Generating Matrix Identities and Proof Complexity

Fu Li* Iddo Tzameret†

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Abstract

Motivated by the fundamental lower bounds questions in proof complexity, we initiate the study of matrix identities as hard instances for *strong* proof systems. A *matrix identity* of $d \times d$ matrices over a field $\mathbb{F}$, is a non-commutative polynomial $f(x_1, \ldots, x_n)$ over $\mathbb{F}$ such that f vanishes on every $d \times d$ matrix assignment to its variables.

We focus on *arithmetic proofs*, which are proofs of polynomial identities operating with arithmetic circuits and whose axioms are the polynomial-ring axioms (these proofs serve as an algebraic analogue of the Extended Frege propositional proof system; and over $GF(2)$ they constitute formally a sub-system of Extended Frege [9]). We introduce a decreasing in strength hierarchy of proof systems within arithmetic proofs, in which the dth level is a sound and complete proof system for proving $d \times d$ matrix identities (over a given field). For each level $d > 2$ in the hierarchy, we establish a proof-size lower bound in terms of the number of variables in the matrix identity proved: we show the existence of a family of matrix identities f_n with n variables, such that any proof of $f_n = 0$ requires $\Omega(n^{2d})$ number of lines.

The lower bound argument uses fundamental results from the theory of algebras with polynomial identities together with a generalization of the arguments in [7]. Specifically, we establish an unconditional lower bound on the minimal number of generators needed to generate a matrix identity, where the generators are substitution instances of elements from any given finite basis of the matrix identities; a result that might be of independent interest.

We then set out to study matrix identities as hard instances for (*full*) arithmetic proofs. We present two conjectures, one about non-commutative arithmetic circuit complexity and the other about proof complexity, under which up to *exponential-size* lower bounds on arithmetic proofs (in terms of the arithmetic circuit size of the identities proved) hold. Finally, we discuss the applicability of our approach to strong *propositional* proof systems such as Extended Frege.

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†Department of Computer Science, Royal Holloway, University of London. Email: iddo.tzameret@gmail.com Supported in part by the NSFC Grant 61373002.

1 Background

Proving super-polynomial size lower bounds on strong propositional proof systems, like the Extended Frege system, is a major open problem in proof complexity, and in general is among a handful of fundamental hardness questions in computational complexity theory. An Extended Frege proof is simply a textbook logical proof system for establishing Boolean tautologies, in which one starts from basic tautological axioms written as Boolean formulas, and derives, step by step, new tautological formulas from previous ones by using a finite set of logical sound derivation rules; including the so-called *extension axiom* enabling one to denote a possibly big formula by a *single* new variable (where the variable is used neither before in the proof nor in the last line of the proof). It is not hard to show (see [11]) that Extended Frege can equivalently be defined as a logical proof system operating with Boolean *circuits* (and without the extension axiom[1]).

Lower bounds on Extended Frege proofs can be viewed as lower bounds on certain non-deterministic algorithms for establishing the unsatisfiability of Boolean formulas (and thus as a progress towards separating $\boldsymbol{NP}$ from $\boldsymbol{coNP}$). It is also usually considered (somewhat informally) as related to establishing (explicit) Boolean circuit size lower bounds. In fact, it has also another highly significant consequence, that places such a lower bound as a step towards separating $\boldsymbol{P}$ from $\boldsymbol{NP}$: showing any super-polynomial lower bound on the size of Extended Frege proofs implies that, at least with respect to "polynomial-time reasoning" (namely, reasoning in the formal theory of arithmetic denoted S^1_2), it is not possible to prove that $\boldsymbol{P} = \boldsymbol{NP}$; or in other words, it is consistent with S^1_2 that $\boldsymbol{P} \neq \boldsymbol{NP}$ (cf. [15]).

Accordingly, proving Extended Frege lower bounds is considered a very hard problem. In fact, even *conditional* lower bounds on strong proof systems, including Extended Frege, are not known and are considered very interesting;[2] here, we mean a condition that is different from $\boldsymbol{NP} \neq \boldsymbol{coNP}$ (see [18]; the latter condition immediately implies that any propositional proof system admits a family of tautologies with no polynomial-size proofs [4]). The only size lower bound on Extended Frege proofs that is known to date is linear $\Omega(n)$ (where n is the size of the tautological formula proved; see [14] for a proof). Establishing *super-linear* size lower bounds on Extended Frege proofs is thus a highly interesting open problem.

That said, although proving Extended Frege lower bounds is a fundamental open problem in complexity, it is quite unclear whether such lower bounds are indeed far from reach or beyond current techniques (in contrast to other fundamental hardness problems in complexity, such as strong explicit Boolean circuit lower bounds, for which formal so-called barriers are known).

Another feature of proof complexity is that, in contrast to circuit complexity, even the

[1] An additional simple technical axiom is needed to formally define this proof system ([11]).

[2] Informally, we call a proof system *strong* if there are no known (non-trivial) size lower bounds on proofs in the system and further such lower bounds are believed to be outside the realm of current techniques.

existence of non-explicit hard instances for strong propositional proof systems, including Extended Frege, are unknown. For instance, simple counting arguments cannot establish super-linear size lower bounds on Extended Frege proofs (in contrast to Shannon's counting argument which gives non-explicit lower bounds on circuit size, but does not in itself yield complexity class separations). Thus, the existence of non-explicit hard instances in proof complexity is sufficient for the purpose of lower bounding the size of strong proof systems.

Furthermore, for strong proof systems there are almost no hard candidates, namely, tautologies that are believed to require long proofs in these systems (see Bonet, Buss and Pitassi [2]); except, perhaps for random k-CNF formulas near the satisfiability threshold. But for the latter instances, even lower bounds on Frege proofs of constant-depth are unknown. It is worth noting also that Razborov [19] and especially Krajíček (see e.g., [16]) had proposed some tautologies as hard candidates for strong proof systems.

Due to the lack of progress on establishing lower bounds on strong propositional proof systems, it is interesting, and potentially helpful, to turn our eyes to an *algebraic analogue* of strong propositional proof systems, and try first to prove nontrivial size lower bounds in such settings. Quite recently, such algebraic analogues of Extended Frege (and Frege, which is Extended Frege without the extension axiom) were investigated by Hrubeš and the second author [8, 9]. These proof systems denoted $\mathbb{P}_c(\mathbb{F})$, called simply *arithmetic proofs*, operate with algebraic equations of the form $F = G$, where F and G are algebraic circuits over a given field $\mathbb{F}$. An arithmetic proof of a polynomial identity is a sequence of identities between algebraic circuits derived by means of simple syntactic manipulation representing the polynomial-ring axioms (e.g., associativity, distributivity, unit element, field identities, etc.; see Definition 16). Although arithmetic proof systems are not propositional proof systems, namely they do not prove propositional tautologies, they can be regarded nevertheless as *fragments* of the propositional Extended Frege proof system when the field considered is $GF(2)$. That is, every arithmetic proof over $GF(2)$ of a polynomial identity (considered as a propositional tautology) can formally be viewed also as an Extended Frege proof.[3]

Apart from the hope that arithmetic proofs would shed light on propositional proof systems, the study of arithmetic proofs is motivated by the Polynomial Identity Testing (PIT) problem, namely the problem of deciding if a given algebraic circuit computes the zero polynomial. As a decision problem, polynomial identity testing can be solved by an efficient randomized algorithm [21, 22], but no efficient deterministic algorithm is known. In

[3]In fact, it is probably true (but was not formally verified) that arithmetic proofs are fragments of propositional proofs also over any other finite field, as well as over the ring of integers (when restricted to up to exponentially big integers). That is, it is probably true that every polynomial identity proved with an arithmetic proof over the given field or ring, can be proved with at most a polynomial increase in size in Extended Frege when we fix a certain natural translation between polynomial identities over the field or ring and propositional tautologies. The reason for this is that one could plausibly polynomially simulate arithmetic proofs over such fields or rings with propositional proofs in which numbers are encoded as bit-strings.

fact, it is not even known whether there is a polynomial time non-deterministic algorithm or, equivalently, whether PIT is in ***NP***. An arithmetic proof system can thus be interpreted as a specific non-deterministic algorithm for PIT: in order to verify that an arithmetic circuit C computes the zero polynomial, it is sufficient to guess an arithmetic proof of $C = 0$. Hence, if every true equality has a polynomial-size proof then PIT is in ***NP***. Conversely, the arithmetic proof system captures the common syntactic procedures used to establish equality between algebraic expressions. Thus, showing the existence of identities that require super-polynomial arithmetic proofs would imply that those syntactic procedures are not enough to solve the PIT problem efficiently.[4]

The emphasis in [8, 9] was mainly on demonstrating non-trivial *upper bounds* for arithmetic proofs (as well as lower bounds in very restricted settings). Since arithmetic proofs (at least over $GF(2)$), can also be considered as propositional proofs, arithmetic proofs were found very useful in establishing short propositional proofs for the determinant identities and other statements from linear algebra [9]. As for *lower bounds* on arithmetic proofs (operating with arithmetic circuits), the same basic linear size lower bound known for Extended Frege [14] can be shown to hold for $\mathbb{P}_c$. But any super-linear size lower bound, explicit or not, on $\mathbb{P}_c(\mathbb{F})$ proof size (for any field $\mathbb{F}$) is open. In [8] it was argued that proving lower bounds even on very restricted fragments of arithmetic proofs is a highly nontrivial open problem.

The state of affairs we have described up to now shows how little is known about strong propositional (and arithmetic) proof systems, and why it is highly interesting to introduce and develop novel approaches for lower bounding proofs such as arithmetic proofs, even if these approaches yield only conditional and possibly non-explicit lower bounds; and further, to propose new kinds of hard candidates for strong proof systems.

2 Overview of our results

In this work we initiate the study of matrix identities as hard instances for strong proof systems in various settings and under different assumptions. The term *strong* here stands for proof systems that operate with (Boolean or arithmetic) *circuits*, for which we do not know any non trivial lower bound (see Sec. A.2 for the definitions of arithmetic circuits and non-commutative arithmetic circuits).

The ultimate goal of our suggested approach is proving Extended Frege lower bounds; however, in this work we focus for most part on the seemingly (and relatively) easier task of proving arithmetic proofs $\mathbb{P}_c(\mathbb{F})$ lower bounds, namely lower bounds on arithmetic proofs establishing polynomial identities between arithmetic circuits over a field $\mathbb{F}$.

[4] It is worth emphasizing again that arithmetic proofs are different than algebraic *propositional* proof systems like the Polynomial Calculus [3] and related systems. The latter prove propositional tautologies (a ***coNP*** language) while the former prove formal polynomial identities written as equations between algebraic circuits (a ***coRP*** language).

We introduce a new decreasing hierarchy of proof systems establishing matrix identities of a given dimension, within arithmetic proofs (and whose first level coincides with arithmetic proofs). We obtain unconditional (polynomial) lower bounds on proof systems for matrix identities in terms of the number of variables in the identities proved. We then present two natural conjectures from arithmetic circuit complexity and proof complexity, respectively, based on which one can obtain up to exponential-size lower bounds on arithmetic proofs $\mathbb{P}_c(\mathbb{F})$ in terms of the size of the identities proved.

We start by explaining what matrix identities are, as well as providing some necessary background from algebra.

2.1 Matrix identities

For a field $\mathbb{F}$ let A be a non-commutative (associative) $\mathbb{F}$-algebra; e.g., the algebra $\mathrm{Mat}_d(\mathbb{F})$ of $d \times d$ matrices over $\mathbb{F}$. (Formally, A is an $\mathbb{F}$-algebra, if A is a vector space over $\mathbb{F}$ together with a distributive multiplication operation; where multiplication in A is associative (but it need not be commutative) and there exists a multiplicative unity in A.)

We shall always assume, unless explicitly stated otherwise, that the field $\mathbb{F}$ has characteristic 0.

A ***non-commutative polynomial*** over the field $\mathbb{F}$ and with the variables $X := \{x_1, x_2, \ldots\}$ is a formal sum of monomials where the product of variables is non-commuting. Since most polynomials in this work are non-commutative **when we talk about *polynomials* we shall mean *non-commutative polynomials,*** unless otherwise stated. The set of (non-commutative) polynomials with variables X and over the field $\mathbb{F}$ is denoted $\mathbb{F}\langle X\rangle$.

We say that f is a ***matrix identity*** *of* $\mathrm{Mat}_d(\mathbb{F})$ simply whenever f is a non-commutative polynomial (with coefficients from $\mathbb{F}$) that is equal to zero under any assignment of matrices from $\mathrm{Mat}_d(\mathbb{F})$ to its variables. In other words, the polynomial $f(x_1, \ldots, x_n)$ over $\mathbb{F}$ is *an identity of the algebra* A (and specifically, the matrix algebra $\mathrm{Mat}_d(\mathbb{F})$), if for all $\bar{c} \in A^n$, $f(\bar{c}) = 0$.

2.2 Stratification

A matrix identity is a non-commutative polynomial vanishing over all assignments of matrices. If we consider the "matrix" algebra of 1×1 matrices $\mathrm{Mat}_1(\mathbb{F})$, its set of identities consists of all the non-commutative polynomials that vanish over field elements. Since, by definition, the field is commutative, the identities of $\mathrm{Mat}_1(\mathbb{F})$ are all non-commutative polynomials such that when the product is considered as *commutative* we obtain the zero polynomial; in other words, *we can consider the identities of* $\mathrm{Mat}_1(\mathbb{F})$ *as the set of (standard, i.e., commutative) polynomial identities.* Further, in our application we shall write all polynomials as non-commutative arithmetic circuits, and since a non-commutative arithmetic circuit is

equivalent to a (commutative) arithmetic circuit (except that product gates have order on their children) *we can consider the set of identities of* $\mathrm{Mat}_1(\mathbb{F})$ *written as non-commutative circuits, as the set of (commutative) polynomial identities written as (commutative) arithmetic circuits.*

Using matrix identities of increasing dimensions d we obtain a stratification of the language of (commutative) polynomial identities. Namely, we obtain the following strictly decreasing (with respect to containment) chain of identities:

$$\begin{aligned} \text{(commutative) polynomial identities} &\supsetneq \mathrm{Mat}_2(\mathbb{F})\text{-identities} \\ &\supsetneq \mathrm{Mat}_3(\mathbb{F})\text{-identities} \\ &\supsetneq \dots \\ &\supsetneq \mathrm{Mat}_d(\mathbb{F}) \\ &\supsetneq \mathrm{Mat}_{d+1}(\mathbb{F}) \supsetneq \dots \end{aligned} \tag{1}$$

The fact that the identities of $\mathrm{Mat}_{d+1}(\mathbb{F})$ are also identities of $\mathrm{Mat}_d(\mathbb{F})$ is easy to show. The fact that the chain above is *strictly* decreasing can be proved either by a elementary methods [12] or as a corollary of [1].

2.3 Corresponding proof systems and the main lower bound

We now introduce a novel hierarchy of proof systems within arithmetic proofs $\mathbb{P}_c(\mathbb{F})$. For this we need the concept of a *basis* of a set of identities of a given $\mathbb{F}$-algebra A (e.g., the matrix algebra $\mathrm{Mat}_d(\mathbb{F})$) .

Basis. We say that a set of non-commutative polynomials $\mathcal{B}$ forms a ***basis*** for the identities of A, in the following sense: for every identity f of A there exist non-commutative polynomials $g_1, ..., g_k$, for some k, that are *substitution instances* of polynomials from $\mathcal{B}$, such that f is in the two-sided ideal $\langle g_1, ..., g_k \rangle$ (a ***substitution instance*** of a polynomial $g(x_1, \dots, x_n) \in \mathbb{F}\langle X \rangle$ is a polynomial $g(h_1, \dots, h_n)$, for some $h_i \in \mathbb{F}\langle X \rangle$, $i \in [n]$).

Recall that ***arithmetic proofs*** $\mathbb{P}_c(\mathbb{F})$ (see Definition 16) are proofs that start from basic axioms like associativity, commutativity of addition and product, distributivity of product over addition, unit element axioms, etc., in which we derive new equations between arithmetic circuits $F = G$ using rules for adding and multiplying two previous identities. Arithmetic proofs are sound and complete proof systems for the set of (commutative) polynomial identities, written as equations between arithmetic circuits.

Notice that if one takes out the Commutativity Axiom $f \cdot g = g \cdot f$ from arithmetic proofs, we get a proof system for establishing non-commutative polynomial identities written as non-commutative arithmetic circuits (we can assume that product gates appearing in arithmetic proofs have order on their children).

The proof systems $\mathbb{P}_{\mathrm{Mat}_d}(\mathbb{F})$. For any field $\mathbb{F}$ (of characteristic 0), any $d \geq 1$, and any basis $\mathcal{B}$ of the identities of $\mathrm{Mat}_d(\mathbb{F})$, we define the following proof system $\mathbb{P}_{\mathrm{Mat}_d}(\mathbb{F})$, which is sound and complete for the identities of $\mathrm{Mat}_d(\mathbb{F})$ (written as equations of non-commutative circuits): consider the proof systems $\mathbb{P}_c(\mathbb{F})$ (Definition 16) and *replace* the commutativity axiom $h \cdot g = g \cdot h$ by a finite basis $\mathcal{B}$ of the identities of $\mathrm{Mat}_d(\mathbb{F})$ (namely, add a new axiom $H = 0$ for each polynomial h in the basis, where H is a non-commutative algebraic circuit computing h).[5] Additionally, add the axioms of distributivity of product over addition from *both* left and right (this is needed because we do not have anymore the commutativity axiom in our system to simulate both distributivity axioms).

Note that $\mathbb{P}_c(\mathbb{F})$ can be considered as $\mathbb{P}_{\mathrm{Mat}_1}(\mathbb{F})$, since the commutator $[g, h]$ is an axiom of $\mathbb{P}_c(\mathbb{F})$ and the commutator is a basis of the identities of $\mathrm{Mat}_1(\mathbb{F})$.

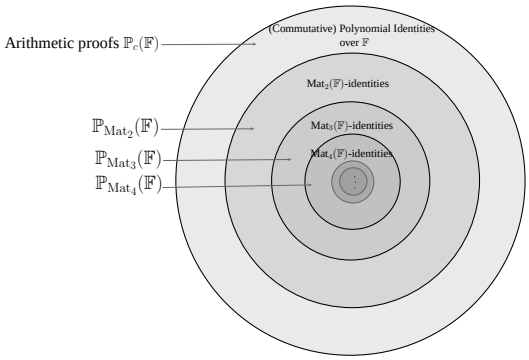

Figure 1: Illustration of the stratification of the language of polynomial identities and the corresponding proof systems for each language.

Our main result is an unconditional lower bound on the size (in fact the number of lines[6]) of $\mathbb{P}_{\mathrm{Mat}_d}(\mathbb{F})$ proofs, for any d, *in terms of the number of variables* n in the matrix identity proved:

Theorem 1 (Main lower bound)**.** *Let $\mathbb{F}$ be any field of characteristic 0. For any natural number $d > 2$ and every finite basis $\mathcal{B}$ of the identities of $\mathrm{Mat}_d(\mathbb{F})$, there exists an identity f over $\mathrm{Mat}_d(\mathbb{F})$ of degree $2d+1$ with n variables, such that any $\mathbb{P}_{\mathrm{Mat}_d}(\mathbb{F})$-proof of f requires $\Omega(n^{2d})$ lines.*

The proof of the main lower bound—which is the main technical contribution of our work—is explained in the following subsection, and is based on a complexity measure defined on matrix identities and their generation in a (two-sided) ideal. The complexity measure is interesting by itself, and can be applied to identities of any algebra with polynomial identities (PI-algebras; see [20, 6] for the theory of PI-algebras), and not only matrix identities.

[5] Formally, we should fix a specific finite basis $\mathcal{B}$ for the sake of definiteness of $\mathbb{P}_{\mathrm{Mat}_d}(\mathbb{F})$. However, different choices of bases can only increase the number of lines in a $\mathbb{P}_{\mathrm{Mat}_d}(\mathbb{F})$-proof by a constant factor.

[6] A *proof-line* is any equation $F = G$ between arithmetic circuits appearing in the proof.

Comments. (i) When $d = 2$, our proof, showing the lower bound for *every* basis $\mathcal{B}$ of the identities of $\mathrm{Mat}_2(\mathbb{F})$, does *not* hold (see Sec. C.1.3 for an explanation).

(ii) The hard instance in the main lower bound theorem is *non-explicit*. Thus, we do not know if there are small non-commutative circuits computing the hard instances. This is the reason the lower bound holds only with respect to the number of variables n in the hard-instances and not with respect to its circuit size—the latter is the more desired result in proof complexity. Section 3 sets out an approach to achieve this latter result.

(iii) The proof-systems $\mathbb{P}_{\mathrm{Mat}_d}(\mathbb{F})$ are defined using a finite basis of the identities of $\mathrm{Mat}_d(\mathbb{F})$. A very interesting feature of our lower bound argument is that it is in fact an open problem to find explicit finite bases for the identities of $\mathrm{Mat}_d(\mathbb{F})$ (for $d > 2$; see the next sub-Section 2.3.1 on this).

(iv) We do not know if the hierarchy of proof systems $\mathbb{P}_{\mathrm{Mat}_d}(\mathbb{F})$ for increasing d's is a *strictly* decreasing hierarchy (since we do not know if $\mathbb{P}_{\mathrm{Mat}_{d-1}}(\mathbb{F})$ has any speed-up over $\mathbb{P}_{\mathrm{Mat}_d}(\mathbb{F})$ for identities of $\mathrm{Mat}_d(\mathbb{F})$).

In the following subsection we give a detailed overview of the lower bound argument.

2.3.1 Proving the main lower bound: generative complexity lower bounds

Here we explain in details the complexity measure we define and how we obtain the lower bound on this measure. It is simple to show that our complexity measure is a lower bound on the minimal number of lines in a corresponding $\mathbb{P}_{\mathrm{Mat}_d}(\mathbb{F})$-proof (for the case $d = 1$ this was observed in [7]).

The complexity measure. Given an $\mathbb{F}$-algebra A (e.g., $\mathrm{Mat}_d(\mathbb{F})$) and an identity f of A, define

$$Q_{\mathcal{B}}(f)$$

as the minimal number k such that there exist $g_1, \ldots, g_k \in \mathbb{F}\langle X\rangle$ for which $f \in \langle g_1, ..., g_k\rangle$, and every g_i is a substitution instance of some polynomial from $\mathcal{B}$. (Note that each substitution instance, even of the same polynomial from $\mathcal{B}$, adds to $Q_{\mathcal{B}}(f)$.) We sometimes call $Q_{\mathcal{B}}(f)$ ***the generative complexity of*** f (with respect to $\mathcal{B}$).

Example: Let $\mathbb{F}$ be an infinite field and consider the field $\mathbb{F}$ itself as an $\mathbb{F}$-*algebra*, denoted $\mathscr{A}$. Then the identities of $\mathscr{A}$ are all the polynomials from $\mathbb{F}\langle X\rangle$ that evaluate to 0 under every assignment from $\mathbb{F}$ to the variables X. Namely, these are the (non-commutative) polynomials that are identically zero polynomials *when considered as commutative polynomials*. For instance, $x_1x_2 - x_2x_1$ is a non-zero polynomial from $\mathbb{F}\langle X\rangle$ which is an identity over $\mathscr{A}$.

It is not hard to show that the *basis* of the algebra $\mathscr{A}$ is the *commutator* $x_1x_2 - x_2x_1$, denoted $[x_1, x_2]$. In other words, every identity of $\mathscr{A}$ is generated (in the two-sided ideal) by substitution instances of the commutator. Considering $Q_{\{[x_1,x_2]\}}$, we can now ask what is

$Q_{\{[x_1,x_2]\}}(x_1x_3 - x_3x_1 + x_2x_3 - x_3x_2)$? The answer is 1 since we need only one substitution instance of the commutator: $(x_1 + x_2)x_3 - x_3(x_1 + x_2) = x_1x_3 - x_3x_1 + x_2x_3 - x_3x_2$.

Hrubeš [7] showed the following lower bound (using a slightly different terminology):

Theorem 2 (Hrubeš [7])**.** *For any field and every n, there exists an identity $f \in \mathbb{F}\langle X\rangle$ of $\mathscr{A}$ with n variables, such that $Q_{\{[x_1,x_2]\}}(f) = \Omega(n^2)$.*

It is also not hard to show that $Q_{\{[x_1,x_2]\}}(f) = O(n^2)$ for any identity f.

Lower bound on the complexity of generating matrix identities. An *algebra with polynomial identities,* or in short a ***PI-algebra*** (PI stands for Polynomial Identities), is simply an $\mathbb{F}$-algebra that has a non-trivial identity, that is, there is a nonzero $f \in \mathbb{F}\langle X\rangle$ that is an identity of the algebra.

Let us treat (the $\mathbb{F}$-algebra) $\mathbb{F}$ as the matrix algebra $\mathrm{Mat}_1(\mathbb{F})$ of 1×1 matrices with entries from $\mathbb{F}$. We shall exploit results about the structure of the identities of matrix algebras and the general theory of PI-algebras to completely generalize Hrubeš [7] lower bound above (excluding the case $d = 2$), from a lower bound of $\Omega(n^2)$ for generating identities of $\mathrm{Mat}_1(\mathbb{F})$ to a lower bound of $\Omega(n^{2d})$ for generating identities of $\mathrm{Mat}_d(\mathbb{F})$, for any $d > 2$ and any field $\mathbb{F}$ of characteristic 0:

Theorem 5 (Lower bound on generative complexity)**.** *Let $\mathbb{F}$ be any field of characteristic 0. For every natural number $d > 2$ and every finite basis $\mathcal{B}$ of the identities of $\mathrm{Mat}_d(\mathbb{F})$, there exists an identity f over $\mathrm{Mat}_d(\mathbb{F})$ of degree $2d+1$ with n variables, such that $Q_{\mathcal{B}}(f) = \Omega(n^{2d})$.*

Notice that similar to [7], the lower bound in this theorem is *non-explicit.* We do not know of an upper bound (in terms of n) that holds on $Q_{\mathcal{B}}(f)$, for every identity f with n variables.

The main lower bound (Theorem 1) is a corollary of the following theorem (proved by simple induction on the number of lines in a $\mathbb{P}_{\mathrm{Mat}_d}(\mathbb{F})$-proof):

Theorem. *For every identity $F = 0$, where F is a non-commutative circuit that computes a non-commutative polynomial f which is an identity of $\mathrm{Mat}_d(\mathbb{F})$, the number of lines of a $\mathbb{P}_{\mathrm{Mat}_d}(\mathbb{F})$-proof of $F = 0$ is lower bounded up to a constant factor (depending on the choice of finite basis $\mathcal{B}$) by $Q_{\mathcal{B}}(f)$.*

Overview of the proof of Theorem 5. The study of algebras with polynomial identities is a fairly developed subject in algebra (see the monographs by Drensky [6] and Rowen [20] on this topic). Within it, perhaps the most well studied topic is about the identities of matrix algebras. In particular, the well-known theorem of Amitsur and Levitzky from 1950 [1] is the following:

Amitsur-Levitzki Theorem ([1]). *Let $\mathcal{S}_d$ be the permutation group on d elements and let $S_d(x_1, x_2, \ldots, x_d)$ denote the* ***standard identity*** *of degree d as follows:*

$$S_d(x_1, x_2, \ldots, x_d) := \sum_{\sigma \in \mathcal{S}_d} sgn(\sigma) \prod_{i=1}^{d} x_{\sigma(i)}.$$

Then, for any natural number d and any field $\mathbb{F}$ (in fact, any commutative ring) the standard identity $S_{2d}(x_1, x_2, \ldots, x_{2d})$ of degree $2d$ is an identity of $\mathrm{Mat}_d(\mathbb{F})$.

Theorem 5 is proved in several steps, but the main argument can be divided into two main parts, described as follows:

Part 1: Here we use the Amitsur-Levitzki Theorem: we show that when $\mathcal{E} = \{S_{2d}(x_1, \ldots, x_{2d})\}$ there exists an $f \in \mathbb{F}\langle X\rangle$ with $2n$ variables and degree $2d+1$, such that $Q_{\mathcal{E}}(f) = \Omega(n^{2d})$. To this end, we generalize the method in [7] to "higher dimensional commutativity axioms": using a counting argument we show the existence of n special polynomials (we call *s-polynomials*; see Definition 10) $P_1, P_2, \ldots, P_n$ over n variables and each of degree $2n$ such that $Q_{S_{2d}}(P_1, \ldots, P_n) = \Omega(n^{2d})$ (see Lemma 9). Then, we combine the n s-polynomials into a single polynomial $P^\star$ with degree $2d+1$, by adding n new variables, such that $Q_{S_{2d}}(P^\star) = \Omega(Q_{S_{2d}}(P_1, \ldots, P_n))$.

While [7] uses the commutator $[x, y]$ to define the s-polynomials, we consider the higher order commutativity axiom S_{2d} instead. It is possible to show that S_{2d} has sufficient properties for the lower bound as the commutator $[x, y]$ (see Lemmas 7, 8, 12).

Part 2: Note that $\mathcal{E} = \{S_{2d}(x_1, \ldots, x_{2d})\}$ is *not* a basis of $\mathrm{Mat}_d(\mathbb{F})$, namely there are identities of $\mathrm{Mat}_d(\mathbb{F})$ that are not generated by substitution instances of S_{2d} (also notice that $Q_{\mathcal{B}}(f)$ can be defined for any $\mathcal{B} \subseteq \mathbb{F}\langle X\rangle$). The second part in the proof of Theorem 5 is dedicated to showing that when $d > 2$, for *all finite bases $\mathcal{B}$ of the identities of* $\mathrm{Mat}_d(\mathbb{F})$ the following holds for the hard identity f considered in the theorem: $Q_{\mathcal{B}}(f) < c \cdot Q_{\mathcal{E}}(f)$ for some constant c.

For this purpose, we find a special set $\mathcal{B}' \subseteq \mathbb{F}\langle X\rangle$ which serves as an "intermediate" set between $\mathcal{B}$ and $\mathcal{E}$, such that $\mathcal{B}$ is generated by $\mathcal{B}'$, and all the polynomials in $\mathcal{B}'$ that contribute to the generation of the hard instance f can be generated already by $\mathcal{E}$. We then show (Lemma 17) that for any basis $\mathcal{B}$, there is a specific set $\mathcal{B}'$ of polynomials of a special form, namely, *multi-homogenous commutator polynomials* (Definition 11), that can generate $\mathcal{B}$. Based on the properties of multi-homogenous commutator polynomials, we show that, for the hard instance f, only the generators of degree at most $2d+1$ in $\mathcal{B}'$ can contribute to the generation of f (Lemma 21). We then prove that when $d > 2$, all the generators of degree at most $2d+1$ in $\mathcal{B}'$ can be generated by $\mathcal{E}$ (this is where we use the assumption that $d > 2$ (see Lemma 20)). We thus get the conclusion $Q_{\mathcal{B}'}(f) < c \cdot Q_{\mathcal{E}}(f)$, when $d > 2$.

A very interesting feature of our proof (and theorem), is that it is in fact an *open problem* to describe bases of the identities of $\mathrm{Mat}_d(\mathbb{F})$, for any $d > 2$. For the case $d = 2$ the basis is known by a result of Drensky [5] (see Section E.3). However, a highly nontrivial result of Kemer [13], shows that for any natural d *there exists* a finite basis for $\mathrm{Mat}_d(\mathbb{F})$. Our proof shows roughly that for the hard instances f in Theorem 5 no generators different from the S_{2d} generators can contribute to the generation of f.

3 Towards strong lower bounds on (full) arithmetic proofs

Here we continue the study of matrix identities as hard proof complexity instances, and set out a program to establish lower bounds on arithmetic proofs. We present two conjectures, interesting by themselves: one about non-commutative arithmetic circuit complexity and the other about proof-complexity, based on which up to exponential-size lower bounds on arithmetic proofs (in terms of the non-commutative circuit-size of the identity proved) follow. We discuss in details these conjectures and the parameters the are needed for different kinds of lower-bounds.

Informally, the two conjectures are as follows (recall the complexity measure $Q_{\mathcal{B}}(f)$ from Sec. 2.3.1, counting the minimal number of substitution instances of generators from a basis $\mathcal{B}$ needed to generate an identity f):

Conjecture I. *(Informal) There exist non-commutative arithmetic circuits of small size that compute matrix identities of high generative complexity.*

Conjecture II. *(Informal) Proving matrix identities by reasoning with polynomials whose variables $X_1, \ldots, X_n$ range over matrices is* as efficient *as proving matrix identities using polynomials whose variables range over the* entries *of the matrices $X_1, \ldots, X_n$?*

3.1 Towards lower bounds on $\mathbb{P}_{\mathrm{Mat}_d}(\mathbb{F})$ in terms of arithmetic-circuit size

Recall that a non-commutative arithmetic circuit is an arithmetic circuit that has an order on the children of product gates and the product is performed according to this order (see Sec. A.2). To get a size lower bound on $\mathbb{P}_{\mathrm{Mat}_d}(\mathbb{F})$ proofs in terms of the circuit equations proved, we need to assume the existence of non-commutative arithmetic circuits of small size that compute matrix identities of high generative complexity:

Conjecture I. *For some fixed $d \geq 1$, there exists a family of identities $f_n \in \mathbb{F}\langle X\rangle$ of $\mathrm{Mat}_d(\mathbb{F})$, with n variables, such that $Q_{\mathcal{B}}(f_n) = \Omega(n^d)$, for some basis $\mathcal{B}$ of the identities of $\mathrm{Mat}_d(\mathbb{F})$, and such that f_n has a non-commutative arithmetic circuit of size $O(n^r)$, for some constant $r < d$.*

Assuming the veracity of the above conjecture we obtain the following lower bound:

Polynomial lower bounds on $\mathbb{P}_{\mathrm{Mat}_d}(\mathbb{F})$-proofs (assuming Conjecture I): *There exists a family of identities f_n of $\mathrm{Mat}_d(\mathbb{F})$ whose non-commutative arithmetic circuit-size is s_n but every $\mathbb{P}_{\mathrm{Mat}_d}(\mathbb{F})$-proof of f_n has size $\Omega(s_n^{d-r})$.*

Note that we know by Theorem 5 that the lower bound in Conjecture I is true for any $d > 2$ and for some specific family f_n. But we do not know whether this specific f_n has small circuits, as required in Conjecture I.

3.2 Towards polynomial-size lower bounds on full arithmetic proofs

Here we consider the possibility that the arbitrary polynomial-size lower bounds on matrix identities proofs $\mathbb{P}_{\mathrm{Mat}_d}(\mathbb{F})$ transfer to arithmetic proofs $\mathbb{P}_c(\mathbb{F})$ lower bounds.

The natural way to formalize Conjecture II mentioned informally above is via the following translation: consider a nonzero identity f of $\mathrm{Mat}_d(\mathbb{F})$, for some $d > 1$. Then f is a nonzero non-commutative polynomial in $\mathbb{F}\langle X\rangle$. If we substitute each (matrix) variable x_i in f by a $d \times d$ matrix of *entry-variables* $\{x_{ijk}\}_{j,k\in[d]}$, then f corresponds to d^2 *commutative* zero polynomials: $f = 0$ says that for every (i, j) and for every possible assignment of field $\mathbb{F}$ elements to the (i, j)-entry of each of the matrix variables in f (when the product and addition of matrices are done in the standard way) the (i, j)-entry evaluates to 0. Accordingly, let F be a non-commutative circuit computing f. Then under the above substitution of d^2 entry-variables to each variable in F, we get d^2 non-commutative circuits, each computing the zero polynomial *when considered as commutative polynomials* (see Definition 15). We denote the set of d^2 circuits corresponding to the identity F by $[\![F]\!]_d$ (and we extend it naturally to equations between circuits: $[\![F = G]\!]_d$).

Example: Let $d = 2$ and let $f = xy - yx$ (it is obviously not an identity of $\mathrm{Mat}_2(\mathbb{F})$, but we use it only for the sake of example). And let $F = xy - yx$ be the corresponding circuit (in fact, formula) computing f. Then we substitute matrices for x, y to get:

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \cdot \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} - \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} \cdot \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}.$$

And the $(1,1)$-entry non-commutative circuit (in fact formula) in $[\![F]\!]_d$, is:

$$(x_{11}y_{11} + x_{12}y_{21}) - (y_{11}x_{11} + y_{12}x_{21}).$$

It is not hard to show that $\left|[\![F]\!]_d\right| = O\left(d^3|F|\right)$, for every non-commutative circuit F (where $\left|[\![F]\!]_d\right|$ is the total sizes of all circuits in $[\![F]\!]_d$ and $|F|$ is the size of F). We denote by

$$\left| \vdash_{\mathbb{P}_c(\mathbb{F})} [\![F = 0]\!]_d \right|$$

the minimal size of a $\mathbb{P}_c(\mathbb{F})$ proof that contains (as proof-lines) all the circuit-equations in $[\![F = 0]\!]_d$.

Conjecture II. *Let d be a positive natural number and let $\mathcal{B}$ be a (finite) basis of the identities of $\mathrm{Mat}_d(\mathbb{F})$. Assume that $f \in \mathbb{F}\langle X\rangle$ is an identity of $\mathrm{Mat}_d(\mathbb{F})$, and let F be a non-commutative arithmetic circuit computing f. Then, the minimal number of lines in a $\mathbb{P}_c(\mathbb{F})$ proof of the collection of d^2 (entry-wise) equations $[\![F = 0]\!]_d$ corresponding to F, is lower bounded (up to a constant factor) by $Q_{\mathcal{B}}(f)$. And in symbols:*

$$\left| \vdash_{\mathbb{P}_c(\mathbb{F})} [\![F = 0]\!]_d \right| = \Omega(Q_{\mathcal{B}}(f)). \tag{2}$$

The conditional lower bound we get is now similar to that in Section 3.1, except that it holds for $\mathbb{P}_c(\mathbb{F})$ and not only for *fragments* of $\mathbb{P}_c(\mathbb{F})$:

Polynomial lower bounds on arithmetic proofs $\mathbb{P}_c(\mathbb{F})$ (assuming Conjectures I and II): *There exists a family of identities f_n of $\mathrm{Mat}_d(\mathbb{F})$ whose non-commutative arithmetic circuit-size is s_n but every $\mathbb{P}_c(\mathbb{F})$-proof of f_n has size $\Omega(s_n^{d-r})$.*

We also present a *propositional version* of Conjecture II, by considering $\mathbb{F}$ to be $GF(2)$, adding to $\mathbb{P}_c(\mathbb{F})$ the Boolean axioms $x_i^2 + x_i = 0$ and considering matrix identities for $\mathrm{Mat}_d(\mathbb{F})$ (see Section E.2).

3.3 Towards *exponential-size* lower bounds on arithmetic proofs

Assuming Conjecture II above holds (i.e., Equation 2), we show under which further conditions one gets *exponential-size* lower bounds on arithmetic proofs $\mathbb{P}_c(\mathbb{F})$. The idea is to take *the dimension d of the matrix algebras as a parameter by itself.* For this we need to set up the assumptions more carefully:

Assumptions:

1. **Refinement of Conjecture II**: Assume that for any d and any basis $\mathcal{B}_d$ of the identities of $\mathrm{Mat}_d(\mathbb{F})$ the number of lines in any $\mathbb{P}_c(\mathbb{F})$ proof of $[\![F = 0]\!]_d$ is at least

$\mathcal{C}_{\mathcal{B}_d} \cdot Q_{\mathcal{B}_d}(f)$, where $\mathcal{C}_{\mathcal{B}_d}$ is a number depending on $\mathcal{B}_d$ and F is a non-commutative arithmetic circuit computing f (this is the same as Conjecture II except that now $\mathcal{C}_{\mathcal{B}_d}$ is not a constant).

2. Assume that for any sufficiently large d and any basis $\mathcal{B}_d$ of the identities of $\mathrm{Mat}_d(\mathbb{F})$, there exists a number $c_{\mathcal{B}_d}$ such that for all sufficiently large n there exists an identity $f_{n,d}$ with $Q_{\mathcal{B}_d}(f_{n,d}) \geq c_{\mathcal{B}_d} \cdot n^{2d}$. (The existence of such identities are known from our unconditional lower bound.)

3. Assume that for the $c_{\mathcal{B}_d}$ in item 2 above: $c_{\mathcal{B}_d} \cdot \mathcal{C}_{\mathcal{B}_d} = \Omega\left(\frac{1}{\mathrm{poly}(d)}\right)$.

4. **(Variant of) Conjecture I**: Assume that the non-commutative arithmetic circuit size of $f_{n,d}$ is at most $\mathrm{poly}(n,d)$.

Corollary (assuming Assumptions 1-4 above): There exists a polynomial size (in n) family of identities between non-commutative arithmetic circuits, for which any $\mathbb{P}_c(\mathbb{F})$ proof requires exponential $2^{\Omega(n)}$ number of proof-lines.

Proof. By the assumptions, every $\mathbb{P}_c(\mathbb{F})$-proof of $[\![f_{n,d} = 0]\!]_d$ has size at least $c_{\mathcal{B}_d} \cdot \mathcal{C}_{\mathcal{B}_d} \cdot n^{2d}$. Consider the family $\{f_{n,d}\}_{n=1}^{\infty}$, *where d is a function of n*, and we take $d = n/4$. Then, we get the following lower bound on the number of lines in any $\mathbb{P}_c(\mathbb{F})$-proof of the family $\{f_{n,d}\}_{n=1}^{\infty}$:

$$c_{\mathcal{B}_d} \cdot \mathcal{C}_{\mathcal{B}_d} \cdot n^{2d} = \frac{1}{\mathrm{poly}(n/4)} n^{n/2} = 2^{\Omega(n)},$$

which (by Assumption 4) is *exponential* in the arithmetic circuit-size of the identities $f_{n,d}$ proved. QED

Justification of assumptions. We wish to justify to a certain extent the new Assumptions 3 above (which lets us obtain the exponential lower bound). We shall use the special hard polynomials f that we proved exist in Theorem 5 for this purpose.

First, note that Assumption 2 holds for these f's, by Theorem 5. In Section E.1 we show that the function $c_{\mathcal{B}_d}$ for these f's does not decrease too fast. And we use this fact to get the following (conditional exponential lower bound):

Proposition. *Suppose* Assumption 1 *above holds (refinement of Conjecture II) and assume that $\mathcal{C}_{\mathcal{B}_{n/4}} = \Omega(1/\mathrm{poly}(n))$. Then, there exists a family of non-commutative circuits $\{F_n\}_{n=1}^{\infty}$ (computing the family of polynomials $\{f_{n,\frac{n}{4}}\}_{n=1}^{\infty}$) such that the number of lines in any $\mathbb{P}_c(\mathbb{F})$-proof of $[\![F_n = 0]\!]_{n/4}$ is at least $2^{\Omega(n)}$.*

Note that this will give us an exponential-size lower bound on $\mathbb{P}_c(\mathbb{F})$ proofs only if moreover the arithmetic circuit size of $\{F_n\}_{n=1}^{\infty}$ is small enough (e.g., if Assumption 4 above holds).

4 Concluding remarks

This work originates from the fundamental goal of establishing lower bounds on strong proof systems. Our focus was on arithmetic proofs which serve as a useful [9] analogue of propositional Extended Frege proofs. Along the way, we have discovered an interesting hierarchy within arithmetic proofs: a hierarchy of sound and complete proof systems for matrix identities of increasing dimensions. In this hierarchy we have been able to establish unconditional nontrivial size-lower bounds (in terms of the number of variables in the identities proved).

We then used these results, together with two seemingly natural conjectures about non-commutative arithmetic circuits and proof complexity, to propose matrix identities as hard candidates for strong proof systems. We showed that using these two conjectures, one can obtain up to exponential-size lower bounds (in terms of the circuit-size of the identities proved).

Proving lower bounds on strong (propositional) proof systems is a fundamental open problem in the theory of computing; nevertheless, it is in fact not clear whether such lower bounds are beyond current techniques (in contrast to other fundamental hardness problems in complexity, such as explicit Boolean circuits lower bounds). In light of this, and the fact that almost no hard candidates for strong proof systems are currently known (see [2, 16]), it seems that an important *conceptual*, so to speak, contribution of this paper, is to supply such new hard candidates in the form of matrix identities. Moreover, as our work partially demonstrates, such matrix identities have structure that is helpful in proving proof complexity lower bounds.

5 Relation to previous work

Relation to previous work by Hrubeš [7]. The problem of proving *quadratic* size lower bounds on arithmetic proofs $\mathbb{P}_c$ was considered by Hrubeš in [7]. The work in [7] gave several conditions and open problems, under which, quadratic size lower bounds on arithmetic proofs would follow (and further, showed that the general framework suggested may have potential, at least in theory, to yield Extended Frege quadratic-size lower bounds). The current work can be viewed as an attempt to extend the approach suggested in Hrubeš [7], from an approach suitable for proving up to $\Omega(n^2)$ size lower bounds on $\mathbb{P}_c$ proofs, (and potentially Extended Frege proofs) to an approach for proving much stronger lower bounds, namely an $\Omega(n^d)$ lower bound on $\mathbb{P}_c(\mathbb{F})$ proofs, for every positive $d > 2$ and for every zero characteristic field $\mathbb{F}$; and under stronger assumptions, exponential $2^{\Omega(n)}$ lower bounds on $\mathbb{P}_c(\mathbb{F})$ proofs (and similarly, potentially on Extended Frege proofs).

Relation to other previous works. Apart from the connection to [7], we may consider the relation of the current work to the work of Hrubeš and Tzameret [9] that obtained

polynomial-size (arithmetic and propositional) proofs for certain identities concerning matrices. As far as we see, there are no direct relations between these two works: in the current work we are studying matrix identities whose number of matrices (i.e., variables) grows with the number of variables n (if the number of matrices in the matrix identities over $\mathrm{Mat}_d(\mathbb{F})$ is m then the number of variables in the translation of the identities to a set of d^2 identities is $d^2 \cdot n$). Whereas in [9] the number of matrices was fixed and only the dimension of the matrices grows.

Note also that the matrix identities studied in [9] are not even translations (via $[\![\cdot]\!]$) of matrix identities over $\mathrm{Mat}_d(\mathbb{F})$. For instance consider the identity $\det(A)\cdot\det(B) = \det(AB)$ from [9], where A and B are 2×2 matrices. Then we get that:

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \det\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \det\begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$$

is equal to $(ad-bc)\cdot(eh-fg) = (ae+bg)(cf+dh) - (af+bh)(ce+dg)$. But notice that, e.g., in our translation of a matrix identity over $\mathrm{Mat}_d(\mathbb{F})$, two variables that correspond to the same matrix cannot multiply each other, while in the example above, a multiplies c and b multiplies d, though they are entries of the same matrix.

Technical appendix

A Formal preliminaries

A.1 Algebras with polynomial identities

For a natural number n, put $[n] := \{1, 2, ..., n\}$. We use lower case letters a, b, c for constants from the underlying field, x, y, z for variables and $\overline{x}, \overline{y}, \overline{z}$ for vectors of variables, f, g, h, ℓ or upper case letters such as A, B, P, Q for polynomials and $\overline{f}, \overline{g}, \overline{h}, \overline{\ell}, \overline{A}, \overline{B}, \overline{P}, \overline{Q}$, for vectors of polynomials (when the arity of the vector is clear from the context).

A polynomial is a formal sum of monomials, where a monomial is a product of (possibly non-commuting) variables and a constant from the underlying field. For two polynomials $f(x_1, \ldots, x_n)$ and g we say that g *is a substitution instance of* f if $g = f(h_1, \ldots, h_n)$ for some polynomials $h_1, \ldots, h_n$; and we sometimes denote $f(h_1, \ldots, h_n)$ by $f(\overline{h})$. For a polynomial $f(x_1, \ldots, x_n) \in \mathbb{F}\langle X\rangle$, $f\big|_{x_{i_1} \leftarrow g_{i_1}, \ldots, x_{i_k} \leftarrow g_{i_k}}$ denotes the polynomial that replaces $x_{i_1}, \ldots, x_{i_k}$ by $g_{i_1}, \ldots, g_{i_k}$ in f, respectively, where $g_{i_1}, \ldots, g_{i_k} \in \mathbb{F}\langle X\rangle, i_1, \ldots, i_k$ are distinct numbers from $[n]$ and $k \in [n]$.

For a vector $\overline{H}$ of polynomials $H_1, \ldots, H_k \in \mathbb{F}\langle X\rangle$ where k is positive integer, we also use the notation $\overline{H}|_{H_j \leftarrow f}$, to denote the vector of polynomials that replace the j^{th} coordinate H_j in $\overline{H}$ by a polynomial $f \in \mathbb{F}\langle X\rangle$, where $j \in [k]$.

Definition 1. *Let A be a vector space over a field $\mathbb{F}$ and $\cdot : A \times A \to A$ be a distributive multiplication operation. If $\cdot$ is associative, that is, $a_1 \cdot (a_2 \cdot a_3) = (a_1 \cdot a_2) \cdot a_3$ for all a_1, a_2, a_3 in A, then the pair $(A, \cdot)$ is called an* ***associative algebra over*** *$\mathbb{F}$, or an* ***$\mathbb{F}$-algebra****, for short.*[7]

Perhaps the most prominent example of an $\mathbb{F}$-algebra is the algebra of $d \times d$ matrices, for some positive natural number d, with entries from $\mathbb{F}$ (with the usual addition and multiplication of matrices). We denote this algebra by $\mathrm{Mat}_d(\mathbb{F})$. Note indeed that $\mathrm{Mat}_d(\mathbb{F})$ is an associative algebra but not a commutative one (i.e., the multiplication of matrices is non-commutative because AB does not necessarily equal BA, for two $d \times d$ matrices A, B).

Definition 2. *Let $\mathbb{F}\langle X\rangle$ denote the associative algebra of all polynomials such that the variables $X := \{x_1, x_2, \ldots\}$ are non-commutative with respect to multiplication. We call $\mathbb{F}\langle X\rangle$ the* ***free algebra (over X)****.*

For example, $x_1x_2 - x_2x_1 + x_3x_2x_3^2 - x_2x_3^3$, $x_1x_2 - x_2x_1$ and 0 are three distinct polynomials in $\mathbb{F}\langle X\rangle$.

[7] In general an $\mathbb{F}$-algebra can be non-associative, but since we only talk about associative algebras in this paper we use the notion of $\mathbb{F}$-algebra to imply that the algebra is associative.

Note that the set $\mathbb{F}\langle X\rangle$ forms a non-commutative ring. We sometimes call $\mathbb{F}\langle X\rangle$ *the ring of non-commutative polynomials* and call the polynomials from $\mathbb{F}\langle X\rangle$ *non-commutative polynomials*. Throughout this paper, unless otherwise stated, a polynomial is meant to be a non-commutative polynomial, namely a polynomial from the free algebra $\mathbb{F}\langle X\rangle$.

We now introduce the concept of a *polynomial identity algebra*, PI-algebra for short:

Definition 3. *Let A be an $\mathbb{F}$-algebra. An **identity of** A is a polynomial $f(x_1, ..., x_n) \in \mathbb{F}\langle X\rangle$ such that:*

$$f(a_1, ..., a_n) = 0, \text{ for all } a_1, ..., a_n \in A.$$

*A **PI-algebra** is simply an algebra that has a non-trivial identity, that is, there is a nonzero $f \in \mathbb{F}\langle X\rangle$ that is an identity of the algebra.*

For example, every *commutative* $\mathbb{F}$-algebra A is also a PI-algebra: for any $a, b \in A$, it holds that $ab - ba = 0$, and so $x_ix_j - x_jx_i$ is a nonzero polynomial identity of A, for any positive $i \neq j \in \mathbb{N}$. A concrete example of a commutative algebra is the usual ring of (*commutative*) polynomials with coefficients from a field $\mathbb{F}$ and variables $X = \{x_1, x_2, \ldots\}$, denoted usually $\mathbb{F}[X]$.

An example of an algebra that is *not* a PI-algebra is the free algebra $\mathbb{F}\langle X\rangle$ itself. This is because a nonzero polynomial $f \in \mathbb{F}\langle X\rangle$ cannot be an identity of $\mathbb{F}\langle X\rangle$ (since the assignment that maps each variable to itself does not nullify f).

A *two-sided ideal* I of an $\mathbb{F}$-algebra A is a subset of A such that for any (not necessarily distinct) elements $f_1, ..., f_n$ from I we have $\sum_{i=1}^{n} g_i \cdot f_i \cdot h_i \in I$, for all $g_1, ..., g_n, h_1, ..., h_n \in A$.

Definition 4. *A **T-ideal** $\mathcal{T}$ is a two-sided ideal of $\mathbb{F}\langle X\rangle$ that is closed under all endomorphisms[8], namely, is closed under all substitutions of variables by polynomials.*

In other words, a T-ideal is a two-sided ideal $\mathcal{T}$, such that if $f(x_1, ..., x_n) \in \mathcal{T}$ then $f(g_1, ..., g_n) \in \mathcal{T}$, for any $g_1, ..., g_n \in \mathbb{F}\langle X\rangle$.

It is easy to see the following:

Fact 3. *The set of identities of an (associative) algebra is a T-ideal.*

A basis of a T-ideal $\mathcal{T}$ is a set of polynomials whose substitution instances generate $\mathcal{T}$ *as an ideal*:

Definition 5. *Let $B \subseteq \mathbb{F}\langle X\rangle$ be a set of polynomials and let $\mathcal{T}$ be a T-ideal in $\mathbb{F}\langle X\rangle$. We say that B is **a basis for** $\mathcal{T}$ or that $\mathcal{T}$ **is generated as a T-ideal by** B, if every $f \in \mathcal{T}$ can be written as:*

$$f = \sum_{i \in I} h_i \cdot B_i(g_{i1}, ..., g_{in_i}) \cdot \ell_i \,,$$

for $h_i, \ell_i, g_{i1}, ..., g_{in_i} \in \mathbb{F}\langle X\rangle$ and $B_i \in B$ (for all $i \in I$).

[8] An algebra endomorphism of A is an (algebra) homomorphism $A \to A$.

Given $B \subseteq \mathbb{F}\langle X\rangle$, we write $T(B)$ to denote the T-ideal generated by B. Thus, a T-ideal $\mathcal{T}$ is generated by $B \subseteq \mathbb{F}\langle X\rangle$ if $\mathcal{T} = T(B)$.

Examples: $T(x_1)$ is simply the set of all polynomials from $\mathbb{F}\langle X\rangle$. $T(x_1x_2 - x_2x_1)$ is the set of all non-commutative polynomials that are zero if considered as commutative polynomials.

Note that the concept of a T-ideal is already somewhat reminiscent of logical proof systems, where generators of the T-ideal $\mathcal{T}$ are like axioms schemes and generators of a two-sided ideal containing f are like substitution instances of the axioms.

A polynomial is *homogenous* if all its monomials have the same total degree. Given a polynomial f, the *homogenous part of degree j* of f, denoted $f^{(j)}$ is the sum of all monomials with total degree j. We write $(C)^{(j)}$ to denote the jth-homogeneous part of the circuit C and the vector $\left(\overline{C}\right)^{(j)}$ denotes the vector consisting of the jth-homogeneous parts of the circuits $C_1, C_2, \ldots, C_{2d}$.

Definition 6. *$S_d(x_1, x_2, \ldots, x_d)$ denotes the **standard identity** of degree d as follows:*

$$S_d(x_1, x_2, \ldots, x_d) := \sum_{\sigma \in \mathcal{S}_d} sgn(\sigma) \prod_{i=1}^{d} x_{\sigma(i)}\,,$$

where $\mathcal{S}_d$ denotes the symmetric group on d elements and $sgn(\sigma)$ is the sign of the permutation σ.

For n polynomials $f_1, \ldots, f_n$ where $n \geq 2, n \in \mathbb{Z}$, we define the ***generalized-commutator*** $[f_1, \ldots, f_n]$ as follows:

$$[f_1, f_2] := f_1f_2 - f_2f_1, \quad \text{(in case } n = 2)$$

$$\text{and} \quad [f_1, \ldots, f_{n-1}, f_n] := [[f_1, \ldots, f_{n-1}], f_n], \quad \text{for } n > 2.$$

A polynomial $f \in \mathbb{F}\langle X\rangle$ with n variables is homogenous *with degrees* $(1, \ldots, 1)$ (n times) if in every monomial the power of every variable $x_1, \ldots, x_n$ is precisely 1. In other words, every monomial is of the form $\alpha \cdot \prod_{i=1}^{n} x_{\sigma(i)}$, for some permutation σ of order n and some scalar α. For the sake of simplicity, we shall talk in the sequel about ***polynomial of degree*** n, when referring to polynomial with degrees $(1, \ldots, 1)$ (n times). Thus, any polynomial with n variables is homogenous of total-degree n.

A.2 Arithmetic circuits

Definition 7. *Let $\mathbb{F}$ be a field, and let $X = \{x_1, \ldots, x_n\}$ be a set of input variables. An **arithmetic (or algebraic) circuit** is a directed acyclic graph, where the in-degree of nodes is at most 2. Every leaf of the graph (namely, a node of in-degree 0) is labelled with either*

an input variable or a field element. Every other node of the graph is labelled with either $+$ *or* $\times$ *(in the first case the node is a sum-gate and in the second case a product-gate). Every edge in the graph is labelled with an arbitrary field element. A node of out-degree* 0 *is called an output-gate of the circuit.*

Every node and every edge in an *arithmetic circuit* computes a polynomial in the commutative polynomial-ring $\mathbb{F}[X]$ in the following way. A leaf just computes the input variable or field element that labels it. the sum of the polynomials computed by the two edges that reach it. A product-gate computes the product of the polynomials computed by the two edges that reach it. We say that a polynomial $g \in \mathbb{F}[X]$ is computed by the circuit if it is computed by one of the circuit's output-gates.

The size of a circuit Φ is defined to be the number of edges in Φ, and is denoted by $|\Phi|$.

Definition 8. *Let* $\mathbb{F}$ *be a field, and let* $X = \{x_1, \ldots, x_n\}$ *be a set of input variables. A* ***noncommutative arithmetic circuits*** *is similarly to the arithmetic circuits defined above, with the following additional feature: given any* $\times$*-gate of fanin* 2*, its children are labeled by a fixed order.*

Every node and every edge in a *non-commutative arithmetic circuit* computes a noncommutative polynomial in the free algebra $\mathbb{F}\langle X\rangle$ in exactly the same way as the arithmetic circuit does, except that at each $\times - gate$, the ordering among the children is taken into account in defining the polynomial computed at the gate.

The size of a noncommutative circuit Φ is also defined to be the number of vertices in Φ, and is denoted by $|\Phi|$.

B The complexity measure

Let A be a PI-algebra (Definition 3) and let $\mathcal{T}$ be the T-ideal (Definition 4) consisting of all identities of A (see Fact 3). Assume that B is a basis for the T-ideal $\mathcal{T}$, that is, $T(B) = \mathcal{T}$. Then every $f \in \mathcal{T}$ is a consequence of B, namely, can be written as a linear combination of substitution instance of polynomials from B as follows:

$$f = \sum_{i \in I} h_i \cdot B_i(g_{i1}, ..., g_{in_i}) \cdot \ell_i \,, \tag{3}$$

for $h_i, \ell_i, g_{i1}, ..., g_{in_i} \in \mathbb{F}\langle X\rangle$ and $B_i \in B$ (for all $i \in I$).

A very natural question, from the complexity point of view, is the following: *What is the minimal number of distinct substitution instances* $B_i(g_{i1}, \ldots, g_{in_i})$ *of generators from* B *that must occur in (3)?* Or in other words, *how many distinct substitution instances of generators are needed to generate* f *above?*

Formally, we have the following:

Definition 9 ($Q_B(f)$)**.** *For a set of polynomials $B \subseteq \mathbb{F}\langle X\rangle$, define $Q_B(f)$ as the smallest (finite) k such that there exist substitution instances $g_1, g_2, \ldots, g_k$ of polynomials from B with*

$$f \in \langle g_1, g_2, \ldots, g_k\rangle,$$

where $\langle g_1, g_2, \ldots, g_k\rangle$ is the two-sided ideal generated by $g_1, g_2, \ldots, g_k$.

If the set B is a singleton $B = \{h\}$, we shall sometimes write $Q_h(\cdot)$ instead of $Q_{\{h\}}(\cdot)$.

Accordingly, we extend Definition 9 to a *sequence* of polynomials and let $Q_B(f_1, \ldots, f_n)$ be the smallest k such that there exist some substitution instances $g_1, g_2, \ldots, g_k$ of polynomials from B with

$$f_i \in \langle g_1, g_2, \ldots, g_k\rangle, \quad \text{for all } i \in [k].$$

Note that $Q_B(f)$ is interesting only if f is not already in the generating set. Hence, we need to make sure that the generating set does not contain f and the easiest way to do this (when considering asymptotic growth of measure) is by stipulating the the generating set is finite. Given an algebra, the question whether there exists a finite generating set of the T-ideal of the identities of the algebra is a highly non-trivial problem, that goes by the name *The Specht Problem*. Fortunately, for matrix algebras we can use the solution of the Specht problem given by Kemer [13]. Kemer showed that for every matrix algebra A there exists a finite basis of the T-ideal of the identities of A. The problem to actually find such a finite basis for most matrix algebras (namely for all values of d, for $\mathrm{Mat}_d(\mathbb{F})$) is open.

We have the following simple proposition (which is analogous to a certain extent to the fact that every two Frege proof systems polynomially simulate each other; see e.g. [14]):

Proposition 4. *Let A be some $\mathbb{F}$-algebra and let B_0 and B_1 be two* finite *bases for the identities of A. Then, there exists a constant c (that depends only on B_0, B_1) such that for any identity f of A:*

$$Q_{B_0}(f) \leq c \cdot Q_{B_1}(f).$$

Proof. Assume that $B_0 = \{A_1, A_2, \ldots, A_k\}$ and $B_1 = \{B_1, B_2, \ldots, B_\ell\}$. And suppose that $Q_{B_1}(f) = q$ and $f \in \left\langle B_{i_1}(\overline{g_1}), \ldots, B_{i_q}(\overline{g_q})\right\rangle$, for $i_j \in [\ell]$ and where $\overline{g_j} \in \mathbb{F}\langle X\rangle$ are the substitutions of polynomials for the variables of B_{i_j}. By assumption that both B_0 and B_1 are bases for A, there exists a constant r such that $B_{i_j} \in \left\langle A_{j_1}(\overline{h_{j_1}}), ..., A_{j_r}(\overline{h_{j_r}})\right\rangle$, for all $j \in [q]$, and where $\overline{h_{j_l}} \in \mathbb{F}\langle X\rangle$ are the substitutions of polynomials for the variables of A_{j_l}, for any $l \in [r]$ (formally, $r = \max\{Q_{B_0}(B_i) \ : \ i \in [\ell]\}$).

Note that if $B_{i_j} \in \left\langle A_{j_1}(\overline{h_{j_1}}), \ldots, A_{j_r}(\overline{h_{j_r}})\right\rangle$, then for any substitution $\overline{g}_j$ (of polynomials to the variables X) we have $B_{i_j}(\overline{g_j}) \in \left\langle \left(A_{j_1}(\overline{h_{j_1}})\right)(\overline{g_j}), \ldots, \left(A_{j_r}(\overline{h_{j_r}})\right)(\overline{g_j})\right\rangle$. Thus, every $B_{i_j}(\overline{g_j})$ is generated by r substitution instances of polynomials from B_0, for any $j \in [q]$. Therefore, f can be generated with at most $r \cdot q$ substitution instances of generators from B_0, that is,

$$Q_{B_0}(f) \leq r \cdot Q_{B_1}(f) \qquad \text{where } r = \max\{Q_{B_0}(B_i) \ : \ i \in [\ell]\}. \tag{4}$$

QED

C Matrix algebras

Hrubeš' work. For an identity f in a commutative algebra, we define the notation $Q_{\{[x,y]\}}(f)$ as the minimal number of substitution instances of the commutativity axioms $[x,y]=0$ we need to generate f in the two-sided ideal.

For example, $Q_{[x,y]}(x_1x_2-x_2x_1)$ is 1. And $Q_{[x,y]}(x_1x_2-x_2x_1+x_1x_3-x_3x_1)$ is also 1 since the formula $x_1x_2-x_2x_1+x_1x_3-x_3x_1$ equals $[x_2+x_3,x_1]$. In [7] it was concluded that there is an identity f with n variables, such that:

$$Q_{[x,y]}(f)=\Omega(n^2).$$

We wish to extend this result to matrix algebras. Let $\mathrm{Mat}_d(\mathbb{F})$ denote the $d\times d$ matrix algebra over $\mathbb{F}$, that is, the set of all $n\times n$ matrices with entries from $\mathbb{F}$, with the usual operations of matrices. First of all, we extend the notation $Q_{[x,y]}(f)$, which only count the instances of one axiom, to the notation $Q_{A_1,A_2,\ldots,A_n}$ which count the instances of n axioms $A_1=0,A_2=0,\ldots,A_n=0$.

Concerning matrix algebras, the following is the famous Amitsur-Levitzky Theorem:

Amitsur-Levitzki Theorem ([1])**.** *For any natural number d and any field $\mathbb{F}$ (in fact, any commutative ring) the standard identity $S_{2d}(x_1,x_2,\ldots,x_{2d})$ of degree $2d$ is an identity of* $\mathrm{Mat}_d(\mathbb{F})$.

Further, it can be shown that $\mathrm{Mat}_d(\mathbb{F})$ does not have identities of degree smaller than $2d$. And that the identities of $\mathrm{Mat}_d(\mathbb{F})$ can be *finitely* generated [13]. That is, there must be a finite generating set for $\mathrm{Mat}_d(\mathbb{F})$. By Proposition 4 no matter which finite generating set $\{A_1,A_2,...,A_k\}$ for $\mathrm{Mat}_d(\mathbb{F})$ we choose, the value $Q_{A_1,A_2,\ldots A_k}$ is the same up to a constant factor.

Our main theorem is the following:

Theorem 5. *Let $\mathbb{F}$ be any field of characteristic 0. For every natural number $d>2$ and for every finite basis $\mathcal{B}$ of the T-ideal of identities of* $\mathrm{Mat}_d(\mathbb{F})$*, there exists an identity P over* $\mathrm{Mat}_d(\mathbb{F})$ *of degree $2d+1$ with n variables, such that $Q_{\mathcal{B}}(P)=\Omega(\binom{n}{2d})=\Omega(n^{2d})$.*

It is interesting to point out that although we do not necessarily know what is the (finite) generating set of $\mathrm{Mat}_d(\mathbb{F})$ we still can lower bound the number of generators needed to generate certain identities.

C.1 The lower bound

We start by proving a lower bound on $Q_{S_{2d}}$, that is, we prove a lower bound on the number of substitution instances of S_{2d} identities needed to generate a certain identity (though S_{2d} is *not* known to be the basis of the T-ideal of the identities over $\mathrm{Mat}_d(\mathbb{F})$) .

Lemma 6. *For any natural $d \geq 1$ and any field $\mathbb{F}$ of characteristic 0 there exists a polynomial $P \in \mathrm{Mat}_d(\mathbb{F})$ of degree $2d+1$ with n variables such that $Q_{S_{2d}}(P) = \Omega(n^{2d})$.*

Comment: It can be shown that the lemma also holds for any finite *field* $\mathbb{F}$. Since in Section C.1.3 we need to assume that the field is of characteristic 0, we prove the lemma only for fields of characteristic 0 .

For proving the lemma, we introduce the following definition:

Definition 10. *A polynomial $P \in \mathbb{F}\langle X\rangle$ with n variables $x_1, \ldots, x_n$ is called an **s-polynomial** if:*

$$P = \sum_{j_1<j_2<\ldots<j_{2d}\in[n]} c_{j_1 j_2 \ldots j_{2d}} \cdot S_{2d}\left(x_{j_1}, x_{j_2} \ldots x_{j_{2d}}\right),$$

for some natural d and constants $c_{j_1 j_2 \ldots j_{2d}} \in \{0,1\}$, for $j_1 < j_2 < \ldots < j_{2d} \in [n]$.

Lemma 7. *For any $P_1, P_2, \ldots, P_{2d} \in \mathbb{F}\langle X\rangle$ where d is a positive integer, $S_{2d}(P_1, P_2, \ldots, P_{2d})$ is the zero polynomial if there exists $i \in [2d]$ such that P_i is a constant.*

Proof. For a fixed $\mathcal{I} \in [2d]$, we have $P_{\mathcal{I}} = c \in \mathbb{F}$.

For convenience, write the set $\{x \in [2d] | x \neq \mathcal{I}\}$ as $[2d]/\mathcal{I}$, the permutation $\begin{pmatrix} 1 & 2 & \ldots & m-1 & m & m+1 & \ldots & 2d \\ i_1 & i_2 & \ldots & i_{m-1} & \mathcal{I} & i_m & \ldots & i_{2d-1} \end{pmatrix}$ as σ_m where $\{i_1, \ldots, i_{2d-1}\} = [2d]/\mathcal{I}$.

Then

$$\begin{aligned}
S_{2d}(P_1, P_2, \ldots, P_{2d}) &= \sum_{\sigma\in\mathcal{S}_{2d}} sgn(\sigma) \prod_{i=1}^{2d} P_{\sigma(i)} \\
&= \prod_{\{i_1,i_2,\ldots,i_{2d-1}\}=[2d]/\mathcal{I}} \sum_{m=1}^{2d} sgn(\sigma_m) \prod_{j=1}^{m-1} P_{i_j} P_{\mathcal{I}} \prod_{j=m}^{2d-1} P_{i_j} \\
&= \prod_{\{i_1,i_2,\ldots,i_{2d-1}\}=[2d]/\mathcal{I}} \sum_{m=1}^{2d} sgn(\sigma_m) c \prod_{j=1}^{2d-1} P_{i_j} \\
&= c \prod_{\{i_1,i_2,\ldots,i_{2d-1}\}=[2d]/\mathcal{I}} \left(\sum_{m=1}^{2d} sgn(\sigma_m)\right) \prod_{j=1}^{2d-1} P_{i_j} \\
&= c \prod_{\{i_1,i_2,\ldots,i_{2d-1}\}=[2d]/\mathcal{I}} \left(\sum_{m=1}^{d} (sgn(\sigma_{2m-1}) + sgn(\sigma_{2m}))\right) \prod_{j=1}^{2d-1} P_{i_j} \\
&= c \prod_{\{i_1,i_2,\ldots,i_{2d-1}\}=[2d]/\mathcal{I}} \left(\sum_{m=1}^{d} 0\right) \prod_{j=1}^{2d-1} P_{i_j} \\
&= 0.
\end{aligned}$$

QED

Any s-polynomial has the following property:

Lemma 8. *Let f be an s-polynomial. If there exist vectors of polynomials $\overline{P_1},\ldots,\overline{P_r}$ with*

$$f \in \left\langle S_{2d}(\overline{P_1}),\ldots,S_{2d}(\overline{P_r})\right\rangle,$$

then

$$f=\sum_{i=1}^{r} c_i S_{2d}\left(\left(\overline{P_i}\right)^{(1)}\right).$$

Proof. Notice that the s-formula f is $2d-$homogenous. Thus,

$$f=(f)^{(2d)} \in \left\{(h)^{(2d)} \;\middle|\; h \in \left\langle S_{2d}(\overline{P_1}),\ldots,S_{2d}(\overline{P_r})\right\rangle\right\}.$$

That is

$$f \in \left\langle S_{2d}(\overline{P_1})^{(2d)},\ldots,S_{2d}(\overline{P_r})^{(2d)}\right\rangle.$$

By Lemma 7, for some $j\in[r], i\in[2d]$, the polynomial $S_{2d}(\overline{P}_j)$ equals to the zero polynomial if some $\overline{P}_{j_i}$ is a constant. Namely $S_{2d}(\overline{P_j})^{(2d)}=S_{2d}\left(\left(\overline{P_j}\right)^{(1)}\right)$, for all $j\in[r]$. Then,

$$f \in \left\langle S_{2d}\left(\left(\overline{P_1}\right)^{(1)}\right),\ldots,S_{2d}\left(\left(\overline{P_r}\right)^{(1)}\right)\right\rangle.$$

That is,

$$f=\sum_{j=1}^{r}\sum_{i=1}^{t_j} A_{ji}S_{2d}\left(\left(\overline{P_j}\right)^{(1)}\right)B_{ji}, \qquad \text{for some } A_{ji},B_{ji}\in\mathbb{F}\langle X\rangle.$$

Moreover,

$$\left(A_{ji}S_{2d}\left(\left(\overline{P_j}\right)^{(1)}\right)B_{ji}\right)^{(2d)}=\left(A_{ji}B_{ji}\right)^{(0)}S_{2d}\left(\left(\overline{P_j}\right)^{(1)}\right).$$

Thus

$$f=\sum_{j=1}^{r} c_j S_{2d}\left(\left(\overline{P_j}\right)^{(1)}\right),$$

where c_j is the constant $\sum_{i=1}^{t_j}\left(A_{ji}B_{ji}\right)^{(0)}$, for any $j\in[r]$. QED

C.1.1 The counting argument

Notation. *If $B\subseteq\mathbb{F}\langle X\rangle$ contains only one polynomial g, then we write $Q_g(\cdot)$ instead of $Q_B(\cdot)$, to simplify the writing. Note that B may not be a basis for the algebra considered (e.g., we may consider identities of the $\mathrm{Mat}_d(\mathbb{F})$ generated by some B, where B is not a basis for (all) the identities of $\mathrm{Mat}_d(\mathbb{F})$).*

Lemma 9. *For any field $\mathbb{F}$ of characteristic 0, there exist s-polynomials $P_1, \ldots, P_n$ which are identities of $\mathrm{Mat}_d(\mathbb{F})$ in n variables, such that $Q_{S_{2d}}(P_1, \ldots, P_n) = \Omega(n^{2d})$ (and $Q_{S_{2d}}(P_1, \ldots, P_n)$ is finite).*

In Section C.1.3 we show that, if $\mathbb{F}$ is of characteristic 0 then this lower bound holds for *any finite basis of* $\mathrm{Mat}_d(\mathbb{F})$, namely for Q_B, where B is any finite basis of $\mathrm{Mat}_d(\mathbb{F})$.

Proof. We prove by a generalization of the counting argument from [7] that there exists a sequence of polynomials $P_1, P_2, \ldots, P_n$ that require $\Omega\left(n^{2d}\right)$ substitution instances of the $S_{2d}(x_1, \ldots, x_{2d})$ identities to generate (all of the polynomials in the sequence) in a two-sided ideal.

Recall that an s-polynomial (Definition 10) is of the following form:

$$\sum_{j_1<j_2<\ldots<j_{2d}\in[n]} c_{i_{j_1 j_2 \cdots j_{2d}}} S_{2d}(x_{j_1}, x_{j_2}, \ldots, x_{j_{2d}}), \quad \text{where } c_{i_{j_1 j_2 \cdots j_{2d}}} \in \{0,1\}\,. \tag{5}$$

Assume that

$$\ell = \max\left\{Q_{S_{2d}}(P_1, \ldots, P_n) \;:\; P_i \text{ is an s-polynomial, for all } i \in [n]\right\}.$$

Then for any choice of n s-polynomials $P_1, \ldots, P_n$ there are ℓ vectors of polynomials $\overline{Q_1}, \ldots, \overline{Q_\ell}$ from $\mathbb{F}\langle X\rangle$, such that

$$P_1, \ldots, P_n \in \left\langle S_{2d}(\overline{Q_1}), \ldots, S_{2d}(\overline{Q_\ell})\right\rangle.$$

By Lemma 8, for any choice of $P_1, \ldots, P_n$ and $\overline{Q}_1, \ldots, \overline{Q}_\ell$, for every $i \in [n]$:

$$P_i = \sum_{j=1}^{\ell} c_{i_j} S_{2d}\left(\overline{Q_j}^{(1)}\right) = \sum_{j=1}^{\ell} c_{i_j} S_{2d}\left(\sum_{m=1}^{n} a_{mj_1} x_m, \sum_{m=1}^{n} a_{mj_2} x_m, \ldots, \sum_{m=1}^{n} a_{mj_{2d}} x_m\right)$$
$$\text{(for some } c_{i_j}, a_{mj_k} \in \mathbb{F}).$$

Consider a vector $(c_{1_j}, \ldots, c_{n_j}, a_{k1m}, \ldots, a_{k(2d)m})$ $(m \in [n], k \in [\ell])$. By linearity of S_{2d}:

$$\sum_{k=1}^{\ell} c_{i_k} S_{2d}\left(\sum_{m=1}^{n} a_{k1m} x_m, \sum_{m=1}^{n} a_{k2m} x_m, \ldots, \sum_{m=1}^{n} a_{k(2d)m} x_m\right) = \tag{6}$$

$$\sum_{j_1<j_2<\ldots<j_{2d}\in[n]} c_{i_{j_1 j_2 \cdots j_{2d}}} S_{2d}(x_{j_1}, x_{j_2}, \ldots, x_{j_{2d}}) \qquad (\text{where } c_{i_{j_1 j_2 \cdots j_{2d}}} \in \mathbb{F}). \tag{7}$$

A *polynomial map* $\mu : \mathbb{F}^n \to \mathbb{F}^m$ of degree $d > 0$, is a map $\mu = (\mu_1, \ldots, \mu_m)$, where each μ_i is a (commutative) polynomial of degree d with n variables.

Claim. *Consider the coefficients* $c_{1_j}, \ldots, c_{n_j}, a_{k1m}, \ldots, a_{k(2d)m}$ *and the coefficients* $c_{i_{j_1 j_2 \cdots j_{2d}}}$ *in Equation 6* as variables. *Then, Equation 6 defines a degree-*$(2d+1)$ *polynomial map* $\phi : \mathbb{F}^{(2d+1)nl} \to \mathbb{F}^{n\binom{n}{2d}}$ *that maps each vector*

$$(c_{1_j}, \ldots, c_{n_j}, a_{k1m}, \ldots, a_{k(2d)m}), \quad \text{for } m \in [n], k \in [\ell],$$

to

$$(c_{1_{j_1 j_2 \cdots j_{2d}}}, \ldots, c_{n_{j_1 j_2 \cdots j_{2d}}}), \quad \text{for } j_1 < j_2 < \ldots < j_{2d} \in [n].$$

We omit the details of the proof of this claim. We have the following lemma:

Lemma 10 ([10], Lemma 5)**.** *For any field* $\mathbb{F}$*, if* $\mu : \mathbb{F}^n \to \mathbb{F}^m$ *is a polynomial map of degree* $d > 0$*, then* $|\mu(\mathbb{F}^n) \bigcap \{0,1\}^m| \leq (2d)^n$.

Thus, for the degree-$(2d+1)$ polynomial map $\phi : \mathbb{F}^{(2d+1)nl} \to \mathbb{F}^{n\binom{n}{2d}}$, we have

$$|\phi(\mathbb{F}^{(2d+1)nl}) \bigcap \{0,1\}^{n\binom{n}{2d}}| \leq (2(2d+1))^{(2d+1)nl}.$$

Recall that for any choice of n s-polynomials $P_1, \ldots, P_n$ there are ℓ vectors of polynomials $\overline{Q_1}, \ldots, \overline{Q_\ell}$ from $\mathbb{F}\langle X \rangle$, such that

$$P_1, \ldots, P_n \in \left\langle S_{2d}(\overline{Q_1}), \ldots, S_{2d}(\overline{Q_\ell}) \right\rangle.$$

For convenience, we use $\overline{\mathcal{C}}$ for the $0-1$ vector $(c_{1_{j_1 j_2 \cdots j_{2d}}}, \ldots, c_{n_{j_1 j_2 \cdots j_{2d}}})$, where $c_{i_{j_1 j_2 \cdots j_{2d}}} \in \{0,1\}, i \in [n], j_1 < j_2 < \ldots < j_{2d} \in [n]$. Since for every possible $\overline{\mathcal{C}}$, the following polynomials are s-polynomials:

$$\sum_{j_1 < j_2 < \ldots < j_{2d} \in [n]} \mathcal{C}_{1_{j_1 j_2 \cdots j_{2d}}} S_{2d}(x_{j_1}, x_{j_2}, \ldots, x_{j_{2d}}), \quad \ldots, \quad \sum_{j_1 < j_2 < \ldots < j_{2d} \in [n]} \mathcal{C}_{n_{j_1 j_2 \cdots j_{2d}}} S_{2d}(x_{j_1}, x_{j_2}, \ldots, x_{j_{2d}}),$$

there exist ℓ vectors of polynomials $\overline{Q_1}, \ldots, \overline{Q_\ell}$ in $\mathbb{F}\langle X \rangle$, such that

$$\sum_{j_1 < j_2 < \ldots < j_{2d} \in [n]} \mathcal{C}_{i_{j_1 j_2 \cdots j_{2d}}} S_{2d}(x_{j_1}, x_{j_2}, \ldots, x_{j_{2d}}) \in \left\langle S_{2d}(\overline{Q_1}), \ldots, S_{2d}(\overline{Q_\ell}) \right\rangle, i \in [n].$$

That is, there exists a vector $(c_{1_j}, \ldots, c_{n_j}, a_{k1m}, \ldots, a_{k(2d)m})$ $(m \in [n], k \in [\ell])$, such that $\phi(c_{1_j}, \ldots, c_{n_j}, a_{k1m}, \ldots, a_{k(2d)m}) = \overline{\mathcal{C}}$.

Therefore, every possible $\overline{\mathcal{C}}$ belongs to $\phi(\mathbb{F}^{(2d+1)nl}) \bigcap \{0,1\}^{n\binom{n}{2d}}$.

Further there are $2^{n\binom{n}{2d}}$ distinct vectors $\overline{\mathcal{C}} = (c_{1_{j_1 j_2 \cdots j_{2d}}}, \ldots, c_{n_{j_1 j_2 \cdots j_{2d}}})$, where $c_{i_{j_1 j_2 \cdots j_{2d}}} \in \{0,1\}, i \in [n], j_1 <, \ldots, < j_{2d} \in [n]$. Hence,

$$|\phi(\mathbb{F}^{(2d+1)nl}) \bigcap \{0,1\}^{n\binom{n}{2d}}| \geq 2^{n\binom{n}{2d}}.$$

This implies that

$$(2(2d+1))^{(2d+1)nl} \geq 2^{n\binom{n}{2d}}. \tag{8}$$

Using the ln function on both sides:

$$(2d+1)nl\ln(2(2d+1)) \geq n\binom{n}{2d}\ln 2.$$

Hence,

$$l > \frac{\binom{n}{2d}\ln 2}{(2d+1)\ln(4d+2)}. \tag{9}$$

Namely

$$l > c\binom{n}{2d} = c\frac{n(n-1)\ldots(n-2d+1)}{d!} = \Omega\left(n^{2d}\right)$$

$$\text{(where } c \in \mathbb{F}\text{), hence}$$

$$l = \Omega\left(n^{2d}\right).$$

QED

C.1.2 Combining the polynomials into one

Here we show that there exists already a *single* polynomial, denoted $P^\star$ such that $Q_{S_{2d}}(P^\star) = \Omega(n^{2d})$. This is done in a manner which is similar to the work of Hrubeš [7]; however, there is a further complication here, which is dealt via the technical Lemma 12.

Lemma 11. *Let $P_1, \ldots, P_n$ be s-polynomials in n variables $x_1, \ldots, x_n$, and let $z_1, \ldots, z_n$ be new variables, different from $x_1, \ldots, x_n$. Let $P^\star := \sum_{i=1}^n z_i P_i$. Then*

$$Q_{S_{2d}}(P^\star) \geq \frac{1}{2d+1} Q_{S_{2d}}(P_1, \ldots, P_n). \tag{10}$$

Specifically, for any field $\mathbb{F}$ of characteristic 0 and every $d \geq 1$, there exists a polynomial with n variables such that $Q_{S_{2d}}(P^\star) = \Omega(n^{2d})$.

Proof. For convenience, call the new variables $z_1, \ldots, z_n$ the Z-variables. Given a polynomial f, the ***Z-homogenous part of degree j of f***, denoted $(f)_Z^{(j)}$, is the sum of all monomials where the total degree of the Z-variables is j. For example if $f = z_1xy + z_2z_1 + z_3x + 1 + x$, then $(f)_Z^1 = z_1xy + z_3x$, $(f)_Z^2 = z_2z_1$, $(f)_Z^0 = 1 + x$. A polynomial that does not contain any Z-variable is said to be *Z-independent*.

First, we claim the $P^\star$ has the following property:

Claim. *For any ℓ Z-independent polynomials $\overline{G}_1, \overline{G}_2, \ldots, \overline{G}_\ell \in \mathbb{F}\langle X\rangle$, if*

$$P^\star \in \left\langle S_{2d}(\overline{G}_1), \ldots, S_{2d}(\overline{G}_\ell)\right\rangle,$$

then

$$P_1, \ldots, P_n \in \left\langle S_{2d}(\overline{G}_1), \ldots, S_{2d}(\overline{G}_\ell)\right\rangle.$$

Proof of claim: Since $P^\star \in \left\langle S_{2d}(\overline{G}_1), \ldots, S_{2d}(\overline{G}_\ell)\right\rangle$,

$$P^\star = \sum_{i=1}^{n} z_i P_i = \sum_{j=1}^{\ell}\sum_{i=1}^{t_j} f_{ji} S_{2d}(\overline{G}_j) g_{ji}, \quad \text{for some } f_{ji}, g_{ji} \in \mathbb{F}\langle X\rangle.$$

Now, assign $z_1 = 1, z_2 = z_3 = \cdots = z_n = 0$ in $P^\star$. Since $\overline{G}_1, \ldots, \overline{G}_\ell$ do not contain $z_1, \ldots, z_n$, the $\overline{G}_1, \ldots, \overline{G}_\ell$ will remain the same. Thus,

$$P_1 = \sum_{j=1}^{\ell}\sum_{i=1}^{t_j} f'_{ji} S_{2d}(\overline{G}_j) g'_{ji},$$

where $f'_{ji} = f_{ji}|_{z_1\leftarrow 1, z_2\leftarrow 0, \ldots, z_n\leftarrow 0}$, $g'_{ji} = g_{ji}|_{z_1\leftarrow 1, z_2\leftarrow 0, \ldots, z_n\leftarrow 0}$. Namely, $P_1 \in \left\langle S_{2d}(\overline{G}_1), \ldots, S_{2d}(\overline{G}_\ell)\right\rangle$.

Similarly, we can show $P_2, \ldots, P_n \in \left\langle S_{2d}(\overline{G}_1), \ldots, S_{2d}(\overline{G}_\ell)\right\rangle$. Therefore,

$$P_1, \ldots, P_n \in \left\langle S_{2d}(\overline{G}_1), \ldots, S_{2d}(\overline{G}_\ell)\right\rangle.$$

■Claim

In the following, assume $Q_{S_{2d}}(P^\star) = \ell$. That is, there are k vectors of polynomials $\overline{G}_1, \overline{G}_2, \ldots, \overline{G}_\ell$ such that

$$P^\star \in \left\langle S_{2d}(\overline{G}_1), \ldots, S_{2d}(\overline{G}_\ell)\right\rangle.$$

Namely

$$P^\star = \sum_{i=1}^{n} z_i P_i = \sum_{j=1}^{\ell}\sum_{i=1}^{t_j} f_{ji} S_{2d}(\overline{G}_j) g_{ji}, \quad \text{for some } f_{ji}, g_{ji} \in \mathbb{F}\langle X\rangle.$$

If we can find $(2d+1)\cdot\ell$ Z-independent vector of polynomials $\overline{G}_1, \overline{G}_2, \ldots, \overline{G}_{(2d+1)\cdot\ell}$ such that

$$P^\star = \sum_{j=1}^{\ell}\sum_{i=1}^{t_j} f_{ji} S_{2d}(\overline{G}_j) g_{ji} \in \left\langle S_{2d}(\overline{G}_1), \ldots, S_{2d}(\overline{G}_{(2d+1)\cdot\ell})\right\rangle,$$

then we can, by the above claim, show that

$$P_1, \ldots, P_n \in \left\langle S_{2d}(\overline{G}_1), \ldots, S_{2d}(\overline{G}_{(2d+1)\cdot\ell})\right\rangle,$$

which is the conclusion we want to prove:

$$Q_{S_{2d}}(P_1,\ldots,P_n) \le (2d+1)\cdot\ell.$$

Now, to find the $(2d+1)\cdot\ell$ Z-independent vectors of polynomials $\overline{G}_1,\overline{G}_2,\ldots,\overline{G}_{(2d+1)\cdot\ell}$ which generate $P^\star$, let $[\cdot]$ be a map that maps a polynomial $P\in\mathbb{F}\langle X\rangle$ to a polynomial $[P]$ that is defined by the following three properties:

1. The map $[\cdot]$ is linear, namely $[\alpha G+\beta H]=\alpha[G]+\beta[H]$ for any polynomials G,H and $\alpha,\beta\in\mathbb{F}$; and
2. Let M be a monomial whose Z-homogenous part is of degree 1. Thus, M can be uniquely written as $M_1 z_i M_2, z_i\in Z$, where M_1,M_2 are Z-independent. Then
$$[M]=[M_1 z M_2]=zM_2M_1\,;\quad \text{and}$$
3. For a monomial M whose Z-homogenous part is not of degree 1, $[M]=0$.

For convenience, in what follows, given the polynomials f,g and the vector of polynomials $\overline{H}$, we denote $(f)^0_Z,(\overline{H})^0_Z,(g)^0_Z$ by $\mathcal{F},\overline{\mathcal{H}},\mathcal{G}$, respectively.

Claim. *For any polynomials $f_1,g_1,\ldots,f_k,g_k$ and vector of polynomials $\overline{H}$ with variables $X_1,\ldots,X_n,z_1,\ldots,z_n$:*

$$\left[\sum_{i=1}^{k} f_i S_{2d}(\overline{H})g_i\right]\in\left\langle S_{2d}(\overline{\mathcal{H}}),S_{2d}(\overline{\mathcal{H}}|_{\mathcal{H}_j\leftarrow\sum_{i=1}^{k}\mathcal{G}_i\mathcal{F}_i})\right\rangle,\qquad \textit{for any } j\in[2d].$$

Proof of claim: Consider the following:

$$\begin{aligned}\left[\sum_{i=1}^{k} f_i S_{2d}(\overline{H})g_i\right] &= \left[(\sum_{i=1}^{k} f_i S_{2d}(\overline{H})g_i)^1_Z\right] \quad \text{by Property 3 of } [\cdot]\\ &= \left[\sum_{i=1}^{k}(f_i)^1_Z S_{2d}(\overline{\mathcal{H}})\mathcal{G}_i+\sum_{i=1}^{k}\sum_{j=1}^{2d}\mathcal{F}_i S_{2d}\left(\overline{\mathcal{H}}|_{\mathcal{H}_j\leftarrow(H_j)^1_Z}\right)\mathcal{G}_i+\sum_{i=1}^{k}\mathcal{F}_i S_{2d}(\overline{\mathcal{H}})(g_i)^1_Z\right]\\ \text{(by linearity of } [\cdot]) \quad &= \sum_{i=1}^{k}\left[(f_i)^1_Z S_{2d}(\overline{\mathcal{H}})\mathcal{G}_i\right]+\sum_{j=1}^{2d}\left[\sum_{i=1}^{k}\mathcal{F}_i S_{2d}\left(\overline{\mathcal{H}}|_{\mathcal{H}_j\leftarrow(H_j)^1_Z}\right)\mathcal{G}_i\right]+\sum_{i=1}^{k}\left[\mathcal{F}_i S_{2d}(\overline{\mathcal{H}})(g_i)^1_Z\right].\end{aligned}$$

For every $i\in[n]$, assume $(f_i)^1_Z=\sum_{i=1}^{n}\sum_j g_{ij}z_i h_{ij}$ where g_{ij},h_{ij} are Z-independent polynomials and $z_1,\ldots,z_n$ are Z-variables, then

$$\left[(f_i)^1_Z S_{2d}(\overline{\mathcal{H}})\mathcal{G}_i\right]=\left[\sum_{i=1}^{n}\sum_j g_{ij}z_i h_{ij}S_{2d}(\overline{\mathcal{H}})\mathcal{G}_i\right]=\sum_{i=1}^{n}\sum_j z_i h_{ij}S_{2d}(\overline{\mathcal{H}})\mathcal{G}_i g_{ij}\in\left\langle S_{2d}(\overline{\mathcal{H}})\right\rangle$$

where the right most equality stems from Property 2 of the map $[\cdot]$. Similarly, for every $i \in [n]$, we can show

$$\left[\mathcal{F}_i S_{2d}(\overline{\mathcal{H}})(g_i)_Z^1\right] \in \left\langle S_{2d}(\overline{\mathcal{H}})\right\rangle.$$

By Lemma 12, which is proved below, we have

$$\left[\sum_{i=1}^{k} \mathcal{F}_i S_{2d}(\overline{\mathcal{H}}|_{\mathcal{H}_j \leftarrow (H_j)_Z^1})\mathcal{G}_i\right] \in \left\langle S_{2d}(\overline{\mathcal{H}}|_{\mathcal{H}_j \leftarrow \sum_{i=1}^{k} \mathcal{G}_i\mathcal{F}_i})\right\rangle, \qquad \text{for any } j \in [2d].$$

Thus $\left[\sum_{i=1}^{k} f_i S_{2d}(\overline{H}) g_i\right] \in \left\langle S_{2d}(\overline{H}), S_{2d}(\overline{\mathcal{H}}|_{\mathcal{H}_j \leftarrow \sum_{i=1}^{k} \mathcal{G}_i\mathcal{F}_i})\right\rangle$ for any $j \in [2d]$. ■Claim

Note that $P^\star = (P^\star)_Z^1$. By the properties of $[\cdot]$ we have:

$$\begin{aligned}
P^\star &= [P^\star] \\
&= \left[\sum_{j=1}^{\ell}\sum_{i=1}^{t_j} f_{ji} S_{2d}(\overline{H}_j) g_{ji}\right] \\
&= \sum_{j=1}^{\ell}\left[\sum_{i=1}^{t_j} f_{ji} S_{2d}(\overline{H}_j) g_{ji}\right] \\
&\in \left\langle S_{2d}(\overline{H}_j), S_{2d}(\overline{H}_j|_{H_{jq} \leftarrow \sum_{m=1}^{t_j} \mathcal{G}_{jm}\mathcal{F}_{jm}})\right\rangle \quad \text{for any } j \in [\ell], q \in [2d].
\end{aligned}$$

Namely for $P^\star = \sum_{j=1}^{\ell}\sum_{i=1}^{t_j} f_{ji} S_{2d}(\overline{H}_j) g_{ji}$, we have $(2d+1)\cdot\ell$ Z-independent polynomials that generate $P^\star$, concluding the theorem. QED

Lemma 12. *Let $X = \{x_1, x_2, \ldots, x_n\}$ and $f_1, g_1, \ldots, f_k, g_k \in \mathbb{F}\langle X\rangle$. Let $Z = \{z, z_1, z_2, \ldots, z_n\}$ and assume that n is an even positive integer, and let $\overline{P}$ be a vector of polynomials $(P_1, P_2, \ldots, P_n)$ with variable set $X \cup Z$. We denote $(\overline{P})_Z^0$, $(f_i)_Z^0, (g_i)_Z^0$ by $\overline{\mathcal{P}}, \mathcal{F}_i, \mathcal{G}_i, i \in [k]$, respectively. Then, for any $j \in [n]$, it holds that*

$$\left[\sum_{i=1}^{k} \mathcal{F}_i S_n(\overline{\mathcal{P}}|_{\mathcal{P}_j \leftarrow (P_j)_Z^1})\mathcal{G}_i\right] \in \left\langle S_n(\overline{\mathcal{P}}|_{\mathcal{P}_j \leftarrow \sum_{i=1}^{k} \mathcal{G}_i\mathcal{F}_i})\right\rangle.$$

For example, when $n = 2$, the above lemma shows the following:

$$\left[\sum_{i=1}^{k} \mathcal{F}_i S_2((P_1)_Z^1, \mathcal{P}_2)\mathcal{G}_i\right] \in \left\langle S_2(\sum_{i=1}^{k} \mathcal{G}_i\mathcal{F}_i, P_2)\right\rangle,$$

$$\left[\sum_{i=1}^{k} \mathcal{F}_i S_2(\mathcal{P}_1, (P_2)_Z^1)\mathcal{G}_i\right] \in \left\langle S_2(P_1, \sum_{i=1}^{k} \mathcal{G}_i\mathcal{F}_i)\right\rangle.$$

Proof. For a fixed $\mathcal{I} \in [n]$, we have $(P_{\mathcal{I}})_Z^1 = \sum_{i=1}^n \sum_j \mathcal{U}_{ij} z_i \mathcal{V}_{ij}$, where $z \in Z, \mathcal{U}_{ij}, \mathcal{V}_{ij} \in \mathbb{F}\langle X \rangle$ and $\mathcal{U}_{ij}, \mathcal{V}_{ij}$ are Z-independent.

For a permutation $\sigma \in \mathcal{S}_n$ and the polynomial vector $\overline{P} = (P_1, \ldots, P_n)$, we let

$$(\overline{P})_{\sigma[i,j]} = \begin{cases} \prod_{m=i}^{j} P_{\sigma(m)}, & i \leq j; \\ 1, & i > j. \end{cases}$$

We write $\mathcal{S}_n/m$ to denote the set $\{\sigma \in \mathcal{S}_n \mid \sigma(m) = \mathcal{I}\}$.

And define

$$\pi_m = \begin{pmatrix} 1 & 2 & \ldots & n-m & n-m+1 & n-m+2 & \ldots & n \\ m+1 & m+2 & \ldots & n & m & 1 & \ldots & m-1 \end{pmatrix} \forall m \in [n].$$

Fact 13. *$sgn(\pi_m) = (-1)^{m(n-m)+m-1} = (-1)^{nm-m(m-1)-1} = -1$.*

Fact 14. *$\overline{P}_{\sigma[m+1,n]} \cdot \overline{P}_{\sigma[1,m-1]} = \overline{P}_{\sigma\pi_m[1,n-m]} \cdot \overline{P}_{\sigma\pi_m[n-m+2,n]}$, for all $\sigma \in \mathcal{S}_n/m$.*

Fact 15. *$(\mathcal{S}_n/m)\pi_m = \mathcal{S}_n/(n-m+1)$.*

So we have the following:

$$\begin{aligned}
&\left[\sum_{i=1}^{k} \mathcal{F}_i s_n(\overline{\mathcal{P}}|_{\mathcal{P}_{\mathcal{I}} \leftarrow \sum_{i=1}^n \sum_j \mathcal{U}_{ij} z_i \mathcal{V}_{ij}}) \mathcal{G}_i\right] \\
=&\left[\sum_{i=1}^{k} \mathcal{F}_i \sum_{\sigma \in \mathcal{S}_n} sgn(\sigma)(\overline{\mathcal{P}}_{\sigma[1,n]})|_{\mathcal{P}_{\mathcal{I}} \leftarrow \sum_{i=1}^n \sum_j \mathcal{U}_{ij} z_i \mathcal{V}_{ij}} \mathcal{G}_i\right] \\
=&\left[\sum_{i=1}^{k} \mathcal{F}_i \sum_{m=1}^{n} \sum_{\substack{\sigma \in \mathcal{S}_n \\ \sigma^{-1}(i) = m}} sgn(\sigma)(-1)^m (\overline{\mathcal{P}}_{\sigma[1,m-1]} \mathcal{P}_{\sigma(m)} \overline{\mathcal{P}}_{\sigma[m+1,n]})|_{\mathcal{P}_{\mathcal{I}} \leftarrow \sum_{i=1}^n \sum_j \mathcal{U}_{ij} z_i \mathcal{V}_{ij}} \mathcal{G}_i\right] \\
=&\left[\sum_{i=1}^{k} \mathcal{F}_i \sum_{m=1}^{n} \sum_{\sigma \in \mathcal{S}_n/m} sgn(\sigma)(-1)^m (\overline{\mathcal{P}}_{\sigma[1,m-1]} \mathcal{P}_{\mathcal{I}} \overline{\mathcal{P}}_{\sigma[m+1,n]})|_{\overline{\mathcal{P}}_{\mathcal{I}} \leftarrow \sum_{i=1}^n \sum_j \mathcal{U}_{ij} z_i \mathcal{V}_{ij}} \mathcal{G}_i\right] \\
=&\left[\sum_{i=1}^{k} \mathcal{F}_i \sum_{m=1}^{n} \sum_{\sigma \in \mathcal{S}_n/m} sgn(\sigma)(-1)^m (\overline{\mathcal{P}}_{\sigma[1,m-1]} \sum_{i=1}^{n} \sum_j \mathcal{U}_{ij} z_i \mathcal{V}_{ij} \overline{\mathcal{P}}_{\sigma[m+1,n]}) \mathcal{G}_i\right] \\
=&\sum_{i=1}^{n} \sum_j z_i \mathcal{V}_{ij} \sum_{m=1}^{n} \sum_{\sigma \in \mathcal{S}_n/m} sgn(\sigma)(-1)^m \overline{\mathcal{P}}_{\sigma[m+1,n]} \left(\sum_{i=1}^{k} \mathcal{G}_i \mathcal{F}_i\right) \overline{\mathcal{P}}_{\sigma[1,m-1]} \mathcal{U}_{ij} \\
=&\sum_{i=1}^{n} \sum_j z_i \mathcal{V}_{ij} \sum_{m=1}^{n} \sum_{\sigma \in \mathcal{S}_n/m} sgn(\sigma)(-1)^m \overline{\mathcal{P}}_{\sigma\pi_m[1,n-m]} \left(\sum_{i=1}^{k} \mathcal{G}_i \mathcal{F}_i\right) \overline{\mathcal{P}}_{\sigma\pi_m[n-m+2,n]} \mathcal{U}_{ij} \qquad \text{by } \textbf{Fact 14}
\end{aligned}$$

$$= \sum_{i=1}^{n} \sum_{j} z_i \mathcal{V}_{ij} \sum_{m=1}^{n} \sum_{\sigma \in \mathcal{S}_n/m} sgn(\sigma\pi_m) sgn(\pi_m) (-1)^m \overline{\mathcal{P}}_{\sigma\pi_m[1,n-m]} \left(\sum_{i=1}^{k} \mathcal{G}_i \mathcal{F}_i \right) \overline{\mathcal{P}}_{\sigma\pi_m[n-m+2,n]} \mathcal{U}_{ij}.$$

let $\pi = \sigma\pi_m$, then $\pi\pi_m^{-1} = \sigma$,

$$= \sum_{i=1}^{n} \sum_{j} z_i \mathcal{V}_{ij} \sum_{m=1}^{n} \sum_{\pi\pi_m^{-1} \in \mathcal{S}_n/m} sgn(\pi)(-1)(-1)^m \overline{\mathcal{P}}_{\pi[1,n-m]} \left(\sum_{i=1}^{k} \mathcal{G}_i \mathcal{F}_i \right) \overline{\mathcal{P}}_{\pi[n-m+2,n]} \mathcal{U}_{ij} \quad \text{by } \textbf{Fact } 13$$

$$= - \sum_{i=1}^{n} \sum_{j} z_i \mathcal{V}_{ij} \sum_{m=1}^{n} \sum_{\pi \in \mathcal{S}_n/(n-m+1)} sgn(\pi)(-1)^m \overline{\mathcal{P}}_{\pi[1,n-m]} \left(\sum_{i=1}^{k} \mathcal{G}_i \mathcal{F}_i \right) \overline{\mathcal{P}}_{\pi[n-m+2,n]} \mathcal{U}_{ij} \quad \text{by } \textbf{Fact } 15$$

let $m' = n - m + 1$, then $m = n - m' - 1$,

$$= - \sum_{i=1}^{n} \sum_{j} z_i \mathcal{V}_{ij} \sum_{m'=1}^{n} \sum_{\pi \in \mathcal{S}_n/m'} sgn(\pi)(-1)^{n-m'+1} \overline{\mathcal{P}}_{\pi[1,m'-1]} \left(\sum_{i=1}^{k} \mathcal{G}_i \mathcal{F}_i \right) \overline{\mathcal{P}}_{\pi[m'+1,n]} \mathcal{U}_{ij}$$

$$= - (-1)^{n+1} \sum_{i=1}^{n} \sum_{j} z_i \mathcal{V}_{ij} \sum_{m'=1}^{n} \sum_{\pi \in \mathcal{S}_n/m'} sgn(\pi)(-1)^{m'} \overline{\mathcal{P}}_{\pi[1,m'-1]} \left(\sum_{i=1}^{k} \mathcal{G}_i \mathcal{F}_i \right) \overline{\mathcal{P}}_{\pi[m'+1,n]} \mathcal{U}_{ij}$$

$$= \sum_{i=1}^{n} \sum_{j} z_i \mathcal{V}_{ij} S_n(\overline{\mathcal{P}}|_{\mathcal{P}_{\mathcal{I}} \leftarrow \sum_{i=1}^{k} \mathcal{G}_i \mathcal{F}_i}) \mathcal{U}_{ij}$$

$$\in \left\langle S_n(\overline{\mathcal{P}}|_{\mathcal{P}_{\mathcal{I}} \leftarrow \sum_{i=1}^{k} \mathcal{G}_i \mathcal{F}_i}) \right\rangle.$$

QED

C.1.3 Concluding the lower bound for every basis of the identities of $\mathrm{Mat}_d(\mathbb{F})$

Here we show that the $\Omega(n^{2d})$ lower bound proved in previous sections holds (for every $d > 2$ and) *every finite basis of the identities of* $\mathrm{Mat}_d(\mathbb{F})$, when $\mathbb{F}$ is of characteristic 0. To this end, we use several results from the theory of PI-algebras (for more on PI-theory see the monographs [20, 6]).

A polynomial $f \in \mathbb{F}\langle X \rangle$ with n variables is ***multi-homogenous with degrees*** $(1, \ldots, 1)$ (n times) if in every monomial the power of every variable $x_1, \ldots, x_n$ is precisely 1. In other words, every monomial is of the form $\alpha \cdot \prod_{i=1}^{n} x_{\sigma(i)}$, for some permutation σ of order n and some scalar α. For the sake of simplicity, we shall talk in the sequel about a ***multi-homogenous polynomial of degree*** n, when referring to a multi-homogenous polynomial with degrees $(1, \ldots, 1)$ (n times). Thus, any multi-homogenous polynomial with n variables is homogenous of total-degree n.

We need the following definition:

Definition 11. *A polynomial $f \in \mathbb{F}\langle X \rangle$ is called a* ***commutator polynomial*** *if it is a linear combination of products of generalized-commutators. (We assume that 1 is a product of an empty set of commutators.)*

For example, $[x_1, x_2] \cdot [x_3, x_4] + [x_1, x_2, x_3]$ is a commutator polynomial.

We need the following proposition:

Proposition 16 (Proposition 4.3.3 in [6]). *If R is a unitary PI-algebra over a field $\mathbb{F}$ of characteristic 0, then every identity of R can be generated by multi-homogenous commutator polynomials.*

Remark. Multi-homogenous *and* commutator polynomials*, in the current paper, are called* multilinear *and* proper polynomials *in [6], respectively.*

Lemma 17. *Let R be a unitary PI-algebra and let $\mathcal{T}$ be the T-ideal consisting of all identities of R. Then $\mathcal{T}$ has a finite basis in which every polynomial is a multi-homogenous commutator polynomial.*

Proof. By Kemer [13], the identities of any $\mathbb{F}$-algebra, for any $\mathbb{F}$, can be generated by a finite set of identities. Namely $\mathcal{T}$ has a finite basis $\{A_1, , \ldots, A_k\}$, for some positive integer k.

By Proposition 16, for a fixed identity of R, we can find finite many multi-homogenous commutator polynomials to generate. Thus, each A_i, $i \in [k]$, can be generated by finite many multi-homogenous commutator polynomials. Then there are finite many multi-homogenous commutator polynomials generating the basis $\{A_1, , \ldots, A_k\}$ of $\mathcal{T}$, and hence, also finite many multi-homogenous commutator identities generating $\mathcal{T}$.

QED

Lemma 18. *Let $f \in \mathbb{F}\langle X\rangle$ be a multi-homogenous commutator polynomial with n variables. If x_i is a constant for some $i \in [n]$, then $f(x_1, \ldots, x_n) \equiv 0$ (that is, f is the zero polynomial).*

Proof. In the proof, when we talk about the commutator, we mean the non-zero polynomial $[x_{t_1}, \ldots, x_{t_s}]$ for all possible $t_1, \ldots, t_s \in [n]$ and some natural number $s \geq 2$. It is easy to check that if we replace a variable by a constant $c \in \mathbb{F}$ in the commutator $[x_{t_1}, \ldots, x_{t_s}]$, then the commutator equals 0.

By the definition of commutator polynomial, we know

$$f = \sum_{i=1}^{m} c_i \prod_{j=1}^{k_i} B_{ij},$$

where $0 \neq c_i \in \mathbb{F}$ and $m, n \in \mathbb{N}$, and B_{ij} is some commutator $[x_{i_1}, \ldots, x_{i_s}]$.

For a fixed $\mathcal{I} \in [n]$, by the definition of multi-homogenous polynomial, f must be linear in $x_{\mathcal{I}}$, namely $c_i \prod_{j=1}^{k_i} B_{ij}$ must be linear in $x_{\mathcal{I}}$ for every $i \in [m]$. Then there must be a $j_0 \in [k]$ such that B_{ij_0} is linear in $x_{\mathcal{I}}$. That is, $B_{ij_0}|_{x_{\mathcal{I}} \leftarrow c} = 0$. Furthermore, $\prod_{j=1}^{k_i} B_{ij}|_{x_{\mathcal{I}} \leftarrow c} = 0$ for all $i \in [m]$. Namely $f|_{x_{\mathcal{I}} \leftarrow c} = 0$. QED

By lemma 9 and lemma 11, we know that there exist s-polynomials $P_1, \ldots, P_n$ in n variables $x_1, \ldots, x_n$ that are identities over $\mathrm{Mat}_d(\mathbb{F})$, such that putting $P^\star := \sum_{i=1}^{n} z_i P_i$, where $z_1, \ldots, z_n$ are new variables, we have:

$$Q_{S_{2d}}(P^\star) \geq \frac{1}{2d+1} \cdot Q_{S_{2d}}(P_1, \ldots, P_n) = \Omega(n^{2d}).$$

The following is the main lemma of this section:

Lemma 19. *Let $d > 2$, and let $\mathcal{B}$ be some basis for the T-ideals of the identities of $\mathrm{Mat}_d(\mathbb{F})$. Then, there are constants c, c' such that for any identity P over $\mathrm{Mat}_d(\mathbb{F})$ of degree $2d+1$:*

$$cQ_{S_{2d}}(P) \leq Q_{\mathcal{B}}(P) \leq c' Q_{S_{2d}}(P).$$

To prove this theorem we need the following two lemmata.

Lemma 20. *For any natural number $d > 2$, every multi-homogenous identity (with any number of variables) over $\mathrm{Mat}_d(\mathbb{F})$ of degree at most $2d+1$ is a consequence of the standard identity S_{2d}.*

Proof. By Leron [17], we know that for any $d > 2$ every multi-homogenous identity of $\mathrm{Mat}_d(\mathbb{F})$ with degree $2d+1$ is a consequence of the standard identity S_{2d}. By Exercise 7.1.2 in [6], there are no identities of degree less than $2d$ in $\mathrm{Mat}_d(\mathbb{F})$ and every multi-homogenous polynomial identity of degree $2d$ in $\mathrm{Mat}_d(\mathbb{F})$ is also a consequence of the standard identity S_{2d}. QED

By Lemma 17, there is a basis $\{A_1, A_2, \ldots, A_m\}$ of $\mathrm{Mat}_d(\mathbb{F})$, where $A_1, \ldots, A_m$ are all multi-homogenous commutator identities (Definition 11).

Lemma 21. *Let P be an identity of $\mathrm{Mat}_d(\mathbb{F})$ of degree $2d+1$ and let G be a basis $\{A_1, A_2, \ldots, A_m\}$ of $\mathrm{Mat}_d(\mathbb{F})$, where $A_1, \ldots, A_m$ are all multi-homogenous commutator identities of $\mathrm{Mat}_d(\mathbb{F})$. And assume $Q_G(P) = k$, that is, k is the minimal number such that exist k substitution instances $B_1, B_2, \ldots, B_k$ of $A_1, A_2, \ldots, A_m$, for which:*

$$P \in \langle B_1, B_2, \ldots, B_k \rangle .$$

Then, no B_ℓ, for $\ell \in [k]$, is a substitution instance of a basis element A_j whose degree is greater than $2d+1$.

Proof. Assume there is A_j (for $j \in [m]$) in the basis G such that the degree of $A_j(\overline{x})$ is greater than $2d+1$. In the following, we show that none of B_ℓ ($\ell \in [k]$) is a substitution instance of A_j.

Assume otherwise. Hence, there is a $B_{\mathcal{I}}$, $\mathcal{I} \in [k]$, such that $B_{\mathcal{I}}$ is the substitution instance of A_j. Since $A_j(\overline{x})$ is homogeneous, every term in $A_j(\overline{x})$ is of degree greater than $2d+1$.

We consider the following two cases:

Case 1: Every term in the $A_j(\overline{Q})$, which is a substitution instances of $A_j(\overline{x})$, is of degree greater than $2d+1$.

For convenience, given a polynomial f, we denote by $f^{\leq j}$ the polynomial $\sum_{i=0}^{j}(f)^{(i)}$, namely the sum of all homogenous parts of f of degree at most j. We consider the $2d+1$ homogenous part, that is:

$$\begin{aligned} P &= (P)^{(2d+1)} \\ &\in \left\{(h)^{(2d+1)} \mid h \in \langle B_1, B_2, \ldots, B_k\rangle\right\} \subset \left\langle (B_1)^{(\leq 2d+1)}, \ldots, (B_k)^{(\leq 2d+1)}\right\rangle. \end{aligned}$$

But $(B_{\mathcal{I}})^{(\leq 2d+1)} = \left(A_j(\overline{Q})\right)^{(\leq 2d+1)} = 0$, because, in this case, every term in $A_j(\overline{Q})$ is of degree greater than $2d+1$. So P can also belong to the ideal generated by $\left\{(B_1)^{(\leq 2d+1)}, (B_2)^{(\leq 2d+1)}, \ldots, (B_k)^{(\leq 2d+1)}\right\} \setminus (B_{\mathcal{I}})^{(\leq 2d+1)}$. This means $Q_G(P) = k-1$ which contradicts $Q_G(P) = k$. Thus the assumption is false.

Case 2: There is a term of degree at most $2d+1$ in $A_j(\overline{Q})$, which is a substitution instance of $A_j(\overline{x})$.

But we assumed that every term in $A_j(\overline{x})$ must be of degree greater than $2d+1$. This means one of the coordinates of $\overline{Q}$ must be a constant. That is, $A_j(\overline{Q}) = 0$ (by Lemma 18). So P can be generated by $\{B_1, B_2, \ldots, B_k\} \setminus B_i$. Hence, $Q_G(P) = k-1$, which contradicts $Q_G(P) = k$. Thus the assumption is false.

Now we can conclude that the assumption that there is a $B_{\mathcal{I}}$, $\mathcal{I} \in [k]$, such that $B_{\mathcal{I}}$ is a substitution instance of A_j is false. So none of B_ℓ ($\ell \in [k]$) is a substitution instance of A_j. QED

We are now back to the proof of Lemma 19:

Proof. Let $\mathcal{B}$ be a basis $\{A_1, A_2, \ldots, A_m\}$ of $\mathrm{Mat}_d(\mathbb{F})$, where $A_1, \ldots, A_m$ are all multi-homogenous commutator identities of $\mathrm{Mat}_d(\mathbb{F})$. Let

$$(\mathcal{B})^{(\leq 2d+1)} := \{A_i \in \mathcal{B} \mid \text{the degree of } A_i \text{ is no more than } 2d+1\}.$$

For any identity P of $\mathrm{Mat}_d(\mathbb{F})$ of degree $2d+1$, by Lemma 21,

$$Q_{(\mathcal{B})^{(\leq 2d+1)}}(P) = Q_{\mathcal{B}}(P).$$

This also means that every identity of $\mathrm{Mat}_d(\mathbb{F})$ of degree at most $2d+1$ can be generated by $(\mathcal{B})^{(\leq 2d+1)}$. Thus, S_{2d} can be generated by $(\mathcal{B})^{(\leq 2d+1)}$. Write $(\mathcal{B})^{(\leq 2d+1)}$ as the set

$\{A'_1, A'_2, \ldots, A'_{m'}\}$, $m' \leq m$, where the degree of A'_i ($\forall i \in [m']$) is less than $2d+1$. By Lemma 20, $A'_1, \ldots, A_{m'}$ is generated by S_{2d}. Then, by Equation 4 in Proposition 4, for any identity P over $\mathrm{Mat}_d(\mathbb{F})$ of degree $2d+1$:

$$\frac{1}{Q_{(\mathcal{B})^{(\leq 2d+1)}}(S_{2d})} Q_{S_{2d}}(P) \leq Q_{(\mathcal{B})^{(\leq 2d+1)}}(P) \leq \left(\max_{B \in \mathcal{B}'} Q_{S_{2d}}(B)\right) Q_{S_{2d}}(P) \quad d > 2. \tag{11}$$

Namely, for every identity P of $\mathrm{Mat}_d(\mathbb{F})$ of degree $2d+1$,there are constants c, c' such that:

$$c Q_{S_{2d}}(P) \leq Q_{\mathcal{B}}(P) \leq c' Q_{S_{2d}}(P) \quad d > 2.$$

QED

We can now conclude the main theorem of this section, Theorem 5, which we restate for convenience:

Theorem 5. *Let $\mathbb{F}$ be any field of characteristic 0. For every natural number $d > 2$ and for every finite basis $\mathcal{B}$ of the T-ideal of identities of $\mathrm{Mat}_d(\mathbb{F})$, there exists an identity P over $\mathrm{Mat}_d(\mathbb{F})$ of degree $2d+1$ with n variables, such that $Q_{\mathcal{B}}(P) = \Omega(n^{2d})$.*

Note on the case of $d = 2$. When $d = 2$, Lemma 19 is not true. For example, the polynomial $f = [[x_1, x_2][x_3, x_4] + [x_3, x_4][x_1, x_2], x_5]$ is an identity over $\mathrm{Mat}_2(\mathbb{F})$, but in [17] it is proved that f cannot be generated by S_4. Namely the restriction $d > 2$ in Lemma 19, and also in Theorem 5, is essential for our proof.

D Relations to tensor-rank

Here we show that in order to make the hard (non-explicit) instances f from Theorem 5 into explicit ones, means finding explicit tensors with high tensor-rank. This generalizes (to any order) a similar observation made in [7] for order 3 tensors. This means that the *specific* hard instances we provide in Theorem 5 are not good candidates for proof complexity hardness, because it is reasonable to assume they do not have small size circuits.

Definition 12. *A tensor $A : [n]^r \to \mathbb{F}$ is a **simple tensor** if there exist r vectors $a_1, \ldots, a_r : [n] \to \mathbb{F}$ such that $A = a_1 \otimes \cdots \otimes a_r$, where $\otimes$ denotes tensor product, that is, A is defined by $A(i_1, i_2, \ldots, i_r) = a_1(i_1) \cdots a_r(i_r)$.*

Definition 13. *For a tensor A, the **tensor rank** $rank(A)$ is the minimal k such that there exist k simple tensors $A_1, A_2, \ldots, A_k : [n]^r \to \mathbb{F}$ such that $A = \sum_{i=1}^{k} A_i$.*

Definition 14. *For a natural number n, let A be a tensor $[n]^{r+1} \to \mathbb{F}$. We define the* ***corresponding polynomials*** *(from $\mathbb{F}\langle X\rangle$)* ***of the tensor*** *$A$ as follows:*

$$f_{j_0} := \sum_{j_1,j_2,\ldots,j_r\in[n]} A(j_0,j_1,\ldots,j_r)S_r(x_{j_1},x_{j_2},\ldots,x_{j_r}), \quad \forall j_0 \in [n].$$

By the following theorem, if we find an collection of *explicit*[9] s-polynomials $f_1,\ldots,f_n$ over $\mathrm{Mat}_d(\mathbb{F})$ such that $Q_{S_{2d}}(f_1,\ldots,f_n)$ is $\Omega(n^{2d})$, then we can find an *explicit*[10] tensor $A : [n]^{2d+1} \to \{0,1\}$ with rank $\Omega(n^{2d})$, where the corresponding polynomials of A are the s-polynomials $f_1,\ldots,f_n$.

Theorem 22. *For a natural number n, let $A_{f_1,\ldots,f_n}$ be a tensor $[n]^{r+1} \to \mathbb{F}$ and let $f_1,\ldots,f_n \in \mathbb{F}\langle X\rangle$ be the corresponding polynomials of $A_{f_1,\ldots,f_n}$, then:*

$$Q_{S_{2d}}(f_1,\ldots,f_n) \le rank(A_{f_1,\ldots,f_n}).$$

Proof. Assume $rank(A_{f_1,\ldots,f_n}) = R$. Namely we can find R simple tensors $A_1,A_2,\ldots,A_R$ such that

$$A_{f_1,\ldots,f_n} = \sum_{i=1}^{R} A_i. \tag{12}$$

For every $i \in [R]$, by simple tensor's definition, there exist $2d+1$ vectors $a_0^{(i)},a_1^{(i)},\ldots,a_{2d}^{(i)} : [n] \to F$ such that $A_i = a_0^{(i)} \otimes a_1^{(i)} \otimes \cdots \otimes a_{2d}^{(i)}$. Namely $A_i(i_0,i_1,i_2,\ldots,i_{2d}) = a_0^{(i)}(i_0)a_1^{(i)}(i_1)\cdots a_{2d}^{(i)}(i_{2d})$, where $i_0,\ldots,i_{2d} \in [n]$.

Concerning the corresponding polynomials $f_1,\ldots,f_n$ of $A_{f_1,\ldots,f_n}$, for every $j_0 \in [n]$,

$$\begin{aligned}
f_{j_0} &= \sum_{j_1,j_2,\ldots,j_r\in[n]} A_{f_1,\ldots,f_n}(j_0,\ldots,j_{2d})S_{2d}(x_{j_1},\ldots,x_{j_{2d}}) \\
&= \sum_{j_1,j_2,\ldots,j_r\in[n]} \sum_{i=1}^{R} A_i(j_0,\ldots,j_{2d})S_{2d}(x_{j_1},\ldots,x_{j_{2d}}) \quad \text{(by 12)} \\
&= \sum_{i=1}^{R} \sum_{j_1,j_2,\ldots,j_r\in[n]} A_i(j_0,\ldots,j_{2d})S_{2d}(x_{j_1},\ldots,x_{j_{2d}}) \\
&= \sum_{i=1}^{R} a_0^{(i)}(j_0) \sum_{j_1,j_2,\ldots,j_r\in[n]} a_1^{(i)}(j_1)\cdots a_{2d}^{(i)}(j_{2d})S_{2d}(x_{j_1},x_{j_2},\ldots,x_{j_{2d}}) \\
&= \sum_{i=1}^{R} a_0^{(i)}(j_0)S_{2d}\left(\sum_{1\le j\le n} a_1^{(i)}(j)x_j, \sum_{1\le j\le n} a_2^{(i)}(j)x_j, \ldots, \sum_{1\le j\le n} a_{2d}^{(i)}(j)x_j\right)
\end{aligned}$$

[9] A polynomial is said to be *explicit* if the coefficient of a monomial of degree d is computable by algebraic circuits of size at most poly(d), where d is a natural number.

[10] A tensor $T : [n]^r \to \mathbb{F}$ is called explicit if $T(i_1,\ldots,i_r)$ can be computed by algebraic circuits of size at most polynomial in poly$(r \lg n)$, that is, at most polynomial in the size of the input $(i_1,\ldots,i_r)$.}

$$= \sum_{i=1}^{R} a_0^{(i)}(j_0) S_{2d}(\overline{P}_i)$$

(For convenience, write $\left(\sum_{1\le j\le n} a_1^{(i)}(j)x_j, \sum_{1\le j\le n} a_2^{(i)}(j)x_j, \ldots, \sum_{1\le j\le n} a_{2d}^{(i)}(j)x_j\right)$ as $\overline{P}_i$, for any $i \in [R]$).

Namely

$$f_1, \ldots, f_n \in \left\langle S_{2d}\left(\overline{P}_1\right), \ldots, S_{2d}\left(\overline{P}_R\right)\right\rangle.$$

Thus $Q_{S_{2d}}(f_1, \ldots, f_n) \le R$, namely $Q_{S_{2d}}(f_1, \ldots, f_n) \le \text{ rank}(A_{f_1,\ldots,f_n})$. QED

By the above theorem, we have the following:

Corollary 23. *If there exists a n explicit collection of s-polynomials $f_1, \ldots, f_n$ (that are all identities of) $\text{Mat}_d(\mathbb{F})$, such that $Q_{S_{2d}}(f_1, \ldots, f_n) = \Omega(n^{2d})$, then there exists an explicit tensor $A : [n]^{2d+1} \to \{0,1\}$ with tensor-rank $\Omega(n^{2d})$.*

E Matrix identities as hard proof complexity candidates

Here we seek to find connections between the work we have done above to the problem of proving lower bounds in proof complexity.

Consider a matrix identity f over $\text{Mat}_d(\mathbb{F})$. It is a non-commutative polynomial. Let f be a nonzero polynomial identity over $\text{Mat}_d(\mathbb{F})$. Then f is a nonzero non-commutative polynomial from $\mathbb{F}\langle X\rangle$. If we substitute each (matrix) variable x_i in f by a $d \times d$ matrix of *entry-variables* $\{x_{ijk}\}_{j,k\in[n]}$, then now f corresponds to d^2 commutative zero polynomials, one for each entry computed by f. Accordingly, let F be a non-commutative circuit computing f. Then under the above substitution of d^2 entry-variables to each variable in F, we get d^2 non-commutative circuits, each computing the zero polynomial *when considered as commutative polynomials*. Formally, we define the set of d^2 non-commutative circuits corresponding to the non-commutative circuit F as follows:

Definition 15 ($[\![F]\!]_d$, $[\![F = 0]\!]_d$)**.** *Let F be a non-commutative circuit computing the polynomial $f \in \mathbb{F}\langle X\rangle$, such that f is an identity of $\text{Mat}_d(\mathbb{F})$. We define $[\![F]\!]_d$ as the set of d^2 circuits which are generated from bottom to top in the circuit of F according to the following rules:*

1. *every variable x in F corresponds to d^2 new variables $x_{ij}, i, j \in [d]$;*
2. *every plus gate $X \oplus Y$, where X, Y represent two circuits, in F corresponds to d^2 plus gates $\oplus_{ij}, i, j \in [d]$ where each plus gate $\oplus_{ij}$ connects the corresponding circuit X_{ij} and Y_{ij} which have been generated before;*

3. *every multiplication gate $X \otimes Y$ in F corresponds to d^2 plus gates $\oplus_{ij}, i, j \in [d]$ where each plus gate $\oplus_{ij}$ is connected to d multiplication gates $\otimes_k, k \in [d]$ which represent the multiplication of two corresponding circuit X_{ik} and Y_{kj} that have been generated before. (Formally, plus gates have* fan-in two*, and so $\oplus_{ij}$ is the root of a binary tree whose internal nodes are all plus gates and whose d leaves are the product gates $\otimes_k$, $k \in [d]$.)*

We define $[\![F = 0]\!]_d$ to be the set of equations between circuits, where each circuit in $[\![F]\!]_d$ equals the circuit 0*.*

Fact 24. *Since every gate in F corresponds to at most d^3 gates in $[\![F]\!]_d$, we have:*

$$\left|[\![F]\!]_d\right| = O\left(d^3|F|\right)$$

(where $|F|$ denotes the size of F, that is the number of nodes in F and $\left|[\![F]\!]_d\right|$ denotes the sum of size of all circuits in $[\![F]\!]_d$). Thus, if we fix the dimension of a matrix as a constant, then we can claim that $|[\![f]\!]_d| = \Theta(|f|)$.

First, we recall the arithmetic proof system $\mathbb{P}_c(\mathbb{F})$ (introduced in [9], and almost similarly in [8]) for deriving (commutative) polynomial identities over a field $\mathbb{F}$. The system manipulate arithmetic equations, that is, expressions of the form $F = G$ where F, G are circuits.

Definition 16 (Arithmetic proofs $\mathbb{P}_c(\mathbb{F})$)**.** *Let $\mathbb{F}$ be a field. The system $\mathbb{P}_c(\mathbb{F})$ proves equations of the form $F = G$, where F, G are non-commutative arithmetic circuits (over $\mathbb{F}$). The inference rules are:*

$$\frac{F = G}{G = F} \qquad \frac{F = G \qquad G = H}{F = H}$$

$$\frac{F_1 = G_1 \qquad F_2 = G_2}{F_1 + F_2 = G_1 + G_2} \qquad \frac{F_1 = G_1 \qquad F_2 = G2}{F_1 \times F_2 = G_1 \times G_2}.$$

The axioms are equations of the following form, with F, G, H ranging over non-commutative circuits:

$$\begin{aligned}
&\textit{Identity}: && F = F \\
&\textit{Product commutativity}: && F \cdot G = G \cdot F \\
&\textit{Addition commutativity}: && F + G = G + F \\
&\textit{Associativity}: && F + (G + H) = (F + G) + H \qquad F \cdot (G \cdot H) = (F \cdot G) \cdot H \\
&\textit{Distributivity}: && F \cdot (G + H) = F \cdot G + F \cdot H \\
&\textit{Zero element}: && F + 0 = F \qquad F \cdot 0 = 0 \\
&\textit{Unit element}: && F \cdot 1 = F \\
&\textit{Field identities}: && c = a + b \qquad d = a' \cdot b'
\end{aligned}$$

where $a, a', b, b', c, d \in \mathbb{F}$*, such that the equations hold in* $\mathbb{F}$*.*

Circuit axiom : $F = F'$ *if* F *and* F' *are (syntactically) identical when both are un-winded into* formulas*.*

Note that the Circuit axiom can be verified in polynomial time (see e.g., [11]).

A proof π *in* $\mathbb{P}_c(\mathbb{F})$ *is a sequence of equations* $F_1 = G_1, F_2 = G_2, \ldots, F_k = G_k$*, with* F_i, G_i *circuits, such that every equation is either an axiom, or was obtained from previous equations by one of the derivation rules. An equation* $F_i = G_i$ *appearing in a proof is also called a proof-line. Denote by* $|\vdash_{\mathbb{P}_c(\mathbb{F})} F|$ *the minimum number of lines in a* $\mathbb{P}_c$ *proof of* $F = 0$*. We say that* π *is a* $\mathbb{P}_c$ *proof of a set of equations if* π *is a* $\mathbb{P}_c$ *and it contains all the equations in the set as proof-lines).*

For $\mathbb{F}$ an infinite field, f is an identity in $\mathrm{Mat}_d(\mathbb{F})$ iff $[\![F = 0]\!]_d$ has a $\mathbb{P}_c(\mathbb{F})$ proof. This is easy to prove as follows: assume by contradiction otherwise, then there must be an assignment A that makes $g \neq 0$. This follows since the field is infinite (and so every non zero polynomial has an assignment that does not nullifies the polynomial). But this assignment A (extended in any way to all entries) makes the matrix identity nonzero, in contradiction to the assumption that it is a matrix identity.

Conjecture II. *Let* d *be a positive natural number and let* $\mathcal{B}$ *be a (finite) basis of the T-ideal of the identities of* $\mathrm{Mat}_d(\mathbb{F})$*. Assume that* $f \in \mathbb{F}\langle X\rangle$ *is an identity over* $\mathrm{Mat}_d(\mathbb{F})$*, and let* F *be a non-commutative algebraic circuit computing* f*. Then, the minimal number of lines in an arithmetic proof of the collection of* d^2 *(entry-wise) equations* $[\![F = 0]\!]_d$ *corresponding to* F *is lower bounded (up to a constant factor) in* $Q_{\mathcal{B}}(f)$*. And in symbols:*

$$\left|\vdash_{\mathbb{P}_c(\mathbb{F})} [\![F = 0]\!]_d\right| = \Omega(Q_{\mathcal{B}}(f)).$$

E.1 Conditions for exponential lower bounds

Can we, even potentially, obtain exponential lower bounds on $\mathbb{P}_c(\mathbb{F})$ proof size using the measure $Q_B(\cdot)$ and assuming Conjecture 1 holds? The answer is yes, under certain further technical assumptions. We write the assumptions formally:

Assumptions:

1. **Refinement of Conjecture II**: Assume that for any d and any basis $\mathcal{B}_d$ of the identities of $\mathrm{Mat}_d(\mathbb{F})$ the number of lines in any $\mathbb{P}_c(\mathbb{F})$ proof of $[\![F = 0]\!]_d$ is at least $\mathcal{C}_{\mathcal{B}_d} \cdot Q_{\mathcal{B}_d}(f)$, where $\mathcal{C}_{\mathcal{B}_d}$ is a number depending on $\mathcal{B}_d$ and F is the non-commutative arithmetic circuit computing f (this is the same as Conjecture 1 except that now $\mathcal{C}_{\mathcal{B}_d}$ is not a constant).

2. Assume that for any sufficiently large d and any basis $\mathcal{B}_d$ of the identities of $\mathrm{Mat}_d(\mathbb{F})$, there exists a number $c_{\mathcal{B}_d}$ such that for all sufficiently large n there exists an identity $f_{n,d}$ with $Q_{\mathcal{B}_d}(f_{n,d}) \geq c_{\mathcal{B}_d} \cdot n^{2d}$. (The existence of such identities are known from our unconditional lower bound.)

3. Assume that for the $c_{\mathcal{B}_d}$ in item 2 above: $c_{\mathcal{B}_d} \cdot \mathcal{C}_{\mathcal{B}_d} = \Omega\left(\frac{1}{\mathrm{poly}(d)}\right)$.

4. **(Variant of) Conjecture I**: Assume that the non-commutative arithmetic circuit size of $f_{n,d}$ is at most $\mathrm{poly}(n,d)$.

Corollary (assuming Assumptions 1-4 above): There exists a polynomial size (in n) family of identities between non-commutative arithmetic circuits, for which any $\mathbb{P}_c$ proof requires exponential $2^{\Omega(n)}$ number of proof-lines.

Proof. By the assumptions, every $\mathbb{P}_c(\mathbb{F})$-proof of $[\![f_{n,d} = 0]\!]_d$ has size at least $c_{\mathcal{B}_d} \cdot \mathcal{C}_{\mathcal{B}_d} \cdot n^{2d}$. Consider the family $\{f_{n,d}\}_{n=1}^{\infty}$, *where d is a function of n*, and we take $d = n/4$. Then, we get the following lower bound on the number of lines in $\mathbb{P}_c(\mathbb{F})$-proofs of the family $\{f_{n,d}\}_{n=1}^{\infty}$:

$$c_{\mathcal{B}_d} \cdot \mathcal{C}_{\mathcal{B}_d} \cdot n^{2d} = \frac{1}{\mathrm{poly}(n/4)} n^{n/2} = 2^{\Omega(n)},$$

which (by Assumption 4) is *exponential* in the arithmetic circuit-size of the identities $f_{n,d}$ proved. QED

Justification of assumptions. We wish to justify to a certain extent the new Assumptions 3 above (which lets us obtain the exponential lower bound). We shall use the s-polynomials for this. First, note that Assumption 2 holds for the case of the s-polynomials, by Theorem 5.

We now show that the function c_{B_d} does not decrease too fast. By Equations 9, 10 and 11 in Section C.1, we know that for any natural number d, there is an s-polynomial f, such that:

$$Q_{B_d}(f) \geq \frac{1}{Q_{(B_d)^{(\leq 2d+1)}}(S_{2d})} \frac{1}{2d+1} \frac{\binom{n}{2d} \ln 2}{(2d+1)\ln(4d+2)}.$$

Let $\mathcal{B}_d$ be a set of identities of $\mathrm{Mat}_d(\mathbb{F})$ that contains the S_{2d} identities. Hence,

$$Q_{(\mathcal{B}_d)^{(\leq 2d+1)}}(S_{2d}) = 1.$$

Thus

$$Q_{\mathcal{B}_d}(f) \geq \frac{1}{2d+1} \frac{\binom{n}{2d} \ln 2}{(2d+1)\ln(4d+2)}.$$

If we let $d = n/4$, then

$$Q_{\mathcal{B}_{n/4}}(f) \geq \frac{1}{n/2+1} \frac{\binom{n}{n/2} \ln 2}{(n/2+1)\ln(n+2)}.$$

By Stirling's formula, we get that $n! \sim \sqrt{2\pi n}(\frac{n}{e})^n$. Hence, $\binom{n}{n/2} \sim \frac{2^{n+1/2}}{\sqrt{n\pi}}$. Then

$$Q_{\mathcal{B}_{n/4}}(f) = \Omega\left(\frac{2^n}{n^{5/2}\ln n}\right).$$

This shows that the function c_{B_d} does not decrease too fast.

We can use the fact that c_{B_d} does not decrease too fast to get the following (conditional exponential lower bound):

Proposition 25. *Suppose* Assumption 1 *above holds (refinement of Conjecture 1) and assume that* $\mathcal{C}_{\mathcal{B}_{n/4}} = \Omega(1/\mathrm{poly}(n))$. *Then, there exists a family of non-commutative circuits* $\{F_n\}_{n=1}^{\infty}$ *(computing the family of polynomials* $\{f_{n,\frac{n}{4}}\}_{n=1}^{\infty}$*) such that the number of lines in any* $\mathbb{P}_c(\mathbb{F})$*-proof of* $[\![F_n = 0]\!]_{n/4}$ *is at least* $\mathcal{C}_{B_{n/4}}\Omega\left(\frac{2^n}{n^{5/2}\ln n}\right) = \Omega\left(\frac{2^n}{poly(n)}\right) = 2^{\Omega(n)}$.

Note that we get only an exponential lower bound in n for the lines of proofs in $\mathbb{P}_c$ in the above consequence. But this does not entail an exponential lower bound in the size of $[\![F_n = 0]\!]_{n/4}$ (the latter is polynomial in the size of the circuit F_n, computing the s-polynomials. So this proposition is presented here in order to show that at least for some identities, the additional requirement (Assumption 3) on parameters, added to get a conditional exponential lower bound, is attainable.

E.2 A propositional version of Conjecture II

We wish to comment on the applicability of our suggested framework, for achieving propositional Extended Frege lower bounds.

It seems that the most natural way to connect the complexity, measure $Q_{\mathcal{B}}(\cdot)$ to the number of lines in an Extended Frege (see, e.g., [14] or [11] for a formal definition of Extended Frege) proof is to require that the Main Open Problem states an *even stronger* statement. Admittedly, this makes the new assumption, shown below, quite speculative at the moment.

Given a commutative algebraic circuit C over $GF(2)$, we can think of the circuit equation $C = 0$ as a *Boolean* circuit computing a tautology, instead of an algebraic circuit: interpreting $+$ as XOR, $\cdot$ as $\wedge$, and $=$ as logical equivalence $\equiv$ (that is, $\leftrightarrow$). Accordingly, we can consider arithmetic proofs over $GF(2)$ augmented with the Boolean axioms $x_i^2 + x_i = 0$, for each variables x_i, to obtain a propositional proof system which formally *is* an Extended Frege proof system (see [9]). Denote this system $\mathbb{P}_c(\mathbb{F}) + \{x_i^2 + x_i = 0 \ : \ x_i \in X\}$.

Then, there is no clear reason to rule out the following:

Conjecture 1 for the propositional case over $GF(2)$. *Let $\mathbb{F} = GF(2)$, let d be a positive natural number and let $\mathcal{B}$ be a (finite) basis of the identities of $\mathrm{Mat}_d(\mathbb{F})$. Assume that $f \in \mathbb{F}\langle X\rangle$ is an identity of $\mathrm{Mat}_d(\mathbb{F})$, and let F be a non-commutative algebraic circuit computing f. Then, the minimal number of lines in a $\mathbb{P}_c(\mathbb{F}) + \{x_i^2 + x_i = 0 : x_i \in X\}$ proof of the collection of d^2 (entry-wise) equations $[\![F = 0]\!]_d$ corresponding to F is lower bounded (up to a constant factor) by $Q_{\mathcal{B}}(f)$. And in symbols:*

$$\left| \vdash_{\mathbb{P}_c(\mathbb{F})+\{x_i^2+x_i=0\,:\,x_i\in X\}} [\![F = 0]\!]_d \right| = \Omega(Q_{\mathcal{B}}(f)). \tag{13}$$

(Where, as before, $\left| \vdash_{\mathbb{P}_c(\mathbb{F})+\{x_i^2+x_i=0:\,x_i\in X\}} [\![F = 0]\!]_d \right|$ is the minimal size of a $\mathbb{P}_c(\mathbb{F}) + \{x_i^2 + x_i = 0 \ : \ x_i \in X\}$ proof containing all the circuit-equations in $[\![F = 0]\!]_d$.)

Comment: One can plausibly consider the same propositional version of the main open problem, with $\mathbb{F}$ being the rational numbers, and hence of characteristic 0 (for we which we have more knowledge about $Q_{\mathcal{B}}(\cdot)$, as obtained in our work). However, the way to translate arithmetic proofs $\mathbb{P}_c$ over the rationals is less immediate than the same translation for the case of $GF(2)$, and we have not verified formally the details of such a translation.

E.3 Hierarchy of proofs for matrix identities

The proof system $\mathbb{P}_c(\mathbb{F})$ works for proving identities over commutative fields. Here we formulate a fragment of $\mathbb{P}_c(\mathbb{F})$ that proves matrix $\mathrm{Mat}_d(\mathbb{F})$ identities, for every given d. In what follows, $\mathbb{F}$ always denotes a field of characteristic 0.

For any field $\mathbb{F}$ (of characteristic 0), any $d \geq 1$, and any basis $\mathcal{B}$ of the identities of $\mathrm{Mat}_d(\mathbb{F})$, we define the following proof system $\mathbb{P}_{\mathrm{Mat}_d}(\mathbb{F})$, which is sound and complete for the identities of $\mathrm{Mat}_d(\mathbb{F})$ (written as equations of non-commutative circuits): consider the proof systems $\mathbb{P}_c(\mathbb{F})$ and *replace* the commutativity axiom $h \cdot g = g \cdot h$ by a finite basis $\mathcal{B}$ of the identities of $\mathrm{Mat}_d(\mathbb{F})$ (namely, add a new axiom $H = 0$ for each polynomial h in the basis, where H is a non-commutative algebraic circuit computing h). Additionally, add the axioms of distributivity of product over addition from *both* left and right (this is needed because we do not have anymore the commutativity axiom in our system).

Since, for $d > 2$, the set of generators for the identities over $\mathrm{Mat}_d(\mathbb{F})$ are still not well understood, we shall give an explicit formulation only of the system $\mathbb{P}_{\mathrm{Mat}_2}(\mathbb{F})$, following the basis of identities of $\mathrm{Mat}_2(\mathbb{F})$ found by Drensky [5].

Definition 17 (The system $\mathbb{P}_{\mathrm{Mat}_2}(\mathbb{F})$: proofs of identities over $\mathrm{Mat}_2(\mathbb{F})$)**.** $\mathbb{P}_{\mathrm{Mat}_2}(\mathbb{F})$ *is the arithmetic proof system whose set of axioms consists of the following equations (ranging over non-commutative arithmetic circuits):*

$$\textit{Addition commutativity}: \qquad f + g = g + f$$

$$
\begin{aligned}
&\textit{Associativity}: && f+(g+h)=(f+g)+f \qquad f\cdot(g\cdot h)=(f\cdot g)\cdot h\\
&\textit{Distributivity}: && f\cdot(g+h)=f\cdot g+f\cdot h\\
& && (g+h)\cdot f=g\cdot\ f+h\cdot f\\
&\textit{Zero element}: && f+0=f \qquad f\cdot 0=0\\
&\textit{Unit element}: && f\cdot 1=f\\
&\textit{Genertators}: && S_4(x,y,z,w)=0 \qquad [[x,y]^2,z]=0\\
&\textit{Field identities}: && c=a+b \qquad d=a'\cdot b'
\end{aligned}
$$

where in the Field identities $a, a', b, b', c, d \in \mathbb{F}$*, such that the equations hold in* $\mathbb{F}$*.*

Circuit axiom: $F = F'$ *if* F *and* F' *are (syntactically) identical when both are un-winded into* formulas.

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