very important implications towards the structure of the SM, as well as that of any feasible GUT model, will be outlined in detail [1, 3–6]. Section 3 will then examine spontaneous symmetry breaking, a paradigm through which the Higgs mechanism operates [3, 7–10], while Section 4 will look in-depth at the Standard Model, its particle content and Lagrangian, as well as the addition of minimal supersymmetry in the form of the Minimal Supersymmetric Standard Model (MSSM) [1, 3, 8, 10–12]. The focus will then shift in Section 5 towards explaining Grand Unified Theories, their advantages, shortcomings, and different permutations [2, 13–17]. Finally, the addition of theoretical particle states to the SM will occur in Section 6, whereby their influence on the running couplings will be computed, and plots of the most successful unification efforts will be made [18–20].

2 Gauge Theories

The SM is a description of the interactions of the fundamental forces of nature in terms of gauge theories, whereby the strong, weak, and electromagnetic interactions are associated with the $SU(3)_C$, $SU(2)_L$, and $U(1)_{\mathrm{EM}}$ gauge groups, respectively. The SM gauge group, $\mathcal{G}_{SM}$, is given by [3]

$$\mathcal{G}_{SM} = SU(3)_C \times SU(2)_L \times U(1)_Y \, , \tag{1}$$

where the gauge group $U(1)_Y$ has quantum number Y, known as the "weak hypercharge". Y relates the electric charge and weak isospin quantum numbers, the latter of which treats left-handed particles with different electric charges, but which are affected equally by the weak force, as different "isospin" states of the same particle. Despite its name, isospin is not related to angular momentum or spin; the etymology of the term stems from the fact that the mathematical formalism of isospin is similar to that of spin.

By definition, a gauge theory is invariant under a set of local transformations, whose parameters have spacetime dependence. In the case of electromagnetism, which is associated with the $U(1)_{\mathrm{EM}}$ group, the local gauge transformations are complex phase transformations of fields of charged particles. We will see in Section 3.3 how a subgroup of $\mathcal{G}_{SM}$, $SU(2)_L \times U(1)_Y$, can be broken down to $U(1)_{\mathrm{EM}}$. To maintain the gauge invariance of the theory, a massless vector (spin-1) particle must exist, with the purpose of mediating electromagnetic interactions. This particle is the photon, and it is introduced through abelian $U(1)$ gauge symmetry. The abelian form, whereby elements of a group commute with one another, will be covered in more depth in Section 2.1.

One can extend gauge invariance to include non-abelian (non-commutative) transformations, such as those in $SU(2)$ and $SU(3)$, whereby sets of $N^2 - 1$ vector fields are required for each $SU(N)$. Therefore, there are 3 massless vector fields associated with $SU(2)_L$ (which are interpreted as the weak gauge bosons), and 8 massless vector fields associated with $SU(3)_C$, corresponding to each colour of gluon. The weak gauge bosons and gluons mediate weak and strong interactions, respectively, in the same way that the photon mediates electromagnetic interactions.

Weak interactions are short-ranged, and require the intervention of a massive vector boson in order to proceed. Non-abelian gauge theories were therefore initially rejected by the scientific community, as they necessarily predict all particle fields to be massless to preserve gauge invariance. However, in 1964, Peter Higgs proposed that the mass problem could be circumvented via a spontaneous symmetry breaking mechanism, a consequence of which is the existence of a scalar (spin-0) particle, known as the Higgs boson, which completes the particle spectrum of the Standard Model as we know it today.

The Higgs mechanism (as well as spontaneous symmetry breaking in general, which will be discussed further in Section 3) also fulfils the vital requirement that a physical quantum field theory must be renormalisable, such that the infinities yielded in the calculation of higher order perturbations can be cancelled via their reabsorption into the Lagrangian itself. As a result of the renormalisability of the Standard Model, it is possible to perform perturbative calculations, which predict with great accuracy the decay rates and cross-sections of processes which proceed via strong and weak interactions.

The Standard Model has 19 free parameters, including the lepton masses, quark masses, CKM matrix parameters, gauge couplings, QCD vacuum angle, and the Higgs mass and vacuum expectation value (vev). Though the SM remarkably predicts quantities such as the ratio M_Z/M_W correctly, the number of free parameters remains unfavourably high. This is another reason that the realisation of a Grand Unified Theory is desirable, such that the SM can be embedded into a larger gauge group with a minimal number of free parameters.

2.1 Abelian form

Consider the set of U(1) global complex phase transformations, given by

$$\mathcal{U} = e^{i\omega}\,, \tag{2}$$

which is abelian, as

$$[e^{i\omega_1}, e^{i\omega_2}] = 0\,. \tag{3}$$

The Lagrangian density of a free Dirac field is

$$\mathcal{L} = \overline{\psi}(i\gamma^\mu\partial_\mu - m)\psi\,. \tag{4}$$

Upon applying $\mathcal{U}$ to the field ψ, it transforms as

$$\psi \to e^{i\omega}\psi = \psi'\,, \tag{5}$$

$$\overline{\psi} = \psi^\dagger\gamma^0 \to \psi^\dagger\gamma^0 e^{-i\omega} = \overline{\psi}'\,, \tag{6}$$

and the Lagrangian is trivially invariant:

$$\begin{aligned} \mathcal{L}' &= \overline{\psi}'(i\gamma^\mu\partial_\mu - m)\psi' &(7)\\ &= e^{-i\omega}\overline{\psi}(i\gamma^\mu\partial_\mu - m)e^{i\omega}\psi &(8)\\ &= \overline{\psi}(i\gamma^\mu\partial_\mu - m)\psi &(9)\end{aligned}$$

$$\Rightarrow \mathcal{L}' = \mathcal{L}\,. \tag{10}$$

Now consider the set of U(1) local complex phase transformations, $\tilde{\mathcal{U}}$, whereby the parameter ω is spacetime dependent,

$$\tilde{\mathcal{U}} = e^{i\omega(x)}\,. \tag{11}$$

Infinitesimally close to the identity, $\tilde{\mathcal{U}}$ can be expressed as a Taylor expansion,

$$\tilde{\mathcal{U}} \approx \mathbb{1} + i\omega(x)\,, \tag{12}$$

and the Lagrangian transforms as follows:

$$\begin{aligned}
\mathcal{L}' &= (\mathbb{1} - i\omega(x))\overline{\psi}(i\gamma^\mu\partial_\mu - m)(\mathbb{1} + i\omega(x))\psi & (13)\\
&= (\overline{\psi} - i\omega(x)\overline{\psi})(i\gamma^\mu\partial_\mu - m)(\psi + i\omega(x)\psi) & (14)\\
&= (\overline{\psi} - i\omega(x)\overline{\psi})((i\gamma^\mu\partial_\mu)[\psi + i\omega(x)\psi] - m[\psi + i\omega(x)\psi]) & (15)\\
&= (\overline{\psi} - i\omega(x)\overline{\psi})(i\gamma^\mu\partial_\mu\psi + i\gamma^\mu i\omega(x)\partial_\mu\psi + i\gamma^\mu i(\partial_\mu\omega(x))\psi - m\psi - mi\omega(x)\psi) & (16)\\
&= (\overline{\psi} - i\omega(x)\overline{\psi})((\mathbb{1} + i\omega(x))(i\gamma^\mu\partial_\mu - m)\psi - \gamma^\mu(\partial_\mu\omega(x))\psi) & (17)\\
&= \overline{\psi}(\mathbb{1} - i\omega(x))((\mathbb{1} + i\omega(x))(i\gamma^\mu\partial_\mu - m)\psi - \gamma^\mu(\partial_\mu\omega(x))\psi)\,. & (18)
\end{aligned}$$

Terms which are quadratic or above in $\omega(x)$ are neglected in the expansion to first order:

$$\begin{aligned}
&\Rightarrow(\mathbb{1} - i\omega(x))((\mathbb{1} + i\omega(x)) = \mathbb{1} + \omega^2(x) \approx \mathbb{1}\,, & (19)\\
&i\omega(x)\gamma^\mu(\partial_\mu\omega(x)) \sim \omega^2(x) \approx 0\,, & (20)
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \mathcal{L}' &= \overline{\psi}(i\gamma^\mu\partial_\mu - m)\psi - \overline{\psi}(\mathbb{1} - i\omega(x))\gamma^\mu(\partial_\mu\omega(x))\psi & (21)\\
&= \overline{\psi}(i\gamma^\mu\partial_\mu - m)\psi - \overline{\psi}\gamma^\mu(\partial_\mu\omega(x))\psi & (22)\\
\Rightarrow \mathcal{L}' &= \overline{\psi}(i\gamma^\mu\partial_\mu - m)\psi - \overline{\psi}\gamma^\mu(\partial_\mu\omega(x))\psi \neq \mathcal{L}\,, & (23)
\end{aligned}$$

$$\Rightarrow \delta\mathcal{L} = \mathcal{L}' - \mathcal{L} = -\overline{\psi}(x)\gamma^\mu(\partial_\mu\omega(x))\psi(x)\,. \tag{24}$$

Therefore, we see that the Lagrangian is no longer invariant when the transformation applied is instead a spacetime-dependent transformation, known as a gauge transformation. In other words, the Lagrangian is not gauge invariant, which we require within a gauge theory, and we must therefore find a way to restore this invariance. One way of doing this is to modify the Lagrangian by replacing the partial derivatives with covariant ones,

$$\partial_\mu \to D_\mu = \partial_\mu + ieA_\mu\,, \tag{25}$$

where e is interpreted as the electric charge of the fermion field, and A_μ is the vector four-potential of the electromagnetic field. The modified Lagrangian then becomes

$$\tilde{\mathcal{L}} = \overline{\psi}(i\gamma^\mu\partial_\mu - e\gamma^\mu A_\mu - m)\psi\,. \tag{26}$$

If we also demand that A_μ transforms as

$$A_\mu \to A_\mu - \frac{1}{e}\partial_\mu\omega(x) = A'_\mu\,, \tag{27}$$

then $\tilde{\mathcal{L}}$ transforms as follows under a gauge transformation:

$$\tilde{\mathcal{L}} \to (\overline{\psi} - i\omega(x)\overline{\psi})\Big(i\gamma^\mu\partial_\mu[\psi + i\omega(x)\psi] - m[\psi + i\omega(x)\psi] - e\gamma^\mu[\mathbb{1} + i\omega(x)]\Big[A_\mu - \frac{1}{e}\partial_\mu\omega(x)\Big]\psi\Big) \tag{28}$$

$$= \overline{\psi}\Big([\mathbb{1} - i\omega(x)][i\gamma^\mu\partial_\mu][\psi + i\omega(x)\psi] - m[\mathbb{1} - i\omega(x)][\mathbb{1} + i\omega(x)]\psi \tag{29}$$

$$- e[\mathbb{1} - i\omega(x)]\gamma^\mu[\mathbb{1} + i\omega(x)]A_\mu\psi - [\mathbb{1} - i\omega(x)]\frac{e}{e}\gamma^\mu\partial_\mu\omega(x)\psi + \frac{e}{e}[\mathbb{1} - i\omega(x)]\gamma^\mu i\omega(x)(\partial_\mu\omega(x))\psi\Big) \tag{30}$$

Using the approximations from Equations (19) and (20):

$$\tilde{\mathcal{L}} \to \overline{\psi}\Big([\mathbb{1} - i\omega(x)]\gamma^\mu[i\partial_\mu\psi - (\partial_\mu\omega(x))\psi - \omega(x)(\partial_\mu\psi)] - m\psi - e\gamma^\mu A_\mu\psi + \gamma^\mu(\partial_\mu\omega(x))\psi\Big) \tag{31}$$

$$= \overline{\psi}\Big(i\gamma^\mu\partial_\mu\psi - \gamma^\mu(\partial_\mu\omega(x))\psi - \gamma^\mu\omega(x)(\partial_\mu\psi) + \gamma^\mu\omega(x)\partial_\mu\psi - m\psi - e\gamma^\mu A_\mu\psi + \gamma^\mu(\partial_\mu\omega(x))\psi\Big) \tag{32}$$

$$= \overline{\psi}\Big(i\gamma^\mu\partial_\mu\psi - e\gamma^\mu A_\mu\psi - m\psi\Big) = \tilde{\mathcal{L}}' \tag{33}$$

$$\Rightarrow \tilde{\mathcal{L}}' = \overline{\psi}(i\gamma^\mu\partial_\mu - e\gamma^\mu A_\mu - m)\psi = \tilde{\mathcal{L}} \tag{34}$$

$$\Rightarrow \delta\tilde{\mathcal{L}} = \tilde{\mathcal{L}}' - \tilde{\mathcal{L}} = 0 \tag{35}$$

From Equation (35), we see that gauge invariance has been restored.

To formulate a quantum field theory with a proper physical interpretation, one must add gauge invariant (and Lorentz invariant) kinetic terms for photons, in the form

$$\mathcal{L}_{\text{kin}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\,, \tag{36}$$

where $F_{\mu\nu}$, the electromagnetic field tensor, is given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,, \tag{37}$$

and the prefactor of $-\frac{1}{4}$ in $\mathcal{L}_{\text{kin}}$ ensures that the equations of motion generated by the Lagrangian match those of Maxwell. Adding the $\mathcal{L}_{\text{kin}}$ piece to the final Lagrangian gives

$$\mathcal{L}_{\text{QED}} = \overline{\psi}(i\gamma^\mu\partial_\mu - e\gamma^\mu A_\mu - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{38}$$

$$\Rightarrow \mathcal{L}_{\text{QED}} = \overline{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\,, \tag{39}$$

which is the complete QED Lagrangian, with the exception of gauge fixing terms, which are used to maintain the gauge symmetry in the observables through breaking that of the Lagrangian, such that the theory can be quantised, and the photon propagator thereby becomes calculable. However, gauge fixing terms will be neglected here, and will not be discussed further.

Notice that there is no explicit mass term for the photon, which would be of form $M_\gamma^2 A_\mu A^\mu$. If such a term were to be added to $\mathcal{L}_{\text{QED}}$, then using Equation (27), it would transform as follows under a $U(1)$ transformation:

$$M_\gamma^2 A_\mu A^\mu \to M_\gamma^2\Big(A_\mu - \frac{1}{e}\partial_\mu(\omega(x))\Big)\Big(A^\mu - \frac{1}{e}\partial^\mu(\omega(x))\Big) \tag{40}$$

$$= M_\gamma^2\Big(A_\mu A^\mu - \frac{1}{e}\partial_\mu(\omega(x))A^\mu - \frac{1}{e}A_\mu\partial^\mu(\omega(x)) + \frac{1}{e^2}\partial_\mu(\omega(x))\partial^\mu(\omega(x))\Big) \tag{41}$$

The term $\partial_\mu(\omega(x))\partial^\mu(\omega(x)$ is quadratic in ω and can hence be neglected in the expansion:

$$\Rightarrow M_\gamma^2 A_\mu A^\mu \to M_\gamma^2\Big(A_\mu A^\mu - \frac{1}{e}\partial_\mu(\omega(x))A^\mu - \frac{1}{e}A_\mu\partial^\mu(\omega(x))\Big) \tag{42}$$

The derivatives and four-potential terms can be written in four-vector form as

$$A^\mu = (\phi, \vec{A})\,,\; A_\mu = (\phi, -\vec{A})\,, \tag{43}$$

$$\partial^\mu = (\partial_t, \vec{\nabla})\,,\; \partial_\mu = (\partial_t, -\vec{\nabla})\,, \tag{44}$$

$$\Rightarrow A^\mu \partial_\mu = A_\mu \partial^\mu\,, \tag{45}$$

$$\Rightarrow \frac{1}{e} A^\mu \partial_\mu(\omega(x)) = \frac{1}{e} A_\mu \partial^\mu(\omega(x))\,, \tag{46}$$

where, ϕ is the electric potential, and $\vec{A}$ is the magnetic vector potential. Therefore, the mass term transforms as

$$M_\gamma^2 A_\mu A^\mu \rightarrow M_\gamma^2 A_\mu A^\mu - \frac{2M_\gamma^2}{e} \partial_\mu(\omega(x)) A^\mu\,, \tag{47}$$

$$\Rightarrow \delta\mathcal{L}_{\mathrm{QED}} = -\frac{2M_\gamma^2}{e} \partial_\mu(\omega(x)) A^\mu\,, \tag{48}$$

in which case, the only way to maintain gauge invariance in the Lagrangian to require that $M_\gamma = 0$, and thus the photon must be massless in the theory.

2.2 Non-abelian form

One can extend the concepts outlined in Section 2.1 to the cases of the strong and weak gauge bosons, which, unlike the photon, may self-interact. Therefore, different elements of each gauge group will not commute with any other element within the same group. As a result, we must this time utilise non-abelian gauge theories.

In the non-abelian form, N free fermion fields, ψ_i, are embedded within a multiplet ψ, as follows:

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_N \end{pmatrix} \tag{49}$$

The Lagrangian density will then be

$$\mathcal{L} = \overline{\psi}(i\gamma^\mu \partial_\mu - m)\psi = \sum_{i=1}^{N} \overline{\psi}_i(i\gamma^\mu \partial_\mu - m)\psi_i \,, \tag{50}$$

and it is invariant under global complex transformations of the form

$$\psi \to \mathbf{U}\psi = \psi' \,, \tag{51}$$

$$\overline{\psi} \to \overline{\psi}\mathbf{U}^\dagger = \overline{\psi}' \,, \tag{52}$$

where $\mathbf{U}$ is an $N \times N$ matrix of the $SU(N)$ group. $\mathbf{U}$ has two specific properties, denoted by the "SU" part of the group name; the "S" in SU(N) stands for "special", meaning that $\det(\mathbf{U}) = 1$, and the "U" stands for "unitary", whereby $\mathbf{U}^\dagger\mathbf{U} = \mathbf{U}\mathbf{U}^\dagger = \mathbb{1}$.

$SU(N)$ matrices are specified by $N^2 - 1$ real parameters, ω^a, where the index a runs from 1 to $N^2 - 1$. An $SU(N)$ matrix has the following general form:

$$\mathbf{U} = e^{-i\sum_{a=1}^{N^2-1} \omega^a \mathbf{T}^a} \,, \tag{53}$$

where $\mathbf{T}^a$ are Hermitian, traceless matrices, which are known as generators of the $SU(N)$ group. Throughout this paper, the generators of the $SU(2)$ and $SU(3)$ groups will subsequently be denoted by τ^a and T^a, respectively, unless explicitly stated otherwise.

Two $SU(N)$ transformations do not commute:

$$\left(e^{-i\omega_1^a \mathbf{T}^a}\right)\left(e^{-i\omega_2^b \mathbf{T}^b}\right) \neq \left(e^{-i\omega_2^b \mathbf{T}^b}\right)\left(e^{-i\omega_1^a \mathbf{T}^a}\right) . \tag{54}$$

A representation $\rho(\mathcal{G})$ of a group $\mathcal{G}$ is a homomorphism from $\mathcal{G}$ to the space of linear maps acting on representation space $\mathcal{V}_{\text{rep}}$:

$$\rho : \mathcal{G} \to GL(\mathcal{V}_{\text{rep}}) \tag{55}$$

where $GL(\mathcal{V}_{\text{rep}})$ is the general linear group of $\mathcal{V}_{\text{rep}}$, that is the group of all automorphisms of $\mathcal{V}_{\text{rep}}$, in other words the group of all morphisms of each object to itself, a morphism being a structure-preserving map from one mathematical structure to another [4]. Each linear map can be considered to be a matrix, and as such, $\rho(\mathcal{G})$ can be regarded as a group of matrix objects, which contains within it all of the structure of the group $\mathcal{G}$, and thus is useful for describing the transformations of states under particular symmetries.

Each element of ρ is a square matrix of size $\dim(\mathcal{V}_{\text{rep}}) \times \dim(\mathcal{V}_{\text{rep}})$, and acts on $\mathcal{V}_{\text{rep}}$ as

$$[\rho(\mathcal{G})]_i : \mathcal{V}_{\text{rep}} \to \mathcal{V}_{\text{rep}} \tag{56}$$

When considering representations of $SU(N)$ as $m \times m$ matrices, it becomes clear that there are certain values of m which are particularly significant. Disregarding the trivial representation, where $m = 1$, which is a singlet and does not transform under any $SU(N)$, there are two main representations. The first, the fundamental representation, where $m = N$, is the representation which is equal to the group, such that $\rho(SU(N)) = SU(N)$. The other, the adjoint representation, where $m = \dim(SU(N))$, is a representation whereby the components of the generators have the form $(T^a_{ad}) = f^a_{bc}$, where f^a_{bc} are structure constants, antisymmetric in the indices a, b, c.

The generators $\mathbf{T}$ are normalised in the fundamental representation via the relation

$$\text{Tr}[\mathbf{T}^a\mathbf{T}^b] = \frac{1}{2}\delta_{ab}\,, \tag{57}$$

where δ_{ab} is the Kronecker delta function. The commutator of two generators is

$$[\mathbf{T}^a, \mathbf{T}^b] = if^{abc}T^c\,, \tag{58}$$

where f^{abc} are the structure functions encountered previously. We will use these relations later when considering the gauge boson kinetic terms within the Lagrangian.

As with abelian transformations, the Lagrangian is not invariant under local $SU(N)$ transformations:

$$\begin{aligned}
\mathcal{L} &\to \overline{\psi}\mathbf{U}^\dagger(i\gamma^\mu\partial_\mu - m)\mathbf{U}\psi && (59)\\
&= \overline{\psi}\mathbf{U}^\dagger(i\gamma^\mu[\psi(\partial_\mu\mathbf{U}) + \mathbf{U}(\partial_\mu\psi)]) - m\overline{\psi}\underbrace{\mathbf{U}^\dagger\mathbf{U}}_{=\mathbb{1}}\psi && (60)\\
&= \overline{\psi}\mathbf{U}^\dagger i\gamma^\mu\mathbf{U}(\partial_\mu\psi) - m\overline{\psi}\psi + \overline{\psi}\mathbf{U}^\dagger i\gamma^\mu\psi(\partial_\mu\mathbf{U}) && (61)
\end{aligned}$$

Since $\mathbf{U}$ is a constant, $\mathbf{U}$ and $\mathbf{U}^\dagger$ terms can be moved freely to eliminate one another:

$$\begin{aligned}
\Rightarrow \mathcal{L} \to &= \overline{\psi}\underbrace{\mathbf{U}^\dagger\mathbf{U}}_{=\mathbb{1}}i\gamma^\mu(\partial_\mu\psi) - m\overline{\psi}\psi + \overline{\psi}\mathbf{U}^\dagger i\gamma^\mu\psi(\partial_\mu\mathbf{U}) && (62)\\
&= \overline{\psi}(i\gamma^\mu\partial_\mu - m) + \overline{\psi}\mathbf{U}^\dagger i\gamma^\mu\psi(\partial_\mu\mathbf{U}) = \mathcal{L}' && (63)
\end{aligned}$$

$$\Rightarrow \delta\mathcal{L} = \mathcal{L}' - \mathcal{L} = \overline{\psi}\mathbf{U}^\dagger i\gamma^\mu\psi(\partial_\mu\mathbf{U}) \tag{64}$$

which indicates that the Lagrangian is clearly not gauge invariant. Analogously to the abelian case, we will replace partial derivatives with covariant ones,

$$\mathbf{D}_\mu = \partial_\mu + ig\mathbf{A}_\mu\,, \tag{65}$$

where $\mathbf{A}_\mu = \mathbf{T}^a A^a_\mu$. The covariant derivative contains $N^2 - 1$ gauge bosons, where each gauge boson corresponds to a particular generator.

If we now demand that under a gauge transformation, $\mathbf{A}_\mu$ transforms as

$$\mathbf{A}_\mu \to \mathbf{U}\mathbf{A}_\mu\mathbf{U}^\dagger + \frac{i}{g}(\partial_\mu \mathbf{U})\mathbf{U}^\dagger = \mathbf{A}'_\mu\,, \tag{66}$$

where g is interpreted as the weak or strong coupling constant, for $SU(2)_L$ and $SU(3)_C$ respectively, then we can restore gauge invariance in the Lagrangian as follows:

$$\tilde{\mathcal{L}} = \overline{\psi}(i\gamma^\mu \mathbf{D}_\mu - m)\psi \tag{67}$$

$$= \overline{\psi}(i\gamma^\mu \partial_\mu - g\gamma^\mu \mathbf{A}_\mu - m)\psi \tag{68}$$

$$\tilde{\mathcal{L}} \to \overline{\psi}\mathbf{U}^\dagger(i\gamma^\mu\partial_\mu - g\gamma^\mu\mathbf{A}'_\mu - m)\mathbf{U}\psi = \tilde{\mathcal{L}}' \tag{69}$$

$$= \overline{\psi}\mathbf{U}^\dagger\Big(i\gamma^\mu\partial_\mu - g\gamma^\mu\Big[\mathbf{U}\mathbf{A}_\mu\mathbf{U}^\dagger + \frac{i}{g}(\partial_\mu\mathbf{U})\mathbf{U}^\dagger\Big] - m\Big)\mathbf{U}\psi \tag{70}$$

$$= \overline{\psi}\mathbf{U}^\dagger i\gamma^\mu\partial_\mu(\mathbf{U}\psi) - \overline{\psi}m\underbrace{\mathbf{U}^\dagger\mathbf{U}}_{=\mathbb{1}}\psi - g\overline{\psi}\mathbf{U}^\dagger\gamma^\mu[\mathbf{U}\mathbf{A}_\mu\mathbf{U}^\dagger]\mathbf{U}\psi - \overline{\psi}\mathbf{U}^\dagger\gamma^\mu\frac{ig}{g}(\partial_\mu\mathbf{U})\underbrace{\mathbf{U}^\dagger\mathbf{U}}_{=\mathbb{1}}\psi \tag{71}$$

$$= \overline{\psi}\mathbf{U}^\dagger i\gamma^\mu(\partial_\mu\mathbf{U})\psi - \overline{\psi}\mathbf{U}^\dagger i\gamma^\mu(\partial_\mu\mathbf{U})\psi + \overline{\psi}\mathbf{U}^\dagger i\gamma^\mu(\partial_\mu\psi)\mathbf{U} - \overline{\psi}m\psi - g\overline{\psi}\mathbf{U}^\dagger\gamma^\mu[\mathbf{U}\mathbf{A}_\mu\mathbf{U}^\dagger]\mathbf{U}\psi \tag{72}$$

$$= \overline{\psi}\underbrace{\mathbf{U}^\dagger\mathbf{U}}_{=\mathbb{1}}i\gamma^\mu(\partial_\mu\psi) - \overline{\psi}m\psi - g\overline{\psi}\underbrace{\mathbf{U}^\dagger\mathbf{U}}_{=\mathbb{1}}\gamma^\mu\mathbf{A}_\mu\underbrace{\mathbf{U}^\dagger\mathbf{U}}_{=\mathbb{1}}\psi \tag{73}$$

$$= \overline{\psi}i\gamma^\mu(\partial_\mu\psi) - \overline{\psi}m\psi - g\overline{\psi}\gamma^\mu\mathbf{A}_\mu\psi \tag{74}$$

$$= \overline{\psi}(i\gamma^\mu\partial_\mu - g\gamma^\mu\mathbf{A}_\mu - m)\psi = \tilde{\mathcal{L}} \tag{75}$$

$$\Rightarrow \delta\tilde{\mathcal{L}} = \tilde{\mathcal{L}}' - \tilde{\mathcal{L}} = 0 \tag{76}$$

Next we will consider the kinetic term for gauge bosons, which is comprised of field strengths $F^a_{\mu\nu}$. The boldface version (matrix form) of the field strength, $\mathbf{F}_{\mu\nu}$, again representing the format

$$\mathbf{F}_{\mu\nu} = \mathbf{T}^a F^a_{\mu\nu}\,, \tag{77}$$

is defined as the commutator

$$\mathbf{F}_{\mu\nu} = -\frac{i}{g}[\mathbf{D}_\mu, \mathbf{D}_\nu]\,. \tag{78}$$

By manipulating this commutator, one can find an expression for $F^a_{\mu\nu}$ in terms of the four-potentials $\mathbf{A}_\mu$ and $\mathbf{A}_\nu$:

$$\mathbf{F}_{\mu\nu} = -\frac{i}{g}\Big[(\partial_\mu + ig\mathbf{A}_\mu)(\partial_\nu + ig\mathbf{A}_\nu) - (\partial_\nu + ig\mathbf{A}_\nu)(\partial_\mu + ig\mathbf{A}_\mu)\Big] \tag{79}$$

$$= -\frac{i}{g}\Big[\partial_\mu\partial_\nu - g^2\mathbf{A}_\mu\mathbf{A}_\nu + ig\mathbf{A}_\mu\partial_\nu + ig\partial_\mu\mathbf{A}_\nu - \partial_\nu\partial_\mu + g^2\mathbf{A}_\nu\mathbf{A}_\mu - ig\partial_\nu\mathbf{A}_\mu - ig\mathbf{A}_\nu\partial_\mu\Big]\mathbb{1} \tag{80}$$

$$\partial_\mu\mathbb{1} = \partial_\nu\mathbb{1} = 0 \tag{81}$$

$$\partial_\mu\partial_\nu = \partial_\nu\partial_\mu \tag{82}$$

$$\Rightarrow \mathbf{F}_{\mu\nu} = -\frac{i}{g}\Big[-g^2\mathbf{A}_\mu\mathbf{A}_\nu + ig\partial_\mu\mathbf{A}_\nu + g^2\mathbf{A}_\nu\mathbf{A}_\mu - ig\partial_\nu\mathbf{A}_\mu\Big] \tag{83}$$

$$\Rightarrow \mathbf{F}_{\mu\nu} = -\frac{i}{g}\Big[g^2(\mathbf{A}_\nu\mathbf{A}_\mu - \mathbf{A}_\mu\mathbf{A}_\nu) + ig(\partial_\mu\mathbf{A}_\nu - \partial_\nu\mathbf{A}_\mu)\Big] \tag{84}$$

$$= g^2\frac{i}{g}\Big(\mathbf{A}_\mu\mathbf{A}_\nu - \mathbf{A}_\nu\mathbf{A}_\mu\Big) - ig\frac{i}{g}\Big(\partial_\mu\mathbf{A}_\nu - \partial_\nu\mathbf{A}_\mu\Big) \tag{85}$$

$$\Rightarrow \mathbf{F}_{\mu\nu} = ig(\mathbf{A}_\mu\mathbf{A}_\nu - \mathbf{A}_\nu\mathbf{A}_\mu) + (\partial_\mu\mathbf{A}_\nu - \partial_\nu\mathbf{A}_\mu) \tag{86}$$

Inspecting Equations (66) and (86), it becomes clear that $\mathbf{F}_{\mu\nu}$, which is a function only of four-potential matrices and their derivatives, should transform as

$$\mathbf{F}_{\mu\nu} \to \mathbf{U}\mathbf{F}_{\mu\nu}\mathbf{U}^\dagger = \mathbf{F}'_{\mu\nu}\,, \tag{87}$$

as upon expansion, all terms which include elements $\sim \frac{i}{g}(\partial_\mu\mathbf{U})\mathbf{U}^\dagger$ either cancel due to relative minus signs, or are eliminated by derivatives.

Using Equation (77) in conjunction with Equation (86), one can obtain an expression for the field strength $F^a_{\mu\nu}$:

$$\mathbf{T}^a F^a_{\mu\nu} = (\partial_\mu\mathbf{T}^a A^a_\nu - \partial_\nu\mathbf{T}^a A^a_\mu) + ig(\mathbf{T}^b A^b_\mu \mathbf{T}^c A^c_\nu - \mathbf{T}^c A^c_\nu \mathbf{T}^b A^b_\mu) \tag{88}$$

$$= \mathbf{T}^a(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) + ig(\mathbf{T}^b\mathbf{T}^c - \mathbf{T}^c\mathbf{T}^b)A^b_\mu A^c_\nu \tag{89}$$

$$= \mathbf{T}^a(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) + ig[\mathbf{T}^b, \mathbf{T}^c]A^b_\mu A^c_\nu \tag{90}$$

Computing the commutation relation via Equation (58) gives

$$\Rightarrow \mathbf{T}^a F^a_{\mu\nu} = \mathbf{T}^a(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) + i^2 f^{bca}\mathbf{T}^a g(A^b_\mu A^c_\nu) \tag{91}$$

$$= \mathbf{T}^a(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{bca} A^b_\mu A^c_\nu) \tag{92}$$

$$\Rightarrow F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{bca} A^b_\mu A^c_\nu \tag{93}$$

The structure function f is antisymmetric in its indices, hence $f^{bca} = f^{abc}$. Therefore, the field strength is

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu \tag{94}$$

The gauge invariant kinetic Lagrangian term for gauge bosons is [3]

$$\mathcal{L}_{\text{kin}} = -\frac{1}{2}\,\text{Tr}[\mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu}] \tag{95}$$

$$= -\frac{1}{2}F^a_{\mu\nu}F^{b\mu\nu}\,\text{Tr}[\mathbf{T}^a\mathbf{T}^b]\,, \tag{96}$$

and evaluating the trace using Equation (57) gives

$$\mathcal{L}_{\text{kin}} = -\frac{1}{2} F^a_{\mu\nu} F^{b\mu\nu} \frac{1}{2} \delta_{ab} \tag{97}$$

$$\Rightarrow \mathcal{L}_{\text{kin}} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} \,. \tag{98}$$

An important exercise, which helps gain insight into how abelian and non-abelian theories have differing physical implications, is to expand Equation (98) in terms of four-potentials using Equation (94):

$$\mathcal{L}_{\text{kin}} = -\frac{1}{4} \Big[(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) - g f^{abc} A^b_\mu A^c_\nu \Big] \Big[(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu}) - g f^{ade} A^{d\mu} A^{e\nu} \Big] \tag{99}$$

$$\Rightarrow \mathcal{L}_{\text{kin}} = -\frac{1}{4} \Big[(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu}) - g f^{abc} A^b_\mu A^c_\nu (\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu}) \tag{100a}$$

$$- g f^{ade} A^{d\mu} A^{e\nu} (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) + g^2 f^{abc} f^{ade} A^b_\mu A^c_\nu A^{d\mu} A^{e\nu} \Big] \tag{100b}$$

Unlike abelian gauge theories, the kinetic part of the Lagrangian for the gauge bosons therefore contains within it three- and four-point interaction terms (the terms which are cubic and quartic in the different four-potentials, respectively), which implies that non-abelian gauge bosons interact with themselves, in contrast to photons.

Combining the results from this section, the full Lagrangian density for an $SU(N)$ gauge theory is

$$\mathcal{L}_{SU(N)} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \overline{\psi}(i\gamma^\mu \mathbf{D}_\mu - m)\psi + \mathcal{L}_{\text{GF}} + \mathcal{L}_{\text{ghost}} \,, \tag{101}$$

where $\mathcal{L}_{\text{GF}}$ is the gauge fixing term, and $\mathcal{L}_{\text{ghost}}$ is for ghost interactions in additional higher-order loop diagrams, needed as a consequence of fixing the gauge.

As with the abelian case, mass terms are forbidden in a non-abelian gauge theory, as a term of the form $M^2 A^a_\mu A^{a\mu}$ is not gauge invariant. This would be a problem in the theory due to the existence of massive physical gauge bosons, but it can be circumvented via the Higgs mechanism, an example of spontaneous symmetry breaking, which will be discussed in Section 3.

2.3 Gauge anomalies

At the classical level, the assignments of hypercharges (the quantum numbers Y of $U(1)_Y$) to fermion fields are arbitrary, but at the quantum level, the so-called Adler-Bell-Jackiw (ABJ) anomaly arises, spoiling the renormalisability of a theory whenever the conservation of a gauged current is violated. However, Adler-Bardeen theorem asserts that this anomaly occurs only for diagrams at one-loop order. Therefore, one only needs to check that the anomalies of all one-loop diagrams vanish, in order for the theory to be renormalisable. Though this means that some theories are left renormalisable, there is still the implication that hypercharges are restricted at the quantum level, thereby restricting the particle content of theories.

Neglecting quantum effects, the vector and axial vector currents,

$$j^\mu(x) = \overline{\psi}\gamma^\mu\psi\,, \tag{102}$$

$$j^{\mu 5}(x) = \overline{\psi}\gamma^\mu\gamma^5\psi\,, \tag{103}$$

defined by Equations (102) and (103) respectively, are conserved in the massless limit, i.e. $\partial_\mu j^\mu(x) = \partial_\mu j^{\mu 5}(x) = 0$, and γ^5 is defined by the following product of Dirac gamma matrices:

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3\,. \tag{104}$$

However, when quantum effects are present, then, in the case of a single massless fermion, the axial vector current is not conserved. To show this, consider the QED Lagrangian for one massless fermion, and using Equations (107) and (114) (Euler-Lagrange equations), solve for the equations of motion with respect to the fermion field:

$$\mathcal{L}_{\text{QED}} = \overline{\psi}(i\gamma^\mu D_\mu)\psi - \frac{1}{4}(F_{\mu\nu})^2 \tag{105}$$

$$\Rightarrow \mathcal{L}_{\text{QED}} = \overline{\psi}(i\gamma^\mu\partial_\mu - e\gamma^\mu A_\mu)\psi - \frac{1}{4}(F_{\mu\nu})^2\,, \tag{106}$$

$$\frac{\partial\mathcal{L}_{\text{QED}}}{\partial\psi} - \partial_\mu\left(\frac{\partial\mathcal{L}_{\text{QED}}}{\partial(\partial_\mu\psi)}\right) = 0\,, \tag{107}$$

$$\frac{\partial\mathcal{L}_{\text{QED}}}{\partial\psi} = -\overline{\psi}e\gamma^\mu A_\mu\,, \tag{108}$$

$$\frac{\partial\mathcal{L}_{\text{QED}}}{\partial(\partial_\mu\psi)} = \overline{\psi}i\gamma^\mu\,, \tag{109}$$

$$\Rightarrow -\overline{\psi}e\gamma^\mu A_\mu = \partial_\mu[\overline{\psi}i\gamma^\mu] \tag{110}$$

$$\Rightarrow -\overline{\psi}e\gamma^\mu A_\mu = i(\partial_\mu\overline{\psi})\gamma^\mu \tag{111}$$

$$\Rightarrow (\partial_\mu\overline{\psi})\gamma^\mu = ie\overline{\psi}\gamma^\mu A_\mu \tag{112}$$

$$\Rightarrow (\partial_\mu\overline{\psi})\gamma^\mu = ie\overline{\psi}\not{A}\,, \tag{113}$$

$$\frac{\partial\mathcal{L}_{\text{QED}}}{\partial\overline{\psi}} - \partial_\mu\left(\frac{\partial\mathcal{L}_{\text{QED}}}{\partial(\partial_\mu\overline{\psi})}\right) = 0\,, \tag{114}$$

$$\frac{\partial \mathcal{L}_{\text{QED}}}{\partial \overline{\psi}} = i\gamma^{\mu}\partial_{\mu}\psi - e\gamma^{\mu}A_{\mu}\psi\,, \tag{115}$$

$$\frac{\partial \mathcal{L}_{\text{QED}}}{\partial(\partial_{\mu}\overline{\psi})} = 0\,, \tag{116}$$

$$\Rightarrow i\gamma^{\mu}\partial_{\mu}\psi - e\gamma^{\mu}A_{\mu}\psi = 0 \tag{117}$$

$$\Rightarrow i\gamma^{\mu}\partial_{\mu}\psi = e\gamma^{\mu}A_{\mu}\psi \tag{118}$$

$$\Rightarrow \gamma^{\mu}\partial_{\mu}\psi = -ie\gamma^{\mu}A_{\mu}\psi \tag{119}$$

$$\Rightarrow \not{\partial}\psi = -ie\not{A}\psi\,, \tag{120}$$

where we have used Feynman slash notation, $\not{\partial}\psi = \gamma^{\mu}\partial_{\mu}\psi$.

The axial vector current $j^{\mu 5}$ is a product of local operators, which can often be singular. The solution to this is to place the two fermion fields ψ and $\overline{\psi}$ at an infinitesimal distance ε apart, which can later be evaluated in the limit that $\varepsilon \to 0$. This limit must be taken symmetrically to ensure that $j^{\mu 5}$ behaves properly when it is Lorentz transformed. In order to maintain gauge invariance, it must also contain the Wilson line, $\exp[-ie\int_{x-\frac{\varepsilon}{2}}^{x+\frac{\varepsilon}{2}} \mathrm{d}zA(z)]$, as follows [3]:

$$j^{\mu 5} = \lim_{\varepsilon\to 0}\left(\overline{\psi}\Big(x+\frac{\varepsilon}{2}\Big)\gamma^{\mu}\gamma^{5}\exp\left[-ie\int_{x-\frac{\varepsilon}{2}}^{x+\frac{\varepsilon}{2}}\mathrm{d}zA(z)\right]\psi\Big(x-\frac{\varepsilon}{2}\Big)\right), \tag{121}$$

where $\lim_{\varepsilon\to 0}$ here denotes the symmetric limit. Taking the derivative of $j^{\mu 5}$ and substituting Equations (113) and (120) gives

$$\partial_{\mu}j^{\mu 5} = \lim_{\varepsilon\to 0}\left(\overline{\psi}\Big(x+\frac{\varepsilon}{2}\Big)\left[ie\not{A}\Big(x+\frac{\varepsilon}{2}\Big) - ie\not{A}\Big(x-\frac{\varepsilon}{2}\Big) - ie\varepsilon^{\nu}\gamma^{\mu}\partial_{\mu}A_{\nu}(x)\right]\gamma^{5}\psi\Big(x-\frac{\varepsilon}{2}\Big)\right) \tag{122}$$

$$= \lim_{\varepsilon\to 0}\left(\overline{\psi}\Big(x+\frac{\varepsilon}{2}\Big)[-ie\gamma^{\mu}\varepsilon^{\nu}(\partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu})]\gamma^{5}\psi\Big(x-\frac{\varepsilon}{2}\Big)\right) \tag{123}$$

Despite the singular nature of the fermion bilinear term, Equation (123) actually goes to zero, due to tracing over $\gamma^{\mu}\gamma^{5}$. Therefore, one must look at higher orders of the expansion in products of the operators. Doing so yields an expectation value of [3]

$$\left\langle\overline{\psi}\Big(x+\frac{\varepsilon}{2}\Big)\gamma^{\mu}\gamma^{5}\psi\Big(x-\frac{\varepsilon}{2}\Big)\right\rangle \sim 2e\varepsilon^{\mu\alpha\beta\gamma}F_{\alpha\beta}(x)\Big(\frac{-i}{8\pi^{2}}\frac{\varepsilon_{\gamma}}{\varepsilon^{2}}\Big)\,, \tag{124}$$

where $\varepsilon^{\mu\alpha\beta\gamma}$ is the 4D Levi-Civita function. Substituting Equation (124) into Equation (123), and using $\partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} = F_{\mu\nu}$, one finds that

$$\partial_{\mu}j^{\mu 5} = \lim_{\varepsilon\to 0}\left[\varepsilon^{\mu\alpha\beta\gamma}F_{\alpha\beta}(x)\Big(\frac{e}{4\pi^{2}}\Big)\Big(\frac{-i\varepsilon_{\gamma}}{\varepsilon^{2}}\Big)(-ie\varepsilon^{\nu}F_{\mu\nu})\right] \tag{125}$$

$$= \lim_{\varepsilon\to 0}\left[\Big(\frac{-e^{2}}{4\pi^{2}}\Big)\varepsilon^{\mu\alpha\beta\gamma}F_{\alpha\beta}(x)\Big(\frac{\varepsilon_{\gamma}\varepsilon^{\nu}}{\varepsilon^{2}}\Big)F_{\mu\nu}(x)\right]. \tag{126}$$

Using the result [3]

$$\partial_{\mu}j^{\mu 5} = \lim_{\varepsilon\to 0}\Big[\frac{\varepsilon_{\gamma}\varepsilon^{\nu}}{\varepsilon^{2}}\Big] = \frac{1}{4}g_{\gamma}^{\nu}\,, \tag{127}$$

where g^{ν}_{γ} is the metric tensor, which is only non-zero for $\gamma = \nu$, Equation (127) becomes

$$\partial_\mu j^{\mu 5} = \left[\left(\frac{-e^2}{16\pi^2} \right) \varepsilon^{\mu\alpha\beta\nu} F_{\alpha\beta}(x) F_{\mu\nu}(x) \right] . \tag{128}$$

The Levi-Civita function is antisymmetric, and as such $\varepsilon^{\alpha\beta\mu\nu} = -\varepsilon^{\mu\alpha\beta\nu}$, so the final result is

$$\partial_\mu j^{\mu 5} = \left[\left(\frac{e^2}{16\pi^2} \right) \varepsilon^{\alpha\beta\mu\nu} F_{\alpha\beta}(x) F_{\mu\nu}(x) \right] \neq 0 . \tag{129}$$

This is the ABJ anomaly, whereby the axial vector current is not conserved. Therefore, we must check by hand that anomalous terms from triangle Feynman diagrams cancel exactly. These triangle Feynman diagrams have a triangular fermion loop with one external gauge boson protruding from each vertex of the loop, as shown in Figure (1), and there are 10 of them in total in the Standard Model, after omitting diagrams which have left-right symmetric couplings, such as $(SU(3))^3$ and $SU(3) \times \text{grav}^2$, where grav denotes an external graviton.

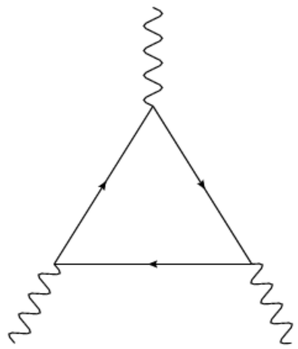

Figure 1: An example of a one-loop triangle diagram describing anomalies (taken from [5]).

Each diagram has an anomaly index $\mathcal{A}_{abc}$ associated with it, in terms of the generators T^i of the particular gauge groups of the bosons involved in the diagram, which is given by [6]

$$\mathcal{A}_{abc} = \text{Tr}[T^a\{T^b, T^c\}] . \tag{130}$$

For all of the anomalies to cancel, every $\mathcal{A}_{abc}$ index associated with a triangle diagram must be equal to zero. The 10 diagrams with possible gauge anomalies within the SM are shown in Figure (2).

Of these 10 diagrams, 6 are trivially zero. The anomalies of the 5 diagrams containing one $SU(2)$ and/or one $SU(3)$ are zero, as $\mathcal{A}_{abc} \propto Tr[T^a] = 0$, as the generators of either gauge group are traceless. The anomaly of the $(SU(2))^3$ diagram is

$$\mathcal{A}^{(SU(2))^3}_{abc} = \text{Tr}[\tau^a\{\tau^b, \tau^c\}] , \tag{131}$$

where τ^i is one of the $SU(2)$ generators, which are equal to exactly half of their respective Pauli spin matrix σ^i. Therefore,

$$\mathcal{A}_{abc}^{(SU(2))^3} = \mathrm{Tr}\left[\frac{1}{2}\sigma^a \frac{1}{4}\{\sigma^b, \sigma^c\}\right] \tag{132}$$

$$\{\sigma^b, \sigma^c\} = 2\delta_{bc} \tag{133}$$

$$\Rightarrow \mathcal{A}_{abc}^{(SU(2))^3} = \frac{1}{8}\,\mathrm{Tr}[\sigma^a 2\delta_{bc}] \tag{134}$$

$$\Rightarrow \mathcal{A}_{abc}^{(SU(2))^3} \propto \mathrm{Tr}[\sigma^a] = 0 \tag{135}$$

The remaining 4 diagrams are non-trivial, and as such must be calculated by hand on a model-by-model basis. The equations for the anomalies of each diagram will be given below [3]; these will then be used in Section 4.3 during the discussion of gauge anomalies specific to the Standard Model, and again in Section 6.3, where calculations of the anomalies of groups of theoretical particle states added by hand to the Standard Model will be made.

$(U(1))^3$

$$\mathcal{A}_{abc}^{(U(1))^3} \propto \mathrm{Tr}[Y^3]\,. \tag{136}$$

$U(1) \times \mathrm{grav}^2$

$$\mathcal{A}_{abc}^{U(1)\times\mathrm{grav}^2} \propto \mathrm{Tr}[Y]\,. \tag{137}$$

$U(1) \times (SU(2))^2$

$$\mathcal{A}_{abc}^{U(1)\times(SU(2))^2} = \mathrm{Tr}[Y\tau^a\tau^b] \tag{138}$$

$$= \frac{1}{2}\delta_{ab}\,\mathrm{Tr}[Y_L] \tag{139}$$

$$\Rightarrow \mathcal{A}_{abc}^{U(1)\times(SU(2))^2} \propto \sum_L Y_L\,, \tag{140}$$

where $\sum_L Y_L$ is the sum of the hypercharges of all left-handed fermions.

$U(1) \times (SU(3))^2$

$$\mathcal{A}_{abc}^{U(1)\times(SU(3))^2} = \mathrm{Tr}[YT^aT^b] \tag{141}$$

$$= \frac{1}{2}\delta_{ab}\,\mathrm{Tr}[Y_q] \tag{142}$$

$$\Rightarrow \mathcal{A}_{abc}^{U(1)\times(SU(3))^2} \propto \sum_q Y_q\,, \tag{143}$$

where $\sum_q Y_q$ is the sum of the hypercharges of all quarks, with a relative minus sign between left- and right-handed quarks.

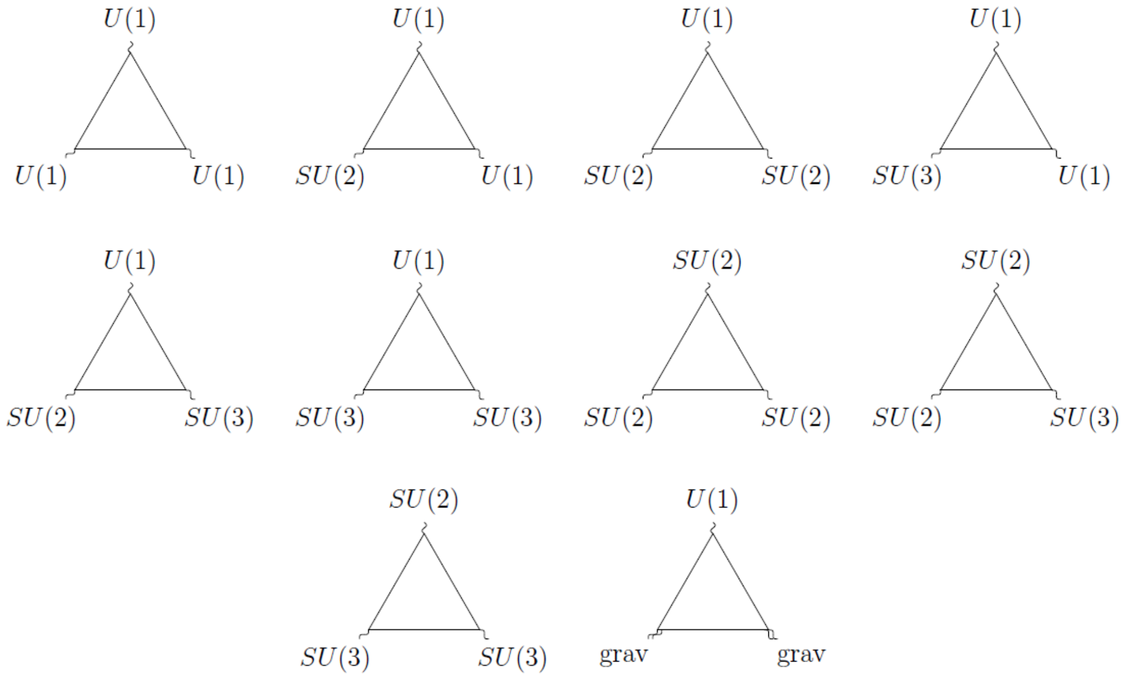

Figure 2: Full set of Standard Model diagrams with potentially anomalous terms (taken from [3]).

3 Spontaneous Symmetry Breaking

As discussed previously in Section 2, all gauge bosons are required to be massless within an unbroken gauge theory. This poses a dilemma, as the only massless gauge bosons observed are photons; gluons are also massless, but they are manifest within hadrons and do not appear as free particles, due to the premise of confinement. The weak gauge bosons, however, are massive, as weak interactions are very short-ranged, and so one must find a way to break the symmetry of the gauge theory to allow massive weak gauge bosons.

This could be done simply by adding a mass term for the weak bosons by hand, but this would directly violate the gauge symmetry, rendering the theory unrenormalisable. Another possibility is to invoke a gauge symmetry that is spontaneously broken as a result of the lack of respect the vacuum state has for this symmetry.

In this case, the symmetry of the Lagrangian under local gauge transformations is still conserved, while the vacuum state is not a singlet of the gauge symmetry. Of the infinite number of possible states with the same ground-state energy, one is chosen by Nature to represent the "true vacuum"; this choice arises in analogy with a perfectly symmetric situation being disturbed in a perfectly symmetric fashion, but with the result that the symmetry of the situation is broken. A classical example of such a situation would occur if an object, symmetric with respect to rotations about its axis, is initially at rest, and then has a force of a sufficient magnitude applied directly along its axis of symmetry, the object will subsequently bend or break in a seemingly random direction, breaking the rotational symmetry of the situation [7].

3.1 Classical and abelian examples

Consider a simple case in classical field theory, for a scalar field ϕ, which has the Lagrangian [8]

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2}\mu^2\phi^2 - \frac{\lambda}{4!}\phi^4 \,, \tag{144}$$

which is symmetric in $\phi \to -\phi$, and where we have replaced the usual mass term m^2 with a negative one, $-\mu^2$.

The minimum-energy configuration is the field $\phi(x) = \phi_0$, which is chosen to minimise the potential

$$V(\phi) = -\frac{1}{2}\mu^2\phi^2 + \frac{\lambda}{4!}\phi^4 \,, \tag{145}$$

$$\frac{\partial V(\phi)}{\partial \phi} \overset{\text{set}}{=} 0 = -\mu^2\phi_0 + \frac{4\lambda}{24}\phi_0^3 \,, \tag{146}$$

$$\Rightarrow \frac{\lambda}{6}\phi_0^2 = \mu^2 \tag{147}$$

$$\Rightarrow \phi_0 = \pm\sqrt{\frac{6}{\lambda}}\mu = \pm v \,, \tag{148}$$

where v is the vacuum expectation value (vev) of ϕ. Suppose the system is in the vicinity of the positive minimum, such that

$$\phi(x) = v + \sigma(x) \,, \tag{149}$$

where $\sigma(x)$ is an infinitesimal spacetime-dependent perturbation. If one now rewrite $\mathcal{L}$ in terms of $\sigma(x)$:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\sigma)^2 - \frac{1}{2}(2\mu^2)\sigma^2 - \sqrt{\frac{\lambda}{6}}\mu\sigma^3 - \frac{\lambda}{4!}\sigma^4 \,, \tag{150}$$

which is no longer invariant under the symmetric transformation $\phi \to -\phi$. This is a quintessential example of spontaneous symmetry breaking (SSB). Here, the Lagrangian density $\mathcal{L}$ describes a scalar field of mass $\mu\sqrt{2}$, which has interactions which are cubic and quartic in $\sigma(x)$.

Consider now an abelian case, where a complex scalar field is coupled to both itself and to an electromagnetic field, such that the Lagrangian $\mathcal{L}$ is

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu})^2 + |D_\mu\phi|^2 - V(\phi) \,, \tag{151}$$

where D_μ is the covariant derivative, defined in Section 2.1. $\mathcal{L}$ is invariant under the local $U(1)$ transformation

$$\phi(x) \to e^{i\omega(x)}\phi(x) \,, \tag{152}$$

$$\phi(x)^* \to \phi(x)^* e^{-i\omega(x)} \,, \tag{153}$$

$$A_\mu(x) \to A_\mu(x) - \frac{1}{e}\partial_\mu\omega(x) \,, \tag{154}$$

as outlined in Section 2.1, but given again here for clarity and convenience.

The potential $V(\phi)$ is chosen to be of the form

$$V(\phi) = -\mu^2\phi^*\phi + \frac{\lambda}{2}(\phi^*\phi)^2 \,, \tag{155}$$

such that for $\mu^2 > 0$, the field ϕ acquires a vev, and the $U(1)$ symmetry is spontaneously broken.

Under the $U(1)$ transformation,

$$V(\phi) = -\mu^2 \underbrace{e^{-i\omega(x)}e^{i\omega(x)}}_{=1}\phi^*\phi + \frac{\lambda}{2}\underbrace{(e^{-i\omega(x)}e^{i\omega(x)})^2}_{=1}(\phi^*\phi)^2 \quad . \tag{156}$$

Finding the minimum of the potential [9]:

$$\frac{\partial V(\phi)}{\partial \phi} \stackrel{\text{set}}{=} 0 = -\mu^2\phi_0^* + \frac{2\lambda}{2}\phi_0^{*2}\phi_0 \tag{157}$$

$$\Rightarrow \mu^2 = \lambda\phi_0^*\phi_0 = \lambda|\phi_0|^2 \tag{158}$$

$$\Rightarrow \phi_0 = \frac{\mu}{\sqrt{\lambda}}\,. \tag{159}$$

If we expand about the vacuum state, expressing the complex field $\phi(x)$ as

$$\phi(x) = \phi_0 + \frac{1}{\sqrt{2}}(\phi_1(x) + i\phi_2(x))\,, \tag{160}$$

then if we choose the unitarity gauge, whereby $\omega(x)$ is chosen such that $\phi(x)$ is real-valued for all x (i.e. $\phi_2 = 0$), then the necessary-but-unphysical Goldstone boson associated with ϕ_2 does not appear as an independent particle, and the potential becomes

$$V(\phi) = -\mu^2\Big(\phi_0 + \frac{1}{\sqrt{2}}\phi_1(x)\Big)^2 + \frac{\lambda}{2}\Big(\phi_0 + \frac{1}{\sqrt{2}}\phi_1(x)\Big)^4 \tag{161}$$

$$\Rightarrow V(\phi) = -\mu^2\Big(\phi_0^2 + \frac{1}{2}\phi_1^2(x) + \sqrt{2}\phi_0\phi_1(x)\Big) \tag{162a}$$

$$+\frac{\lambda}{2}\Big(\phi_0^4 + \frac{1}{4}\phi_1^4(x) + 2\phi_1^2(x)\phi_0^2 + \frac{1}{2}\phi_1^2(x)\phi_0^2 + \sqrt{2}\phi_0^3\phi_1(x) \tag{162b}$$

$$+\frac{1}{2}\phi_1^2(x)\phi_0^2 + \frac{\sqrt{2}}{2}\phi_1^3(x)\phi_0 + \sqrt{2}\phi_0^3\phi_1(x) + \frac{\sqrt{2}}{2}\phi_1^3(x)\phi_0\Big)\,. \tag{162c}$$

Neglecting terms linear in $\phi_1(x)$, which do not describe interactions between two or more fields, and collecting together all terms cubic and above in $\phi_1(x)$, we have

$$V(\phi) = -\mu^2\Big(\phi_0^2 + \frac{1}{2}\phi_1^2(x)\Big) + \frac{\lambda}{2}\Big(\phi_0^4 + 2\phi_1^2(x)\phi_0^2 + \frac{1}{2}\phi_1^2(x)\phi_0^2 + \frac{1}{2}\phi_1^2(x)\phi_0^2 + \mathcal{O}(\phi_1^3(x))\Big)\,. \tag{163}$$

Substituting Equation (159) yields

$$V(\phi) = -\mu^2\Big(\frac{\mu^2}{\lambda} + \frac{\phi_1^2}{2}\Big) + \frac{\lambda}{2}\Big(\frac{\mu^4}{\lambda^2} + 3\phi_1^2\frac{\mu^2}{\lambda}\Big) + \ldots \tag{164}$$

$$\Rightarrow V(\phi) = \mu^2\phi_1^2 - \frac{\mu^4}{2\lambda} + \ldots\,. \tag{165}$$

Therefore, the field ϕ_1 acquires a mass $m_1 = \mu\sqrt{2}$.

Transforming the kinetic energy term, omitting field terms beyond second order in A_μ, ϕ_1 and ϕ_2, yields

$$|D_\mu\phi|^2 = \frac{1}{2}(\partial_\mu\phi_1)^2 + \frac{1}{2}(\partial_\mu\phi_2)^2 + e\sqrt{2}\phi_0 A_\mu\partial^\mu\phi_2 + e^2\phi_0^2 A_\mu A^\mu + \ldots\,, \tag{166}$$

where $e^2\phi_0^2 A_\mu A^\mu$ is the photon mass term, $\Delta\mathcal{L} = \frac{1}{2}m_A^2 A_\mu A^\mu$, leading to $m_A^2 = 2e^2\phi_0^2$. In four dimensions, a gauge boson cannot acquire a mass, unless this mass term is associated with a pole in the vacuum polarisation amplitude, which can be created only by a massless scalar particle. The Goldstone boson supplies the pole needed, and though it therefore plays an important role in the theory, it does not appear as an independent physical particle.

If we again choose the unitarity gauge, ϕ_2 drops out of the theory, and the Lagrangian becomes

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu})^2 + (\partial_\mu \phi)^2 + e^2\phi^2 A_\mu A^\mu - V(\phi)\,. \tag{167}$$

If $V(\phi)$ is such that the vev $\langle\phi\rangle \neq 0$, the gauge field acquires a mass, and also retains a coupling to the remaining physical field ϕ_1. This mechanism, whereby SSB generates mass for a gauge boson, is known as the Higgs mechanism.

The Higgs mechanism can also be extended to non-abelian cases; however, though the examples in the following section will be non-abelian, the discussion will instead focus on the ways in which the generators of $SU(2)$ and $SU(3)$ gauge theories can be broken.

3.2 Extension to non-abelian gauge theory

Consider now a model whereby an $SU(2)$ gauge field is coupled to a doublet of scalar fields, Φ_i, where each Φ_i transforms as an $SU(2)$ spinor. The covariant derivative of Φ is

$$\mathbf{D}_\mu \Phi = (\partial_\mu + ig\mathbf{A}_\mu)\Phi \,, \tag{168}$$

where $\mathbf{A}_\mu = \tau^a A_\mu^a$. Here, $\tau^a = \frac{\sigma^a}{2}$ are the $SU(2)$ generators.

If Φ acquires a vev, v, then using the rotational freedom of $SU(2)$ transformations, which is such that the vev can be chosen to be in the $\tau^3 = -\frac{1}{2}$ direction, one can write

$$\langle \Phi \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} \,. \tag{169}$$

The kinetic energy term for Φ is then equal to

$$|D_\mu \Phi|^2 = (ig\tau^a A_\mu^a)(ig\tau^b A^{\mu b})\Phi^\dagger \Phi \tag{170}$$

$$\Rightarrow |D_\mu \Phi|^2 = \frac{1}{2} g^2 \begin{pmatrix} 0 & v \end{pmatrix} \tau^a \tau^b \begin{pmatrix} 0 \\ v \end{pmatrix} A_\mu^a A^{\mu b} + ... \tag{171}$$

To find the gauge boson mass term $\Delta\mathcal{L}$, we symmetrise the matrix product via interchanging indices a and b through the utilisation of Equation (133), the anticommutation relation for the Pauli matrices. This yields

$$\Delta\mathcal{L} = \frac{g^2 v^2}{8} A_\mu^a A^{\mu a} \,, \tag{172}$$

and all three $SU(2)$ gauge bosons therefore obtain a mass of

$$m_{SU(2)} = \frac{gv}{2} \,, \tag{173}$$

which means that all three $SU(2)$ generators are broken equally by the non-vanishing vev in Equation (169).

In terms of the number of degrees of freedom, we started with a two-component complex scalar, each of which has a real and imaginary component, leading to four total degrees of freedom. Three generators are then broken, corresponding to the three Goldstone bosons, which are each “eaten” by one of the three gauge bosons, through which each gauge boson acquires a mass. Thereafter, one degree of freedom remains, corresponding to the massive Higgs scalar field.

Expanding Φ^i about its vev yields

$$\Phi^i = \frac{1}{\sqrt{2}} \begin{pmatrix} G_2 - iG_3 \\ v + H + iG_1 \end{pmatrix} \,, \tag{174}$$

where the three G fields are associated with the Goldstone bosons, and each acquire a mass of zero, while H is the physical Higgs scalar, which acquires a mass term of $m_H = \mu\sqrt{2}$, despite all four fields having zero vev.

If we instead take Φ to be a real-valued vector, which transforms in the adjoint of $SU(2)$, and whose covariant derivative is

$$(D_\mu \Phi)_a = \partial_\mu \Phi_a + g f_{abc} A^b_\mu \phi_c \,, \tag{175}$$

then upon its acquisition of a vev, ϕ_0, the gauge boson mass term is

$$\Delta\mathcal{L} = \frac{1}{2}(D_\mu \Phi)^2 = \frac{g^2}{2}(f_{abc} A^b_\mu (\phi_0)_c)^2 + \dots \,. \tag{176}$$

We may again use the rotational freedom of $SU(2)$ to choose a vector $\vec{V}_{SU(2)}$ of $SU(2)$ to be in the z-direction (where the x, y, and z directions will correspond to the indices 1, 2, and 3, respectively), such that the vev of Φ_c then becomes

$$\langle \Phi_c \rangle = (\phi_0)_c = |\vec{V}_{SU(2)}| \delta_{c3} \,, \tag{177}$$

which can be substituted into Equation (176) to give

$$\Delta\mathcal{L} = \frac{g^2}{2} |\vec{V}_{SU(2)}|^2 (f_{ab3} A^b_\mu)^2 \tag{178}$$

$$= \frac{g^2}{2} |\vec{V}_{SU(2)}|^2 ((A^1_\mu)^2 + (A^2_\mu)^2) \,, \tag{179}$$

leading to mass terms m_1 and m_2, for the gauge bosons corresponding to the generators τ^1 and τ^2, respectively, of

$$m_1 = m_2 = g |\vec{V}_{SU(2)}| \tag{180}$$

while the gauge boson for τ^3 remains massless, as the vev of Φ_c destroys the rotational symmetry of the system about the x- and y-directions, but preserves it in the z-direction.

It is tempting to interpret the massive gauge bosons as weak $W^\pm$ bosons, and the massless boson as the photon, in an apparent theory of electroweak unification. However, this is not the real physical model we observe within Nature; the actual model will be discussed in Section 3.3.

Let us consider a further example, in which the gauge symmetry of an $SU(3)$ theory, with a scalar field ϕ in the adjoint representation, is broken spontaneously. In this case, the covariant derivative of ϕ and the mass term $\Delta\mathcal{L}$ have the forms outlined in Equations (175) and (176), respectively. Using the previous boldface convention, whereby $\boldsymbol{\phi} = \mathbf{T}^c \phi^c$, the mass term can be rewritten as

$$\Delta\mathcal{L} = -g^2 \,\mathrm{Tr}([T^a, \boldsymbol{\phi}][T^b, \boldsymbol{\phi}]) A^a_\mu A^{\mu b} \,. \tag{181}$$

If $\boldsymbol{\phi}$ then acquires a vev of $\boldsymbol{\phi}_0$, which is a traceless, Hermitian matrix, with three arbitrary eigenvalues that sum to zero, then $SU(3)$ can be broken to $SU(2) \times U(1)$ if

$$\boldsymbol{\phi}_0 = |\boldsymbol{\phi}| \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \,, \tag{182}$$

which commutes with four of the eight $SU(3)$ generators, each of which thus remain massless, while the other four generators acquire masses of

$$m^2 = (3g|\boldsymbol{\phi}|)^2\,. \tag{183}$$

$SU(3)$ can also be broken to $U(1) \times U(1)$, if

$$\boldsymbol{\phi}_0 = |\boldsymbol{\phi}| \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\,, \tag{184}$$

in which case only two of the eight $SU(3)$ generators commute with $\boldsymbol{\phi}_0$ and remain massless, while the other six generators each obtain a mass $m^2 \propto (g|\boldsymbol{\phi}|)^2$.

This type of breaking is also applicable to larger gauge groups, such as $SU(5)$ and $SO(10)$, which are (or have previously been) candidates for Grand Unified Theories (GUTs), as they offer an abundance of possible symmetry-breaking patterns, giving a wide variety of options when attempting to construct physical theories.

3.3 Breaking of $SU(2)_L \times U(1)_Y \to U(1)_{\mathbf{EM}}$

Similarly to the first example of Section 3.2, we will consider a doublet of scalar fields, Φ, in the spinor representation of $SU(2)$, i.e. a Higgs doublet, which has the form

$$\Phi = \begin{pmatrix} H^+ \\ H^0 \end{pmatrix} , \tag{185}$$

but also this time introduce an additional $U(1)$ gauge symmetry with quantum number Y, corresponding to the hypercharge of weak isospin. The complete $SU(2) \times U(1)$ gauge transformation is then

$$\Phi^a \to e^{i\omega^a \tau^a} e^{i\theta/2} \Phi^a . \tag{186}$$

If Φ acquires a vev of

$$\langle \Phi \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} , \tag{187}$$

and if

$$\omega^1 = \omega^2 = 0 \tag{188}$$

$$\text{and } \omega^3 = \theta , \tag{189}$$

then $\langle \Phi \rangle$ is invariant under the transformation, and the linear combination of generators $\tau^3 + Y$ is left unbroken, annihilating the vacuum state as follows:

$$(\tau^3 + Y)\langle \Phi \rangle = \left[\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} + \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \right] \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix} \tag{190}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix} \tag{191}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} . \tag{192}$$

$\tau^3 + Y$ therefore corresponds to a massless gauge boson, which we identify as the photon. The broken $SU(2)$ generators, τ^1 and τ^2, as well as the linear combination $\tau^3 + Y$, each subsequently obtain a mass via the Higgs mechanism, and are thereby interpreted as the $W^\pm$ and Z weak bosons. From the previous section, we also know the Higgs scalar field to be massive.

This spontaneously broken gauge theory, whereby the unification of the weak and electromagnetic interactions occurs through $SU(2)_L \times U(1)_Y \to U(1)_{\text{EM}}$, is known as the Glashow-Weinberg-Salam theory of weak interactions, and gives the experimentally correct description of weak interactions.

Using similar procedures to those found at the beginning of this section, it is quite straightforward to obtain the mass spectrum of the $SU(2)_L$ and $U(1)_Y$ gauge bosons.

The covariant derivative of Φ is

$$D_\mu \Phi = \Big(\partial_\mu - igA^a_\mu \tau^a - i\frac{1}{2}g' B_\mu\Big)\Phi\,, \tag{193}$$

where A^a_μ and B_μ are the $SU(2)_L$ and $U(1)_Y$ gauge boson fields respectively, and g and g' are the two distinct coupling constants, which appear in the covariant derivative term associated with the field for their particular gauge group.

The mass term $\Delta\mathcal{L}$ is then found by mod-squaring Equation (193), evaluated at the vev from Equation (187), such that

$$\begin{aligned}\Delta\mathcal{L} &= \frac{1}{2}\begin{pmatrix}0 & v\end{pmatrix}\Big(gA^a_\mu\tau^a + \frac{1}{2}g'B_\mu\Big)\Big(gA^{b\mu}\tau^b + \frac{1}{2}g'B_\mu\Big)\begin{pmatrix}0\\ v\end{pmatrix} &(194)\\ &= \frac{1}{2}\frac{v^2}{4}\Big[g^2(A^1_\mu)^2 + g^2(A^2_\mu)^2 + (-gA^3_\mu + g'B_\mu)^2\Big]\,. &(195)\end{aligned}$$

The three massive vector boson fields are then written in the form

$$\begin{aligned}W^\pm_\mu &= \frac{1}{\sqrt{2}}\Big(A^1_\mu \mp iA^2_\mu\Big)\,, &(196)\\ Z^0_\mu &= \frac{1}{\sqrt{g^2+g'^2}}\Big(gA^3_\mu - g'B_\mu\Big)\,, &(197)\end{aligned}$$

and have respective masses of [10]

$$\begin{aligned}m_W &= \frac{gv}{2}\,, &(198)\\ m_Z &= \frac{v\sqrt{g^2+g'^2}}{2}\,, &(199)\end{aligned}$$

where m_W and m_Z correspond to the $W^\pm$ and Z boson masses. The fourth vector boson field, C_μ, corresponding to the photon, remains massless, and is given by

$$C_\mu = \frac{1}{\sqrt{g^2+g'^2}}\Big(g'A^3_\mu + gB_\mu\Big)\,, \tag{200}$$

which is orthogonal to Z^0_μ.

In the following section on the Standard Model, we will briefly return to the ideas outlined in Section 3.3, in order to see explicitly how the $W^\pm$ and Z boson masses are directly related to one another.

4 The Standard Model

The Standard Model of particle physics is quite remarkable in its ability to describe almost all phenomena currently encountered. It is, in essence, the quintessential 'theory of almost everything'. In spite of its successes, it falls short of explaining in full the fundamental force of gravitation, when general relativity is considered. It also fails to provide any viable dark matter candidate, which would be required to explain a number of otherwise ambiguous cosmological observations, including discrepancies between measured and calculated galaxy rotation curves and velocity dispersions. In addition, the Standard Model neither accounts for the fact that the expansion of the Universe is accelerating, nor incorporates neutrino oscillations. Therefore, it seems fair to conclude that the Standard Model is incomplete [1].

However, in terms of experimental verification, the SM is the closest theory we have to a complete picture of the fundamental constituents of the Universe. It describes both the strong force and electroweak force (as unified in Section 3.3) under a single formalism. The SM will be used as a starting point in Section 6 when attempting to unify the fundamental forces at high energies through the addition of theoretical particle states, such that the running of the couplings with energy is altered; this phenomenon of "running" will be detailed in Section 4.3.

Supersymmetric models offer a solution to most of the main flaws of the SM [12]. Supersymmetry, or SUSY, is a theory whereby there exists a symmetry between fermions and bosons, such that each SM particle has a "superpartner" whose spin differs from its own by $\frac{1}{2}$. This symmetry is, however, broken so that the superpartners are more massive than their SM counterparts, which is necessary due to the lack of observation of the former. The key aspects of the formulation of SUSY theory will be addressed in Section 4.2.

SUSY is also a step towards Grand Unification, as introducing the superpartners at an energy scale of $\mathcal{O}(1\,\text{TeV})$ drastically changes the gradients of the running couplings, such that they almost unify at an energy scale of $\mathcal{O}(10^{16}\,\text{GeV})$. This will be discussed further in Section 4.3, alongside the equivalent SM case.

4.1 SM particle content and Lagrangian

As touched upon in Section 2, the Standard Model describes the fundamental interactions within Nature by relating them to the gauge group $\mathcal{G}_{\text{SM}} = SU(3)_C \times SU(2)_L \times U(1)_Y$, where the first factor is associated with strong interactions, and the other two factors account for electroweak interactions. Due to parity violation in the weak sector, components of fermions with left-handed chirality couple to both the $SU(2)_L$ and $U(1)_Y$ gauge bosons, whereas those with right-handed chirality couple only to the $U(1)_Y$ bosons.

To this end, one can embed left- and right-handed fermions into different representations of electroweak symmetry, whereby left-handed fermion fields are located in the doublet representation of $SU(2)_L$, and right-handed fields are located in the singlet representation.

The fermionic content of the Standard Model can therefore be split into the following representations of $SU(3)_C \times SU(2)_L \times U(1)_Y$:

$$\{\mathbf{3}, \mathbf{2}, \tfrac{1}{6}\} \leftrightarrow \begin{pmatrix} u^r & u^g & u^b \\ d^r & d^g & d^b \end{pmatrix}_i \equiv Q_i \,, \tag{201}$$

$$\{\mathbf{1}, \mathbf{2}, -\tfrac{1}{2}\} \leftrightarrow \begin{pmatrix} \nu_l \\ l \end{pmatrix}_i \equiv L_i \,, \tag{202}$$

$$\{\overline{\mathbf{3}}, \mathbf{1}, -\tfrac{2}{3}\} \leftrightarrow \begin{pmatrix} \overline{u}^r & \overline{u}^g & \overline{u}^b \end{pmatrix}_i \equiv \overline{u}_i \,, \tag{203}$$

$$\{\overline{\mathbf{3}}, \mathbf{1}, \tfrac{1}{3}\} \leftrightarrow \begin{pmatrix} \overline{d}^r & \overline{d}^g & \overline{d}^b \end{pmatrix}_i \equiv \overline{d}_i \,, \tag{204}$$

$$\{\mathbf{1}, \mathbf{1}, 1\} \leftrightarrow \begin{pmatrix} \bar{l}_i \end{pmatrix} \,, \tag{205}$$

where the subscript indices i run from 1 to 3, and denote the generation of the quark or lepton, the superscript indices r, g, and b are the colour charges of the quarks, and bars above particles represent their component of conjugate chirality; for example, if u is a left-handed up-type quark, then $\overline{u}$ is the corresponding right-handed up-type antiquark. The former state transforms under $SU(2)$ transformations, whereas the latter state does not.

In Equations (201-205), the notation used for representations of $\mathcal{G}_{\mathrm{SM}}$ is such that in the SM representation $\{\mathbf{a}, \mathbf{b}, c\}$, $\mathbf{a}$ is the $SU(3)_C$ representation, $\mathbf{b}$ is the $SU(2)_L$ representation, and c is the weak hypercharge, Y, of the SM representation, chosen such that the electric charge $Q = Y + T^3$, where T^3 is the third component of the $SU(2)_L$ generator, as previously discussed in Section 3.3.

The boldface numerals are representations of a particular gauge group. For example, $\mathbf{3}$, $\overline{\mathbf{3}}$, and $\mathbf{1}$ are the fundamental triplet, conjugate triplet, and singlet representations of $SU(3)_C$, respectively, whilst $\mathbf{2}$ and $\mathbf{1}$ are the fundamental doublet and singlet representations of $SU(2)_L$.

The gauge boson content of the SM is made up of the $SU(3)$ octet, $\{\mathbf{8}, \mathbf{1}, 0\}$, the $SU(2)$ triplet, $\{\mathbf{1}, \mathbf{3}, 0\}$, and the singlet state $\{\mathbf{1}, \mathbf{1}, 0\}$, which encompass gluons, weak bosons, and photons, respectively. The SM Higgs boson is embedded into the $\{\mathbf{1}, \mathbf{2}, \frac{1}{2}\}$ $SU(2)$ doublet representation.

The mass spectrum of the Standard Model can be found via the identification of quadratic field terms (of canonical form $m^2\phi^2$, $\overline{\psi} m \psi$, and $m^2 A_\mu A^\mu$, for scalar boson, fermionic, and gauge boson fields, respectively) within the SM Lagrangian, given below [11]:

$$\mathcal{L}_{\rm SM} = -\frac{1}{2}\,\mathrm{Tr}(\mathbf{G}_{\mu\nu}\mathbf{G}^{\mu\nu}) - \frac{1}{8}\,\mathrm{Tr}(\mathbf{A}_{\mu\nu}\mathbf{A}^{\mu\nu}) - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} \tag{206a}$$

$$+ \begin{pmatrix}\overline{\nu}_L & \overline{e}_L\end{pmatrix}\tilde{\sigma}^\mu iD_\mu \begin{pmatrix}\overline{\nu}_L \\ \overline{e}_L\end{pmatrix} + \overline{e}_R\sigma^\mu iD_\mu e_R + \overline{\nu}_R\sigma^\mu iD_\mu \nu_R + (\text{h.c.}) \tag{206b}$$

$$- \frac{\sqrt{2}}{v}\left[\begin{pmatrix}\overline{\nu}_L & \overline{e}_L\end{pmatrix}\phi M^e e_R + \overline{e}_R\overline{M}^e\overline{\phi}\begin{pmatrix}\nu_L \\ e_L\end{pmatrix}\right] \tag{206c}$$

$$+ \begin{pmatrix}\overline{u}_L & \overline{d}_L\end{pmatrix}\tilde{\sigma}^\mu iD_\mu \begin{pmatrix}\overline{u}_L \\ \overline{d}_L\end{pmatrix} + \overline{u}_R\sigma^\mu iD_\mu u_R + \overline{d}_R\sigma^\mu iD_\mu d_R + (\text{h.c.}) \tag{206d}$$

$$- \frac{\sqrt{2}}{v}\left[\begin{pmatrix}\overline{u}_L & \overline{d}_L\end{pmatrix}\phi M^d d_R + \overline{d}_R\overline{M}^d\overline{\phi}\begin{pmatrix}u_L \\ d_L\end{pmatrix}\right] \tag{206e}$$

$$- \frac{\sqrt{2}}{v}\left[\begin{pmatrix}-\overline{d}_L & \overline{u}_L\end{pmatrix}\phi^* M^u u_R + \overline{u}_R\overline{M}^u\phi^T\begin{pmatrix}-d_L \\ u_L\end{pmatrix}\right] \tag{206f}$$

$$+ \overline{(D_\mu\phi)}D^\mu\phi \tag{206g}$$

$$- \frac{m_H^2}{2v^2}\left[\phi^\dagger\phi - \frac{v^2}{2}\right]^2, \tag{206h}$$

where h.c. stands for the Hermitian conjugate of the preceding terms, and where the derivative operators are:

$$D_\mu\begin{pmatrix}\nu_L \\ e_L\end{pmatrix} = \left[\partial_\mu - \frac{ig'}{2}B_\mu + \frac{ig}{2}\mathbf{A}_\mu\right]\begin{pmatrix}\nu_L \\ e_L\end{pmatrix}, \tag{207}$$

$$D_\mu\begin{pmatrix}u_L \\ d_L\end{pmatrix} = \left[\partial_\mu + \frac{ig'}{6}B_\mu + \frac{ig}{2}\mathbf{A}_\mu + ig_s\mathbf{G}_\mu\right]\begin{pmatrix}u_L \\ d_L\end{pmatrix}, \tag{208}$$

$$D_\mu e_R = [\partial_\mu - ig'B_\mu]e_R\,, \tag{209}$$

$$D_\mu u_R = \left[\partial_\mu + \frac{2ig'}{3}B_\mu + ig_s\mathbf{G}_\mu\right]u_R\,, \tag{210}$$

$$D_\mu d_R = \left[\partial_\mu - \frac{ig'}{3}B_\mu + ig_s\mathbf{G}_\mu\right]d_R\,, \tag{211}$$

$$D_\mu\phi = \left[\partial_\mu + \frac{ig'}{2}B_\mu + \frac{ig}{2}\mathbf{A}_\mu\right]\phi\,. \tag{212}$$

Here, ϕ is the two-component complex Higgs fields, $\mathbf{G}_\mu$, $\mathbf{A}_\mu$ and B_μ are the $SU(3)_C$, $SU(2)_L$, and $U(1)_Y$ vector potentials, respectively, M^e, M^u, and M^d are the fermion mass matrices, g_s is the $SU(3)_C$ coupling constant, and $\tilde{\sigma}^\mu$ is a permutation of the vector of Pauli matrices, such that

$$\tilde{\sigma}^\mu = (\mathbb{1}, -\vec{\sigma})\,, \tag{213}$$

$$\vec{\sigma} = (\sigma^1, \sigma^2, \sigma^3)\,. \tag{214}$$

Let us now examine the individual contributions to $\mathcal{L}_{\mathrm{SM}}$, via the inspection of each of its components. The parts of the Lagrangian which are bilinear in field terms describe the interactions between the two fields in that particular piece, and as mentioned previously, parts of the Lagrangian quadratic in field terms give the masses acquired by the fields. The component (206a) contains $SU(3)_C$, $SU(2)_L$, and $U(1)_Y$ gauge terms, while Equation (206b) and (206c) are the lepton dynamical and mass terms, respectively. Quark interactions are characterised by Equation (206d), and the mass terms for the up- and down- type quarks are located in Equations (206e) and (206f), respectively. The terms concerning the Higgs' interactions and mass, respectively, are contained in Equations (206g) and (206h).

While the fermion and Higgs masses are free parameters in their own right, the $W^\pm$ and Z boson masses are explicitly related to one another, though by only one other parameter, the weak mixing angle, θ_w, which is the angle that appears in the change of basis from (A^3_μ, B_μ) to (Z^0_μ, C_μ), which proceeds through the following rotation [8]:

$$\begin{pmatrix} Z^0_\mu \\ C_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_w & -\sin\theta_w \\ \sin\theta_w & \cos\theta_w \end{pmatrix} \begin{pmatrix} A^3_\mu \\ B_\mu \end{pmatrix} . \tag{215}$$

Using Equations (197) and (200), one finds that

$$\cos\theta_w = \frac{g}{\sqrt{g^2 + g'^2}} , \tag{216}$$

$$\sin\theta_w = \frac{g'}{\sqrt{g^2 + g'^2}} , \tag{217}$$

and by noting the form of Equations (198) and (199), the ratio m_W/m_Z is

$$\frac{m_W}{m_Z} = \frac{g}{\sqrt{g^2 + g'^2}} = \cos\theta_w \tag{218}$$

$$\Rightarrow m_W = m_Z \cos\theta_w , \tag{219}$$

the $W^\pm$ and Z masses thereby being related by just one parameter.

If one rewrites the covariant derivative of a fermion field charged under $SU(2)_L$ and $U(1)_Y$,

$$D_\mu = \partial_\mu - igA^a_\mu \tau^a - ig'YB_\mu , \tag{220}$$

in terms of mass eigenstate fields, one can see that the couplings of all weak bosons are described by only two parameters:

$$D_\mu = \partial_\mu - i\frac{g}{\sqrt{2}}\Big(W^+_\mu(\tau^1 + i\tau^2) + W^-_\mu(\tau^1 - i\tau^2)\Big) \tag{221a}$$

$$- i\frac{1}{\sqrt{g^2 + g'^2}} Z^0_\mu\Big(g^2\tau^3 - g'^2 Y\Big) - i\frac{gg'}{\sqrt{g^2 + g'^2}} C_\mu(\tau^3 + Y) . \tag{221b}$$

Identifying the coefficient of electromagnetic interaction (in the final term of Equation (221b) as the electric charge, e, yields

$$e = \frac{gg'}{\sqrt{g^2 + g'^2}} , \tag{222}$$

and using Equation (217), we see that

$$g = \frac{e}{\sin\theta_w}\,, \tag{223}$$

and so, at tree level, all weak processes whereby $W^{\pm}/Z$ are exchanged are dependent only on two parameters, e and θ_w.

4.2 Adding minimal supersymmetry

One of the primary motivations for supersymmetry stems from the fact that the measured mass of the SM Higgs boson is relatively small, at $m_H^2 \sim (100\,\mathrm{GeV})^2$, in comparison with the large quantum corrections to the Higgs mass, which arise from the virtual effects of every particle that couples, whether directly or indirectly, to the Higgs field [1].

For example, if there exists a Dirac fermion, f, which has mass m_f, and couples to the Higgs field with a contribution to the Lagrangian $\mathcal{L}$, of $-\lambda_f H f \bar{f}$, then the quantum loop correction to m_H^2 will grow quadratically with the physical momentum cutoff Λ (the energy scale above which the SM should be replaced with an alternative theory), as follows:

$$\Delta m_H^2 = -\frac{|\lambda_f|^2}{8\pi^2}\Lambda^2 + \dots\,, \tag{224}$$

where λ_f is the Yukawa coupling of the fermion, which is proportional to m_f. Here, the largest possible correction is over thirty orders of magnitude larger than the desired value of the scalar Higgs boson mass. While this appears only to be a problem for the Higgs boson mass itself, the fermions and gauge bosons within the SM all obtain masses via coupling to the Higgs field, $\langle H\rangle \propto m_H$, and are therefore all sensitive to the cutoff Λ, which can be on the order of the Planck scale, $M_P \sim 10^{19}\,\mathrm{GeV}$.

Additionally, contributions to the quantum corrections, of the form of Equation (224), may also arise from the existence of any arbitrarily massive scalar particle, S, giving a term within the Lagrangian of $-\lambda_S |H|^2 |S|^2$, and yielding a correction [12]:

$$\Delta m_H^2 = \frac{\lambda_S}{16\pi^2}(\Lambda^2 - 2{m_S}^2 \ln(\Lambda/m_S)) + \dots\,. \tag{225}$$

The dependence on m_S, which could be on the order of M_P, cannot be removed without introducing an unjustifiable counterterm, hence resulting in the persistence of a large correction to the Higgs mass. As a result, we see that m_H^2 is sensitive to the mass of the heaviest particle that the Higgs couples to.

Noticing the relative minus sign between fermionic and bosonic contributions, one could postulate a "symmetry" between fermions and bosons, known as a "supersymmetry", such that fermion and boson loop contributions cancel exactly.

If, for each quark and lepton in the SM, there exist two associated complex scalars, such that $\lambda_s = |\lambda_f|^2$, then the quantum loop corrections from a heavy fermion are cancelled by those from a two-loop arrangement of a heavy boson, as shown in Figure (3).

In the supersymmetric regime, we introduce an operator Q, capable of performing SUSY transformations between fermions and bosons, such that

$$\begin{aligned} Q\,|B\rangle &\propto |F\rangle\ , \\ Q\,|F\rangle &\propto |B\rangle\ , \end{aligned} \tag{226}$$

for fermionic and bosonic states, denoted by $|F\rangle$ and $|B\rangle$, respectively.

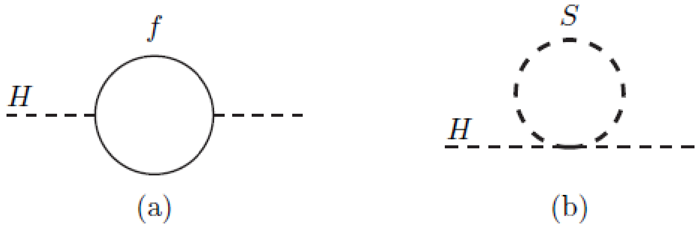

Figure 3: One-loop quantum corrections to m_H^2 due to (a) a Dirac fermion, and (b) a scalar (taken from [1]).

The single-particle states of a supersymmetric theory fall into irreducible supermultiplets. For two states, $|\Omega\rangle$ and $|\Omega'\rangle$, which are both within the same supermultiplet, one can be transformed into the other via a combination of the supersymmetric operators, Q and $Q^\dagger$, which respectively lower and raise the helicity of a state by $\frac{1}{2}$.

To construct the spin states of a supermultiplet, one must define the minimum spin state which can be annihilated by the annihilation operator, Q. Defining this state as $|\lambda_0\rangle$ gives the relation

$$Q\,|\lambda_0\rangle = 0\,, \tag{227}$$

and acting the creation operator, $Q^\dagger$, on $|\lambda_0\rangle$ as many times as is allowed by the anticommutation relations of $Q^\dagger$, as well as including any conjugate spin states (such that CPT invariance is maintained), yields the supermultiplet spin states.

Specific sets of each type of supermultiplet then manifest themselves into the following arrangement of particles, which make up the Minimal Supersymmetric Standard Model (MSSM), shown in Tables 1 and 2:

Table 1: Chiral supermultiplets within the MSSM (taken from [12])

Chiral Superfield		Spin 0	Spin 1/2	$(SU(3)_C, SU(2)_L, U(1)_Y)$
Quarks/Squarks with 3 families	Q	$(\tilde{u}_L\ \tilde{d}_L) \equiv \tilde{Q}_L$	$(u_L\ d_L)$	$(\mathbf{3},\mathbf{2},\frac{1}{6})$
	U	$\tilde{u}_R^*$	$\overline{u_R}$	$(\overline{\mathbf{3}},\mathbf{1},-\frac{2}{3})$
	D	$\tilde{d}_R^*$	$\overline{d_R}$	$(\overline{\mathbf{3}},\mathbf{1},\frac{1}{3})$
Leptons/Sleptons with 3 families	L	$(\tilde{\nu}_L\ \tilde{e}_L) \equiv \tilde{L}_L$	$(\nu_L\ e_L)$	$(\mathbf{1},\mathbf{2},-\frac{1}{2})$
	E	$\tilde{e}_R^*$	$\overline{e_R}$	$(\mathbf{1},\mathbf{1},1)$
Higgs/Higgsinos	H_u	$(H_u^+\ H_u^0)$	$(\tilde{H}_u^+\ \tilde{H}_u^0) \equiv \tilde{H}_u$	$(\mathbf{1},\mathbf{2},\frac{1}{2})$
	H_d	$(H_d^0\ H_d^-)$	$(\tilde{H}_d^0\ \tilde{H}_d^-) \equiv \tilde{H}_d$	$(\mathbf{1},\mathbf{2},-\frac{1}{2})$

Table 2: Gauge supermultiplets within the MSSM (taken from [12])

Gauge Superfield	Spin 1/2	Spin 1	$(SU(3)_C, SU(2)_L, U(1)_Y)$
Gluino/Gluon	$\tilde{g}$	g	$(\mathbf{8},\mathbf{1},\mathbf{0})$
Wino/W bosons	$\tilde{W}^\pm\ \tilde{W}^0$	$W^\pm\ W^0$	$(\mathbf{1},\mathbf{3},\mathbf{0})$
Bino/B Boson	$\tilde{B}^0$	B^0	$(\mathbf{1},\mathbf{1},\mathbf{0})$

Note that there are two Higgs chiral supermultiplets in the MSSM, resulting in five different Higgs particles, including a charged Higgs state, $H^{\pm}$, a neutral CP-odd Higgs state, A^0, and two neutral CP-even Higgs states, (h^0, H^0).

The addition of extra states in the transition from the SM to the MSSM will also have a profound effect on the way the couplings of the fundamental interactions run with energy, as will be seen in the following section.

4.3 Running couplings and anomaly cancellation

The coupling "constants" describing the strength of the fundamental interactions are not, in actuality, constant, but instead evolve logarithmically with energy scale. This is because if one probes a quantum field theory over a very short timescale, one sees off-shell (virtual) particles participating in every process, apparently violating conservation of energy. However, the uncertainty relation,

$$\Delta E \Delta t \geq \hbar \,, \tag{228}$$

allows such violations for very short times [8]. Processes involving virtual particles renormalise the coupling, making it dependent on the energy scale, Q, at which it is observed. This is known as the "running" of the coupling.

The one-loop equations, known as beta functions, for the running of the gauge couplings g_a are

$$\beta_{g_a} \equiv \frac{\mathrm{d}}{\mathrm{d}\ln(Q/Q_0)} g_a = \frac{1}{16\pi^2} b_a g_a^3 \,, \tag{229}$$

where a runs from 1 to 3, b_a is a coefficient of proportionality, and Q_0 is an input energy scale, placed at $\mathcal{O}(100\,\mathrm{GeV})$, such that the beta functions can be calculated exactly using boundary conditions from experimental data. The gauge couplings are $g_3 = g_s$, $g_2 = g = \frac{e}{\sin\theta_w}$, and $g_1 = g'\sqrt{\frac{5}{3}} = \frac{e}{\cos\theta_w}\sqrt{\frac{5}{3}}$, for $SU(3)_C$, $SU(2)_L$, and $U(1)_Y$, respectively.

Using the convention $\alpha_a = \frac{g_a^2}{4\pi}$, Equation (229) can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}\ln(Q/Q_0)}(\alpha_a^{-1}) = -\frac{b_a}{2\pi} \,, \tag{230}$$

meaning b_a is simply proportional to the gradient of the coupling α_a^{-1} with respect to $\ln Q$.

Upon calculating the b_a values for the MSSM, the couplings almost unify exactly at an energy scale $M_U \sim 1.5 \times 10^{16}\,\mathrm{GeV}$. Their unification is not perfect, and $\alpha_1^{-1}(M_U) = \alpha_2^{-1}(M_U)$ is often taken to be the unified coupling, α_u^{-1}. However, the discrepancy can be attributed to corrections due to any new particles which may come into existence once M_U is reached.

When computing the b_a values, gauge bosons charged under the specific gauge group relating to b_a contribute $\Delta b_a = -\frac{11}{3} n_r$, Weyl fermions contribute $\Delta b_a = \frac{2}{3} n_r$, and complex scalars contribute $\Delta b_a = \frac{1}{3} n_r$, where n_r is the index of a group for a particular representation. For the fundamental representation of a $SU(N)$, $n_r = \frac{1}{2}$, and for the adjoint representation, $n_r = N$. For Weyl fermions and scalars in $U(1)$, $n_r = \frac{3}{5} Y^2$.

The remainder of this section will be devoted to computing the non-trivial anomaly cancellations (see Section 2.3) and running couplings for the SM and MSSM, which will be labelled with appropriate subscripts in the calculations that follow. These calculations make use of the particle contents given in Sections 4.1 and 4.2. N_G will be used to stand for the number of generations of quarks/leptons, which needs to be factored into the calculations. The number of colours, N_C, is equal to three.

Anomaly cancellations

$$\mathcal{A}_{\text{SM}}^{(U(1))^3} \propto N_G\Big[\underbrace{-2(-\tfrac{1}{2})^3}_{L_i}+\underbrace{(-1)^3}_{l_i}+\underbrace{3}_{N_C}\Big[\underbrace{-2(\tfrac{1}{6})^3}_{Q_i}+\underbrace{(\tfrac{2}{3})^3}_{u_i}+\underbrace{(-\tfrac{1}{3})^3}_{d_i}\Big]\Big]=0 \tag{231}$$

$$\mathcal{A}_{\text{SM}}^{U(1)\times\text{grav}^2} \propto N_G\Big[\underbrace{-2(-\tfrac{1}{2})}_{L_i}+\underbrace{(-1)}_{l_i}+\underbrace{3}_{N_C}\Big[\underbrace{-2(\tfrac{1}{6})}_{Q_i}+\underbrace{(\tfrac{2}{3})}_{u_i}+\underbrace{(-\tfrac{1}{3})}_{d_i}\Big]\Big]=0 \tag{232}$$

$$\mathcal{A}_{\text{SM}}^{U(1)\times(SU(2))^2} \propto N_G\Big[\underbrace{-(-\tfrac{1}{2})}_{L_i}+\underbrace{3}_{N_C}\cdot\underbrace{-(\tfrac{1}{6})}_{Q_i}\Big]=0 \tag{233}$$

$$\mathcal{A}_{\text{SM}}^{U(1)\times(SU(3))^2} \propto N_G\Big[\underbrace{-2(\tfrac{1}{6})}_{Q_i}+\underbrace{(\tfrac{2}{3})}_{u_i}+\underbrace{(-\tfrac{1}{3})}_{d_i}\Big]=0 \tag{234}$$

$$\mathcal{A}_{\text{MSSM}}^{(U(1))^3} \propto N_G\Big[\underbrace{-2(-\tfrac{1}{2})^3}_{L_i}+\underbrace{(-1)^3}_{\overline{E}_i}\underbrace{-2(\tfrac{1}{2})^3}_{H_u}\underbrace{-2(-\tfrac{1}{2})^3}_{H_d}+\underbrace{3}_{N_C}\Big[\underbrace{-2(\tfrac{1}{6})^3}_{Q_i}+\underbrace{(\tfrac{2}{3})^3}_{\overline{U}_i}+\underbrace{(-\tfrac{1}{3})^3}_{\overline{D}_i}\Big]\Big]=0 \tag{235}$$

$$\mathcal{A}_{\text{MSSM}}^{U(1)\times\text{grav}^2} \propto N_G\Big[\underbrace{-2(-\tfrac{1}{2})}_{L_i}+\underbrace{(-1)}_{\overline{E}_i}\underbrace{-2(\tfrac{1}{2})}_{H_u}\underbrace{-2(-\tfrac{1}{2})}_{H_d}+\underbrace{3}_{N_C}\Big[\underbrace{-2(\tfrac{1}{6})}_{Q_i}+\underbrace{(\tfrac{2}{3})}_{\overline{U}_i}+\underbrace{(-\tfrac{1}{3})}_{\overline{D}_i}\Big]\Big]=0 \tag{236}$$

$$\mathcal{A}_{\text{MSSM}}^{U(1)\times(SU(2))^2} \propto N_G\Big[\underbrace{-(-\tfrac{1}{2})}_{L_i}\underbrace{-2(\tfrac{1}{2})}_{H_u}\underbrace{-2(-\tfrac{1}{2})}_{H_d}+\underbrace{3}_{N_C}\cdot\underbrace{-(\tfrac{1}{6})}_{Q_i}\Big]=0 \tag{237}$$

$$\mathcal{A}_{\text{MSSM}}^{U(1)\times(SU(3))^2} \propto N_G\Big[\underbrace{-2(\tfrac{1}{6})}_{Q_i}+\underbrace{(\tfrac{2}{3})}_{\overline{U}_i}+\underbrace{(-\tfrac{1}{3})}_{\overline{D}_i}\Big]=0 \tag{238}$$

From these calculations, one can also deduce that the number of generations of quarks/lepton, N_G, in the SM or MSSM does not have any adverse effect their anomaly cancellations. It does, however, affect the running couplings.

Calculation of b_a coefficients

$\underline{b_3^{\text{SM}}}$

Gluons, $\{\mathbf{8},\mathbf{1},0\}:\ \Delta b_3^{\text{SM}} = \underbrace{-\tfrac{11}{3}}_{\text{gauge boson}} \cdot \underbrace{3}_{n_r=N} = -11$

Quarks, $\left[\{\mathbf{3},\mathbf{2},\tfrac{1}{6}\}\oplus\{\mathbf{3},\mathbf{1},\tfrac{2}{3}\}\oplus\{\mathbf{3},\mathbf{1},-\tfrac{1}{3}\}\right]:\ \Delta b_3^{\text{SM}} = \underbrace{\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{\tfrac{1}{2}}_{n_r} \cdot \underbrace{4}_{SU(3)\text{ triplets}} \cdot \underbrace{3}_{N_G} = +4$

$$\Rightarrow b_3^{\text{SM}} = -7 \tag{239}$$

$\underline{b_2^{\text{SM}}}$

Weak bosons, $\{\mathbf{1},\mathbf{3},0\}:\ \Delta b_2^{\text{SM}} = \underbrace{-\tfrac{11}{3}}_{\text{gauge boson}} \cdot \underbrace{2}_{n_r=N} = -\dfrac{22}{3}$

Quarks, $\{\mathbf{3},\mathbf{2},\tfrac{1}{6}\}:\ \Delta b_2^{\text{SM}} = \underbrace{\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{\tfrac{1}{2}}_{n_r} \cdot \underbrace{3}_{N_C} \cdot \underbrace{3}_{N_G} = +3$

Higgs, $\{\mathbf{1},\mathbf{2},\tfrac{1}{2}\}:\ \Delta b_2^{\text{SM}} = \underbrace{\tfrac{1}{3}}_{\text{scalar}} \cdot \underbrace{\tfrac{1}{2}}_{n_r} = +\dfrac{1}{6}$

Leptons, $\{\mathbf{1},\mathbf{2},-\tfrac{1}{2}\}:\ \Delta b_2^{\text{SM}} = \underbrace{\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{\tfrac{1}{2}}_{n_r} \cdot \underbrace{3}_{N_G} = +1$

$$\Rightarrow b_2^{\text{SM}} = -\frac{19}{6} \tag{240}$$

$\underline{b_1^{\text{SM}}}$

Quarks, $\left[\{\mathbf{3},\mathbf{2},\tfrac{1}{6}\}\oplus\{\mathbf{3},\mathbf{1},\tfrac{2}{3}\}\oplus\{\mathbf{3},\mathbf{1},-\tfrac{1}{3}\}\right]$:

$$\Delta b_1^{\text{SM}} = \underbrace{\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{3}_{N_C} \cdot \underbrace{3}_{N_G} \left[\left(\underbrace{2}_{SU(2)\text{ doublet}} \cdot \underbrace{\tfrac{3}{5}(\tfrac{1}{6})^2}_{(n_r)_Q}\right) + \underbrace{\tfrac{3}{5}(\tfrac{2}{3})^2}_{(n_r)_u} + \underbrace{\tfrac{3}{5}(-\tfrac{1}{3})^2}_{(n_r)_d}\right] = +\frac{11}{5}$$

Leptons, $\left[\{\mathbf{1},\mathbf{2},-\tfrac{1}{2}\}\oplus\{\mathbf{1},\mathbf{1},-1\}\right]$:

$$\Delta b_1^{\text{SM}} = \underbrace{\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{3}_{N_G} \left[\left(\underbrace{2}_{SU(2)\text{ doublet}} \cdot \underbrace{\tfrac{3}{5}(-\tfrac{1}{2})^2}_{(n_r)_L}\right) + \underbrace{\tfrac{3}{5}(-1)^2}_{(n_r)_l}\right] = +\frac{9}{5}$$

Higgs, $\{\mathbf{1},\mathbf{2},\tfrac{1}{2}\}$: $\Delta b_1^{\text{SM}} = \underbrace{\tfrac{1}{3}}_{\text{scalar}} \cdot \underbrace{2}_{SU(2)\text{ doublet}} \cdot \underbrace{\tfrac{3}{5}(\tfrac{1}{2})^2}_{(n_r)_H} = +\dfrac{1}{10}$

$$\Rightarrow b_1^{\text{SM}} = +\frac{41}{10} \tag{241}$$

$$\Rightarrow b_a^{SM} = \{\tfrac{41}{10}, -\tfrac{19}{6}, -7\} \tag{242}$$

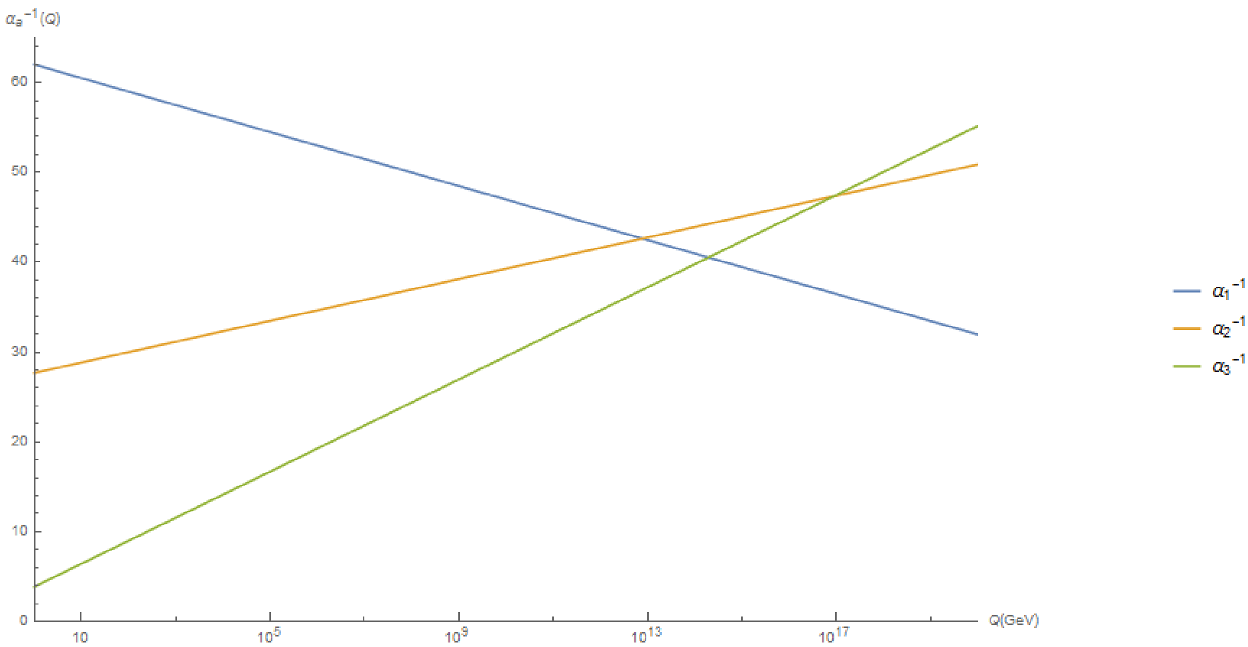

Figure 4: The running of the SM couplings α_a^{-1} with energy Q.

$\underline{b_3^{\text{MSSM}}}$

$$\text{Gluons and gluinos, } \{\mathbf{8},\mathbf{1},0\}: \ \Delta b_3^{\text{MSSM}} = \overbrace{\underbrace{-\tfrac{11}{3}}_{\text{gauge boson}} \cdot \underbrace{3}_{n_r=N}}^{\text{gluons}} \overbrace{\underbrace{+\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{3}_{n_r=N}}^{\text{gluinos}} = -9$$

$$\text{Quarks and squarks, } \left[\{\mathbf{3},\mathbf{2},\tfrac{1}{6}\} \oplus \{\mathbf{3},\mathbf{1},\tfrac{2}{3}\} \oplus \{\mathbf{3},\mathbf{1},-\tfrac{1}{3}\}\right]:$$

$$\Delta b_3^{\text{MSSM}} = \Big[\underbrace{\tfrac{1}{2}}_{n_r} \cdot \underbrace{4}_{SU(3)\text{ triplets each}} \cdot \underbrace{3}_{N_G}\Big] \cdot \Big[\overbrace{\underbrace{\tfrac{2}{3}}_{\text{fermion}}}^{\text{quarks}} + \overbrace{\underbrace{\tfrac{1}{3}}_{\text{scalar}}}^{\text{squarks}}\Big] = +6$$

$$\Rightarrow b_3^{\text{MSSM}} = -3 \tag{243}$$

$\underline{b_2^{\text{MSSM}}}$

$$\text{Weak bosons and gauginos, } \{\mathbf{1},\mathbf{3},0\}: \ \Delta b_2^{\text{MSSM}} = \overbrace{\underbrace{-\tfrac{11}{3}}_{\text{gauge boson}} \cdot \underbrace{2}_{n_r=N}}^{\text{weak bosons}} \overbrace{\underbrace{+\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{2}_{n_r=N}}^{\text{weak gauginos}} = -6$$

$$\text{Quarks and squarks, } \{\mathbf{3},\mathbf{2},\tfrac{1}{6}\}: \ \Delta b_2^{\text{MSSM}} = \Big[\underbrace{\tfrac{1}{2}}_{n_r} \cdot \underbrace{3}_{N_C} \cdot \underbrace{3}_{N_G}\Big] \cdot \Big[\overbrace{\underbrace{\tfrac{2}{3}}_{\text{fermion}}}^{\text{quarks}} + \overbrace{\underbrace{\tfrac{1}{3}}_{\text{scalar}}}^{\text{squarks}}\Big] = +\frac{9}{2}$$

$$\text{Higgs and Higgsinos, } \{\mathbf{1},\mathbf{2},\tfrac{1}{2}\}: \ \Delta b_2^{\text{MSSM}} = \underbrace{\tfrac{1}{2}}_{n_r} \cdot \underbrace{2}_{N_H} \cdot \Big[\overbrace{\underbrace{\tfrac{1}{3}}_{\text{scalar}}}^{\text{Higgs}} + \overbrace{\underbrace{\tfrac{2}{3}}_{\text{fermion}}}^{\text{Higgsinos}}\Big] = +1$$

$$\text{Leptons and sleptons, } \{\mathbf{1},\mathbf{2},-\tfrac{1}{2}\}: \ \Delta b_2^{\text{MSSM}} = \Big[\underbrace{\tfrac{1}{2}}_{n_r} \cdot \underbrace{3}_{N_G}\Big] \cdot \Big[\overbrace{\underbrace{\tfrac{2}{3}}_{\text{fermion}}}^{\text{leptons}} + \overbrace{\underbrace{\tfrac{1}{3}}_{\text{scalar}}}^{\text{sleptons}}\Big] = +\frac{3}{2}$$

$$\Rightarrow b_2^{\text{MSSM}} = +1 \tag{244}$$

$\underline{b_1^{\text{MSSM}}}$

Quarks and squarks, $\left[\{\mathbf{3},\mathbf{2},\tfrac{1}{6}\}\oplus\{\mathbf{3},\mathbf{1},\tfrac{2}{3}\}\oplus\{\mathbf{3},\mathbf{1},-\tfrac{1}{3}\}\right]$:

$$\Delta b_1^{\text{MSSM}} = \underbrace{3}_{N_C}\cdot\underbrace{3}_{N_G}\Bigg[\overbrace{\Big[\Big(\underbrace{2}_{SU(2)\text{ doublet}}\cdot\underbrace{\tfrac{3}{5}(\tfrac{1}{6})^2}_{(n_r)_Q}\Big)+\underbrace{\tfrac{3}{5}(\tfrac{2}{3})^2}_{(n_r)_{\overline{U}}}+\underbrace{\tfrac{3}{5}(-\tfrac{1}{3})^2}_{(n_r)_{\overline{D}}}\Big]}^{\text{quarks}}$$
$$+\underbrace{\Big[\Big(\underbrace{2}_{SU(2)\text{ doublet}}\cdot\underbrace{\tfrac{3}{5}(\tfrac{1}{6})^2}_{(n_r)_{\tilde{Q}}}\Big)+\underbrace{\tfrac{3}{5}(\tfrac{2}{3})^2}_{(n_r)_{\overline{\tilde{U}}}}+\underbrace{\tfrac{3}{5}(-\tfrac{1}{3})^2}_{(n_r)_{\overline{\tilde{D}}}}\Big]}_{\text{squarks}}\Bigg] = +\frac{33}{10}$$

Leptons and sleptons, $\left[\{\mathbf{1},\mathbf{2},-\tfrac{1}{2}\}\oplus\{\mathbf{1},\mathbf{1},-1\}\right]$:

$$\Delta b_1^{\text{MSSM}} = \underbrace{3}_{N_G}\Bigg[\overbrace{\underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\Big[\Big(\underbrace{2}_{SU(2)\text{ doublet}}\cdot\underbrace{\tfrac{3}{5}(-\tfrac{1}{2})^2}_{(n_r)_L}\Big)+\underbrace{\tfrac{3}{5}(-1)^2}_{(n_r)_{\overline{E}}}\Big]}^{\text{leptons}}$$
$$+\underbrace{\Big[\underbrace{\tfrac{1}{3}}_{\text{scalar}}\cdot\Big[\Big(\underbrace{2}_{SU(2)\text{ doublet}}\cdot\underbrace{\tfrac{3}{5}(-\tfrac{1}{2})^2}_{(n_r)_{\tilde{L}}}\Big)+\underbrace{\tfrac{3}{5}(-1)^2}_{(n_r)_{\overline{\tilde{E}}}}\Big]}_{\text{sleptons}}\Bigg] = +\frac{27}{10}$$

Higgs and Higgsinos, $\{\mathbf{1},\mathbf{2},\tfrac{1}{2}\}$:

$$\Delta b_1^{\text{MSSM}} = \underbrace{2}_{N_H}\cdot\underbrace{2}_{SU(2)\text{ doublet}}\Big[\underbrace{\tfrac{1}{3}}_{\text{scalar}}\cdot\underbrace{\tfrac{3}{5}(\tfrac{1}{2})^2}_{(n_r)_{H_{u,d}}}+\underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{\tfrac{3}{5}(-\tfrac{1}{2})^2}_{(n_r)_{\tilde{H}_{u,d}}}\Big] = +\frac{3}{10}$$

$$\Rightarrow b_1^{\text{MSSM}} = +\frac{33}{5} \tag{245}$$

$$\Rightarrow b_a^{MSSM} = \{\tfrac{33}{5}, 1, -3\} \tag{246}$$

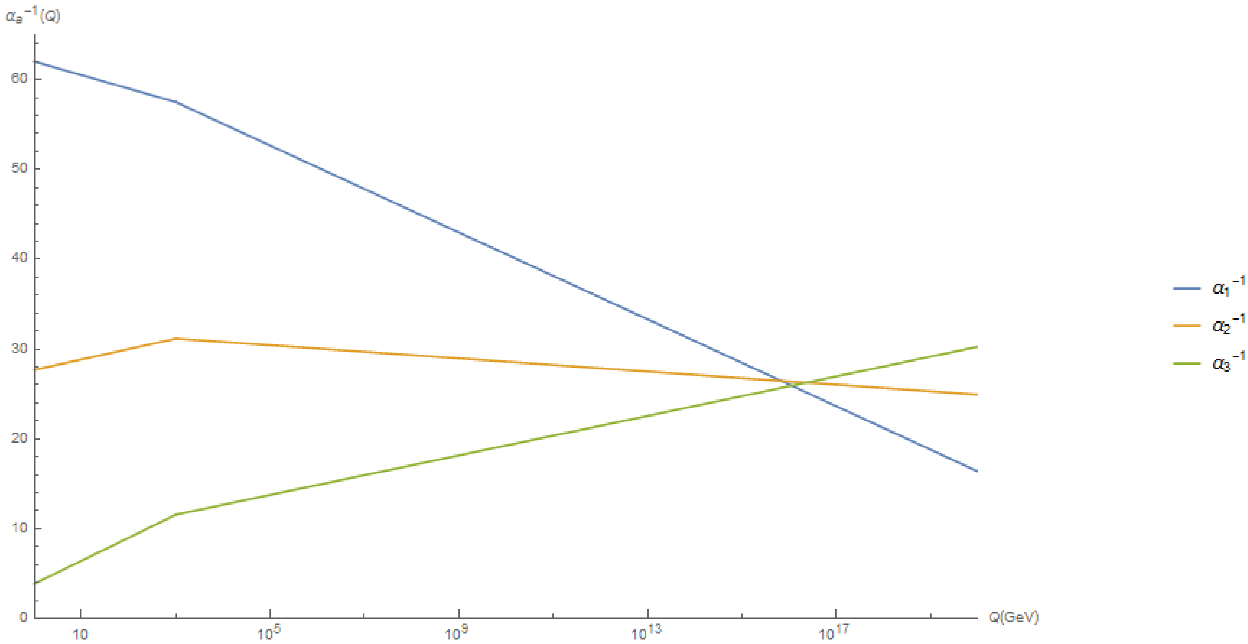

Figure 5: The running of the MSSM couplings α_a^{-1} with energy Q, which almost unify at $Q \approx 10^{16}$ GeV. Here, the SUSY states come into existence at $M_{\text{SUSY}} = 10^3$ GeV.

5 Grand Unified Theories

Grand Unified Theories (GUTs) postulate that three of the fundamental forces in the SM, the strong, weak, and electromagnetic interactions, are in fact low-energy descendants of a single force, such that over the course of the evolution of the SM gauge couplings, the different α_a^{-1} values merge into a single coupling, α_U^{-1}, once some very large energy scale, M_{GUT} (usually theorised to be $\sim 10^{16}\,\mathrm{GeV}$), is reached.

The motivation for such a paradigm comes from the relatively near-miss in the trajectories of the SM couplings, as can be seen in Figure (5), which implies that larger, beyond-the-SM (BSM) theories, such as GUTs, could be capable of containing the SM group, $\mathcal{G}_{\mathrm{SM}}$, within some broader structure, $\mathcal{G}_{\mathrm{GUT}}$, whilst also having the potential to realise a Grand Unification of the SM couplings. In addition, GUTs are able to solve many of the problems with the SM, such as those mentioned in Section 4.

Other advantages of GUTs include the ability to reduce the unfavourably high number of SM free parameters, via the unification of the couplings, and to predict the quantised nature of all elementary particles (for example, their electric charges, masses, and interaction strengths) within a single elegant model.

The choice of $\mathcal{G}_{\mathrm{SM}} = SU(3)_C \times SU(2)_L \times U(1)_Y$ was made on the basis of observational evidence, but is, however, seemingly arbitrary. This arouses some suspicion as to why Nature would make such a random choice, and leads to the idea that there may be a larger group which was spontaneously broken to the SM in the early universe, lessening the arbitrariness of the theory.

Unfortunately, GUTs cannot easily incorporate gravitation, the fourth and final force in the SM. This is mainly due to the fact that Quantum Field Theory is probabilistic, whereas General Relativity is purely geometric. Forces other than gravitation do not directly bend spacetime, and their quantum fluctuations are also meaningful, calculable, and testable by experiment.

In spite of this particular shortcoming, the powerful benefits of GUTs largely outweigh their flaws, and they remain one of the most likely BSM theories to be confirmed as a part of Nature. They also offer many testable BSM predictions, such as the decay of protons, which will be discussed in Section 5.3.

To build a GUT model, one must first choose a gauge group, $\mathcal{G}_{\mathrm{GUT}}$, which must have one or more possible symmetry breaking patterns to reach $\mathcal{G}_{\mathrm{SM}}$; there must also exist exotic fields which can trigger each stage of the symmetry breaking. The sheer range of models with potential unification capabilities is truly vast, and in this paper, the focus will primarily be on two different gauge groups, $SU(5)$ and $SO(10)$, the latter of which is thought to be one of the strongest candidates for the realisation of a GUT.

5.1 $SU(5)$

The SM symmetries are represented by $\mathcal{G}_{\text{SM}}$, which is a Lie group[1] of rank 4 , and hence, $\mathcal{G}_{\text{GUT}} \supset \mathcal{G}_{\text{SM}}$ must be of rank 4 or higher. $\mathcal{G}_{\text{GUT}}$ should also account for the chiral structure of the SM, such that, for example, left-handed particles and left-handed antiparticles are embedded in separate, conjugate representations of the group, due to the fact that they have opposite overall chirality, as left-handed antiparticles actually have right-handed chirality. As a result of the requirement that conjugate representations need to exist to accommodate this chiral structure, $\mathcal{G}_{\text{GUT}}$ must have complex representations, in addition to its having a rank ≥ 4.

The only simple rank 4 group able to meet the two conditions outline above, as well as giving the correct hypercharges for the SM fields, is the $SU(5)$ group. The $SO(8)$, $SO(9)$, $Sp(8)$, and F_4 simple groups do not have complex representations, while the semisimple groups $SU(3) \times SU(3)$, $SU(3) \times SU(2) \times SU(2)$, and $SU(4) \times SU(2)$ do not yield the correct SM hypercharges.

In the $SU(5)$ model, the left-handed fermions are embedded into the $SU(5)$ representations $\overline{\mathbf{5}} \oplus \mathbf{10}$, whose contents are as follows [13]:

$$\overline{\mathbf{5}} \leftrightarrow \begin{pmatrix} \overline{d}^r \\ \overline{d}^g \\ \overline{d}^b \\ e \\ -\nu \end{pmatrix}_L , \tag{247}$$

$$\mathbf{10} \leftrightarrow \begin{pmatrix} 0 & \overline{u}^b & -\overline{u}^g & u^r & d^r \\ -\overline{d}^b & 0 & \overline{u}^r & u^g & d^g \\ u^g & -\overline{u}^r & 0 & u^b & d^b \\ -u^r & -u^g & -u^b & 0 & \overline{e} \\ -d^r & -d^g & -d^b & -\overline{e} & 0 \end{pmatrix}_L , \tag{248}$$

where the subscript L of each matrix means that every element has left-handed overall chirality, and overbars denote chirality conjugates, which do not transform under $SU(2)_L$. The minus signs included in the representations are simply a matter of convention.

To quantise electric charge in an $SU(5)$ theory, one can write down the traceless charge generator

$$Q = \begin{pmatrix} \alpha & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & 0 & -(3\alpha+\beta) \end{pmatrix} . \tag{249}$$

[1] For more information on Lie groups, please consult [14].

Acting Q on the $\overline{\mathbf{5}}$ and $\mathbf{10}$ representations yields

$$Q(u) = \alpha + \beta\,, \tag{250}$$
$$Q(\overline{u}) = 2\alpha\,, \tag{251}$$
$$Q(d) = -(2\alpha + \beta)\,, \tag{252}$$
$$Q(\overline{d}) = -\alpha\,, \tag{253}$$
$$Q(e) = -\beta\,, \tag{254}$$
$$Q(\overline{e}) = -3\alpha\,, \tag{255}$$
$$Q(\nu) = 3\alpha + \beta\,, \tag{256}$$

which replicates the charge spectrum of the SM particles if $Q(\nu) \overset{\text{set}}{=} 0$ and $Q(e) \overset{\text{set}}{=} -1$.

The adjoint $\mathbf{24}$ representation of $SU(5)$ contains the gauge bosons of the SM, and decomposes into $\mathcal{G}_{\text{SM}}$ representations as

$$\mathbf{24} \to \{\mathbf{8}, \mathbf{1}, 0\} \oplus \{\mathbf{1}, \mathbf{3}, 0\} \oplus \{\mathbf{1}, \mathbf{1}, 0\} \oplus \{\mathbf{3}, \mathbf{2}, \tfrac{1}{6}\} \oplus \{\overline{\mathbf{3}}, \mathbf{2}, -\tfrac{1}{6}\}\,, \tag{257}$$

where $\{\mathbf{8}, \mathbf{1}, 0\}$, $\{\mathbf{1}, \mathbf{3}, 0\}$, and $\{\mathbf{1}, \mathbf{1}, 0\}$ represent the $SU(3)_C$, $SU(2)_L$, and $U(1)_Y$ gauge bosons, respectively, while the representation $\{\mathbf{3}, \mathbf{2}, \frac{1}{6}\} \oplus \{\overline{\mathbf{3}}, \mathbf{2}, -\frac{1}{6}\}$ contains exotic coloured vector bosons not found in the SM.

The Higgs sector of the $SU(5)$ model must include a scalar field, S, that breaks $SU(5) \to SU(3)_C \times SU(2)_L \times U(1)_Y$ (in a similar fashion to the way in which the breaking of $SU(2)_L \times U(1)_Y \to U(1)_{\text{EM}}$ proceeds), producing another scalar state in the process.

S must be in the $\mathbf{24}$ representation in order to preserve the rank of the Lie algebra when breaking to $\mathcal{G}_{\text{SM}}$. It must also acquire a vev in the $(2, 2, 2, -3, -3)$ direction, such that $SU(5)$ is broken to $\mathcal{G}_{\text{SM}}$ as desired, rather than to the other maximal subgroup, $SU(4) \times U(1)$. There must also exist a scalar field containing the electroweak Higgs, which lies in the $\{\mathbf{1}, \mathbf{2}, -\frac{1}{2}\}$ representation of the SM; this could be embedded into the $\overline{\mathbf{5}}$ of $SU(5)$ [15], which decomposes to SM representations as

$$\overline{\mathbf{5}} \to \{\overline{\mathbf{3}}, \mathbf{1}, \tfrac{1}{3}\} \oplus \{\mathbf{1}, \mathbf{2}, -\tfrac{1}{2}\}\,. \tag{258}$$

As we shall see from the symmetry breaking chains outlined in the next section, $SU(5)$ is a maximal subgroup of $SO(10)$, and therefore any representation of $SU(5)$ can also be embedded into an $SO(10)$ theory.

5.2 $SO(10)$

One need not be restricted only to rank 4 groups, such as $SU(5)$. Candidate GUTs of rank 5 include $SO(10)$, $SO(11)$, and $Sp(10)$, of which, only $SO(10)$ has complex representations, despite being an orthogonal group, as denoted by the "O" in $SO(10)$.

$SO(10)$ is a highly appealing group, which can unify all SM fermions, as well as right-handed neutrinos, within the same representation, the **16**; the fermions are embedded as

$$\mathbf{16} \leftrightarrow \begin{pmatrix} \overline{u}^r & \overline{d}^r & d^r & u^r & \overline{\nu} & \overline{e} & d^g & u^g & \overline{u}^g & \overline{d}^g & d^b & u^b & \overline{u}^b & \overline{d}^b & e & \nu \end{pmatrix} \tag{259}$$

The adjoint of $SO(10)$, the **45**, contains the 12 SM gauge bosons, as well as 33 additional exotic gauge bosons, which have various colour and weak charges. All of the coloured exotic bosons mediate baryon and lepton number violating processes, enabling protons to decay; the main proton decay process will be addressed in Section 5.3.

The decomposition of **45** to representations of $\mathcal{G}_{\text{SM}}$ occurs as follows:

$$\mathbf{45} \to \left.\{\mathbf{8},\mathbf{1},0\} \oplus \{\mathbf{1},\mathbf{3},0\} \oplus \{\mathbf{1},\mathbf{1},0\}\right\} \leftrightarrow \text{ SM gauge bosons} \tag{260a}$$

$$\oplus \{\mathbf{3},\mathbf{2},\tfrac{1}{6}\} \oplus \{\overline{\mathbf{3}},\mathbf{2},-\tfrac{1}{6}\} \oplus \{\mathbf{3},\mathbf{2},\tfrac{1}{6}\} \tag{260b}$$

$$\oplus \{\overline{\mathbf{3}},\mathbf{2},-\tfrac{1}{6}\} \oplus \{\mathbf{3},\mathbf{1},\tfrac{2}{3}\} \oplus \{\overline{\mathbf{3}},\mathbf{1},-\tfrac{2}{3}\} \tag{260c}$$

$$\oplus \{\mathbf{1},\mathbf{1},1\} \oplus \{\mathbf{1},\mathbf{1},0\} \oplus \{\mathbf{1},\mathbf{1},-1\}\,. \tag{260d}$$

There are a large number of ways in which symmetry breaking from $SO(10)$ to $\mathcal{G}_{\text{SM}}$ can occur, each of which is determined by the vevs of the scalar fields within the theory. The $SO(10)$ Higgs sector must be large enough to be able to break the symmetry for every step of the breaking path, until $\mathcal{G}_{\text{SM}}$ is reached. The various breaking patterns of $SO(10)$ to $\mathcal{G}_{\text{SM}}$ are shown in Figure (6). Of the many different combinations, this paper will concern itself mainly with the simplest paths, labelled (a) and (b) in Figure (6), corresponding to the breaking chains $SO(10) \to SU(5) \to \mathcal{G}_{\text{SM}}$, and $SO(10) \to \mathcal{G}_{\text{SM}}$ (with no intermediate steps), respectively.

The $SO(10) \to SU(5)$ stage of breaking in path (a) reduces the rank of the group from 5 to 4, and thus requires a non-adjoint representation, for example, **16**, **126**, or their conjugates. The decomposition of these $SO(10)$ representations proceeds as follows:

$$\mathbf{16} \to \mathbf{10} \oplus \overline{\mathbf{5}} \oplus \mathbf{1}\,, \tag{261}$$

$$\mathbf{126} \to \mathbf{50} \oplus \mathbf{45} \oplus \overline{\mathbf{15}} \oplus \mathbf{10} \oplus \overline{\mathbf{5}} \oplus \mathbf{1}\,. \tag{262}$$

An additional Higgs state, which will lie in the **24** of $SU(5)$, is required to appear at some intermediate energy scale, such that the final stage of the breaking chain, $SU(5) \to \mathcal{G}_{\text{SM}}$, can be completed. The Higgs field corresponding to this Higgs state may come from the adjoint of $SO(10)$, the **45**, or alternatively from an adjoint-like representation of $SO(10)$, such as the **54** or the **210**.

In contrast, breaking directly from $SO(10) \to \mathcal{G}_{\rm SM}$ via path (b) is triggered by a Higgs state residing in a representation which has no singlets, that is, no **1** representations, in the directions of any intermediate subgroups. In this case, the smallest possible representation meeting all of the required criteria is the **144** of $SO(10)$, which contains no singlets when decomposed to any of the intermediate maximal subgroups of $SO(10)$, such as $SU(5) \times U(1)$, $SU(4)_C \times SU(2)_L \times SU(2)_R$, $SU(3)_C \times SU(2)_L \times SU(2)_R \times U(1)_{B-L}$, as well as all of the others shown in Figure (6).

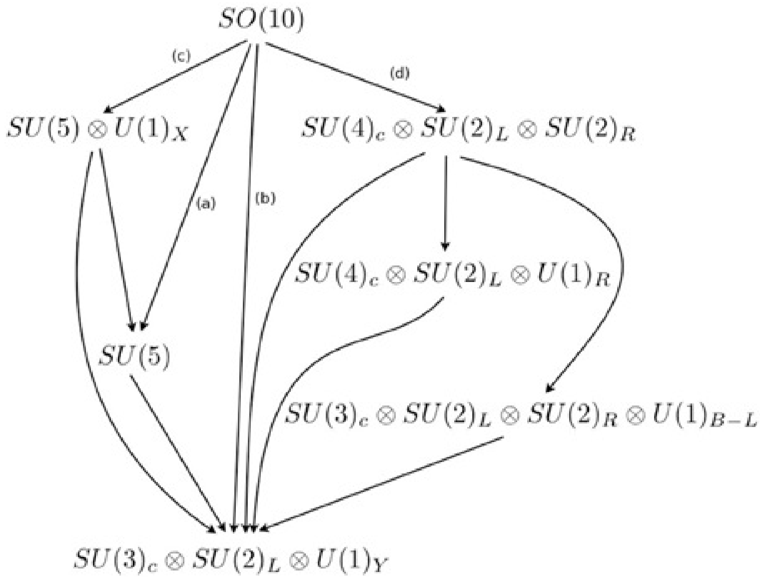

Figure 6: A diagram showing possible symmetry breaking patterns from $SO(10)$ to $\mathcal{G}_{\rm SM}$ (taken from [16]).

The high appeal of $SO(10)$ models stems from their huge abundance of breaking options, in conjunction with the favourability of the subgroups to which $SO(10)$ decomposes, many of which exhibit a high proportion of the characteristics required or desirable for a GUT to have viability.

5.3 Proton decay

A general property of GUTs is that they necessarily predict rapid proton decay, via exotic "X" and "Y" bosons, of masses $\sim M_{\text{GUT}}$, which are charged under both the strong and electroweak symmetries simultaneously, and which therefore couple to both quarks and leptons, such that processes which violate baryon and lepton number conservation are able to proceed.

The main decay mode of protons in a non-SUSY GUT is $p \to e^+\pi^0$, as shown in Figure (7).

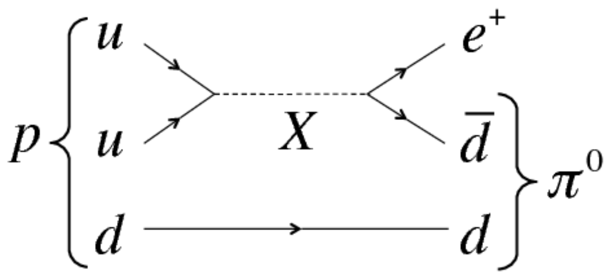

Figure 7: A Feynman diagram of the decay $p \to e^+\pi^0$ (taken from [17]).

The mean lifetime of the proton within a GUT, for this particular decay channel, is given by

$$\tau_p \sim \alpha_U^{-2} \frac{M_X^4}{m_p^5} \tag{263}$$

$$\Rightarrow \tau_p \sim \alpha_U^{-2} \frac{M_{\text{GUT}}^4}{m_p^5} \tag{264}$$

where M_X is the mass of the intermediate X boson, m_p is the mass of the proton, and the α_U^{-1} is the dimensionless unified gauge coupling.

The most typical SUSY GUT proton decay mode is $p \to K^+\overline{\nu}$, which proceeds via a very heavy coloured Higgsino, $\tilde{\phi}_3$, as shown in Figure (8).

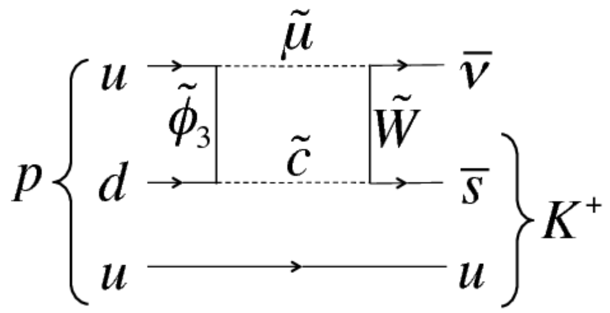

Figure 8: A Feynman diagram of the decay $p \to K^+\overline{\nu}$ (taken from [17]).

The current experimental limit for the mean lifetime of a proton is $\tau_p \gtrsim 1.3 \times 10^{34}$ yr, as set by the Super-Kamiokande experiment, a large water Čerenkov detector. Using Equation (264) given typical values of the unified coupling, α_U^{-1}, this value of τ_p would require a GUT scale of $M_{\text{GUT}} \gtrsim 10^{16}\,\text{GeV}$, ruling out non-SUSY GUTs, which in general have lower unification scales.

The future Hyper-Kamiokande experiment is projected to reach a limit of at least $\tau_p \gtrsim 1.3 \times 10^{35}$ yr for the $p \to e^+\pi^0$ decay mode, further constraining the overall viability of GUT models, whilst ruling some out entirely [17].

6 Introducing additional states to the SM

In this section, theoretical particle states in the form of both fermionic and scalar $SU(5)$ and $SO(10)$ representations, which embed readily into an $SO(10)$ GUT, will be added to the Standard Model, at various energy levels between 10^4 and 10^{13} GeV, in an attempt to unify the couplings of the strong, weak and EM interactions.

Such representations are able to alter the gradients of the running SM couplings, through their influence on the b_a coefficients (which were introduced in Section 4.3), in an identical fashion to the way in which the SM representations themselves contribute to these coefficients. The additional representations may be split into arbitrary pieces, and placed at different energies, provided that subgroups which are conjugate to one another are positioned at the same energy, in order to ensure the cancellation of gauge anomalies across all energies.

An example of a way in which a representation could be split would be to have one piece which contains those subgroups which are adjoints of $SU(3)$, one which contains adjoints of $SU(2)$, and third and fourth pieces which contain coloured and non-coloured non-adjoint representations, respectively.

As with the SM and MSSM, the gauge anomalies of the added representations must be equal to zero, either through the representations being self-adjoint (and hence, anomaly-free), or through adding two conjugate representations, one of which cancels exactly the anomalies generated by the other.

A Python program has been written to iterate over many combinations of energy scales at which additional states may appear; the code for this program can be found in Appendix A. The implications of different combinations of added representations can be studied by inputting the changes in b_a into the program, which then incorporates these values into the evaluation of the solution to the β function, expressed previously in Equation (230). After running the program, plots of the running couplings, α_a^{-1}, against Q, the energy scale, for energy configurations which the computer deems as having most successfully unified the couplings, are outputted, in addition to the energy scale at which each piece of the additional representation resides.

To decide which combinations of energy scale best unify the couplings, the computer utilises a simple algorithm, which determines the width (given in orders of magnitude of Q) of the triangle of $(\alpha_a^{-1}, \alpha_b^{-1}$ intersections, which will be known from here onwards as the variable $\Delta(\log_{10} Q)$. As with any computer program, an element of human intervention can be needed, as some energy configurations, which do not unify the couplings at all (and yet manage to escape explicitly generating an error), occasionally have a tendency to give small, favourable values for $\Delta(\log_{10} Q)$ ($\sim 10^{-3}$), and therefore must be checked by hand for validity.

After narrowing down the field of potential candidates which, to a good degree, achieve unification, via the implementation of an upper $\Delta(\log_{10} Q)$ threshold, whose value is set manually to suit the overall general success of individual representations, the "best" result is then chosen. This choice will be made such that the criterion of having a high unification scale is met, preferably whereby $M_{\text{GUT}} \sim 2 \times 10^{16}\,\text{GeV}$, in analogy with SUSY. If different results have the same M_{GUT}, the one with α_U^{-1} closest to the SUSY value of ~ 26 will be chosen.

The influence on the b_a coefficients for fermionic $SU(5)$ and $SO(10)$ representations will be computed in Section 6.1. The equivalent scalar representations will introduce a prefactor of $\frac{1}{3}$ to their contributions to b_a, instead of $\frac{2}{3}$ for fermionic representations (see Section 4.3), and thus, the changes in b_a for scalar representations are simply half those of the equivalent fermionic representations.

In Section 6.2, plots of the best unification achieved by each representation (or combination thereof) will be shown, as well as a table summarising the key values for each plot, including α_U^{-1}, M_{GUT}, and $\Delta(\log_{10} Q)$.

The representations investigated will be the **5**, **10**, and **24** of $SU(5)$, and the **10**, **16**, **45**, **54**, **120**, and **126** of $SO(10)$. The breaking of these representations to $\mathcal{G}_{\text{SM}}$ representations is given in Appendix A of [18], and will be used throughout Section 6.

The **24**, $\mathbf{10}_{SO(10)}$, **45**, **54**, and **120** are known to be anomaly-free, while the **5**, $\mathbf{10}_{SU(5)}$, **16**, and **126** do introduce gauge anomalies, and will hence need to be added alongside their conjugates for anomaly cancellation to occur. Nevertheless, the anomalies of each representation will be calculated explicitly in Section 6.3.

6.1 Influence on running couplings

Throughout this section (and the rest of the paper), the subscript F affixed to a representation denotes a fermionic representation, where all particle states which reside within the representation have spin-$\frac{1}{2}$. A subscript S means instead that the representation contains scalars, of spin-0. As mentioned in the previous section, the changes to the b_a coefficients from a scalar representation are simply half those of the equivalent fermionic representation, and therefore, only gradients for fermionic representations will be calculated here. These will then be used in Section 6.2 to make plots for both fermionic and scalar additions.

The conjugate of each representation has opposite hypercharge (hence leading to cancellation of anomalies), but gives an identical contribution to b_a. Therefore, when investigating $\mathbf{5}_F \oplus \overline{\mathbf{5}}_F$, for example, one must simply double up $(b_a)_{\mathbf{5}_F}$ when inputting values into the Python program.

<u>b_a calculations</u>

<u>$\mathbf{5}_F$</u>

<u>Δb_3</u>

$$\{\mathbf{3}, \mathbf{1}, -\tfrac{1}{3}\}: \ (\Delta b_3)_{\mathbf{5}_F} = \underbrace{\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{\tfrac{1}{2}}_{n_r} = \frac{1}{3}$$

$$\{\mathbf{1}, \mathbf{2}, \tfrac{1}{2}\}: \ (\Delta b_3)_{\mathbf{5}_F} = \underbrace{0}_{n_r}$$

<u>Δb_2</u>

$$\{\mathbf{3}, \mathbf{1}, -\tfrac{1}{3}\}: \ (\Delta b_2)_{\mathbf{5}_F} = \underbrace{0}_{n_r}$$

$$\{\mathbf{1}, \mathbf{2}, \tfrac{1}{2}\}: \ (\Delta b_2)_{\mathbf{5}_F} = \underbrace{\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{\tfrac{1}{2}}_{n_r} = \frac{1}{3}$$

$\underline{\Delta b_1}$

$$\{\mathbf{3},\mathbf{1},-\tfrac{1}{3}\}:\ (\Delta b_1)_{\mathbf{5}_F} = \underbrace{\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{3}_{SU(3)\text{ triplet}} \cdot \underbrace{\tfrac{3}{5}(-\tfrac{1}{3})^2}_{n_r} = \frac{2}{15}$$

$$\{\mathbf{1},\mathbf{2},\tfrac{1}{2}\}:\ (\Delta b_1)_{\mathbf{5}_F} = \underbrace{\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{2}_{SU(2)\text{ doublet}} \cdot \underbrace{\tfrac{3}{5}(\tfrac{1}{2})^2}_{n_r} = \frac{1}{5}$$

$\underline{\mathbf{10}^{SU(5)}{}_F}$

$\underline{\Delta b_3}$

$$\{\mathbf{3},\mathbf{2},\tfrac{1}{6}\}:\ (\Delta b_3)_{\mathbf{10}_F^{SU(5)}} = \underbrace{\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{\tfrac{1}{2}}_{n_r} \cdot \underbrace{2}_{SU(2)\text{ doublet}} = \frac{2}{3}$$

$$\{\overline{\mathbf{3}},\mathbf{1},-\tfrac{2}{3}\}:\ (\Delta b_3)_{\mathbf{10}_F^{SU(5)}} = \underbrace{\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{\tfrac{1}{2}}_{n_r} = \frac{1}{3}$$

$$\{\mathbf{1},\mathbf{1},1\}:\ (\Delta b_3)_{\mathbf{10}_F^{SU(5)}} = \underbrace{0}_{n_r}$$

$\underline{\Delta b_2}$

$$\{\mathbf{3},\mathbf{2},\tfrac{1}{6}\}:\ (\Delta b_2)_{\mathbf{10}_F^{SU(5)}} = \underbrace{\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{\tfrac{1}{2}}_{n_r} \cdot \underbrace{3}_{SU(3)\text{ triplet}} = 1$$

$$\{\overline{\mathbf{3}},\mathbf{1},-\tfrac{2}{3}\}:\ (\Delta b_2)_{\mathbf{10}_F^{SU(5)}} = \underbrace{0}_{n_r}$$

$$\{\mathbf{1},\mathbf{1},1\}:\ (\Delta b_2)_{\mathbf{10}_F^{SU(5)}} = \underbrace{0}_{n_r}$$

$\underline{\Delta b_1}$

$$\{\mathbf{3},\mathbf{2},\tfrac{1}{6}\}:\ (\Delta b_1)_{\mathbf{10}_F^{SU(5)}} = \underbrace{\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{3}_{SU(3)\text{ triplet}} \cdot \underbrace{2}_{SU(2)\text{ doublet}} \cdot \underbrace{\tfrac{3}{5}(\tfrac{1}{6})^2}_{n_r} = \frac{1}{15}$$

$$\{\overline{\mathbf{3}},\mathbf{1},-\tfrac{2}{3}\}:\ (\Delta b_1)_{\mathbf{10}_F^{SU(5)}} = \underbrace{\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{3}_{SU(3)\text{ triplet}} \cdot \underbrace{\tfrac{3}{5}(-\tfrac{2}{3})^2}_{n_r} = \frac{8}{15}$$

$$\{\mathbf{1},\mathbf{1},1\}:\ (\Delta b_1)_{\mathbf{10}_F^{SU(5)}} = \underbrace{\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{\tfrac{3}{5}(1)^2}_{n_r} = \frac{2}{5}$$

$\underline{\mathbf{24}_F}$

$\underline{\Delta b_3}$

$$\{\mathbf{8},\mathbf{1},0\}:\ (\Delta b_3)_{\mathbf{24}_F} = \underbrace{\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{3}_{n_r=N} = 2$$

$$\{\mathbf{1},\mathbf{3},0\}:\ (\Delta b_3)_{\mathbf{24}_F} = \underbrace{0}_{n_r}$$

$$\{\mathbf{1},\mathbf{1},0\}:\ (\Delta b_3)_{\mathbf{24}_F} = \underbrace{0}_{n_r}$$

$$\{\mathbf{3},\mathbf{2},\tfrac{1}{6}\}:\ (\Delta b_3)_{\mathbf{24}_F} = \underbrace{\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{\tfrac{1}{2}}_{n_r} \cdot \underbrace{2}_{SU(2)\text{ doublet}} = \frac{2}{3}$$

$$\{\overline{\mathbf{3}},\mathbf{2},-\tfrac{1}{6}\}:\ (\Delta b_3)_{\mathbf{24}_F} = \underbrace{\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{\tfrac{1}{2}}_{n_r} \cdot \underbrace{2}_{SU(2)\text{ doublet}} = \frac{2}{3}$$

$\underline{\Delta b_2}$

$$\{\mathbf{8},\mathbf{1},0\}:\ (\Delta b_2)_{\mathbf{24}_F} = \underbrace{0}_{n_r}$$

$$\{\mathbf{1},\mathbf{3},0\}:\ (\Delta b_2)_{\mathbf{24}_F} = \underbrace{\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{2}_{n_r=N} = \frac{4}{3}$$

$$\{\mathbf{1},\mathbf{1},0\}:\ (\Delta b_2)_{\mathbf{24}_F} = \underbrace{0}_{n_r}$$

$$\{\mathbf{3},\mathbf{2},\tfrac{1}{6}\}:\ (\Delta b_2)_{\mathbf{24}_F} = \underbrace{\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{\tfrac{1}{2}}_{n_r} \cdot \underbrace{3}_{SU(3)\text{ triplet}} = 1$$

$$\{\overline{\mathbf{3}},\mathbf{2},-\tfrac{1}{6}\}:\ (\Delta b_2)_{\mathbf{24}_F} = \underbrace{\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{\tfrac{1}{2}}_{n_r} \cdot \underbrace{3}_{SU(3)\text{ triplet}} = 1$$

$\underline{\Delta b_1}$

$$\{\mathbf{8},\mathbf{1},0\}:\ (\Delta b_1)_{\mathbf{24}_F} = \underbrace{0}_{n_r}$$

$$\{\mathbf{1},\mathbf{3},0\}:\ (\Delta b_1)_{\mathbf{24}_F} = \underbrace{0}_{n_r}$$

$$\{\mathbf{1},\mathbf{1},0\}:\ (\Delta b_1)_{\mathbf{24}_F} = \underbrace{0}_{n_r}$$

$$\{\mathbf{3},\mathbf{2},\tfrac{1}{6}\}:\ (\Delta b_1)_{\mathbf{24}_F} = \underbrace{\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{3}_{SU(3)\text{ triplet}} \cdot \underbrace{2}_{SU(2)\text{ doublet}} \cdot \underbrace{\tfrac{3}{5}(\tfrac{1}{6})^2}_{n_r} = \frac{1}{15}$$

$$\{\overline{\mathbf{3}},\mathbf{2},-\tfrac{1}{6}\}:\ (\Delta b_1)_{\mathbf{24}_F} = \underbrace{\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{3}_{SU(3)\text{ triplet}} \cdot \underbrace{2}_{SU(2)\text{ doublet}} \cdot \underbrace{\tfrac{3}{5}(\tfrac{1}{6})^2}_{n_r} = \frac{1}{15}$$

$\underline{\mathbf{10}^{SO(10)}{}_F}$

$\underline{\Delta b_3}$

$$\{\mathbf{3},\mathbf{1},-\tfrac{1}{3}\}:\ (\Delta b_3)_{\mathbf{10}_F^{SO(10)}} = \underbrace{\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{\tfrac{1}{2}}_{n_r} = \frac{1}{3}$$

$$\{\overline{\mathbf{3}},\mathbf{1},\tfrac{1}{3}\}:\ (\Delta b_3)_{\mathbf{10}_F^{SO(10)}} = \underbrace{\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{\tfrac{1}{2}}_{n_r} = \frac{1}{3}$$

$$\{\mathbf{1},\mathbf{2},\tfrac{1}{2}\}:\ (\Delta b_3)_{\mathbf{10}_F^{SO(10)}} = \underbrace{0}_{n_r}$$

$$\{\mathbf{1},\mathbf{2},-\tfrac{1}{2}\}:\ (\Delta b_3)_{\mathbf{10}_F^{SO(10)}} = \underbrace{0}_{n_r}$$

$\underline{\Delta b_2}$

$$\{\mathbf{3},\mathbf{1},-\tfrac{1}{3}\}:\ (\Delta b_2)_{\mathbf{10}_F^{SO(10)}} = \underbrace{0}_{n_r}$$

$$\{\overline{\mathbf{3}},\mathbf{1},\tfrac{1}{3}\}:\ (\Delta b_2)_{\mathbf{10}_F^{SO(10)}} = \underbrace{0}_{n_r}$$

$$\{\mathbf{1},\mathbf{2},\tfrac{1}{2}\}:\ (\Delta b_2)_{\mathbf{10}_F^{SO(10)}} = \underbrace{\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{\tfrac{1}{2}}_{n_r} = \frac{1}{3}$$

$$\{\mathbf{1},\mathbf{2},-\tfrac{1}{2}\}:\ (\Delta b_2)_{\mathbf{10}_F^{SO(10)}} = \underbrace{\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{\tfrac{1}{2}}_{n_r} = \frac{1}{3}$$

$\underline{\Delta b_1}$

$$\{\mathbf{3},\mathbf{1},-\tfrac{1}{3}\}:\ (\Delta b_1)_{\mathbf{10}_F^{SO(10)}} = \underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{3}_{SU(3)\text{ triplet}}\cdot\underbrace{\tfrac{3}{5}(-\tfrac{1}{3})^2}_{n_r} = \frac{2}{15}$$

$$\{\overline{\mathbf{3}},\mathbf{1},\tfrac{1}{3}\}:\ (\Delta b_1)_{\mathbf{10}_F^{SO(10)}} = \underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{3}_{SU(3)\text{ triplet}}\cdot\underbrace{\tfrac{3}{5}(\tfrac{1}{3})^2}_{n_r} = \frac{2}{15}$$

$$\{\mathbf{1},\mathbf{2},\tfrac{1}{2}\}:\ (\Delta b_1)_{\mathbf{10}_F^{SO(10)}} = \underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{2}_{SU(2)\text{ doublet}}\cdot\underbrace{\tfrac{3}{5}(\tfrac{1}{2})^2}_{n_r} = \frac{1}{5}$$

$$\{\mathbf{1},\mathbf{2},-\tfrac{1}{2}\}:\ (\Delta b_1)_{\mathbf{10}_F^{SO(10)}} = \underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{2}_{SU(2)\text{ doublet}}\cdot\underbrace{\tfrac{3}{5}(-\tfrac{1}{2})^2}_{n_r} = \frac{1}{5}$$

$\underline{\mathbf{16}_F}$

$\underline{\Delta b_3}$

$$\{\overline{\mathbf{3}},\mathbf{1},\tfrac{1}{3}\}:\ (\Delta b_3)_{\mathbf{16}_F} = \underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{\tfrac{1}{2}}_{n_r} = \frac{1}{3}$$

$$\{\mathbf{1},\mathbf{2},-\tfrac{1}{2}\}:\ (\Delta b_3)_{\mathbf{16}_F} = \underbrace{0}_{n_r}$$

$$\{\mathbf{3},\mathbf{2},\tfrac{1}{6}\}:\ (\Delta b_3)_{\mathbf{16}_F} = \underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{\tfrac{1}{2}}_{n_r}\cdot\underbrace{2}_{SU(2)\text{ doublet}} = \frac{2}{3}$$

$$\{\overline{\mathbf{3}},\mathbf{1},-\tfrac{2}{3}\}:\ (\Delta b_3)_{\mathbf{16}_F} = \underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{\tfrac{1}{2}}_{n_r} = \frac{1}{3}$$

$$\{\mathbf{1},\mathbf{1},1\}:\ (\Delta b_3)_{\mathbf{16}_F} = \underbrace{0}_{n_r}$$

$$\{\mathbf{1},\mathbf{1},0\}:\ (\Delta b_3)_{\mathbf{16}_F} = \underbrace{0}_{n_r}$$

$\underline{\Delta b_2}$

$$\{\overline{\mathbf{3}},\mathbf{1},\tfrac{1}{3}\}:\ (\Delta b_2)_{\mathbf{16}_F}=\underbrace{0}_{n_r}$$

$$\{\mathbf{1},\mathbf{2},-\tfrac{1}{2}\}:\ (\Delta b_2)_{\mathbf{16}_F}=\underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{\tfrac{1}{2}}_{n_r}=\frac{1}{3}$$

$$\{\mathbf{3},\mathbf{2},\tfrac{1}{6}\}:\ (\Delta b_2)_{\mathbf{16}_F}=\underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{\tfrac{1}{2}}_{n_r}\cdot\underbrace{3}_{SU(3)\ \text{triplet}}=1$$

$$\{\overline{\mathbf{3}},\mathbf{1},-\tfrac{2}{3}\}:\ (\Delta b_2)_{\mathbf{16}_F}=\underbrace{0}_{n_r}$$

$$\{\mathbf{1},\mathbf{1},1\}:\ (\Delta b_2)_{\mathbf{16}_F}=\underbrace{0}_{n_r}$$

$$\{\mathbf{1},\mathbf{1},0\}:\ (\Delta b_2)_{\mathbf{16}_F}=\underbrace{0}_{n_r}$$

$\underline{\Delta b_1}$

$$\{\overline{\mathbf{3}},\mathbf{1},\tfrac{1}{3}\}:\ (\Delta b_1)_{\mathbf{16}_F}=\underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{3}_{SU(3)\ \text{triplet}}\cdot\underbrace{\tfrac{3}{5}(\tfrac{1}{3})^2}_{n_r}=\frac{2}{15}$$

$$\{\mathbf{1},\mathbf{2},-\tfrac{1}{2}\}:\ (\Delta b_1)_{\mathbf{16}_F}=\underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{2}_{SU(2)\ \text{doublet}}\cdot\underbrace{\tfrac{3}{5}(-\tfrac{1}{2})^2}_{n_r}=\frac{1}{5}$$

$$\{\mathbf{3},\mathbf{2},\tfrac{1}{6}\}:\ (\Delta b_1)_{\mathbf{16}_F}=\underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{3}_{SU(3)\ \text{triplet}}\cdot\underbrace{2}_{SU(2)\ \text{doublet}}\cdot\underbrace{\tfrac{3}{5}(\tfrac{1}{6})^2}_{n_r}=\frac{1}{15}$$

$$\{\overline{\mathbf{3}},\mathbf{1},-\tfrac{2}{3}\}:\ (\Delta b_1)_{\mathbf{16}_F}=\underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{3}_{SU(3)\ \text{triplet}}\cdot\underbrace{\tfrac{3}{5}(-\tfrac{2}{3})^2}_{n_r}=\frac{8}{15}$$

$$\{\mathbf{1},\mathbf{1},1\}:\ (\Delta b_1)_{\mathbf{16}_F}=\underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{\tfrac{3}{5}(1)^2}_{n_r}=\frac{6}{15}$$

$$\{\mathbf{1},\mathbf{1},0\}:\ (\Delta b_1)_{\mathbf{16}_F}=\underbrace{0}_{n_r}$$

$\underline{\mathbf{45}_F}$

$\underline{\Delta b_3}$

$$\{\mathbf{3},\mathbf{1},\tfrac{2}{3}\}:\ (\Delta b_3)_{\mathbf{45}_F} = \underbrace{\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{\tfrac{1}{2}}_{n_r} = \frac{1}{3}$$

$$\{\overline{\mathbf{3}},\mathbf{1},-\tfrac{2}{3}\}:\ (\Delta b_3)_{\mathbf{45}_F} = \underbrace{\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{\tfrac{1}{2}}_{n_r} = \frac{1}{3}$$

$$\{\mathbf{3},\mathbf{2},\tfrac{1}{6}\}:\ (\Delta b_3)_{\mathbf{45}_F} = \underbrace{\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{\tfrac{1}{2}}_{n_r} \cdot \underbrace{2}_{SU(2)\text{ doublet}} = \frac{2}{3}$$

$$\{\overline{\mathbf{3}},\mathbf{2},-\tfrac{1}{6}\}:\ (\Delta b_3)_{\mathbf{45}_F} = \underbrace{\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{\tfrac{1}{2}}_{n_r} \cdot \underbrace{2}_{SU(2)\text{ doublet}} = \frac{2}{3}$$

$$\{\mathbf{3},\mathbf{2},-\tfrac{5}{6}\}:\ (\Delta b_3)_{\mathbf{45}_F} = \underbrace{\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{\tfrac{1}{2}}_{n_r} \cdot \underbrace{2}_{SU(2)\text{ doublet}} = \frac{2}{3}$$

$$\{\overline{\mathbf{3}},\mathbf{2},\tfrac{5}{6}\}:\ (\Delta b_3)_{\mathbf{45}_F} = \underbrace{\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{\tfrac{1}{2}}_{n_r} \cdot \underbrace{2}_{SU(2)\text{ doublet}} = \frac{2}{3}$$

$$\{\mathbf{8},\mathbf{1},0\}:\ (\Delta b_3)_{\mathbf{45}_F} = \underbrace{\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{3}_{n_r=N} = 2$$

$$\{\mathbf{1},\mathbf{3},0\}:\ (\Delta b_3)_{\mathbf{45}_F} = \underbrace{0}_{n_r}$$

$$\{\mathbf{1},\mathbf{1},0\}:\ (\Delta b_3)_{\mathbf{45}_F} = \underbrace{0}_{n_r}$$

$$\{\mathbf{1},\mathbf{1},1\}:\ (\Delta b_3)_{\mathbf{45}_F} = \underbrace{0}_{n_r}$$

$$\{\mathbf{1},\mathbf{1},0\}:\ (\Delta b_3)_{\mathbf{45}_F} = \underbrace{0}_{n_r}$$

$$\{\mathbf{1},\mathbf{1},-1\}:\ (\Delta b_3)_{\mathbf{45}_F} = \underbrace{0}_{n_r}$$

$\underline{\Delta b_2}$

$$\{\mathbf{3},\mathbf{1},\tfrac{2}{3}\}:\ (\Delta b_2)_{\mathbf{45}_F} = \underbrace{0}_{n_r}$$

$$\{\overline{\mathbf{3}},\mathbf{1},-\tfrac{2}{3}\}:\ (\Delta b_2)_{\mathbf{45}_F} = \underbrace{0}_{n_r}$$

$$\{\mathbf{3},\mathbf{2},\tfrac{1}{6}\}:\ (\Delta b_2)_{\mathbf{45}_F} = \underbrace{\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{\tfrac{1}{2}}_{n_r} \cdot \underbrace{3}_{SU(3)\text{ triplet}} = 1$$

$$\{\overline{\mathbf{3}},\mathbf{2},-\tfrac{1}{6}\}:\ (\Delta b_2)_{\mathbf{45}_F} = \underbrace{\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{\tfrac{1}{2}}_{n_r} \cdot \underbrace{3}_{SU(3)\text{ triplet}} = 1$$

$$\{\mathbf{3},\mathbf{2},-\tfrac{5}{6}\}:\ (\Delta b_2)_{\mathbf{45}_F} = \underbrace{\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{\tfrac{1}{2}}_{n_r} \cdot \underbrace{3}_{SU(3)\text{ triplet}} = 1$$

$$\{\overline{\mathbf{3}},\mathbf{2},\tfrac{5}{6}\}:\ (\Delta b_2)_{\mathbf{45}_F} = \underbrace{\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{\tfrac{1}{2}}_{n_r} \cdot \underbrace{3}_{SU(3)\text{ triplet}} = 1$$

$$\{\mathbf{8},\mathbf{1},0\}:\ (\Delta b_2)_{\mathbf{45}_F} = \underbrace{0}_{n_r}$$

$$\{\mathbf{1},\mathbf{3},0\}:\ (\Delta b_2)_{\mathbf{45}_F} = \underbrace{\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{2}_{n_r=N} = \frac{4}{3}$$

$$\{\mathbf{1},\mathbf{1},0\}:\ (\Delta b_2)_{\mathbf{45}_F} = \underbrace{0}_{n_r}$$

$$\{\mathbf{1},\mathbf{1},1\}:\ (\Delta b_2)_{\mathbf{45}_F} = \underbrace{0}_{n_r}$$

$$\{\mathbf{1},\mathbf{1},0\}:\ (\Delta b_2)_{\mathbf{45}_F} = \underbrace{0}_{n_r}$$

$$\{\mathbf{1},\mathbf{1},-1\}:\ (\Delta b_2)_{\mathbf{45}_F} = \underbrace{0}_{n_r}$$

$\underline{\Delta b_1}$

$$\{\mathbf{3},\mathbf{1},\tfrac{2}{3}\}:\ (\Delta b_1)_{\mathbf{45}_F}=\underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{3}_{SU(3)\text{ triplet}}\cdot\underbrace{\tfrac{3}{5}(\tfrac{2}{3})^2}_{n_r}=\frac{8}{15}$$

$$\{\overline{\mathbf{3}},\mathbf{1},-\tfrac{2}{3}\}:\ (\Delta b_1)_{\mathbf{45}_F}=\underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{3}_{SU(3)\text{ triplet}}\cdot\underbrace{\tfrac{3}{5}(-\tfrac{2}{3})^2}_{n_r}=\frac{8}{15}$$

$$\{\mathbf{3},\mathbf{2},\tfrac{1}{6}\}:\ (\Delta b_1)_{\mathbf{45}_F}=\underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{3}_{SU(3)\text{ triplet}}\cdot\underbrace{2}_{SU(2)\text{ doublet}}\cdot\underbrace{\tfrac{3}{5}(\tfrac{1}{6})^2}_{n_r}=\frac{1}{15}$$

$$\{\overline{\mathbf{3}},\mathbf{2},-\tfrac{1}{6}\}:\ (\Delta b_1)_{\mathbf{45}_F}=\underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{3}_{SU(3)\text{ triplet}}\cdot\underbrace{2}_{SU(2)\text{ doublet}}\cdot\underbrace{\tfrac{3}{5}(-\tfrac{1}{6})^2}_{n_r}=\frac{1}{15}$$

$$\{\mathbf{3},\mathbf{2},-\tfrac{5}{6}\}:\ (\Delta b_1)_{\mathbf{45}_F}=\underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{3}_{SU(3)\text{ triplet}}\cdot\underbrace{2}_{SU(2)\text{ doublet}}\cdot\underbrace{\tfrac{3}{5}(-\tfrac{5}{6})^2}_{n_r}=\frac{5}{3}$$

$$\{\overline{\mathbf{3}},\mathbf{2},\tfrac{5}{6}\}:\ (\Delta b_1)_{\mathbf{45}_F}=\underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{3}_{SU(3)\text{ triplet}}\cdot\underbrace{2}_{SU(2)\text{ doublet}}\cdot\underbrace{\tfrac{3}{5}(\tfrac{5}{6})^2}_{n_r}=\frac{5}{3}$$

$$\{\mathbf{8},\mathbf{1},0\}:\ (\Delta b_1)_{\mathbf{45}_F}=\underbrace{0}_{n_r}$$

$$\{\mathbf{1},\mathbf{3},0\}:\ (\Delta b_1)_{\mathbf{45}_F}=\underbrace{0}_{n_r}$$

$$\{\mathbf{1},\mathbf{1},0\}:\ (\Delta b_1)_{\mathbf{45}_F}=\underbrace{0}_{n_r}$$

$$\{\mathbf{1},\mathbf{1},1\}:\ (\Delta b_1)_{\mathbf{45}_F}=\underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{\tfrac{3}{5}(1)^2}_{n_r}=\frac{2}{5}$$

$$\{\mathbf{1},\mathbf{1},0\}:\ (\Delta b_1)_{\mathbf{45}_F}=\underbrace{0}_{n_r}$$

$$\{\mathbf{1},\mathbf{1},-1\}:\ (\Delta b_1)_{\mathbf{45}_F}=\underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{\tfrac{3}{5}(-1)^2}_{n_r}=\frac{2}{5}$$

$\underline{\mathbf{54}_F}$

Calculated using the table from [19], $n_r = \frac{5}{2}$ for the $\mathbf{6}$ of $SU(3)$.

$\underline{\Delta b_3}$

$$\{\mathbf{6}, \mathbf{1}, -\tfrac{2}{3}\}: \ (\Delta b_3)_{\mathbf{54}_F} = \underbrace{\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{\tfrac{5}{2}}_{n_r} = \frac{5}{3}$$

$$\{\overline{\mathbf{6}}, \mathbf{1}, \tfrac{2}{3}\}: \ (\Delta b_3)_{\mathbf{54}_F} = \underbrace{\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{\tfrac{5}{2}}_{n_r} = \frac{5}{3}$$

$$\{\mathbf{3}, \mathbf{2}, \tfrac{1}{6}\}: \ (\Delta b_3)_{\mathbf{54}_F} = \underbrace{\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{\tfrac{1}{2}}_{n_r} \cdot \underbrace{2}_{SU(2)\text{ doublet}} = \frac{2}{3}$$

$$\{\overline{\mathbf{3}}, \mathbf{2}, -\tfrac{1}{6}\}: \ (\Delta b_3)_{\mathbf{54}_F} = \underbrace{\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{\tfrac{1}{2}}_{n_r} \cdot \underbrace{2}_{SU(2)\text{ doublet}} = \frac{2}{3}$$

$$\{\mathbf{3}, \mathbf{2}, -\tfrac{5}{6}\}: \ (\Delta b_3)_{\mathbf{54}_F} = \underbrace{\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{\tfrac{1}{2}}_{n_r} \cdot \underbrace{2}_{SU(2)\text{ doublet}} = \frac{2}{3}$$

$$\{\overline{\mathbf{3}}, \mathbf{2}, \tfrac{5}{6}\}: \ (\Delta b_3)_{\mathbf{54}_F} = \underbrace{\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{\tfrac{1}{2}}_{n_r} \cdot \underbrace{2}_{SU(2)\text{ doublet}} = \frac{2}{3}$$

$$\{\mathbf{8}, \mathbf{1}, 0\}: \ (\Delta b_3)_{\mathbf{54}_F} = \underbrace{\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{3}_{n_r = N} = 2$$

$$\{\mathbf{1}, \mathbf{1}, 0\}: \ (\Delta b_3)_{\mathbf{54}_F} = \underbrace{0}_{n_r}$$

$$\{\mathbf{1}, \mathbf{3}, 1\}: \ (\Delta b_3)_{\mathbf{54}_F} = \underbrace{0}_{n_r}$$

$$\{\mathbf{1}, \mathbf{3}, 0\}: \ (\Delta b_3)_{\mathbf{54}_F} = \underbrace{0}_{n_r}$$

$$\{\mathbf{1}, \mathbf{3}, -1\}: \ (\Delta b_3)_{\mathbf{54}_F} = \underbrace{0}_{n_r}$$

$\underline{\Delta b_2}$

$$\{\mathbf{6},\mathbf{1},-\tfrac{2}{3}\}:\ (\Delta b_2)_{\mathbf{54}_F}=\underbrace{0}_{n_r}$$

$$\{\overline{\mathbf{6}},\mathbf{1},\tfrac{2}{3}\}:\ (\Delta b_2)_{\mathbf{54}_F}=\underbrace{0}_{n_r}$$

$$\{\mathbf{3},\mathbf{2},\tfrac{1}{6}\}:\ (\Delta b_2)_{\mathbf{54}_F}=\underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{\tfrac{1}{2}}_{n_r}\cdot\underbrace{3}_{SU(3)\ \text{triplet}}=1$$

$$\{\overline{\mathbf{3}},\mathbf{2},-\tfrac{1}{6}\}:\ (\Delta b_2)_{\mathbf{54}_F}=\underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{\tfrac{1}{2}}_{n_r}\cdot\underbrace{3}_{SU(3)\ \text{triplet}}=1$$

$$\{\mathbf{3},\mathbf{2},-\tfrac{5}{6}\}:\ (\Delta b_2)_{\mathbf{54}_F}=\underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{\tfrac{1}{2}}_{n_r}\cdot\underbrace{3}_{SU(3)\ \text{triplet}}=1$$

$$\{\overline{\mathbf{3}},\mathbf{2},\tfrac{5}{6}\}:\ (\Delta b_2)_{\mathbf{54}_F}=\underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{\tfrac{1}{2}}_{n_r}\cdot\underbrace{3}_{SU(3)\ \text{triplet}}=1$$

$$\{\mathbf{8},\mathbf{1},0\}:\ (\Delta b_2)_{\mathbf{54}_F}=\underbrace{0}_{n_r}$$

$$\{\mathbf{1},\mathbf{1},0\}:\ (\Delta b_2)_{\mathbf{54}_F}=\underbrace{0}_{n_r}$$

$$\{\mathbf{1},\mathbf{3},1\}:\ (\Delta b_2)_{\mathbf{54}_F}=\underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{2}_{n_r=N}=\frac{4}{3}$$

$$\{\mathbf{1},\mathbf{3},0\}:\ (\Delta b_2)_{\mathbf{54}_F}=\underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{2}_{n_r=N}=\frac{4}{3}$$

$$\{\mathbf{1},\mathbf{3},-1\}:\ (\Delta b_2)_{\mathbf{54}_F}=\underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{2}_{n_r=N}=\frac{4}{3}$$

$\underline{\Delta b_1}$

$$\{\mathbf{6},\mathbf{1},-\tfrac{2}{3}\}:\ (\Delta b_1)_{\mathbf{54}_F} = \underbrace{\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{6}_{SU(3)\text{ sextet}} \cdot \underbrace{\tfrac{3}{5}(-\tfrac{2}{3})^2}_{n_r} = \frac{16}{15}$$

$$\{\overline{\mathbf{6}},\mathbf{1},\tfrac{2}{3}\}:\ (\Delta b_1)_{\mathbf{54}_F} = \underbrace{\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{6}_{SU(3)\text{ sextet}} \cdot \underbrace{\tfrac{3}{5}(\tfrac{2}{3})^2}_{n_r} = \frac{16}{15}$$

$$\{\mathbf{3},\mathbf{2},\tfrac{1}{6}\}:\ (\Delta b_1)_{\mathbf{45}_F} = \underbrace{\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{3}_{SU(3)\text{ triplet}} \cdot \underbrace{2}_{SU(2)\text{ doublet}} \cdot \underbrace{\tfrac{3}{5}(\tfrac{1}{6})^2}_{n_r} = \frac{1}{15}$$

$$\{\overline{\mathbf{3}},\mathbf{2},-\tfrac{1}{6}\}:\ (\Delta b_1)_{\mathbf{45}_F} = \underbrace{\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{3}_{SU(3)\text{ triplet}} \cdot \underbrace{2}_{SU(2)\text{ doublet}} \cdot \underbrace{\tfrac{3}{5}(-\tfrac{1}{6})^2}_{n_r} = \frac{1}{15}$$

$$\{\mathbf{3},\mathbf{2},-\tfrac{5}{6}\}:\ (\Delta b_1)_{\mathbf{45}_F} = \underbrace{\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{3}_{SU(3)\text{ triplet}} \cdot \underbrace{2}_{SU(2)\text{ doublet}} \cdot \underbrace{\tfrac{3}{5}(-\tfrac{5}{6})^2}_{n_r} = \frac{5}{3}$$

$$\{\overline{\mathbf{3}},\mathbf{2},\tfrac{5}{6}\}:\ (\Delta b_1)_{\mathbf{45}_F} = \underbrace{\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{3}_{SU(3)\text{ triplet}} \cdot \underbrace{2}_{SU(2)\text{ doublet}} \cdot \underbrace{\tfrac{3}{5}(\tfrac{5}{6})^2}_{n_r} = \frac{5}{3}$$

$$\{\mathbf{8},\mathbf{1},0\}:\ (\Delta b_1)_{\mathbf{54}_F} = \underbrace{0}_{n_r}$$

$$\{\mathbf{1},\mathbf{1},0\}:\ (\Delta b_1)_{\mathbf{54}_F} = \underbrace{0}_{n_r}$$

$$\{\mathbf{1},\mathbf{3},1\}:\ (\Delta b_1)_{\mathbf{54}_F} = \underbrace{\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{3}_{SU(2)\text{ triplet}} \cdot \underbrace{\tfrac{3}{5}(1)^2}_{n_r} = \frac{6}{5}$$

$$\{\mathbf{1},\mathbf{3},0\}:\ (\Delta b_1)_{\mathbf{54}_F} = \underbrace{0}_{n_r}$$

$$\{\mathbf{1},\mathbf{3},-1\}:\ (\Delta b_1)_{\mathbf{54}_F} = \underbrace{\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{3}_{SU(2)\text{ triplet}} \cdot \underbrace{\tfrac{3}{5}(-1)^2}_{n_r} = \frac{6}{5}$$

$\mathbf{120}_F$

$\underline{\Delta b_3}$

$$\{\mathbf{1},\mathbf{2},\tfrac{1}{2}\}:\ (\Delta b_3)_{\mathbf{120}_F}=\underbrace{0}_{n_r}$$

$$\{\mathbf{1},\mathbf{2},-\tfrac{1}{2}\}:\ (\Delta b_3)_{\mathbf{120}_F}=\underbrace{0}_{n_r}$$

$$\{\mathbf{1},\mathbf{2},\tfrac{1}{2}\}:\ (\Delta b_3)_{\mathbf{120}_F}=\underbrace{0}_{n_r}$$

$$\{\mathbf{1},\mathbf{2},-\tfrac{1}{2}\}:\ (\Delta b_3)_{\mathbf{120}_F}=\underbrace{0}_{n_r}$$

$$\{\mathbf{1},\mathbf{1},-1\}:\ (\Delta b_3)_{\mathbf{120}_F}=\underbrace{0}_{n_r}$$

$$\{\mathbf{1},\mathbf{1},1\}:\ (\Delta b_3)_{\mathbf{120}_F}=\underbrace{0}_{n_r}$$

$$\{\mathbf{3},\mathbf{1},-\tfrac{1}{3}\}:\ (\Delta b_3)_{\mathbf{120}_F}=\underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{\tfrac{1}{2}}_{n_r}=\frac{1}{3}$$

$$\{\overline{\mathbf{3}},\mathbf{1},\tfrac{1}{3}\}:\ (\Delta b_3)_{\mathbf{120}_F}=\underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{\tfrac{1}{2}}_{n_r}=\frac{1}{3}$$

$$\{\mathbf{3},\mathbf{1},-\tfrac{1}{3}\}:\ (\Delta b_3)_{\mathbf{120}_F}=\underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{\tfrac{1}{2}}_{n_r}=\frac{1}{3}$$

$$\{\overline{\mathbf{3}},\mathbf{1},\tfrac{1}{3}\}:\ (\Delta b_3)_{\mathbf{120}_F}=\underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{\tfrac{1}{2}}_{n_r}=\frac{1}{3}$$

$$\{\mathbf{3},\mathbf{1},\tfrac{2}{3}\}:\ (\Delta b_3)_{\mathbf{120}_F}=\underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{\tfrac{1}{2}}_{n_r}=\frac{1}{3}$$

$$\{\overline{\mathbf{3}},\mathbf{1},-\tfrac{2}{3}\}:\ (\Delta b_3)_{\mathbf{120}_F}=\underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{\tfrac{1}{2}}_{n_r}=\frac{1}{3}$$

$$\{\mathbf{3},\mathbf{1},-\tfrac{4}{3}\}:\ (\Delta b_3)_{\mathbf{120}_F}=\underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{\tfrac{1}{2}}_{n_r}=\frac{1}{3}$$

$$\{\overline{\mathbf{3}},\mathbf{1},\tfrac{4}{3}\}:\ (\Delta b_3)_{\mathbf{120}_F}=\underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{\tfrac{1}{2}}_{n_r}=\frac{1}{3}$$

$$\{\mathbf{6},\mathbf{1},\tfrac{1}{3}\}:\ (\Delta b_3)_{\mathbf{120}_F}=\underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{\tfrac{5}{2}}_{n_r}=\frac{5}{3}$$

$$\{\overline{\mathbf{6}},\mathbf{1},-\tfrac{1}{3}\}:\ (\Delta b_3)_{\mathbf{120}_F}=\underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{\tfrac{5}{2}}_{n_r}=\frac{5}{3}$$

$$\{\mathbf{3},\mathbf{3},-\tfrac{1}{3}\}:\ (\Delta b_3)_{\mathbf{120}_F}=\underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{3}_{SU(2)\text{ triplet}}\cdot\underbrace{\tfrac{1}{2}}_{n_r}=1$$

$$\{\overline{\mathbf{3}},\mathbf{3},\tfrac{1}{3}\}:\ (\Delta b_3)_{\mathbf{120}_F}=\underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{3}_{SU(2)\text{ triplet}}\cdot\underbrace{\tfrac{1}{2}}_{n_r}=1$$

$$\{\mathbf{3},\mathbf{2},\tfrac{1}{6}\}:\ (\Delta b_3)_{\mathbf{120}_F}=\underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{2}_{SU(2)\text{ doublet}}\cdot\underbrace{\tfrac{1}{2}}_{n_r}=\frac{2}{3}$$

$$\{\overline{\mathbf{3}},\mathbf{2},-\tfrac{1}{6}\}:\ (\Delta b_3)_{\mathbf{120}_F}=\underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{2}_{SU(2)\text{ doublet}}\cdot\underbrace{\tfrac{1}{2}}_{n_r}=\frac{2}{3}$$

$$\{\mathbf{3},\mathbf{2},\tfrac{7}{6}\}:\ (\Delta b_3)_{\mathbf{120}_F}=\underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{2}_{SU(2)\text{ doublet}}\cdot\underbrace{\tfrac{1}{2}}_{n_r}=\frac{2}{3}$$

$$\{\overline{\mathbf{3}},\mathbf{2},-\tfrac{7}{6}\}:\ (\Delta b_3)_{\mathbf{120}_F}=\underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{2}_{SU(2)\text{ doublet}}\cdot\underbrace{\tfrac{1}{2}}_{n_r}=\frac{2}{3}$$

$$\{\mathbf{8},\mathbf{2},\tfrac{1}{2}\}:\ (\Delta b_3)_{\mathbf{120}_F}=\underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{2}_{SU(2)\text{ doublet}}\cdot\underbrace{3}_{n_r=N}=4$$

$$\{\mathbf{8},\mathbf{2},-\tfrac{1}{2}\}:\ (\Delta b_3)_{\mathbf{120}_F}=\underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{2}_{SU(2)\text{ doublet}}\cdot\underbrace{3}_{n_r=N}=4$$

$\underline{\Delta b_2}$

$$\{\mathbf{1},\mathbf{2},\tfrac{1}{2}\}: \ (\Delta b_2)_{\mathbf{120}_F} = \underbrace{\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{\tfrac{1}{2}}_{n_r} = \frac{1}{3}$$

$$\{\mathbf{1},\mathbf{2},-\tfrac{1}{2}\}: \ (\Delta b_2)_{\mathbf{120}_F} = \underbrace{\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{\tfrac{1}{2}}_{n_r} = \frac{1}{3}$$

$$\{\mathbf{1},\mathbf{2},\tfrac{1}{2}\}: \ (\Delta b_2)_{\mathbf{120}_F} = \underbrace{\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{\tfrac{1}{2}}_{n_r} = \frac{1}{3}$$

$$\{\mathbf{1},\mathbf{2},-\tfrac{1}{2}\}: \ (\Delta b_2)_{\mathbf{120}_F} = \underbrace{\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{\tfrac{1}{2}}_{n_r} = \frac{1}{3}$$

$$\{\mathbf{1},\mathbf{1},-1\}: \ (\Delta b_2)_{\mathbf{120}_F} = \underbrace{0}_{n_r}$$

$$\{\mathbf{1},\mathbf{1},1\}: \ (\Delta b_2)_{\mathbf{120}_F} = \underbrace{0}_{n_r}$$

$$\{\mathbf{3},\mathbf{1},-\tfrac{1}{3}\}: \ (\Delta b_2)_{\mathbf{120}_F} = \underbrace{0}_{n_r}$$

$$\{\overline{\mathbf{3}},\mathbf{1},\tfrac{1}{3}\}: \ (\Delta b_2)_{\mathbf{120}_F} = \underbrace{0}_{n_r}$$

$$\{\mathbf{3},\mathbf{1},-\tfrac{1}{3}\}: \ (\Delta b_2)_{\mathbf{120}_F} = \underbrace{0}_{n_r}$$

$$\{\overline{\mathbf{3}},\mathbf{1},\tfrac{1}{3}\}: \ (\Delta b_2)_{\mathbf{120}_F} = \underbrace{0}_{n_r}$$

$$\{\mathbf{3},\mathbf{1},\tfrac{2}{3}\}: \ (\Delta b_2)_{\mathbf{120}_F} = \underbrace{0}_{n_r}$$

$$\{\overline{\mathbf{3}},\mathbf{1},-\tfrac{2}{3}\}: \ (\Delta b_2)_{\mathbf{120}_F} = \underbrace{0}_{n_r}$$

$$\{\mathbf{3},\mathbf{1},-\tfrac{4}{3}\}:\ (\Delta b_2)_{\mathbf{120}_F}=\underbrace{0}_{n_r}$$

$$\{\overline{\mathbf{3}},\mathbf{1},\tfrac{4}{3}\}:\ (\Delta b_2)_{\mathbf{120}_F}=\underbrace{0}_{n_r}$$

$$\{\mathbf{6},\mathbf{1},\tfrac{1}{3}\}:\ (\Delta b_2)_{\mathbf{120}_F}=\underbrace{0}_{n_r}$$

$$\{\overline{\mathbf{6}},\mathbf{1},-\tfrac{1}{3}\}:\ (\Delta b_2)_{\mathbf{120}_F}=\underbrace{0}_{n_r}$$

$$\{\mathbf{3},\mathbf{3},-\tfrac{1}{3}\}:\ (\Delta b_2)_{\mathbf{120}_F}=\underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{3}_{SU(3)\text{ triplet}}\cdot\underbrace{2}_{n_r=N}=4$$

$$\{\overline{\mathbf{3}},\mathbf{3},\tfrac{1}{3}\}:\ (\Delta b_2)_{\mathbf{120}_F}=\underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{3}_{SU(3)\text{ triplet}}\cdot\underbrace{2}_{n_r=N}=4$$

$$\{\mathbf{3},\mathbf{2},\tfrac{1}{6}\}:\ (\Delta b_2)_{\mathbf{120}_F}=\underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{3}_{SU(3)\text{ triplet}}\cdot\underbrace{\tfrac{1}{2}}_{n_r}=1$$

$$\{\overline{\mathbf{3}},\mathbf{2},-\tfrac{1}{6}\}:\ (\Delta b_2)_{\mathbf{120}_F}=\underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{3}_{SU(3)\text{ triplet}}\cdot\underbrace{\tfrac{1}{2}}_{n_r}=1$$

$$\{\mathbf{3},\mathbf{2},\tfrac{7}{6}\}:\ (\Delta b_2)_{\mathbf{120}_F}=\underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{3}_{SU(3)\text{ triplet}}\cdot\underbrace{\tfrac{1}{2}}_{n_r}=1$$

$$\{\overline{\mathbf{3}},\mathbf{2},-\tfrac{7}{6}\}:\ (\Delta b_2)_{\mathbf{120}_F}=\underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{3}_{SU(3)\text{ triplet}}\cdot\underbrace{\tfrac{1}{2}}_{n_r}=1$$

$$\{\mathbf{8},\mathbf{2},\tfrac{1}{2}\}:\ (\Delta b_2)_{\mathbf{120}_F}=\underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{8}_{SU(3)\text{ octet}}\cdot\underbrace{\tfrac{1}{2}}_{n_r}=\frac{8}{3}$$

$$\{\mathbf{8},\mathbf{2},-\tfrac{1}{2}\}:\ (\Delta b_2)_{\mathbf{120}_F}=\underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{8}_{SU(3)\text{ octet}}\cdot\underbrace{\tfrac{1}{2}}_{n_r}=\frac{8}{3}$$

$\underline{\Delta b_1}$

$$\{\mathbf{1},\mathbf{2},\tfrac{1}{2}\}:\ (\Delta b_1)_{\mathbf{120}_F}=\underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{2}_{SU(2)\text{ doublet}}\cdot\underbrace{\tfrac{3}{5}(\tfrac{1}{2})^2}_{n_r}=\frac{1}{5}$$

$$\{\mathbf{1},\mathbf{2},-\tfrac{1}{2}\}:\ (\Delta b_1)_{\mathbf{120}_F}=\underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{2}_{SU(2)\text{ doublet}}\cdot\underbrace{\tfrac{3}{5}(-\tfrac{1}{2})^2}_{n_r}=\frac{1}{5}$$

$$\{\mathbf{1},\mathbf{2},\tfrac{1}{2}\}:\ (\Delta b_1)_{\mathbf{120}_F}=\underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{2}_{SU(2)\text{ doublet}}\cdot\underbrace{\tfrac{3}{5}(\tfrac{1}{2})^2}_{n_r}=\frac{1}{5}$$

$$\{\mathbf{1},\mathbf{2},-\tfrac{1}{2}\}:\ (\Delta b_1)_{\mathbf{120}_F}=\underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{2}_{SU(2)\text{ doublet}}\cdot\underbrace{\tfrac{3}{5}(-\tfrac{1}{2})^2}_{n_r}=\frac{1}{5}$$

$$\{\mathbf{1},\mathbf{1},-1\}:\ (\Delta b_1)_{\mathbf{120}_F}=\underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{\tfrac{3}{5}(-1)^2}_{n_r}=\frac{2}{5}$$

$$\{\mathbf{1},\mathbf{1},1\}:\ (\Delta b_1)_{\mathbf{120}_F}=\underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{\tfrac{3}{5}(1)^2}_{n_r}=\frac{2}{5}$$

$$\{\mathbf{3},\mathbf{1},-\tfrac{1}{3}\}:\ (\Delta b_1)_{\mathbf{120}_F}=\underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{3}_{SU(3)\text{ triplet}}\cdot\underbrace{\tfrac{3}{5}(-\tfrac{1}{3})^2}_{n_r}=\frac{2}{15}$$

$$\{\overline{\mathbf{3}},\mathbf{1},\tfrac{1}{3}\}:\ (\Delta b_1)_{\mathbf{120}_F}=\underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{3}_{SU(3)\text{ triplet}}\cdot\underbrace{\tfrac{3}{5}(\tfrac{1}{3})^2}_{n_r}=\frac{2}{15}$$

$$\{\mathbf{3},\mathbf{1},-\tfrac{1}{3}\}:\ (\Delta b_1)_{\mathbf{120}_F}=\underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{3}_{SU(3)\text{ triplet}}\cdot\underbrace{\tfrac{3}{5}(-\tfrac{1}{3})^2}_{n_r}=\frac{2}{15}$$

$$\{\overline{\mathbf{3}},\mathbf{1},\tfrac{1}{3}\}:\ (\Delta b_1)_{\mathbf{120}_F}=\underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{3}_{SU(3)\text{ triplet}}\cdot\underbrace{\tfrac{3}{5}(\tfrac{1}{3})^2}_{n_r}=\frac{2}{15}$$

$$\{\mathbf{3},\mathbf{1},\tfrac{2}{3}\}:\ (\Delta b_1)_{\mathbf{120}_F}=\underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{3}_{SU(3)\text{ triplet}}\cdot\underbrace{\tfrac{3}{5}(\tfrac{2}{3})^2}_{n_r}=\frac{8}{15}$$

$$\{\overline{\mathbf{3}},\mathbf{1},-\tfrac{2}{3}\}:\ (\Delta b_1)_{\mathbf{120}_F}=\underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{3}_{SU(3)\text{ triplet}}\cdot\underbrace{\tfrac{3}{5}(-\tfrac{2}{3})^2}_{n_r}=\frac{8}{15}$$

$$\{\mathbf{3},\mathbf{1},-\tfrac{4}{3}\}:\ (\Delta b_1)_{\mathbf{120}_F}=\underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{3}_{SU(3)\text{ triplet}}\cdot\underbrace{\tfrac{3}{5}(-\tfrac{4}{3})^2}_{n_r}=\frac{32}{15}$$

$$\{\overline{\mathbf{3}},\mathbf{1},\tfrac{4}{3}\}:\ (\Delta b_1)_{\mathbf{120}_F}=\underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{3}_{SU(3)\text{ triplet}}\cdot\underbrace{\tfrac{3}{5}(\tfrac{4}{3})^2}_{n_r}=\frac{32}{15}$$

$$\{\mathbf{6},\mathbf{1},\tfrac{1}{3}\}:\ (\Delta b_1)_{\mathbf{120}_F}=\underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{6}_{SU(3)\text{ octet}}\cdot\underbrace{\tfrac{3}{5}(\tfrac{1}{3})^2}_{n_r}=\frac{4}{15}$$

$$\{\overline{\mathbf{6}},\mathbf{1},-\tfrac{1}{3}\}:\ (\Delta b_1)_{\mathbf{120}_F}=\underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{6}_{SU(3)\text{ octet}}\cdot\underbrace{\tfrac{3}{5}(-\tfrac{1}{3})^2}_{n_r}=\frac{4}{15}$$

$$\{\mathbf{3},\mathbf{3},-\tfrac{1}{3}\}:\ (\Delta b_1)_{\mathbf{120}_F}=\underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{3}_{SU(3)\text{ triplet}}\cdot\underbrace{3}_{SU(2)\text{ triplet}}\cdot\underbrace{\tfrac{3}{5}(-\tfrac{1}{3})^2}_{n_r}=\frac{2}{5}$$

$$\{\overline{\mathbf{3}},\mathbf{3},\tfrac{1}{3}\}:\ (\Delta b_1)_{\mathbf{120}_F}=\underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{3}_{SU(3)\text{ triplet}}\cdot\underbrace{3}_{SU(2)\text{ triplet}}\cdot\underbrace{\tfrac{3}{5}(\tfrac{1}{3})^2}_{n_r}=\frac{2}{5}$$

$$\{\mathbf{3},\mathbf{2},\tfrac{1}{6}\}:\ (\Delta b_1)_{\mathbf{120}_F}=\underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{3}_{SU(3)\text{ triplet}}\cdot\underbrace{2}_{SU(2)\text{ doublet}}\cdot\underbrace{\tfrac{3}{5}(\tfrac{1}{6})^2}_{n_r}=\frac{1}{15}$$

$$\{\overline{\mathbf{3}},\mathbf{2},-\tfrac{1}{6}\}:\ (\Delta b_1)_{\mathbf{120}_F}=\underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{3}_{SU(3)\text{ triplet}}\cdot\underbrace{2}_{SU(2)\text{ doublet}}\cdot\underbrace{\tfrac{3}{5}(-\tfrac{1}{6})^2}_{n_r}=\frac{1}{15}$$

$$\{\mathbf{3},\mathbf{2},\tfrac{7}{6}\}:\ (\Delta b_1)_{\mathbf{120}_F}=\underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{3}_{SU(3)\text{ triplet}}\cdot\underbrace{2}_{SU(2)\text{ doublet}}\cdot\underbrace{\tfrac{3}{5}(\tfrac{7}{6})^2}_{n_r}=\frac{49}{15}$$

$$\{\overline{\mathbf{3}},\mathbf{2},-\tfrac{7}{6}\}:\ (\Delta b_1)_{\mathbf{120}_F}=\underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{3}_{SU(3)\text{ triplet}}\cdot\underbrace{2}_{SU(2)\text{ doublet}}\cdot\underbrace{\tfrac{3}{5}(-\tfrac{7}{6})^2}_{n_r}=\frac{49}{15}$$

$$\{\mathbf{8},\mathbf{2},\tfrac{1}{2}\}:\ (\Delta b_1)_{\mathbf{120}_F}=\underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{8}_{SU(3)\text{ octet}}\cdot\underbrace{2}_{SU(2)\text{ doublet}}\cdot\underbrace{\tfrac{3}{5}(\tfrac{1}{2})^2}_{n_r}=\frac{8}{5}$$

$$\{\mathbf{8},\mathbf{2},-\tfrac{1}{2}\}:\ (\Delta b_1)_{\mathbf{120}_F}=\underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{8}_{SU(3)\text{ octet}}\cdot\underbrace{2}_{SU(2)\text{ doublet}}\cdot\underbrace{\tfrac{3}{5}(-\tfrac{1}{2})^2}_{n_r}=\frac{8}{5}$$

$\underline{\mathbf{126}_F}$

$\underline{\Delta b_3}$

$$\{\mathbf{1}, \mathbf{2}, \tfrac{1}{2}\}: \ (\Delta b_3)_{\mathbf{126}_F} = \underbrace{0}_{n_r}$$

$$\{\mathbf{1}, \mathbf{2}, -\tfrac{1}{2}\}: \ (\Delta b_3)_{\mathbf{126}_F} = \underbrace{0}_{n_r}$$

$$\{\mathbf{1}, \mathbf{1}, 0\}: \ (\Delta b_3)_{\mathbf{126}_F} = \underbrace{0}_{n_r}$$

$$\{\mathbf{1}, \mathbf{1}, -1\}: \ (\Delta b_3)_{\mathbf{126}_F} = \underbrace{0}_{n_r}$$

$$\{\mathbf{1}, \mathbf{1}, -2\}: \ (\Delta b_3)_{\mathbf{126}_F} = \underbrace{0}_{n_r}$$

$$\{\mathbf{3}, \mathbf{1}, -\tfrac{1}{3}\}: \ (\Delta b_3)_{\mathbf{126}_F} = \underbrace{\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{\tfrac{1}{2}}_{n_r} = \frac{1}{3}$$

$$\{\mathbf{3}, \mathbf{1}, -\tfrac{1}{3}\}: \ (\Delta b_3)_{\mathbf{126}_F} = \underbrace{\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{\tfrac{1}{2}}_{n_r} = \frac{1}{3}$$

$$\{\overline{\mathbf{3}}, \mathbf{1}, \tfrac{1}{3}\}: \ (\Delta b_3)_{\mathbf{126}_F} = \underbrace{\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{\tfrac{1}{2}}_{n_r} = \frac{1}{3}$$

$$\{\mathbf{3}, \mathbf{1}, \tfrac{2}{3}\}: \ (\Delta b_3)_{\mathbf{126}_F} = \underbrace{\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{\tfrac{1}{2}}_{n_r} = \frac{1}{3}$$

$$\{\mathbf{3}, \mathbf{1}, -\tfrac{4}{3}\}: \ (\Delta b_3)_{\mathbf{126}_F} = \underbrace{\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{\tfrac{1}{2}}_{n_r} = \frac{1}{3}$$

$$\{\mathbf{6}, \mathbf{1}, \tfrac{1}{3}\}: \ (\Delta b_3)_{\mathbf{126}_F} = \underbrace{\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{\tfrac{5}{2}}_{n_r} = \frac{5}{3}$$

$$\{\mathbf{6},\mathbf{1},-\tfrac{2}{3}\}:\ (\Delta b_3)_{\mathbf{126}_F}=\underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{\tfrac{5}{2}}_{n_r}=\frac{5}{3}$$

$$\{\mathbf{6},\mathbf{1},\tfrac{4}{3}\}:\ (\Delta b_3)_{\mathbf{126}_F}=\underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{\tfrac{5}{2}}_{n_r}=\frac{5}{3}$$

$$\{\overline{\mathbf{6}},\mathbf{3},-\tfrac{1}{3}\}:\ (\Delta b_3)_{\mathbf{126}_F}=\underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{3}_{SU(2)\text{ triplet}}\cdot\underbrace{\tfrac{5}{2}}_{n_r}=5$$

$$\{\overline{\mathbf{3}},\mathbf{3},\tfrac{1}{3}\}:\ (\Delta b_3)_{\mathbf{126}_F}=\underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{3}_{SU(2)\text{ triplet}}\cdot\underbrace{\tfrac{1}{2}}_{n_r}=1$$

$$\{\mathbf{3},\mathbf{2},\tfrac{1}{6}\}:\ (\Delta b_3)_{\mathbf{126}_F}=\underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{2}_{SU(2)\text{ doublet}}\cdot\underbrace{\tfrac{1}{2}}_{n_r}=\frac{2}{3}$$

$$\{\overline{\mathbf{3}},\mathbf{2},-\tfrac{1}{6}\}:\ (\Delta b_3)_{\mathbf{126}_F}=\underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{2}_{SU(2)\text{ doublet}}\cdot\underbrace{\tfrac{1}{2}}_{n_r}=\frac{2}{3}$$

$$\{\mathbf{1},\mathbf{3},1\}:\ (\Delta b_3)_{\mathbf{126}_F}=\underbrace{0}_{n_r}$$

$$\{\mathbf{3},\mathbf{2},\tfrac{7}{6}\}:\ (\Delta b_3)_{\mathbf{126}_F}=\underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{2}_{SU(2)\text{ doublet}}\cdot\underbrace{\tfrac{1}{2}}_{n_r}=\frac{2}{3}$$

$$\{\overline{\mathbf{3}},\mathbf{2},-\tfrac{7}{6}\}:\ (\Delta b_3)_{\mathbf{126}_F}=\underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{2}_{SU(2)\text{ doublet}}\cdot\underbrace{\tfrac{1}{2}}_{n_r}=\frac{2}{3}$$

$$\{\mathbf{8},\mathbf{2},\tfrac{1}{2}\}:\ (\Delta b_3)_{\mathbf{126}_F}=\underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{2}_{SU(2)\text{ doublet}}\cdot\underbrace{3}_{n_r=N}=4$$

$$\{\mathbf{8},\mathbf{2},-\tfrac{1}{2}\}:\ (\Delta b_3)_{\mathbf{126}_F}=\underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{2}_{SU(2)\text{ doublet}}\cdot\underbrace{3}_{n_r=N}=4$$

$\underline{\Delta b_2}$

$$\{\mathbf{1},\mathbf{2},\tfrac{1}{2}\}:\ (\Delta b_2)_{\mathbf{126}_F} = \underbrace{\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{\tfrac{1}{2}}_{n_r} = \frac{1}{3}$$

$$\{\mathbf{1},\mathbf{2},-\tfrac{1}{2}\}:\ (\Delta b_2)_{\mathbf{126}_F} = \underbrace{\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{\tfrac{1}{2}}_{n_r} = \frac{1}{3}$$

$$\{\mathbf{1},\mathbf{1},0\}:\ (\Delta b_2)_{\mathbf{126}_F} = \underbrace{0}_{n_r}$$

$$\{\mathbf{1},\mathbf{1},-1\}:\ (\Delta b_2)_{\mathbf{126}_F} = \underbrace{0}_{n_r}$$

$$\{\mathbf{1},\mathbf{1},-2\}:\ (\Delta b_2)_{\mathbf{126}_F} = \underbrace{0}_{n_r}$$

$$\{\mathbf{3},\mathbf{1},-\tfrac{1}{3}\}:\ (\Delta b_2)_{\mathbf{126}_F} = \underbrace{0}_{n_r}$$

$$\{\mathbf{3},\mathbf{1},-\tfrac{1}{3}\}:\ (\Delta b_2)_{\mathbf{126}_F} = \underbrace{0}_{n_r}$$

$$\{\overline{\mathbf{3}},\mathbf{1},\tfrac{1}{3}\}:\ (\Delta b_2)_{\mathbf{126}_F} = \underbrace{0}_{n_r}$$

$$\{\mathbf{3},\mathbf{1},\tfrac{2}{3}\}:\ (\Delta b_2)_{\mathbf{126}_F} = \underbrace{0}_{n_r}$$

$$\{\mathbf{3},\mathbf{1},-\tfrac{4}{3}\}:\ (\Delta b_2)_{\mathbf{126}_F} = \underbrace{0}_{n_r}$$

$$\{\mathbf{6},\mathbf{1},\tfrac{1}{3}\}:\ (\Delta b_2)_{\mathbf{126}_F} = \underbrace{0}_{n_r}$$

$$\{\mathbf{6},\mathbf{1},-\tfrac{2}{3}\}:\ (\Delta b_2)_{\mathbf{126}_F} = \underbrace{0}_{n_r}$$

$$\{\mathbf{6},\mathbf{1},\tfrac{4}{3}\}:\ (\Delta b_2)_{\mathbf{126}_F} = \underbrace{0}_{n_r}$$

$$\{\overline{\mathbf{6}},\mathbf{3},-\tfrac{1}{3}\}:\ (\Delta b_2)_{\mathbf{126}_F} = \underbrace{\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{6}_{SU(3)\text{ sextet}} \cdot \underbrace{2}_{n_r=N} = 8$$

$$\{\overline{\mathbf{3}},\mathbf{3},\tfrac{1}{3}\}:\ (\Delta b_2)_{\mathbf{126}_F} = \underbrace{\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{3}_{SU(3)\text{ triplet}} \cdot \underbrace{2}_{n_r=N} = 4$$

$$\{\mathbf{3},\mathbf{2},\tfrac{1}{6}\}:\ (\Delta b_2)_{\mathbf{126}_F} = \underbrace{\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{3}_{SU(3)\text{ triplet}} \cdot \underbrace{\tfrac{1}{2}}_{n_r} = 1$$

$$\{\overline{\mathbf{3}},\mathbf{2},-\tfrac{1}{6}\}:\ (\Delta b_2)_{\mathbf{126}_F} = \underbrace{\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{3}_{SU(3)\text{ triplet}} \cdot \underbrace{\tfrac{1}{2}}_{n_r} = 1$$

$$\{\mathbf{1},\mathbf{3},1\}:\ (\Delta b_2)_{\mathbf{126}_F} = \underbrace{\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{2}_{n_r=N} = \frac{4}{3}$$

$$\{\mathbf{3},\mathbf{2},\tfrac{7}{6}\}:\ (\Delta b_2)_{\mathbf{126}_F} = \underbrace{\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{3}_{SU(3)\text{ triplet}} \cdot \underbrace{\tfrac{1}{2}}_{n_r} = 1$$

$$\{\overline{\mathbf{3}},\mathbf{2},-\tfrac{7}{6}\}:\ (\Delta b_2)_{\mathbf{126}_F} = \underbrace{\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{3}_{SU(3)\text{ triplet}} \cdot \underbrace{\tfrac{1}{2}}_{n_r} = 1$$

$$\{\mathbf{8},\mathbf{2},\tfrac{1}{2}\}:\ (\Delta b_2)_{\mathbf{126}_F} = \underbrace{\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{8}_{SU(3)\text{ octet}} \cdot \underbrace{\tfrac{1}{2}}_{n_r} = \frac{8}{3}$$

$$\{\mathbf{8},\mathbf{2},-\tfrac{1}{2}\}:\ (\Delta b_2)_{\mathbf{126}_F} = \underbrace{\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{8}_{SU(3)\text{ octet}} \cdot \underbrace{\tfrac{1}{2}}_{n_r} = \frac{8}{3}$$

$\underline{\Delta b_1}$

$$\{\mathbf{1},\mathbf{2},\tfrac{1}{2}\}:\ (\Delta b_1)_{\mathbf{126}_F} = \underbrace{\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{2}_{SU(2)\text{ doublet}} \cdot \underbrace{\tfrac{3}{5}(\tfrac{1}{2})^2}_{n_r} = \frac{1}{5}$$

$$\{\mathbf{1},\mathbf{2},-\tfrac{1}{2}\}:\ (\Delta b_1)_{\mathbf{126}_F} = \underbrace{\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{2}_{SU(2)\text{ doublet}} \cdot \underbrace{\tfrac{3}{5}(-\tfrac{1}{2})^2}_{n_r} = \frac{1}{5}$$

$$\{\mathbf{1},\mathbf{1},0\}:\ (\Delta b_1)_{\mathbf{126}_F} = \underbrace{0}_{n_r}$$

$$\{\mathbf{1},\mathbf{1},-1\}:\ (\Delta b_1)_{\mathbf{126}_F} = \underbrace{\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{\tfrac{3}{5}(-1)^2}_{n_r} = \frac{2}{5}$$

$$\{\mathbf{1},\mathbf{1},-2\}:\ (\Delta b_1)_{\mathbf{126}_F} = \underbrace{\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{\tfrac{3}{5}(-2)^2}_{n_r} = \frac{8}{5}$$

$$\{\mathbf{3},\mathbf{1},-\tfrac{1}{3}\}:\ (\Delta b_1)_{\mathbf{126}_F} = \underbrace{\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{3}_{SU(3)\text{ triplet}} \cdot \underbrace{\tfrac{3}{5}(-\tfrac{1}{3})^2}_{n_r} = \frac{2}{15}$$

$$\{\mathbf{3},\mathbf{1},-\tfrac{1}{3}\}:\ (\Delta b_1)_{\mathbf{126}_F} = \underbrace{\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{3}_{SU(3)\text{ triplet}} \cdot \underbrace{\tfrac{3}{5}(\tfrac{1}{3})^2}_{n_r} = \frac{2}{15}$$

$$\{\overline{\mathbf{3}},\mathbf{1},\tfrac{1}{3}\}:\ (\Delta b_1)_{\mathbf{126}_F} = \underbrace{\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{3}_{SU(3)\text{ triplet}} \cdot \underbrace{\tfrac{3}{5}(\tfrac{1}{3})^2}_{n_r} = \frac{2}{15}$$

$$\{\mathbf{3},\mathbf{1},\tfrac{2}{3}\}:\ (\Delta b_1)_{\mathbf{126}_F} = \underbrace{\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{3}_{SU(3)\text{ triplet}} \cdot \underbrace{\tfrac{3}{5}(\tfrac{2}{3})^2}_{n_r} = \frac{8}{15}$$

$$\{\mathbf{3},\mathbf{1},-\tfrac{4}{3}\}:\ (\Delta b_1)_{\mathbf{126}_F} = \underbrace{\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{3}_{SU(3)\text{ triplet}} \cdot \underbrace{\tfrac{3}{5}(-\tfrac{4}{3})^2}_{n_r} = \frac{32}{15}$$

$$\{\mathbf{6},\mathbf{1},\tfrac{1}{3}\}:\ (\Delta b_1)_{\mathbf{126}_F} = \underbrace{\tfrac{2}{3}}_{\text{fermion}} \cdot \underbrace{6}_{SU(3)\text{ sextet}} \cdot \underbrace{\tfrac{3}{5}(\tfrac{1}{3})^2}_{n_r} = \frac{4}{15}$$

$$\{\mathbf{6},\mathbf{1},-\tfrac{2}{3}\}:\ (\Delta b_1)_{\mathbf{126}_F} = \underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{6}_{SU(3)\ \text{sextet}}\cdot\underbrace{\tfrac{3}{5}(-\tfrac{2}{3})^2}_{n_r} = \frac{16}{15}$$

$$\{\mathbf{6},\mathbf{1},\tfrac{4}{3}\}:\ (\Delta b_1)_{\mathbf{126}_F} = \underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{6}_{SU(3)\ \text{sextet}}\cdot\underbrace{\tfrac{3}{5}(\tfrac{4}{3})^2}_{n_r} = \frac{64}{15}$$

$$\{\overline{\mathbf{6}},\mathbf{3},-\tfrac{1}{3}\}:\ (\Delta b_1)_{\mathbf{126}_F} = \underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{6}_{SU(3)\ \text{sextet}}\cdot\underbrace{3}_{SU(2)\ \text{triplet}}\cdot\underbrace{\tfrac{3}{5}(-\tfrac{1}{3})^2}_{n_r} = \frac{4}{5}$$

$$\{\overline{\mathbf{3}},\mathbf{3},\tfrac{1}{3}\}:\ (\Delta b_1)_{\mathbf{126}_F} = \underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{3}_{SU(3)\ \text{triplet}}\cdot\underbrace{3}_{SU(2)\ \text{triplet}}\cdot\underbrace{\tfrac{3}{5}(\tfrac{1}{3})^2}_{n_r} = \frac{2}{5}$$

$$\{\mathbf{3},\mathbf{2},\tfrac{1}{6}\}:\ (\Delta b_1)_{\mathbf{126}_F} = \underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{3}_{SU(3)\ \text{triplet}}\cdot\underbrace{2}_{SU(2)\ \text{doublet}}\cdot\underbrace{\tfrac{3}{5}(\tfrac{1}{6})^2}_{n_r} = \frac{1}{15}$$

$$\{\overline{\mathbf{3}},\mathbf{2},-\tfrac{1}{6}\}:\ (\Delta b_1)_{\mathbf{126}_F} = \underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{3}_{SU(3)\ \text{triplet}}\cdot\underbrace{2}_{SU(2)\ \text{doublet}}\cdot\underbrace{\tfrac{3}{5}(-\tfrac{1}{6})^2}_{n_r} = \frac{1}{15}$$

$$\{\mathbf{1},\mathbf{3},1\}:\ (\Delta b_1)_{\mathbf{126}_F} = \underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{3}_{SU(2)\ \text{triplet}}\cdot\underbrace{\tfrac{3}{5}(1)^2}_{n_r} = \frac{6}{5}$$

$$\{\mathbf{3},\mathbf{2},\tfrac{7}{6}\}:\ (\Delta b_1)_{\mathbf{126}_F} = \underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{3}_{SU(3)\ \text{triplet}}\cdot\underbrace{2}_{SU(2)\ \text{doublet}}\cdot\underbrace{\tfrac{3}{5}(\tfrac{7}{6})^2}_{n_r} = \frac{49}{15}$$

$$\{\overline{\mathbf{3}},\mathbf{2},-\tfrac{7}{6}\}:\ (\Delta b_1)_{\mathbf{126}_F} = \underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{3}_{SU(3)\ \text{triplet}}\cdot\underbrace{2}_{SU(2)\ \text{doublet}}\cdot\underbrace{\tfrac{3}{5}(-\tfrac{7}{6})^2}_{n_r} = \frac{49}{15}$$

$$\{\mathbf{8},\mathbf{2},\tfrac{1}{2}\}:\ (\Delta b_1)_{\mathbf{126}_F} = \underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{8}_{SU(3)\ \text{octet}}\cdot\underbrace{2}_{SU(2)\ \text{doublet}}\cdot\underbrace{\tfrac{3}{5}(-\tfrac{1}{2})^2}_{n_r} = \frac{8}{5}$$

$$\{\mathbf{8},\mathbf{2},-\tfrac{1}{2}\}:\ (\Delta b_1)_{\mathbf{126}_F} = \underbrace{\tfrac{2}{3}}_{\text{fermion}}\cdot\underbrace{8}_{SU(3)\ \text{octet}}\cdot\underbrace{2}_{SU(2)\ \text{doublet}}\cdot\underbrace{\tfrac{3}{5}(-\tfrac{1}{2})^2}_{n_r} = \frac{8}{5}$$

6.2 Combinations which best induce unification

22 different combinations of representations have been investigated, a list of which is given below:

$$\mathbf{5}_F \oplus \overline{\mathbf{5}}_F, \quad 2 \times (\mathbf{5}_F \oplus \overline{\mathbf{5}}_F), \quad \mathbf{5}_S \oplus \overline{\mathbf{5}}_S, \quad (\mathbf{5}_F \oplus \overline{\mathbf{5}}_F \oplus \mathbf{5}_S \oplus \overline{\mathbf{5}}_S),$$

$$(\mathbf{10}_F \oplus \overline{\mathbf{10}}_F)_{SU(5)}, \quad 2 \times (\mathbf{10}_F \oplus \overline{\mathbf{10}}_F)_{SU(5)}, \quad (\mathbf{10}_S \oplus \overline{\mathbf{10}}_S)_{SU(5)},$$

$$\mathbf{24}_F, \quad \mathbf{24}_S, \quad (\mathbf{10}_F)_{SO(10)}, \quad 3 \times (\mathbf{10}_F)_{SO(10)}, \quad 4 \times (\mathbf{10}_F)_{SO(10)},$$

$$\mathbf{16}_F \oplus \overline{\mathbf{16}}_F, \quad \mathbf{16}_S \oplus \overline{\mathbf{16}}_S, \quad (\mathbf{16}_F \oplus \overline{\mathbf{16}}_F \oplus \mathbf{16}_S \oplus \overline{\mathbf{16}}_S),$$

$$\mathbf{45}_F, \quad \mathbf{54}_F, \quad \mathbf{54}_S, \quad \mathbf{120}_F, \quad \mathbf{120}_S,$$

$$(\mathbf{126}_F \oplus \overline{\mathbf{126}}_F), \quad (\mathbf{126}_S \oplus \overline{\mathbf{126}}_S)\,.$$

The best results for each combination of representations are shown in Figures (9) to (29), along with the energies at which the subgroups of the representations are placed, though these can often also be seen by eye if the kink in the trajectory of a particular running coupling is great enough. All plots have dimensionless gauge coupling α_a^{-1} on the y-axis, against energy scale Q in GeV on the x-axis; the blue line represents α_1^{-1} (EM), the red line represents α_2^{-1} (weak), and the green line represents α_3^{-1} (strong).

The results begin overleaf; the values of key parameters for each combination of representations have also been tabulated, and are shown in Table (3). To help make the $\Delta(\log_{10} Q)$ parameter values more meaningful, the $\Delta(\log_{10} Q)$ value for the SM is $\Delta(\log_{10} Q)_{\text{SM}} \approx 4$, while the value for the MSSM is $\Delta(\log_{10} Q)_{\text{MSSM}} \approx 0.5$.

$\mathbf{5}_F \oplus \overline{\mathbf{5}}_F$

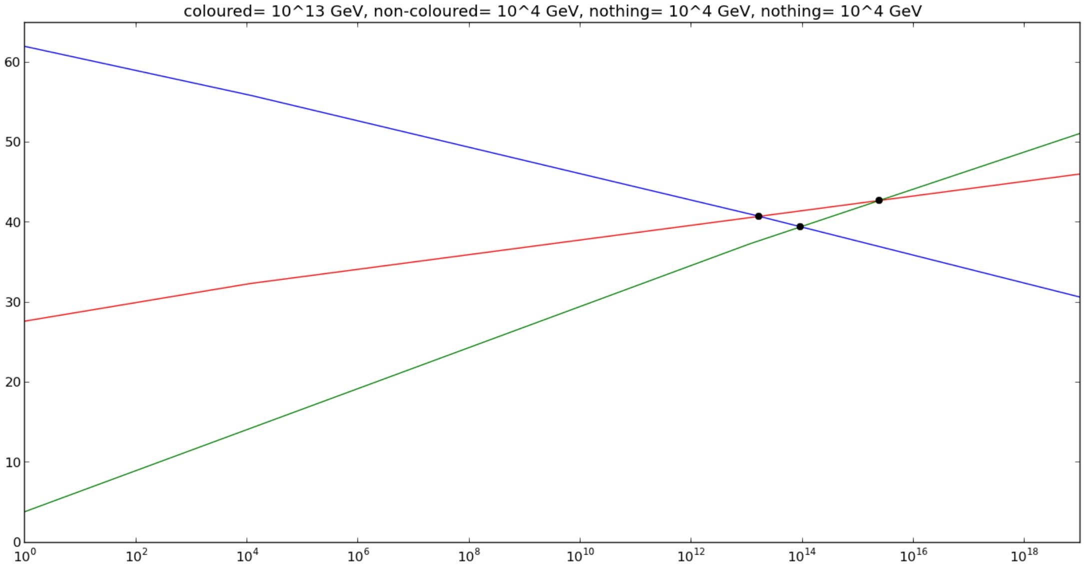

Figure 9: Plot of α_a^{-1} against Q (GeV), showing the running couplings when $\mathbf{5}_F \oplus \overline{\mathbf{5}}_F$ is added to the SM; coloured states inserted at $Q = 10^{13}$ GeV, non-coloured states at $Q = 10^4$ GeV.

$\mathbf{5}_S \oplus \overline{\mathbf{5}}_S$

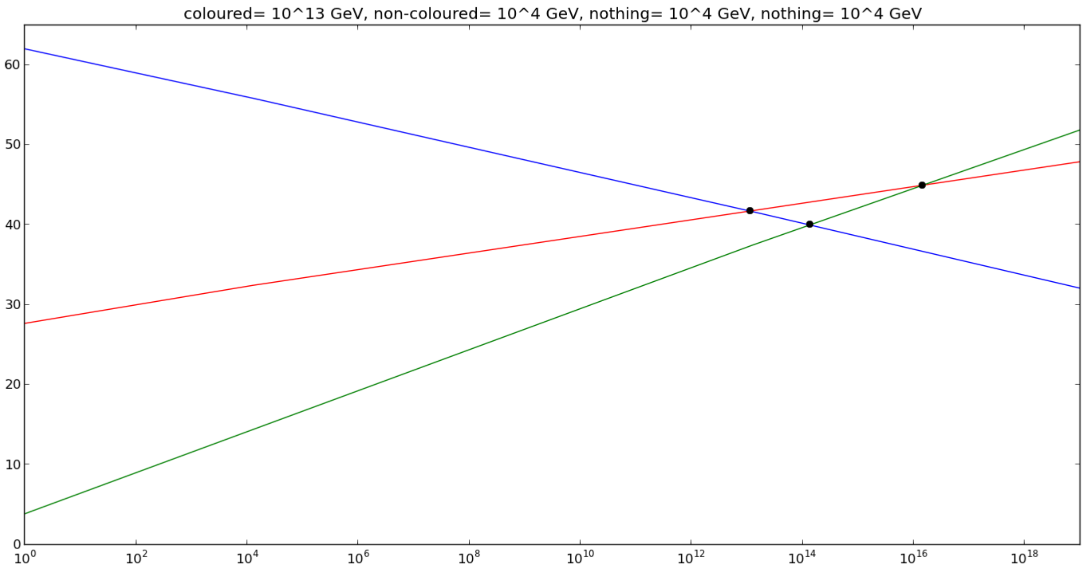

Figure 10: Plot of α_a^{-1} against Q (GeV), showing the running couplings when $\mathbf{5}_S \oplus \overline{\mathbf{5}}_S$ is added to the SM; coloured states inserted at $Q = 10^{13}$ GeV, non-coloured states at $Q = 10^4$ GeV.

$2 \times (\mathbf{5}_F \oplus \overline{\mathbf{5}}_F)$

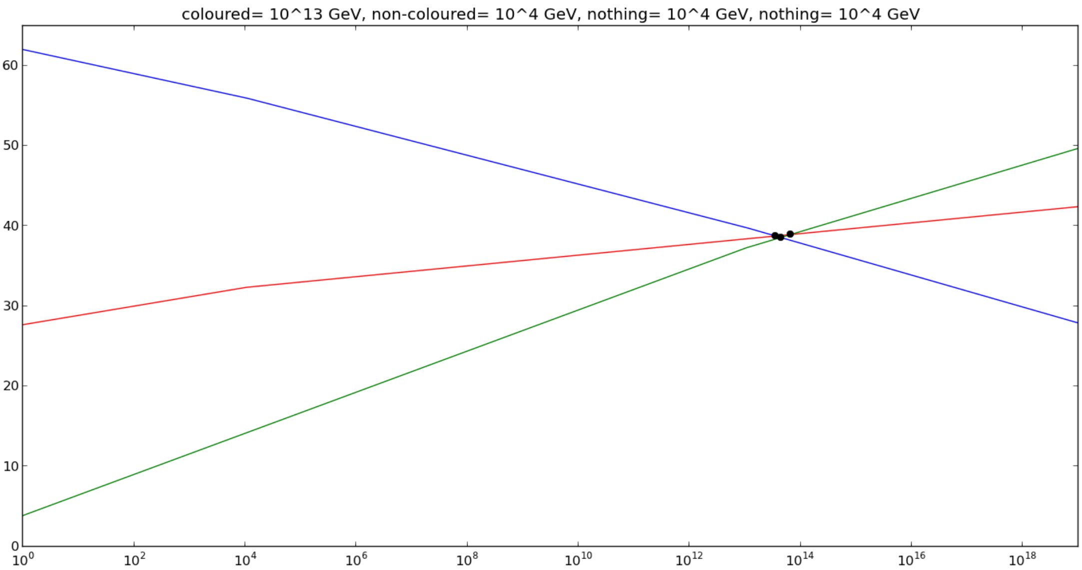

Figure 11: Plot of α_a^{-1} against Q (GeV), showing the running couplings when $2 \times (\mathbf{5}_F \oplus \overline{\mathbf{5}}_F)$ is added to the SM; coloured states inserted at $Q = 10^{13}$ GeV, non-coloured states at $Q = 10^4$ GeV.

$\mathbf{5}_F \oplus \overline{\mathbf{5}}_F \oplus \mathbf{5}_S \oplus \overline{\mathbf{5}}_S$

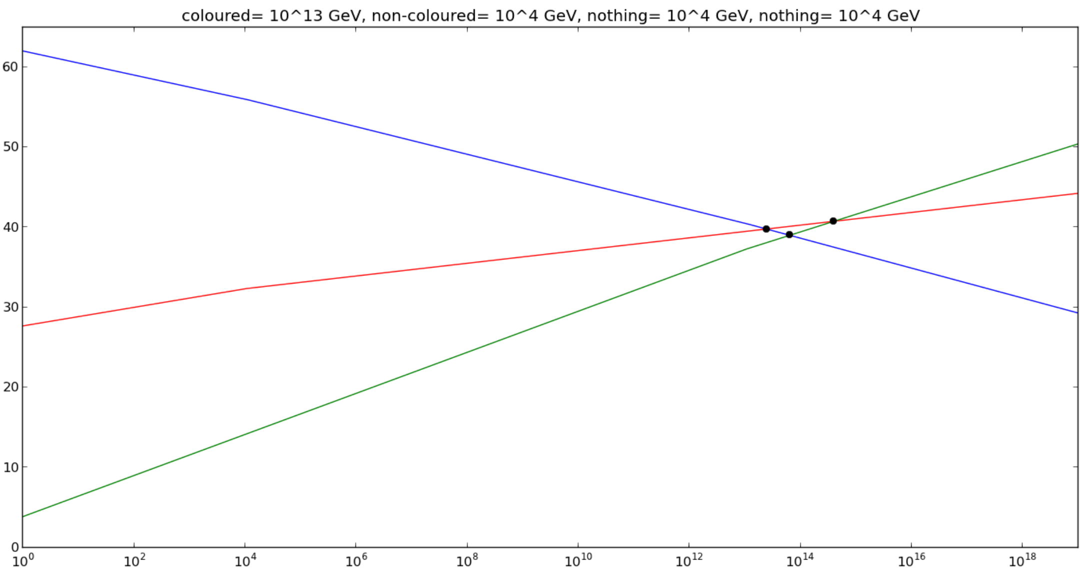

Figure 12: Plot of α_a^{-1} against Q (GeV), showing the running couplings when $(\mathbf{5}_F \oplus \overline{\mathbf{5}}_F \oplus \mathbf{5}_S \oplus \overline{\mathbf{5}}_S)$ is added to the SM; coloured states inserted at $Q = 10^{13}$ GeV, non-coloured states at $Q = 10^4$ GeV.

$(\mathbf{10}_F \oplus \overline{\mathbf{10}}_F)_{SU(5)}$

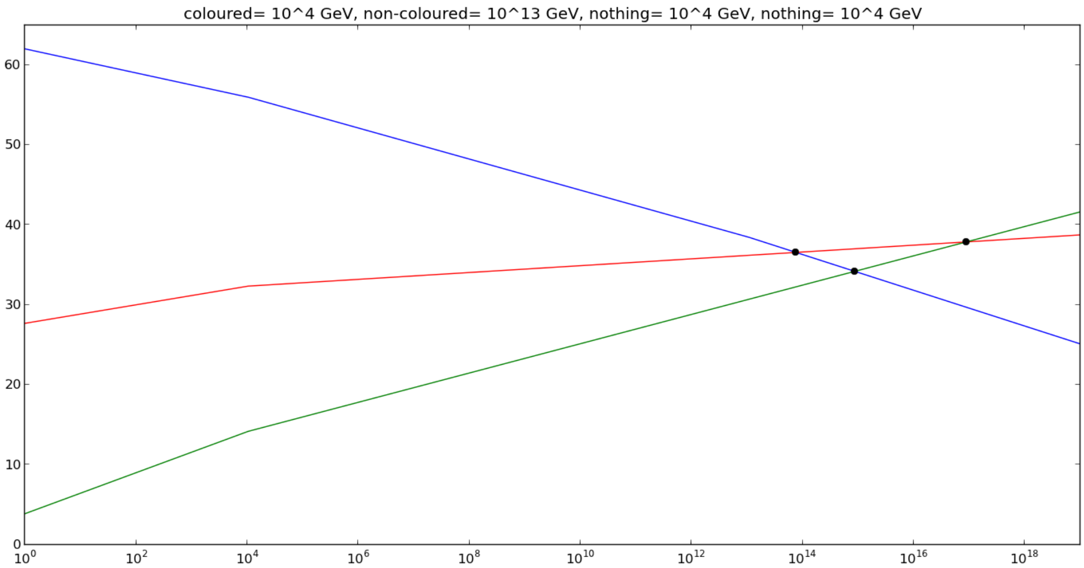

Figure 13: Plot of α_a^{-1} against Q (GeV), showing the running couplings when $(\mathbf{10}_F \oplus \overline{\mathbf{10}}_F)_{SU(5)}$ is added to the SM; coloured states inserted at $Q = 10^4$ GeV, non-coloured states at $Q = 10^{13}$ GeV.

$2 \times (\mathbf{10}_F \oplus \overline{\mathbf{10}}_F)_{SU(5)}$

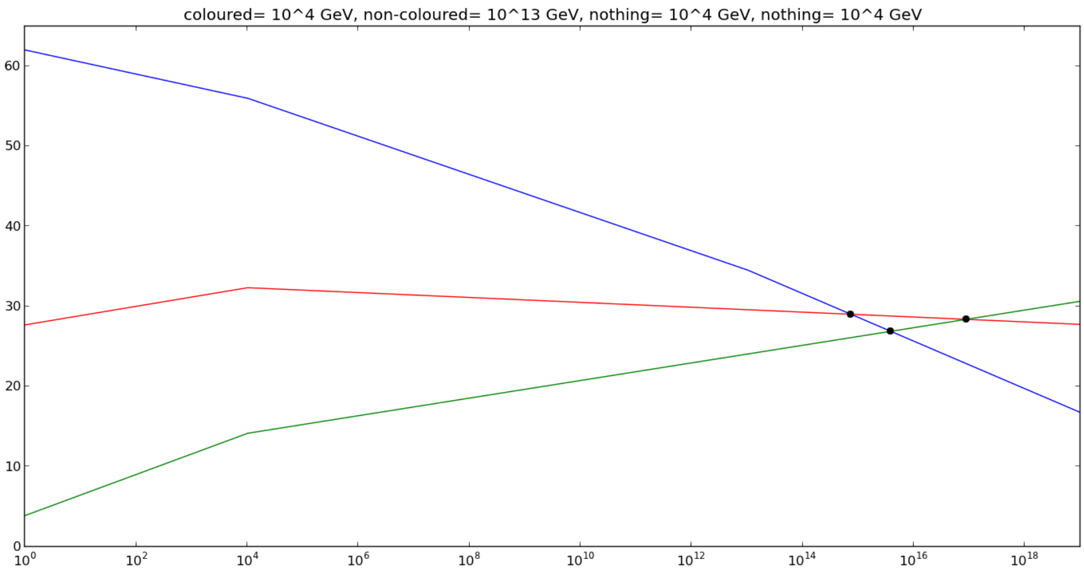

Figure 14: Plot of α_a^{-1} against Q (GeV), showing the running couplings when $2 \times (\mathbf{10}_F \oplus \overline{\mathbf{10}}_F)_{SU(5)}$ is added to the SM; coloured states inserted at $Q = 10^4$ GeV, non-coloured states at $Q = 10^{13}$ GeV.

$(\mathbf{10}_S \oplus \overline{\mathbf{10}}_S)_{SU(5)}$

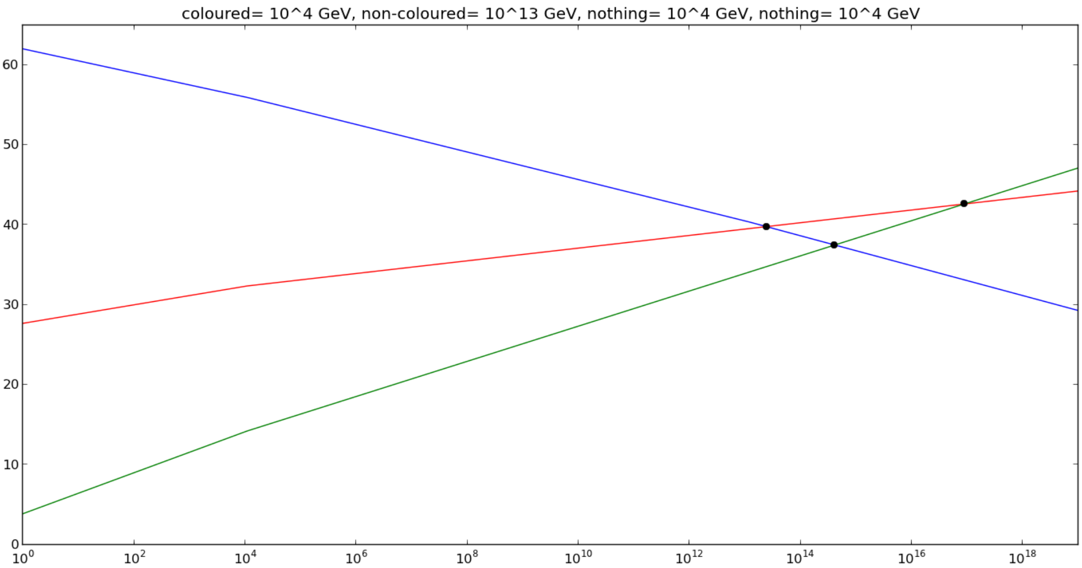

Figure 15: Plot of α_a^{-1} against Q (GeV), showing the running couplings when $(\mathbf{10}_S \oplus \overline{\mathbf{10}}_S)_{SU(5)}$ is added to the SM; coloured states inserted at $Q = 10^4$ GeV, non-coloured states at $Q = 10^{13}$ GeV.

$\mathbf{24}_S$

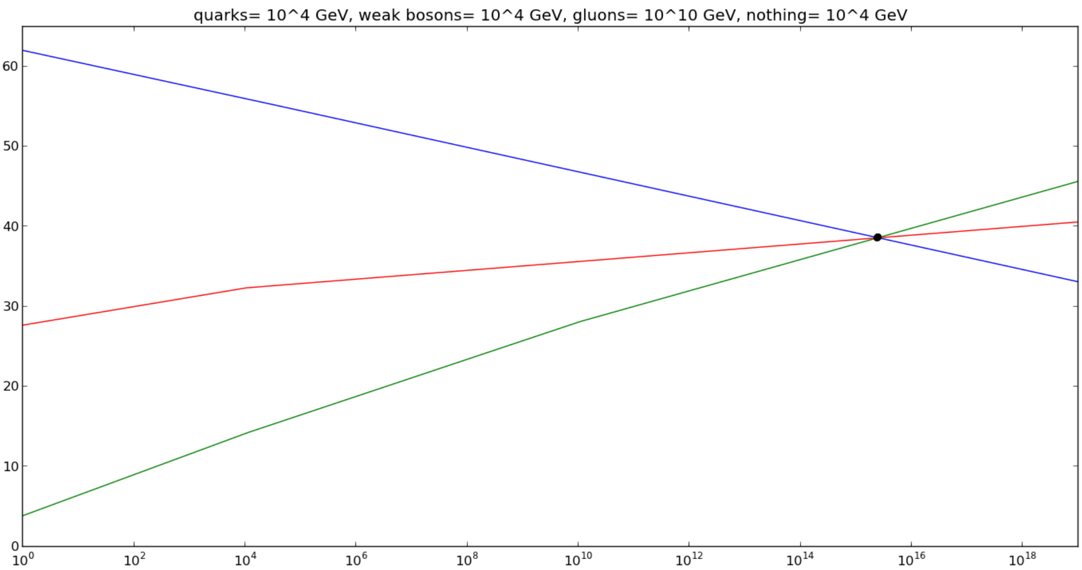

Figure 16: Plot of α_a^{-1} against Q (GeV), showing the running couplings when $\mathbf{24}_S$ is added to the SM; coloured non-adjoint states inserted at $Q = 10^4$ GeV, $SU(2)$ adjoints at $Q = 10^4$ GeV, and $SU(3)$ adjoints at $Q = 10^{10}$ GeV.

$(\mathbf{10}_F)_{SO(10)}$

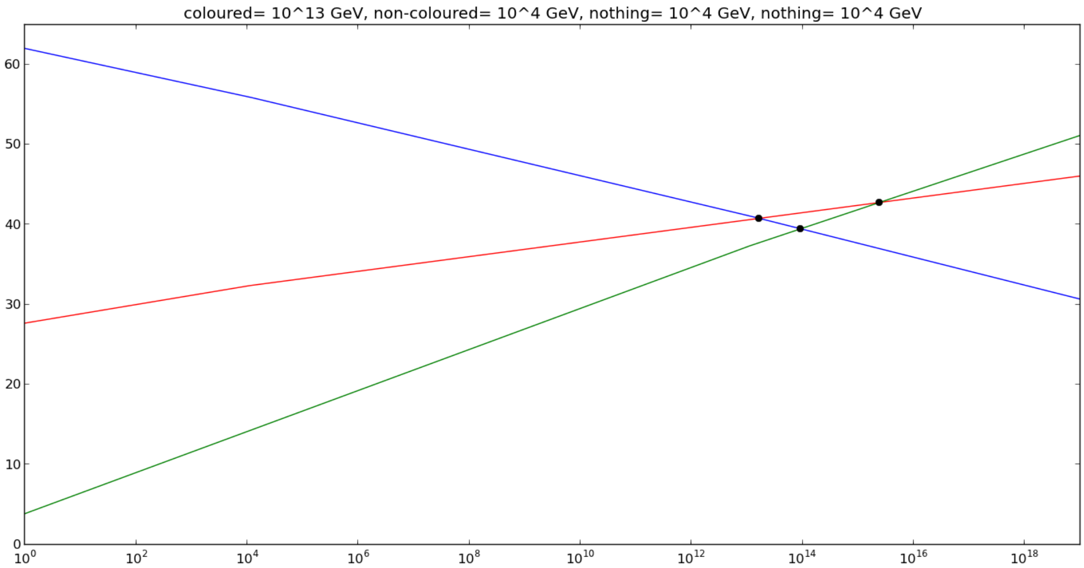

Figure 17: Plot of α_a^{-1} against Q (GeV), showing the running couplings when $(\mathbf{10}_F)_{SO(10)}$ is added to the SM; coloured states inserted at $Q = 10^{13}$ GeV, non-coloured states at $Q = 10^4$ GeV.

$3 \times (\mathbf{10}_F)_{SO(10)}$

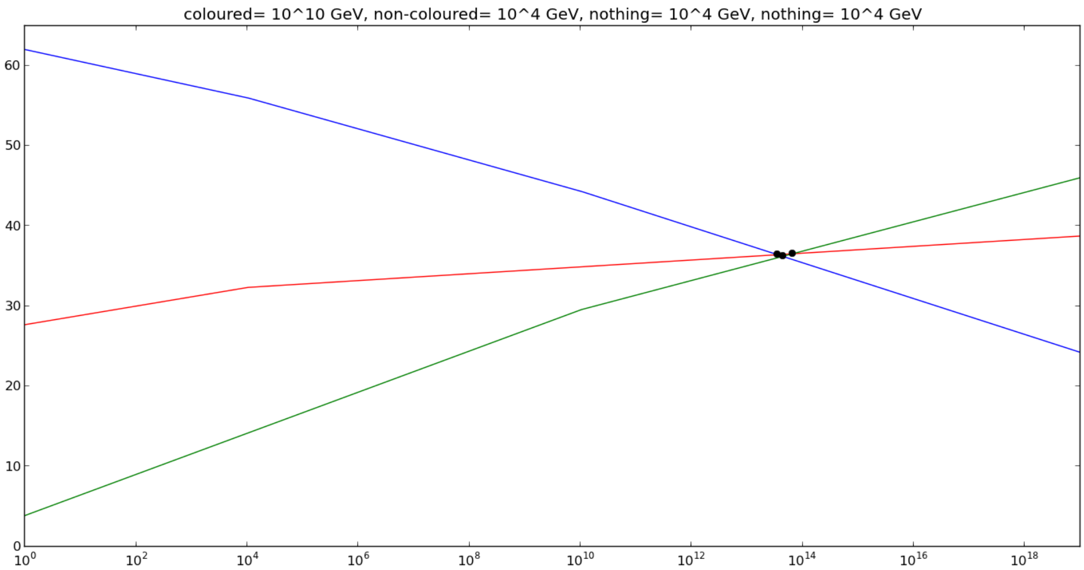

Figure 18: Plot of α_a^{-1} against Q (GeV), showing the running couplings when $3 \times (\mathbf{10}_F)_{SO(10)}$ is added to the SM; coloured states inserted at $Q = 10^{10}$ GeV, non-coloured states at $Q = 10^4$ GeV.

$4 \times (\mathbf{10}_F)_{SO(10)}$

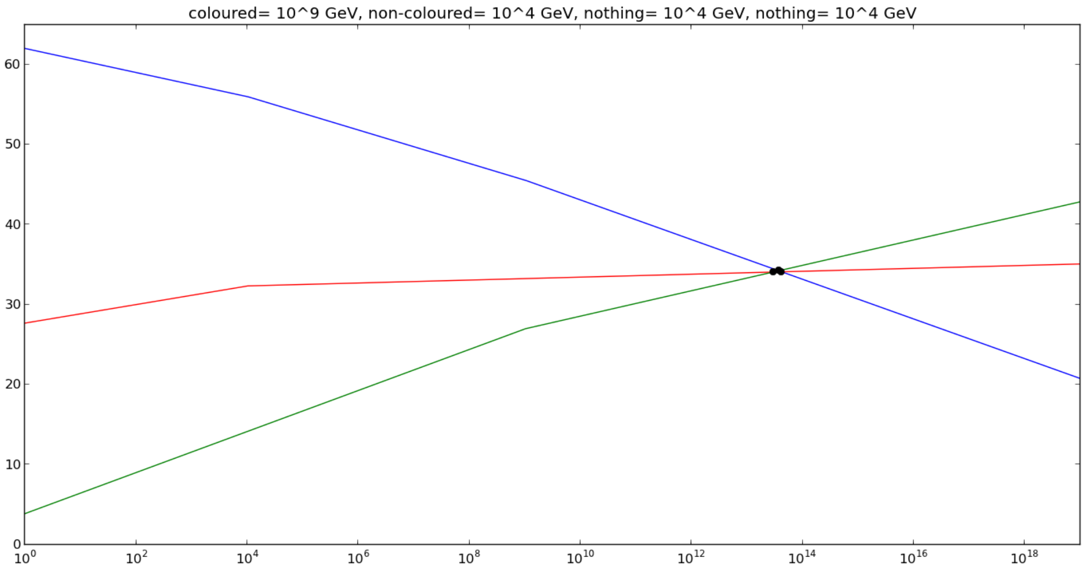

Figure 19: Plot of α_a^{-1} against Q (GeV), showing the running couplings when $4 \times (\mathbf{10}_F)_{SO(10)}$ is added to the SM; coloured states inserted at $Q = 10^9$ GeV, non-coloured states at $Q = 10^4$ GeV.

$\mathbf{16}_F \oplus \overline{\mathbf{16}}_F$

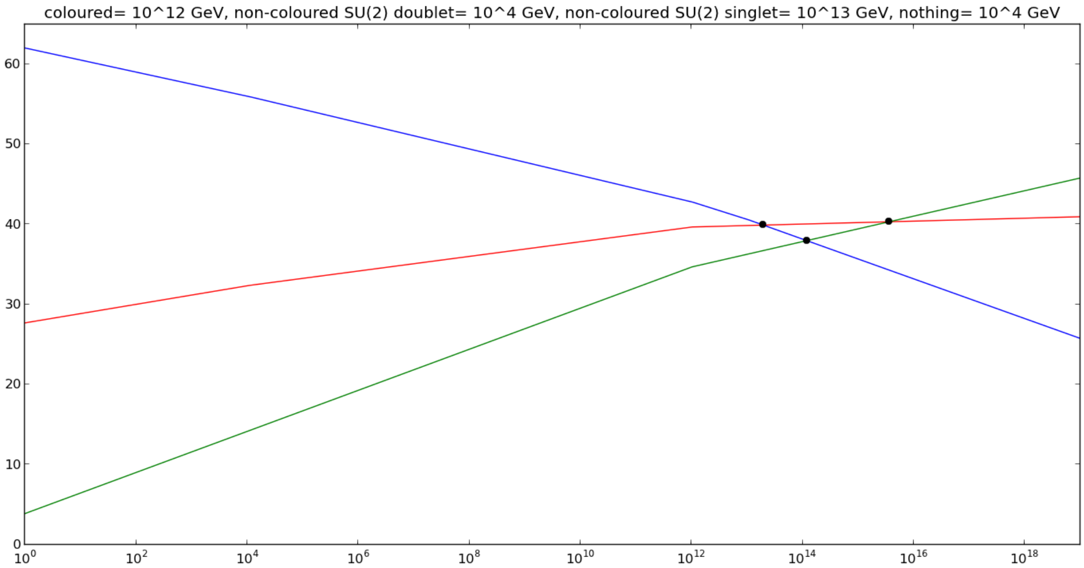

Figure 20: Plot of α_a^{-1} against Q (GeV), showing the running couplings when $\mathbf{16}_F \oplus \overline{\mathbf{16}}_F$ is added to the SM; coloured states inserted at $Q = 10^{12}$ GeV, non-coloured SU(2) doublets at $Q = 10^4$ GeV, and non-coloured SU(2) singlets at $Q = 10^{13}$ GeV.

$\mathbf{16}_S \oplus \overline{\mathbf{16}}_S$

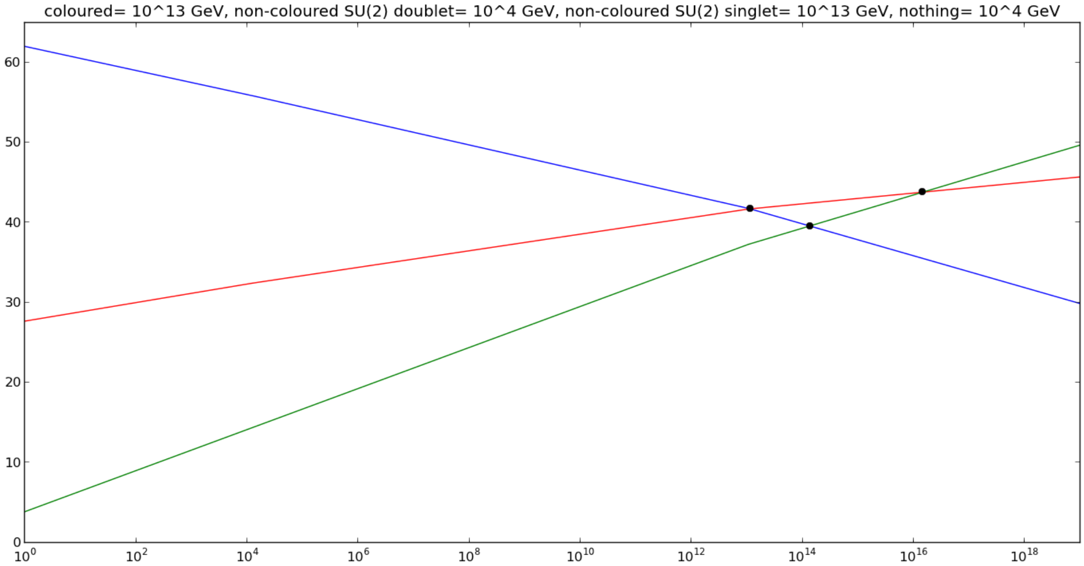

Figure 21: Plot of α_a^{-1} against Q (GeV), showing the running couplings when $\mathbf{16}_S \oplus \overline{\mathbf{16}}_S$ is added to the SM; coloured states inserted at $Q = 10^{13}$ GeV, non-coloured SU(2) doublets at $Q = 10^4$ GeV, and non-coloured SU(2) singlets at $Q = 10^{13}$ GeV.

$\mathbf{16}_F \oplus \overline{\mathbf{16}}_F \oplus \mathbf{16}_S \oplus \overline{\mathbf{16}}_S$

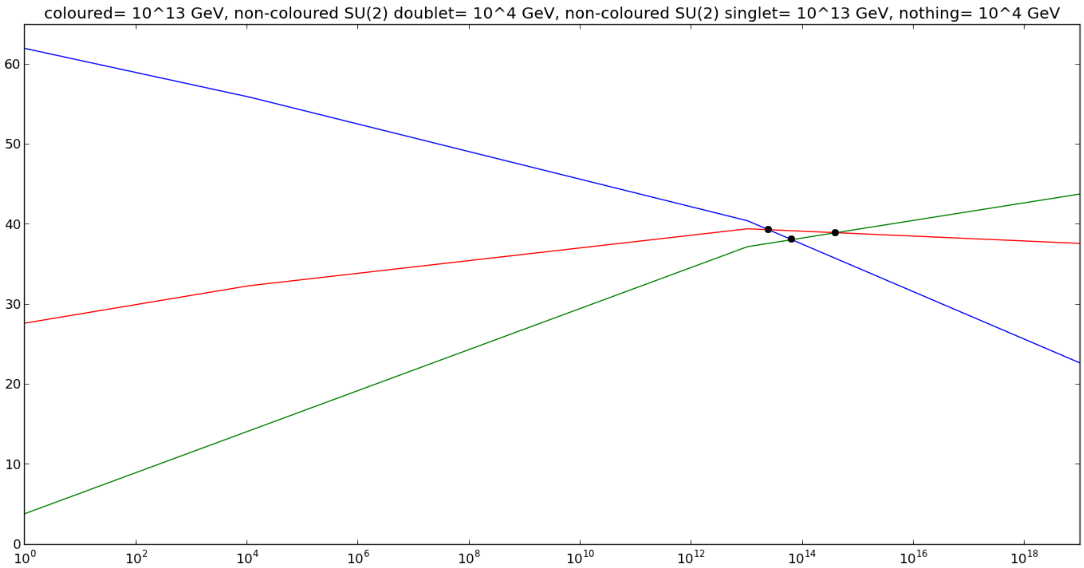

Figure 22: Plot of α_a^{-1} against Q (GeV), showing the running couplings when $\mathbf{16}_F \oplus \overline{\mathbf{16}}_F \oplus \mathbf{16}_S \oplus \overline{\mathbf{16}}_S$ is added to the SM; coloured states inserted at $Q = 10^{13}$ GeV, non-coloured SU(2) doublets at $Q = 10^4$ GeV, and non-coloured SU(2) singlets at $Q = 10^{13}$ GeV.

$\mathbf{45}_F$

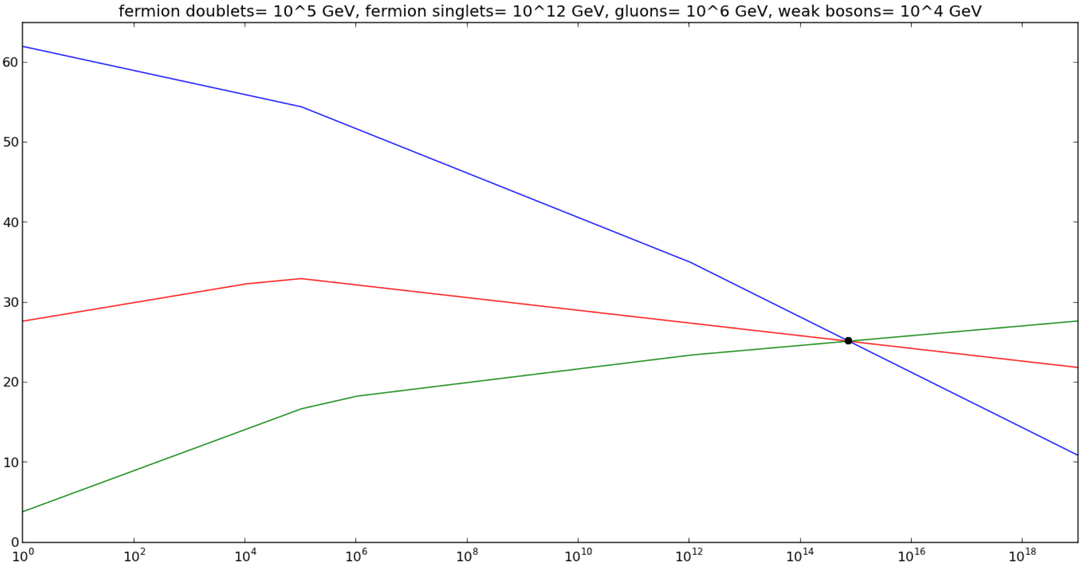

Figure 23: Plot of α_a^{-1} against Q (GeV), showing the running couplings when $\mathbf{45}_F$ is added to the SM; $SU(2)$ doublet states inserted at $Q = 10^5$ GeV, $SU(2)$ singlets at $Q = 10^{12}$ GeV, $SU(2)$ adjoints at $Q = 10^4$ GeV, and $SU(3)$ adjoints at $Q = 10^6$ GeV.

$\mathbf{54}_F$

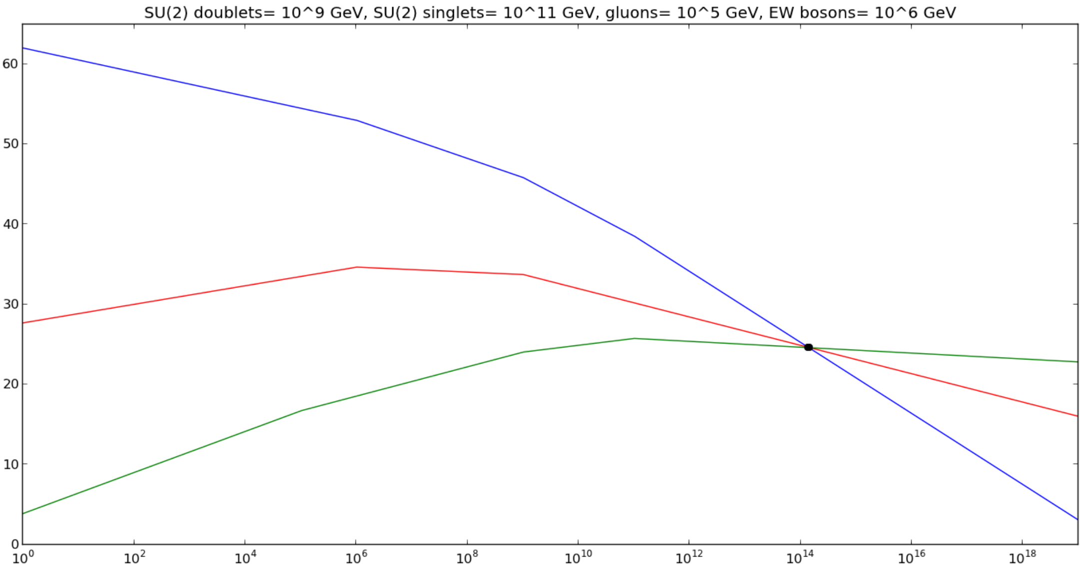

Figure 24: Plot of α_a^{-1} against Q (GeV), showing the running couplings when $\mathbf{54}_F$ is added to the SM; $SU(2)$ doublet states inserted at $Q = 10^9$ GeV, $SU(2)$ singlets at $Q = 10^{11}$ GeV, $SU(2)$ adjoints at $Q = 10^6$ GeV, and $SU(3)$ adjoints at $Q = 10^5$ GeV.

$\mathbf{54}_S$

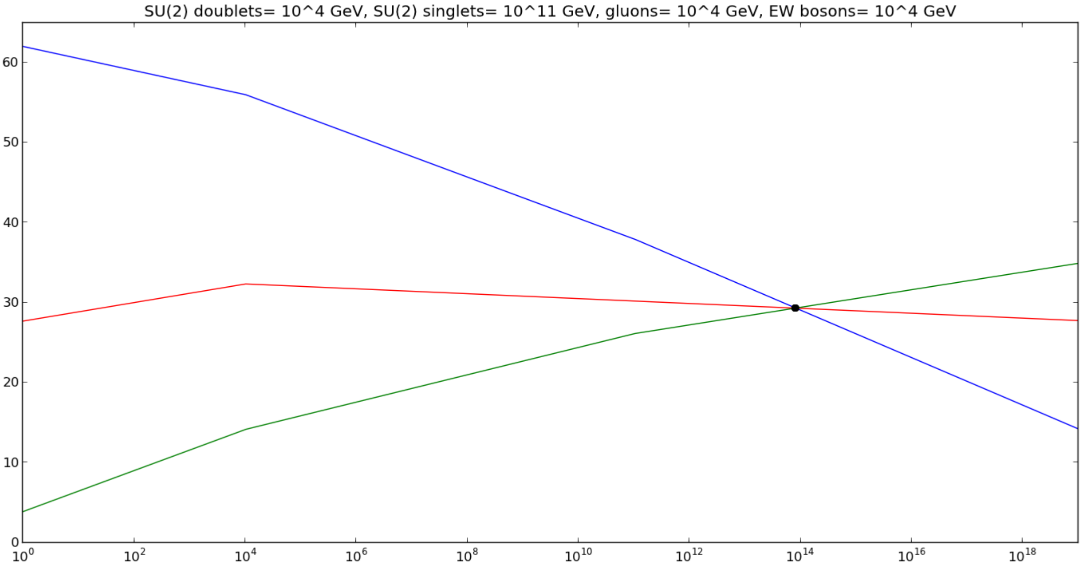

Figure 25: Plot of α_a^{-1} against Q (GeV), showing the running couplings when $\mathbf{54}_S$ is added to the SM; $SU(2)$ doublet states inserted at $Q = 10^4$ GeV, $SU(2)$ singlets at $Q = 10^{11}$ GeV, $SU(2)$ adjoints at $Q = 10^4$ GeV, and $SU(3)$ adjoints at $Q = 10^4$ GeV.

$\mathbf{120}_F$

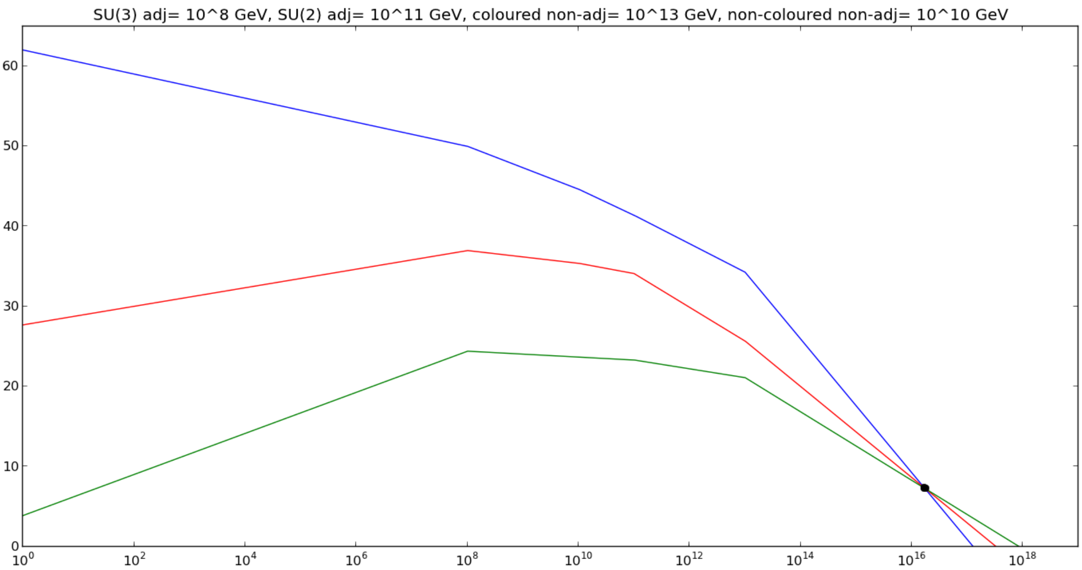

Figure 26: Plot of α_a^{-1} against Q (GeV), showing the running couplings when $\mathbf{120}_F$ is added to the SM; coloured non-adjoint states inserted at $Q = 10^{13}$ GeV, non-coloured non-adjoints at $Q = 10^{10}$ GeV, $SU(2)$ adjoints at $Q = 10^{11}$ GeV, and $SU(3)$ adjoints at $Q = 10^8$ GeV.

$\mathbf{120}_S$

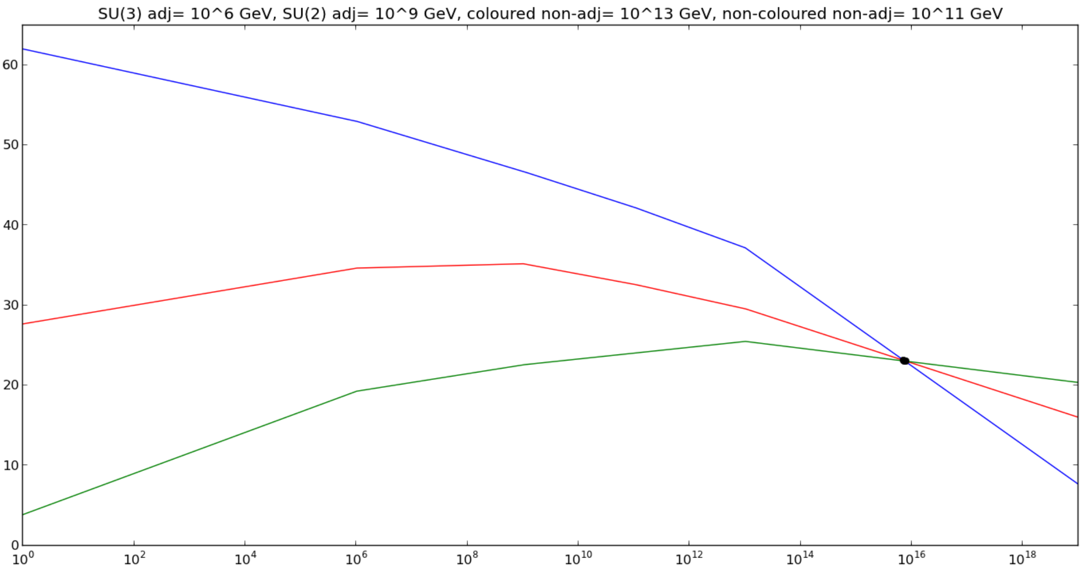

Figure 27: Plot of α_a^{-1} against Q (GeV), showing the running couplings when $\mathbf{120}_S$ is added to the SM; coloured non-adjoint states inserted at $Q = 10^{13}$ GeV, non-coloured non-adjoints at $Q = 10^{11}$ GeV, $SU(2)$ adjoints at $Q = 10^9$ GeV, and $SU(3)$ adjoints at $Q = 10^6$ GeV.

$\mathbf{126}_F \oplus \overline{\mathbf{126}}_F$

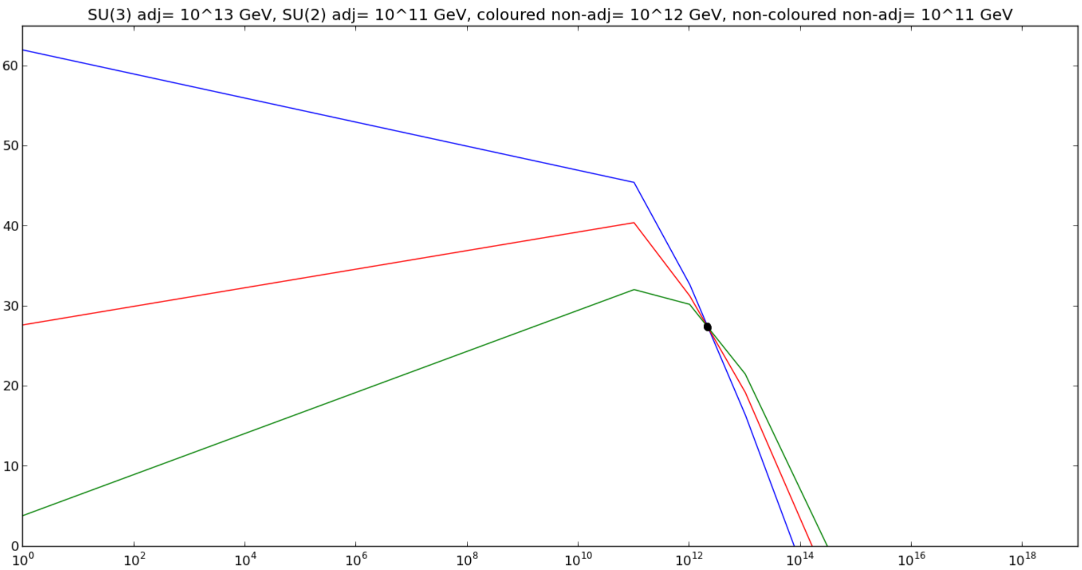

Figure 28: Plot of α_a^{-1} against Q (GeV), showing the running couplings when $\mathbf{126}_F \oplus \overline{\mathbf{126}}_F$ is added to the SM; coloured non-adjoint states inserted at $Q = 10^{12}$ GeV, non-coloured non-adjoints at $Q = 10^{11}$ GeV, $SU(2)$ adjoints at $Q = 10^{11}$ GeV, and $SU(3)$ adjoints at $Q = 10^{13}$ GeV.

$\mathbf{126}_S \oplus \overline{\mathbf{126}}_S$

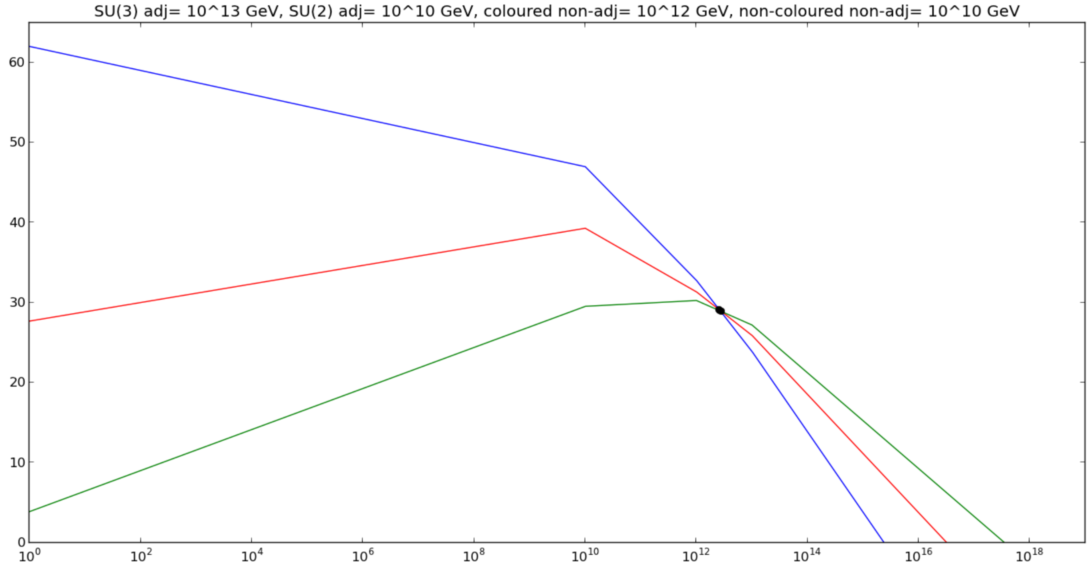

Figure 29: Plot of α_a^{-1} against Q (GeV), showing the running couplings when $\mathbf{126}_S \oplus \overline{\mathbf{126}}_S$ is added to the SM; coloured non-adjoint states inserted at $Q = 10^{12}$ GeV, non-coloured non-adjoints at $Q = 10^{10}$ GeV, $SU(2)$ adjoints at $Q = 10^{10}$ GeV, and $SU(3)$ adjoints at $Q = 10^{13}$ GeV.

Table 3: Table of key parameters for each combination of representations investigated:

Representation	α_a^{-1}	$M_{\rm GUT}$ (GeV)	$\Delta(\log_{10} Q)$	**Energy configuration** [2]
$\mathbf{5}_F \oplus \overline{\mathbf{5}}_F$	40	2×10^{13}	2.14	C=13, NC=4
$\mathbf{5}_S \oplus \overline{\mathbf{5}}_S$	42	10^{13}	3.09	C=13, NC=4
$2 \times (\mathbf{5}_F \oplus \overline{\mathbf{5}}_F)$	38.7	5×10^{13}	0.251	C=13, NC=4
$(\mathbf{5}_F \oplus \overline{\mathbf{5}}_F \oplus \mathbf{5}_S \oplus \overline{\mathbf{5}}_S)$	40	2.5×10^{13}	1.20	C=13, NC=4
$(\mathbf{10}_F \oplus \overline{\mathbf{10}}_F)_{SU(5)}$	36.5	7.5×10^{13}	3.05	C=4, NC=13
$2 \times (\mathbf{10}_F \oplus \overline{\mathbf{10}}_F)_{SU(5)}$	29	7×10^{14}	2.06	C=4, NC=13
$(\mathbf{10}_S \oplus \overline{\mathbf{10}}_S)_{SU(5)}$	40	2×10^{13}	3.54	C=4, NC=13
$\mathbf{24}_S$	38.5	2.5×10^{15}	0.0342	CNA=4, 2A=4, 3A=10
$\mathbf{24}_F$	N/A	N/A	N/A	Did not unify
$\mathbf{10}_F^{SO(10)}$	41	1.5×10^{13}	2.14	C=13, NC=4
$3 \times \mathbf{10}_F^{SO(10)}$	36.4	4×10^{13}	0.251	C=10, NC=4
$4 \times \mathbf{10}_F^{SO(10)}$	34	3×10^{13}	0.169	C=9, NC=4
$\mathbf{16}_F \oplus \overline{\mathbf{16}}_F$	40	2×10^{13}	2.25	C=12, NCD=4, NCS=13
$\mathbf{16}_S \oplus \overline{\mathbf{16}}_S$	42	10^{13}	3.09	C=13, NCD=4, NCS=13
$\mathbf{16}_F \oplus \overline{\mathbf{16}}_F \oplus \mathbf{16}_S \oplus \overline{\mathbf{16}}_S$	39	2×10^{13}	1.20	C=13, NCD=4, NCS=13
$\mathbf{45}_F$	26	7×10^{14}	0.0228	D=5, S=12, 2A=4, 3A=6
$\mathbf{54}_F$	24.5	10^{14}	0.0152	D=9, S=11, 2A=6, 3A=5
$\mathbf{54}_S$	30	10^{14}	0.0285	D=4, S=11, 2A=4, 3A=4
$\mathbf{120}_F$	7	2×10^{16}	0.0209	CNA=13, NCNA=10, 2A=11, 3A=8
$\mathbf{120}_S$	23	7×10^{15}	0.0323	CNA=13, NCNA=11, 2A=9, 3A=6
$\mathbf{126}_F \oplus \overline{\mathbf{126}}_F$	27	2×10^{12}	0.0209	CNA=12, NCNA=11, 2A=11, 3A=13
$\mathbf{126}_S \oplus \overline{\mathbf{126}}_S$	29	2.5×10^{12}	0.0323	CNA=12, NCNA=10, 2A=10, 3A=13

From Table (3), we see that all attempts involving $\mathbf{5}\oplus\overline{\mathbf{5}}$ have an $M_{\rm GUT}$ of only $\mathcal{O}(10^{13}\,{\rm GeV})$, and none unify well, though $2 \times (\mathbf{5}_F \oplus \overline{\mathbf{5}}_F)$ has a $\Delta(\log_{10} Q)$ half that of the MSSM. The $\mathbf{10}_{SU(5)} \oplus \overline{\mathbf{10}}_{SU(5)}$ results have a higher $M_{\rm GUT}$, but do not unify well.

$\mathbf{24}_S$ unifies very precisely, and at a reasonably high $M_{\rm GUT}$; however, adding further $\mathbf{24}_{S/F}$ representations prevents the couplings from meeting at $Q < M_P$, if at all. The $\mathbf{10}_{SU(10)}$ permutations yielded results of a low $M_{\rm GUT}$, but which unify reasonably well; the $\mathbf{16}\oplus\overline{\mathbf{16}}$, however, unified badly. Like $\mathbf{24}_S$, the $\mathbf{45}$ and $\mathbf{54}$ representations also unify very precisely, though with an $M_{\rm GUT}$ of only $\mathcal{O}(10^{14}\,{\rm GeV})$.

$\mathbf{120}_F$ has the most favourable $M_{\rm GUT} \sim 2 \times 10^{16}\,{\rm GeV}$, although its α_U^{-1} is only 7. The $\mathbf{126} \oplus \overline{\mathbf{126}}$ combinations also unify with the same high precision as the $\mathbf{120}_F$, but has a very low $M_{\rm GUT} \sim \mathcal{O}(10^{12}\,{\rm GeV})$.

[2] For each $P = x$ in this column, P is the piece of the representation placed at a given energy, and x is $\log_{10} Q$, where Q is the energy at which the piece is placed. The acronyms for the pieces are as follows: C=coloured; NC=non-coloured; CNA=coloured, non-adjoint; NCNA=non-coloured, non-adjoint; S=$SU(2)$ singlets; D=$SU(2)$ doublets; 2A=$SU(2)$ adjoints; 3A=$SU(3)$ adjoints.

From these results, it becomes clear that there are many choices of $SO(10)$ representation which achieve good to excellent unification of the fundamental forces, reinstating the power of the $SO(10)$ GUT to realise a Grand Unification of the fundamental forces of Nature at high energies.

6.3 Anomaly calculations

The non-trivial gauge anomalies of each representation investigated throughout the paper will be calculated below:

$\underline{\mathbf{5}}$

$$\mathcal{A}_{\mathbf{5}}^{(U(1))^3} \propto \underbrace{3(-\tfrac{1}{3})^3}_{\{\mathbf{3},\mathbf{1},-\frac{1}{3}\}} - \underbrace{2(\tfrac{1}{2})^3}_{\{\mathbf{1},\mathbf{2},\frac{1}{2}\}} = -\frac{13}{36} \neq 0$$

$$\mathcal{A}_{\mathbf{5}}^{U(1)\times \mathrm{grav}^2} \propto \underbrace{3(-\tfrac{1}{3})}_{\{\mathbf{3},\mathbf{1},-\frac{1}{3}\}} - \underbrace{2(\tfrac{1}{2})}_{\{\mathbf{1},\mathbf{2},\frac{1}{2}\}} = -2 \neq 0$$

$$\mathcal{A}_{\mathbf{5}}^{U(1)\times (SU(2))^2} \propto \underbrace{-(\tfrac{1}{2})}_{\{\mathbf{1},\mathbf{2},\frac{1}{2}\}} = -\frac{1}{2} \neq 0$$

$$\mathcal{A}_{\mathbf{5}}^{U(1)\times (SU(3))^2} \propto \underbrace{(-\tfrac{1}{3})}_{\{\mathbf{3},\mathbf{1},-\frac{1}{3}\}} = -\frac{1}{3} \neq 0$$

$$\Rightarrow A(\mathbf{5}) \neq 0 \tag{265}$$

$\underline{\mathbf{10}_{SU(5)}}$

$$\mathcal{A}_{\mathbf{10}_{SU(5)}}^{(U(1))^3} \propto \underbrace{-2\cdot 3(\tfrac{1}{6})^3}_{\{\mathbf{3},\mathbf{2},\frac{1}{6}\}} + \underbrace{3(-\tfrac{2}{3})^3}_{\{\bar{\mathbf{3}},\mathbf{1},-\frac{2}{3}\}} + \underbrace{(1)^3}_{\{\mathbf{1},\mathbf{1},1\}} = \frac{1}{12} \neq 0$$

$$\mathcal{A}_{\mathbf{10}_{SU(5)}}^{U(1)\times \mathrm{grav}^2} \propto \underbrace{-2\cdot 3(\tfrac{1}{6})}_{\{\mathbf{3},\mathbf{2},\frac{1}{6}\}} + \underbrace{3(-\tfrac{2}{3})}_{\{\bar{\mathbf{3}},\mathbf{1},-\frac{2}{3}\}} + \underbrace{(1)}_{\{\mathbf{1},\mathbf{1},1\}} = -2 \neq 0$$

$$\mathcal{A}_{\mathbf{10}_{SU(5)}}^{U(1)\times (SU(2))^2} \propto \underbrace{-\cdot 3(\tfrac{1}{6})}_{\{\mathbf{3},\mathbf{2},\frac{1}{6}\}} = -\frac{1}{2} \neq 0$$

$$\mathcal{A}_{\mathbf{10}_{SU(5)}}^{U(1)\times (SU(3))^2} \propto \underbrace{-2\cdot (\tfrac{1}{6})}_{\{\mathbf{3},\mathbf{2},\frac{1}{6}\}} + \underbrace{(-\tfrac{2}{3})}_{\{\bar{\mathbf{3}},\mathbf{1},-\frac{2}{3}\}} = -1 \neq 0$$

$$\Rightarrow A(\mathbf{10}_{SU(5)}) \neq 0 \tag{266}$$

24

$$\mathcal{A}_{\mathbf{24}}^{(U(1))^3} \propto \underbrace{-2\cdot 3(\tfrac{1}{6})^3}_{\{\mathbf{3},\mathbf{2},\frac{1}{6}\}} - \underbrace{2\cdot 3(-\tfrac{1}{6})^3}_{\{\overline{\mathbf{3}},\mathbf{2},-\frac{1}{6}\}} = 0$$

$$\mathcal{A}_{\mathbf{24}}^{U(1)\times\mathrm{grav}^2} \propto \underbrace{-2\cdot 3(\tfrac{1}{6})}_{\{\mathbf{3},\mathbf{2},\frac{1}{6}\}} - \underbrace{2\cdot 3(-\tfrac{1}{6})}_{\{\overline{\mathbf{3}},\mathbf{2},-\frac{1}{6}\}} = 0$$

$$\mathcal{A}_{\mathbf{24}}^{U(1)\times(SU(2))^2} \propto \underbrace{-3(\tfrac{1}{6})}_{\{\mathbf{3},\mathbf{2},\frac{1}{6}\}} - \underbrace{3(-\tfrac{1}{6})}_{\{\overline{\mathbf{3}},\mathbf{2},-\frac{1}{6}\}} = 0$$

$$\mathcal{A}_{\mathbf{24}}^{U(1)\times(SU(3))^2} \propto \underbrace{-2(\tfrac{1}{6})}_{\{\mathbf{3},\mathbf{2},\frac{1}{6}\}} - \underbrace{2(-\tfrac{1}{6})}_{\{\overline{\mathbf{3}},\mathbf{2},-\frac{1}{6}\}} = 0$$

$$\Rightarrow A(\mathbf{24}) = 0 \tag{267}$$

$\mathbf{10}_{SO(10)}$, 45, 54, 120

All $\mathcal{G}_{\mathrm{SM}}$ representations of the $\mathbf{10}_{SO(10)}$, **45**, **54**, and **120** representations are either a member of a pair of conjugate representations, whose anomaly contributions directly cancel, or are hyperchargeless ($Y = 0$), and are hence anomaly-free. For example, the $\mathcal{G}_{\mathrm{SM}}$ representations of the $\mathbf{10}_{SO(10)}$ are:

$$\mathbf{10}_{SO(10)} \to \underbrace{\{\mathbf{3},\mathbf{1},-\tfrac{1}{3}\}\oplus\{\overline{\mathbf{3}},\mathbf{1},\tfrac{1}{3}\}}_{\text{contributions cancel}} \oplus \underbrace{\{\mathbf{1},\mathbf{2},\tfrac{1}{2}\}\oplus\{\mathbf{1},\mathbf{2},-\tfrac{1}{3}\}}_{\text{contributions cancel}}\,. \tag{268}$$

$$\Rightarrow A(\mathbf{10}_{SO(10)}) = 0\,, \tag{269}$$

$$A(\mathbf{45}) = 0\,, \tag{270}$$

$$A(\mathbf{54}) = 0\,, \tag{271}$$

$$A(\mathbf{120}) = 0\,. \tag{272}$$

16

$$\mathcal{A}_{\mathbf{16}}^{(U(1))^3} \propto \underbrace{3(\tfrac{1}{3})^3}_{\{\overline{\mathbf{3}},\mathbf{1},\frac{1}{3}\}} - \underbrace{2(-\tfrac{1}{2})^3}_{\{\mathbf{1},\mathbf{2},-\frac{1}{2}\}} - \underbrace{3\cdot 2(\tfrac{1}{6})^3}_{\{\mathbf{3},\mathbf{2},\frac{1}{6}\}} + \underbrace{3(-\tfrac{2}{3})^3}_{\{\overline{\mathbf{3}},\mathbf{1},-\frac{2}{3}\}} + \underbrace{(1)^3}_{\{\mathbf{1},\mathbf{1},1\}} = \frac{4}{9} \neq 0$$

$$\mathcal{A}_{\mathbf{16}}^{U(1)\times \mathrm{grav}^2} \propto \underbrace{3(\tfrac{1}{3})}_{\{\overline{\mathbf{3}},\mathbf{1},\frac{1}{3}\}} - \underbrace{2(-\tfrac{1}{2})}_{\{\mathbf{1},\mathbf{2},-\frac{1}{2}\}} - \underbrace{3\cdot 2(\tfrac{1}{6})}_{\{\mathbf{3},\mathbf{2},\frac{1}{6}\}} + \underbrace{3(-\tfrac{2}{3})}_{\{\overline{\mathbf{3}},\mathbf{1},-\frac{2}{3}\}} + \underbrace{(1)}_{\{\mathbf{1},\mathbf{1},1\}} = 0$$

$$\mathcal{A}_{\mathbf{16}}^{U(1)\times (SU(2))^2} \propto - \underbrace{(-\tfrac{1}{2})}_{\{\mathbf{1},\mathbf{2},-\frac{1}{2}\}} - \underbrace{3\cdot(\tfrac{1}{6})}_{\{\mathbf{3},\mathbf{2},\frac{1}{6}\}} = 0$$

$$\mathcal{A}_{\mathbf{16}}^{U(1)\times (SU(3))^2} \propto \underbrace{(\tfrac{1}{3})}_{\{\overline{\mathbf{3}},\mathbf{1},\frac{1}{3}\}} - \underbrace{\cdot 2(\tfrac{1}{6})}_{\{\mathbf{3},\mathbf{2},\frac{1}{6}\}} + \underbrace{(-\tfrac{2}{3})}_{\{\overline{\mathbf{3}},\mathbf{1},-\frac{2}{3}\}} = -\frac{2}{3}$$

$$\Rightarrow A(\mathbf{16}) \neq 0 \tag{273}$$

126

Extrapolating the logic used in the case of the $\mathbf{10}_{SO(10)}$ to that of the $\mathbf{126}$, one finds that only the following SM representations contribute to the anomaly of the $\mathbf{126}$:

$$\begin{aligned} &\{\mathbf{1},\mathbf{1},-1\} \oplus \{\mathbf{1},\mathbf{1},-2\} \oplus \{\mathbf{1},\mathbf{3},1\} \oplus \{\mathbf{3},\mathbf{3},\frac{1}{3}\} \\ &\oplus \{\overline{\mathbf{6}},\mathbf{3},-\frac{1}{3}\} \oplus \{\mathbf{6},\mathbf{1},\frac{4}{3}\} \oplus \{\mathbf{6},\mathbf{1},-\frac{2}{3}\} \oplus \{\mathbf{6},\mathbf{1},\frac{1}{3}\} \\ &\oplus \{\mathbf{3},\mathbf{1},-\frac{4}{3}\} \oplus \{\mathbf{3},\mathbf{1},\frac{2}{3}\} \oplus \{\overline{\mathbf{3}},\mathbf{1},\frac{1}{3}\} \end{aligned}$$

Using the anomaly prefactors for fundamental and adjoint reps, as well as for the $\mathbf{6}$ of $SU(3)$ [20], the anomalies of $\mathbf{126}$ are calculated below:

$$\begin{aligned} \mathcal{A}_{\mathbf{126}}^{(U(1))^3} \propto\ & \underbrace{(-1)^3}_{\{\mathbf{1},\mathbf{1},-1\}} \underbrace{(-2)^3}_{\{\mathbf{1},\mathbf{1},-2\}} + \underbrace{0}_{\{\mathbf{1},\mathbf{3},1\}} + \underbrace{0}_{\{\mathbf{3},\mathbf{3},\frac{1}{3}\}} + \underbrace{0}_{\{\overline{\mathbf{6}},\mathbf{3},-\frac{1}{3}\}} + \underbrace{7\cdot 6(\frac{4}{3})^3}_{\{\mathbf{6},\mathbf{1},\frac{4}{3}\}} + \underbrace{7\cdot 6(-\frac{1}{3})^3}_{\{\mathbf{6},\mathbf{1},-\frac{1}{3}\}} \\ & + \underbrace{7\cdot 6(\frac{2}{3})^3}_{\{\mathbf{6},\mathbf{1},\frac{2}{3}\}} + \underbrace{3(-\frac{4}{3})^3}_{\{\mathbf{3},\mathbf{1},-\frac{4}{3}\}} + \underbrace{3(\frac{2}{3})^3}_{\{\mathbf{3},\mathbf{1},\frac{2}{3}\}} + \underbrace{3(\frac{1}{3})^3}_{\{\overline{\mathbf{3}},\mathbf{1},\frac{1}{3}\}} = \frac{662}{9} \neq 0 \end{aligned}$$

$$\mathcal{A}_{\mathbf{126}}^{(U(1))^3} \propto \underbrace{(-1)}_{\{\mathbf{1},\mathbf{1},-1\}}\ \underbrace{(-2)}_{\{\mathbf{1},\mathbf{1},-2\}} + \underbrace{0}_{\{\mathbf{1},\mathbf{3},1\}} + \underbrace{0}_{\{\mathbf{3},\mathbf{3},\frac{1}{3}\}} + \underbrace{0}_{\{\overline{\mathbf{6}},\mathbf{3},-\frac{1}{3}\}} + \underbrace{7\cdot 6(\frac{4}{3})}_{\{\mathbf{6},\mathbf{1},\frac{4}{3}\}} + \underbrace{7\cdot 6(-\frac{1}{3})}_{\{\mathbf{6},\mathbf{1},-\frac{1}{3}\}}$$

$$+ \underbrace{7\cdot 6(\frac{2}{3})}_{\{\mathbf{6},\mathbf{1},\frac{2}{3}\}} + \underbrace{3(-\frac{4}{3})}_{\{\mathbf{3},\mathbf{1},-\frac{4}{3}\}} + \underbrace{3(\frac{2}{3})}_{\{\mathbf{3},\mathbf{1},\frac{2}{3}\}} + \underbrace{3(\frac{1}{3})}_{\{\overline{\mathbf{3}},\mathbf{1},\frac{1}{3}\}} = 38 \neq 0$$

$$\mathcal{A}_{\mathbf{126}}^{U(1)\times(SU(2))^2} \propto \underbrace{0}_{\{\mathbf{1},\mathbf{3},1\}} + \underbrace{0}_{\{\mathbf{3},\mathbf{3},\frac{1}{3}\}} + \underbrace{0}_{\{\overline{\mathbf{6}},\mathbf{3},-\frac{1}{3}\}} = 0$$

$$\mathcal{A}_{\mathbf{126}}^{U(1)\times(SU(3))^2} \propto \underbrace{0}_{\{\mathbf{3},\mathbf{3},\frac{1}{3}\}} + \underbrace{0}_{\{\overline{\mathbf{6}},\mathbf{3},-\frac{1}{3}\}} + \underbrace{7(\frac{4}{3})}_{\{\mathbf{6},\mathbf{1},\frac{4}{3}\}} + \underbrace{7(-\frac{1}{3})}_{\{\mathbf{6},\mathbf{1},-\frac{1}{3}\}}$$

$$+ \underbrace{7(\frac{2}{3})}_{\{\mathbf{6},\mathbf{1},\frac{2}{3}\}} + \underbrace{(-\frac{4}{3})}_{\{\mathbf{3},\mathbf{1},-\frac{4}{3}\}} + \underbrace{(\frac{2}{3})}_{\{\mathbf{3},\mathbf{1},\frac{2}{3}\}} + \underbrace{(\frac{1}{3})}_{\{\overline{\mathbf{3}},\mathbf{1},\frac{1}{3}\}} = \frac{20}{3} \neq 0$$

$$\Rightarrow A(\mathbf{126}) \neq 0 \tag{274}$$

7 Summary

In this paper, the importance of Grand Unified Theories (GUTs) to the field of BSM physics has been studied. Various topics, spanning gauge theories, the spontaneous symmetry breaking mechanism, the formulation of the SM, and the construction of GUT models, have been addressed, to the end of attempting to find an $SO(10)$ GUT model capable of unifying the strong, weak and electromagnetic interactions of the SM.

GUTs postulate that these three interactions are in actuality low-energy descendants of a single, unified force, such that over the "running" of the interaction strengths (gauge couplings) with energy, the SM couplings merge into one coupling at some large energy scale, M_{GUT}. GUTs can go some length towards solving many problems with the SM, as well as predicting the quantised nature of all elementary particles within a single elegant model. To build a GUT, one must choose a large gauge group, $\mathcal{G}_{\mathrm{GUT}}$, with the ability to contain the SM group, $\mathcal{G}_{\mathrm{SM}}$, as a subgroup.

Theoretical particle states in the form of fermionic and scalar representations, with the property of embedding within an $SO(10)$ GUT, have been added to the SM, in an attempt to achieve Grand Unification through their influence on the running couplings. A Python program was written to iterate over many different energies at which these states may appear.

It was found that there are many different combinations of representations which very precisely unify the couplings, some of which have a favourable $M_{\mathrm{GUT}} \sim 10^{16}\,\mathrm{GeV}$. The combination which achieved the most ideal mixture of precise unification and a high M_{GUT} was the $\mathbf{120}_F$, though the unified coupling strength was relatively low, which may or may not present a disadvantage, depending on the constraints of a particular GUT model.

In the scheme of an $SO(10)$ GUT, given its abundance of breaking options, the number of possibilities was very small indeed, and therefore the fact that several of these successfully achieved unification (many of which were simply a single, self-adjoint representation), demonstrates the power of $SO(10)$ GUTs to unify the fundamental forces of Nature at high energy.

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Appendix A Python code

```python
import numpy as np
import matplotlib.pyplot as plt
from operator import itemgetter
bSM=[41./10,-19./6,-7.]
alphaSM=[59., 30., 9.]
def Qi(E,alphai,bi):
    return np.exp(np.log(E)+(2*np.pi*alphai/bi))

QSM=[Qi(100,alphaSM[0],bSM[0]),Qi(100,alphaSM[1],bSM[1]),Qi(100,alphaSM[2],bSM[2])]

def alphai(Q, Qi, bi):
    return -bi*(np.log(Q)-np.log(Qi))/(2*np.pi)

def fun(x,i,j,o,p,k):
    if np.all(x< 10**i):
        return -bSM[k]*(np.log(x)-np.log(QSM[k]))/(2*np.pi)
    elif np.all(10**i<=x) and np.all(x<10**j):
        return -bk1[k]*(np.log(x)-np.log(Qk1[k]))/(2*np.pi)
    elif np.all(10**j<=x) and np.all(x<10**o):
        return -bk2[k]*(np.log(x)-np.log(Qk2[k]))/(2*np.pi)
    elif np.all(10**o<=x) and np.all(x<10**p):
        return -bk3[k]*(np.log(x)-np.log(Qk3[k]))/(2*np.pi)
    elif np.all(10**p<=x) and np.all (x<10**19):
        return -bk4[k]*(np.log(x)-np.log(Qk4[k]))/(2*np.pi)
x=np.logspace(0,19,10000)
discreps=[]
iterations=0
for m in range (4,14):
    for n in range (4,14):
        for o in range (4,14):
            for p in range (4,14):
                iterations=iterations+1
                print iterations
                leptkink1=[m,48./15,16./3,8.,"SU(3) adj"]
                quarkkink1=[n,36./15,40./3,6.,"SU(2) adj"]
                leptkink2=[o,73./15,4.,28./3,"coloured non-adj"]
                quarkkink2=[p,193./15,2./3,0.,"non-coloured non-adj"]
                kinks=[leptkink1,quarkkink1,leptkink2,quarkkink2]
                sortedkinks=sorted(kinks, key=itemgetter(0))
                bk1=[bSM[0]+sortedkinks[0][1],bSM[1]+sortedkinks[0][2],bSM[2]+sortedkinks[0][3]]
                bk2=[bk1[0]+sortedkinks[1][1],bk1[1]+sortedkinks[1][2],bk1[2]+sortedkinks[1][3]]
                bk3=[bk2[0]+sortedkinks[2][1],bk2[1]+sortedkinks[2][2],bk2[2]+sortedkinks[2][3]]
                bk4=[bk3[0]+sortedkinks[3][1],bk3[1]+sortedkinks[3][2],bk3[2]+sortedkinks[3][3]]
                alphak1=[alphai(10**sortedkinks[0][0], QSM[0],bSM[0]),alphai(10**sortedkinks[0][0], QSM[1],bSM[1]),alphai(10**sortedkinks[0][0], QSM[2],bSM[2])]
                Qk1=[Qi(10**sortedkinks[0][0],alphak1[0],bk1[0]),Qi(10**sortedkinks[0][0],alphak1[1],bk1[1]),Qi(10**sortedkinks[0][0],alphak1[2],bk1[2])]
                alphak2=[alphai(10**sortedkinks[1][0], Qk1[0],bk1[0]),alphai(10**sortedkinks[1][0], Qk1[1],bk1[1]),alphai(10**sortedkinks[1][0], Qk1[2],bk1[2])]
                Qk2=[Qi(10**sortedkinks[1][0],alphak2[0],bk2[0]),Qi(10**sortedkinks[1][0],alphak2[1],bk2[1]),Qi(10**sortedkinks[1][0],alphak2[2],bk2[2])]
                alphak3=[alphai(10**sortedkinks[2][0], Qk2[0],bk2[0]),alphai(10**sortedkinks[2][0], Qk2[1],bk2[1]),alphai(10**sortedkinks[2][0], Qk2[2],bk2[2])]
                Qk3=[Qi(10**sortedkinks[2][0],alphak3[0],bk3[0]),Qi(10**sortedkinks[2][0],alphak3[1],bk3[1]),Qi(10**sortedkinks[2][0],alphak3[2],bk3[2])]
                alphak4=[alphai(10**sortedkinks[3][0], Qk3[0],bk3[0]),alphai(10**sortedkinks[3][0], Qk3[1],bk3[1]),alphai(10**sortedkinks[3][0], Qk3[2],bk3[2])]
                Qk4=[Qi(10**sortedkinks[3][0],alphak4[0],bk4[0]),Qi(10**sortedkinks[3][0],alphak4[1],bk4[1]),Qi(10**sortedkinks[3][0],alphak4[2],bk4[2])]
                vfun = np.vectorize(fun)
                f1=vfun(x,sortedkinks[0][0],sortedkinks[1][0],sortedkinks[2][0],sortedkinks[3][0],0)
                f2=vfun(x,sortedkinks[0][0],sortedkinks[1][0],sortedkinks[2][0],sortedkinks[3][0],1)
                f3=vfun(x,sortedkinks[0][0],sortedkinks[1][0],sortedkinks[2][0],sortedkinks[3][0],2)
                point12=np.argwhere(np.isclose(f1,f2,rtol=0.001))
                point13=np.argwhere(np.isclose(f1,f3,rtol=0.001))
                point23=np.argwhere(np.isclose(f2,f3,rtol=0.001))    #sensitivity has been changed, normally 0.0001, but investigating errors
                index12=int(len(point12)/2)
                index13=int(len(point13)/2)
                index23=int(len(point23)/2)

                try:
                   discrep1213=np.sqrt((np.log10(x[point12[index12]])-np.log10(x[point13[index13]]))**2)
                except IndexError:
                   print "Error"
                try:
                   discrep1223=np.sqrt((np.log10(x[point12[index12]])-np.log10(x[point23[index13]]))**2)
                except IndexError:
                    print "Error"
                try:
                    discrep1323=np.sqrt((np.log10(x[point13[index13]])-np.log10(x[point23[index23]]))**2)
                except IndexError:
```

Figure 30a: Python code for engine written to iterate over many energy configurations of additional states, and then to display those above a certain threshold for the quality of coupling unification.

```
                try:
                    discrepmax=np.maximum(discrep1213,discrep1223,discrep1323)
                    discreps.append([m,n,o,p,discrepmax[0]])
                except NameError:
                    print "At least one combination of couplings does not unify in range"
                    discrepmax[0]=6.
                try:
                    if discrepmax<0.05:
                        plt.figure()
                        plt.plot(x, f1 ,"b")
                        plt.plot(x, f2 ,"r")
                        plt.plot(x, f3 ,"g")
                        try:
                            plt.plot(x[point12[index12]],f1[point12[index12]],"ko")
                        except IndexError:
                            print "The EM and Weak forces do not unify in range"
                        try:
                           plt.plot(x[point13[index13]],f1[point13[index13]],"ko")
                        except IndexError:
                            print "The EM and Strong forces do not unify in range"
                        try:
                            plt.plot(x[point23[index23]],f2[point23[index23]],"ko")
                        except IndexError:
                            print "The Strong and Weak forces do not unify in range"
                        plt.xscale('log')
                        plt.ylim(0,65)
                        plt.xlim(0,10**19.)
                        plt.title(kinks[0][4]+"= 10^"+str(kinks[0][0])+" GeV, "+kinks[1][4]+"= 10^"+str(kinks[1][0])+" GeV, "+kinks[2][4]+"= 10^"+str(kinks[2][0])+
                except IndexError:
                    print "Plotting error"

lowdiscreps=[]
print " "
for i in range (1, len(discreps)):
    if (discreps[i-1][4])<0.05:
        lowdiscreps.append(discreps[i-1])
print lowdiscreps
plt.show()
```

Figure 30b: Python code for engine written to iterate over many energy configurations of additional states, and then to display those above a certain threshold for the quality of coupling unification.