Hereditarily Just Infinite Profinite Groups that are Not Virtually Pro-p

Submitted by

Sarah Elizabeth Anne Phyllis Middleton

for the degree of Doctor of Philosophy

of the

Royal Holloway, University of London

October 2013

Declaration

I, Sarah Elizabeth Anne Phyllis Middleton, hereby declare that this thesis and the work presented in it is entirely my own. Where I have consulted the work of others, this is always clearly stated.

Signed......................(Sarah Elizabeth Anne Phyllis Middleton)
Date:

to my parents

Abstract

A profinite group G is just infinite if it is infinite and every non-trivial closed normal subgroup of G is open, and hereditarily just infinite if every open subgroup is just infinite. Hereditarily just infinite profinite groups that are not virtually pro-p were first described by J. S. Wilson, in his recent paper 'Large hereditarily just infinite groups', in 2010. These profinite groups are inverse limits of finite groups that are iterated wreath products. The iterated wreath products are constructed from finite non-abelian simple groups, using two types of transitive actions; one of which is specified and the other is left unspecified.

The main results of this thesis are the complete characterisation of the closed normal subgroups and the closed subnormal subgroups of such hereditarily just infinite profinite groups, introduced by Wilson. Using positive finite generation work of M. Quick, we see that these profinite groups, in the majority of instances, are positively finitely generated and therefore finitely generated. Recent results by N. Nikolov and D. Segal show that all the normal and subnormal subgroups of such a hereditarily just infinite group, described by Wilson, are automatically closed provided the profinite group is finitely generated. Therefore the characterisations of normal and subnormal subgroups cover all normal and subnormal subgroups of the majority of Wilson's groups.

The characterisation of the subnormal subgroups is interesting because it is dependent on the choices for the unspecified transitive actions, used to construct these profinite groups. A starting point for describing the subnormal subgroups is to make a choice for the unspecified transitive actions. In this way, some restricted constructions of Wilson's groups have all their subnormal subgroups forming chains, where the subnormal subgroups are squeezed between consecutive normal subgroups.

We have examined the possibility of describing maximal subgroups of Wilson's hereditarily just infinite groups. M. Bhattacharjee has worked on maximal subgroups of iterated wreath products of alternating groups with degree ≥ 5, constructed using the natural actions of the alternating groups. We have applied Bhattacharjee's techniques and described maximal subgroups for certain first finite iterated wreath products, in the construction of Wilson's groups. In so doing, we indirectly extend Bhattacharjee's work, whose view point is that of finite generation. This is because we count the exact number of conjugacy classes of maximal subgroups and the exact number of maximal subgroups, for a very small subclass of Bhattacharjee's wreath products.

Acknowledgement

I would like to thank my supervisor, Benjamin Klopsch, who has provided me with excellent supervision. I am also very grateful to him for proofreading this thesis. My thanks also go to Mark Wildon for highlighting significant errors in my upgrade document. This project would not have been possible without the studentship funded by EPSRC.

Contents

Chapter 1

Introduction

This thesis is built on a recent paper by J. S. Wilson, entitled 'Large hereditarily just infinite groups'; see the reference [32]. We are particularly interested in hereditarily just infinite profinite groups. For the general theory of profinite groups there are two books, both of which are entitled 'Profinite Groups'. One book is by J. S. Wilson [30] and the other book is by L. Ribes and P. Zalesskii [26].

A profinite group G is just infinite if it is infinite and every non-trivial closed normal subgroup of G is open. It is hereditarily just infinite if every open subgroup of G is just infinite. The simplest examples of abstract[1] hereditarily just infinite groups are the infinite cyclic group and the infinite dihedral group $D_\infty = \langle x, y : x^2 = y^2 = 1 \rangle$. More complicated examples are the groups $\mathrm{SL}(n, \mathbb{Z})$ modulo their centre, for $n \geq 3$, refer to [20]. Examples of profinite hereditarily just infinite groups are $\mathbb{Z}_p$ and $\mathrm{SL}(n, \mathbb{Z}_p)$ modulo their centre, for $n \geq 3$. Many of the groups introduced by R. I. Grigorchuk [9] and N. Gupta and S. Sidki [10] that act on trees are just infinite. A readable account describing the first Grigorchuk group is given by P. de la Harpe, in [6, Ch. VIII].

For profinite groups, a just infinite group is the analogue of a simple group in the setting of finite groups. Therefore it is natural that we want to classify just infinite profinite groups or, if this is not possible, describe them in some suitable fashion.

It has been shown, refer to [29] and [31], that certain just infinite groups can be embedded, as subgroups of finite index, in permutational wreath products of a hereditarily just infinite group and a finite group. Therefore the study of these just infinite profinite groups reduces to the study of hereditarily just infinite profinite groups.

For some prime p, a profinite group is pro-p if every open normal subgroup has index equal to some power of p. A profinite group is virtually pro-p if it has an open normal subgroup that is pro-p. All hereditarily just infinite profinite groups prior to Wilson's

[1] An abstract group is a group without a topology.

recent construction, in [32], were virtually pro-p groups. The following theorem by Wilson shows that there are hereditarily just infinite profinite groups of a new kind.

Theorem 1.1 (Wilson [32]). *There exists a hereditarily just infinite profinite group with the property that all composition factors of finite continuous images are non-abelian. In particular, the group is not virtually pro-p.*

Every finitely generated profinite group is countably based. See [30, pg. 54] for the general definition of a countably based profinite group. The following corollary is also given by Wilson.

Corollary 1.2 (Wilson [32]). *There exists a hereditarily just infinite profinite group in which every countably based profinite group can be embedded, as a closed subgroup.*

Therefore the groups that arise from Theorem 1.1 are also notable because they are very 'large', since every countably based profinite group can be embedded in at least one of them.

A result similar to Corollary 1.2 has featured, in the pro-p setting, where the Nottingham group already existed as a hereditarily just infinite group. R. Camina [4] proved that every countably based pro-p group can be embedded, as a closed subgroup, in the Nottingham group. The Nottingham group was introduced to group theory by D. L. Johnson [12] and I. O. York [34], themselves motivated by an article of S. A. Jennings [11].

In this thesis, we shall call the groups of Theorem 1.1 Wilson groups and their construction Wilson's construction. Wilson groups and their construction are explained in detail in Section 4.1. The hereditarily just infinite profinite group of Corollary 1.2 is a specific Wilson group, as given in Section 4.2. Wilson groups are new and their construction is very interesting, therefore they deserve further investigation.

1.1 Contributions

This thesis has set about to investigate structural properties of Wilson groups.

1.1.1 Normal and subnormal subgroups of Wilson groups

Our main contribution is a complete classification of the closed subnormal subgroups of an arbitrary Wilson group. As a subcase, we have completely classified the closed normal subgroups of an arbitrary Wilson group. In fact, for a finitely generated[2] Wilson group all subnormal subgroups are automatically closed and therefore, for these groups,

[2] A finitely generated profinite group means it is topologically finitely generated, see Section 2.5.

we have completely classified all their subnormal subgroups. We determine that the majority of Wilson groups are finitely generated, refer to Chapter 9.

In order to lay out these results, we first briefly describe Wilson's construction from which Wilson groups arise; refer to Section 4.1 for further details of Wilson's construction.

Let $X_0, X_1, X_2, \ldots$ be an infinite sequence of finite non-abelian simple groups. Set $G_0 = X_0$. We construct the finite groups G_n, for $n \in \mathbb{N}$, from iterated wreath products of the groups $X_0, X_1, X_2, \ldots$. Each wreath product G_n is formed via two types of actions. One of which is unspecified and the other type of action is specified.

Suppose a group G_{n-1}, for $n \in \mathbb{N}$, with a faithful transitive permutation representation of degree d_n has been constructed. Let $L_n = X_n^{(d_n)}$, for $n \in \mathbb{N}$, the direct product of d_n copies of X_n. Wilson defines a specified transitive permutation representation of the group L_nG_{n-1} on the set L_n (see the action (4.1) in Section 4.1). Let $M_n = X_n^{(|L_n|)}$, for $n \in \mathbb{N}$, the direct product of $|L_n|$ copies of X_n. Form

$$G_n = X_n \wr_{L_n} (X_n \wr_{\Omega_{d_n}} G_{n-1}),$$

where $\Omega_{d_n} = \{1, 2, \ldots, d_n\}$. That is, written as semidirect products,

$$G_n = M_n \rtimes (L_n \rtimes G_{n-1}).$$

A Wilson group is an inverse limit of such finite groups G_n, for $n \geq 0$, as described above. We call the groups G_n Wilson quotients. This becomes evident later when the infinite groups $\ldots M_{n+2}L_{n+2}M_{n+1}L_{n+1}$, for $n \geq 0$, are found to be normal subgroups of a Wilson group.

Corollary 5.3 displays the result of the complete classification of the closed normal subgroups of an arbitrary Wilson group. This is derived from the complete classification of the normal subgroups of the finite groups G_n, as in Theorem 5.1. For the purpose of what follows we define $M_0 = G_0$.

Theorem 5.1. *Let G_n, for $n \geq 0$, be the finite groups as defined above. For $j \in \{0, 1, \ldots, n\}$, define*

$$P_j^n = M_n \rtimes \ldots \rtimes (M_{j+1} \rtimes L_{j+1})$$

and define

$$Q_j^n = M_n \rtimes \ldots \rtimes (M_{j+1} \rtimes (L_{j+1} \rtimes M_j)).$$

Then the normal subgroups of G_n are precisely the groups P_j^n and Q_j^n. In particular, *they form a complete chain*

$$\{1\} = P_n^n \subsetneq Q_n^n \subsetneq P_{n-1}^n \subsetneq \ldots \subsetneq Q_1^n \subsetneq P_0^n \subsetneq Q_0^n = G_n.$$

Corollary 5.3. *Let $G = \varprojlim (G_n)_{n\geq 0}$ be the inverse limit of the groups G_n as defined above. For $j \geq 0$, define*

$$P_j = \varprojlim (P_j^n)_{n\to\infty}$$

and define

$$Q_j = \varprojlim (Q_j^n)_{n\to\infty},$$

regarded as subgroups of G.

Then the non-trivial closed normal subgroups of G are precisely the groups P_j and Q_j. In particular, they form a complete chain

$$\ldots \subsetneq Q_{n+1} \subsetneq P_n \subsetneq Q_n \subsetneq P_{n-1} \subsetneq \ldots \subsetneq Q_1 \subsetneq P_0 \subsetneq Q_0 = G.$$

The normal subgroups of a Wilson group forming such a rigid chain is noteworthy and the same property is shared by the groups $\mathbb{Z}_p$ and $\mathrm{SL}(n, \mathbb{Z}_p)$. The Nottingham group, in comparison, say, has its normal subgroups almost forming a chain (see Remark 5.4 in Section 5.1).

We found that the determination of the subnormal subgroups of a Wilson group depended directly on the nature of the unspecified permutation representations of the groups G_n. That is, whether the subnormal subgroups of the groups G_n, for $n \in \mathbb{N}$, have all their orbits containing at least two elements.

Here we only present the results of an easier situation, where the subnormal subgroups of the groups G_n are guaranteed to have all their orbits containing at least two elements. This is achieved, for instance, by taking the unspecified permutation representations of the groups G_n to be the actions of the groups on themselves by right multiplication. The complete characterisation of the closed subnormal subgroups of these particular Wilson groups is displayed in Corollary 6.6.

The complete characterisation of the closed subnormal subgroups of a general Wilson group has been achieved and the results can be found in Section 6.4. It is not presented here because it involves additional notation that is difficult to read, which indicates orbits containing at least two elements.

Again, the description of the closed subnormal subgroups of these particular Wilson groups relies on the description of the subnormal subgroups of the finite groups G_n. Theorem 6.4 lays out the complete classification of the subnormal subgroups of the groups G_n having the right regular action in Wilson's construction. Their characterisation involves recalling the normal subgroups P_j^n and Q_j^n, for $j \in \{0, 1, \ldots, n\}$, of G_n, as defined above.

Theorem 6.4. *Let G_n, for $n \geq 0$, be the finite groups as defined above. In the Wilson construction, assume that the unspecified action of the group G_n, for $n \geq 0$, is taken to be right multiplication on itself.*

For $j \in \{0, 1, \ldots, n-1\}$, define

$$S_j^n(I_{d_{j+1}}) = Q_{j+1}^n \rtimes X_{j+1}^{I_{d_{j+1}}} \leq P_j^n, \text{ where } \emptyset \neq I_{d_{j+1}} \subseteq \Omega_{d_{j+1}},$$

and define

$$S_n^n = \{1\}.$$

For $j \in \{1, 2, \ldots, n\}$, define

$$T_j^n(I_{L_j}) = P_j^n \rtimes X_j^{I_{L_j}} \leq Q_j^n, \text{ where } \emptyset \neq I_{L_j} \subseteq L_j,$$

and define

$$T_0^n = G_n.$$

Then the subnormal subgroups of G_n are precisely the groups $S_j^n(I_{d_{j+1}})$, S_n^n, $T_j^n(I_{L_j})$ and T_0^n. In particular, for all I_{d_1}, I_{L_1}, $\ldots$, I_{d_n} and I_{L_n}, they form chains

$$\begin{aligned} S_n^n = P_n^n \subsetneq T_n^n(I_{L_n}) \subseteq Q_n^n \subsetneq S_{n-1}^n(I_{d_n}) \subseteq P_{n-1}^n \subsetneq \ldots \\ \subseteq P_1^n \subsetneq T_1^n(I_{L_1}) \subseteq Q_1^n \subsetneq S_0^n(I_{d_1}) \subseteq P_0^n. \end{aligned}$$

The subnormal length in G_n of the group $S_j^n(I_{d_{j+1}})$ is

$$\begin{cases} 1 & \text{if } I_{d_{j+1}} = \Omega_{d_{j+1}} \text{ (implying that } S_j^n(I_{d_{j+1}}) = P_j^n\text{)}, \\ 2 & \text{if } I_{d_{j+1}} \subsetneq \Omega_{d_{j+1}}. \end{cases}$$

The subnormal length in G_n of the group $T_j^n(I_{L_j})$ is

$$\begin{cases} 1 & \text{if } I_{L_j} = L_j \text{ (implying that } T_j^n(I_{L_j}) = Q_j^n\text{)}, \\ 2 & \text{if } I_{L_j} \subsetneq L_j. \end{cases}$$

Recall the normal subgroups P_j and Q_j, for $j \geq 0$, of a Wilson group, as defined above.

Corollary 6.6. *Let $G = \varprojlim (G_n)_{n \geq 0}$ be the inverse limit of the groups G_n as defined* *above. In the Wilson construction, assume that the unspecified action of the group G_n, for $n \geq 0$, is taken to be right multiplication on itself.*

For $j \geq 0$, define

$$S_j(I_{d_{j+1}}) = \varprojlim (S_j^n(I_{d_{j+1}}))_{n\to\infty}, \text{ where } \emptyset \neq I_{d_{j+1}} \subseteq \Omega_{d_{j+1}},$$

regarded as subgroups of G.

For $j \geq 1$, define

$$T_j(I_{L_j}) = \varprojlim (T_j^n(I_{L_j}))_{n\to\infty}, \text{ where } \emptyset \neq I_{L_j} \subseteq L_j,$$

and define

$$T_0 = \varprojlim (T_0^n)_{n\to\infty},$$

regarded as subgroups of G.

Then the non-trivial closed subnormal subgroups of G are precisely the groups $S_j(I_{d_{j+1}})$, $T_j(I_{L_j})$ and T_0. In particular, for all I_{d_1}, I_{L_1}, $\dots$, I_{d_n}, I_{L_n}, $I_{d_{n+1}}$, $\dots$, they form chains

$$\begin{aligned}\dots \subsetneq S_n(I_{d_{n+1}}) \subseteq P_n \subsetneq T_n(I_{L_n}) \subseteq Q_n \subsetneq S_{n-1}(I_{d_n}) \subseteq P_{n-1} \subsetneq \dots \\ \dots \subseteq P_1 \subsetneq T_1(I_{L_1}) \subseteq Q_1 \subsetneq S_0(I_{d_1}) \subseteq P_0.\end{aligned}$$

The subnormal length in G of the group $S_j(I_{d_{j+1}})$ is

$$\begin{cases} 1 & \text{if } I_{d_{j+1}} = \Omega_{d_{j+1}} \text{ (implying that } S_j(I_{d_{j+1}}) = P_j\text{)}, \\ 2 & \text{if } I_{d_{j+1}} \subsetneq \Omega_{d_{j+1}}. \end{cases}$$

The subnormal length in G of the group $T_j(I_{L_j})$ is

$$\begin{cases} 1 & \text{if } I_{L_j} = L_j \text{ (implying that } T_j(I_{L_j}) = Q_j\text{)}, \\ 2 & \text{if } I_{L_j} \subsetneq L_j. \end{cases}$$

For these restricted Wilson groups, the subnormal subgroups form chains where the subnormal subgroups are squeezed between consecutive normal subgroups. A pictorial description of this conclusion is shown in Figure 6.1 of Section 6.2.

1.1.2 Normal and subnormal subgroup growth of Wilson groups

A type of normal subgroup growth and subnormal subgroup growth has been measured for an arbitrary Wilson group, using a lower bound for the size of the finite groups G_n, as follows:

Theorem 7.1. *Let G_n, for $n \geq 0$, be the finite groups as defined in Section 4.1. Suppose there exists a constant c such that $|X_i| \leq c$, for all $i \geq 0$.*

Then

$$\underbrace{4^{4^{\cdot^{\cdot^{\cdot^{4}}}}}}_{n+2} \leq |G_n| \leq \underbrace{\tilde{c}^{\tilde{c}^{\cdot^{\cdot^{\cdot^{\tilde{c}}}}}}}_{2n+2},$$

where $\tilde{c} = 3c$.

The number of normal subgroups of a Wilson group G of index at most $|G_n|$ is

$$S^{\triangleleft}_{|G_n|}(G) = 2n + 2,$$

for $n \geq 0$. This growth is very slow, that is slower than the functions $\underbrace{\log\log\ldots\log}_{r}|G_n|$ for any fixed r.

The number of subnormal subgroups of a Wilson group G of index at most $|L_nG_{n-1}|$, for $n \geq 1$, that is $S^{\triangleleft\triangleleft}_{|L_nG_{n-1}|}(G)$, is less than or equal to the number

$$2^{|X_n|^{d_n}} + \sum_{j=1}^{n} 2^{d_j} + \sum_{j=2}^{n} 2^{d_j-2}(2^{|X_{j-1}|^{d_{j-1}}} - 2),$$

which is roughly the size of the group G_n, although somewhat smaller.

1.1.3 Maximal subgroups of Wilson groups

We now summarise the little information that we have obtained towards maximal subgroups of Wilson groups. In Theorem 8.11, we have described the maximal subgroups of certain Wilson quotients G_1. That is, the first Wilson quotients

$$G_1 = X_1 \wr_{L_1} (X_1 \wr_{\Omega_{d_1}} G_0)$$

such that the finite non-abelian simple groups $G_0 = X_0$ and X_1 are taken to be the alternating group A_m with degree $m \geq 5$, and the unspecified permutation representation of the group G_0 is chosen to be the natural action of the alternating group.

Theorem 8.11. *Let $G_1 = A_m \wr_{A_m^{(m)}} (A_m \wr_{\Omega^{*[1]}} A_m)$, where $\Omega^{*[1]} = \{1, 2, \ldots, m\}$, for some $m \geq 5$. Denote the base group $A_m^{(|A_m|^m)} =: B$ and the permuting top group*

$A_m^{(m)} A_m =: T$. The group T acts on the set $A_m^{(m)}$ according to the action defined in (4.1) *(see Section 4.1). Therefore $G_1 = B \rtimes T$.*

Define

$$M_0(K) = B \rtimes K, \text{ where } K \text{ is a maximal subgroup of } T.$$

Consider the normaliser

$$N_{G_1}(D_1 \times D_2 \times \ldots \times D_s),$$

with the equivalence classes Ω_i, for $1 \le i \le s$ and $s \ne |A_m|^m$, of a T-congruence on $A_m^{(m)}$ having $|\Omega_i| = l$, and where

$$D_i = \{(x_i, \varphi_{(i-1)l+2}(x_i), \varphi_{(i-1)l+3}(x_i), \ldots, \varphi_{il}(x_i)) : x_i \in A_m\}, \text{ for } 1 \le i \le s,$$

and

$$\varphi_j \in \mathrm{Aut}(A_m), \text{ for } (i-1)l + 2 \le j \le il.$$

Define

$$M_2(L) = L^{(|A_m|^m)} \rtimes T, \text{ where } L \text{ is a maximal subgroup of } A_m.$$

Then the groups $M_0(K)$ and $M_2(L)^g$, where $g \in B$, are maximal subgroups of G_1 and every maximal subgroup of G_1 is one of the groups $M_0(K)$, $N_{G_1}(D_1 \times D_2 \times \ldots \times D_s)$ or $M_2(L)^g$, where $g \in B$.

This initial step was taken with a view to describing the maximal subgroups of Wilson groups where the finite non-abelian simple groups X_i, for $i \ge 0$, are taken to be the alternating group A_m with degree $m \ge 5$, and the unspecified permutation representations of the groups G_n, for $n \ge 0$, are chosen to be the natural actions of the alternating groups. We do have some idea of what these maximal subgroups look like even though this work has been left unfinished.

The techniques used to find these maximal subgroups has lain in M. Bhattacharjee's work on maximal subgroups of iterated wreath products of alternating groups of degree $m \ge 5$, constructed using the natural actions of the alternating groups; see the reference [3]. M. Quick generalised Bhattacharjee's work to iterated wreath products of arbitrary finite non-abelian simple groups; refer to papers [24] and [25].

Bhattacharjee's work required her to obtain upper bounds for the number of conjugacy classes of maximal subgroups of the wreath products that she considers. In studying the wreath product $W_1 = A_m \wr_{\Omega^{*[1]}} A_m$, where $\Omega^{*[1]} = \{1, 2, \ldots, m\}$ and $m \ge 5$, which is a small subcase of Bhattacharjee's wreath products, we contribute a little more information regarding counting the precise number of conjugacy classes of maximal subgroups of W_1 .

In Theorem 8.3, we have classified the maximal subgroups of the wreath product W_1 up to conjugation. They are conjugates of three types of subgroups and it is enough to conjugate by elements of the base group. The proof of this theorem uses a result by C. Parker and M. Quick [23] to exclude the possibility of maximal subgroups of W_1 which complement the base group.

Theorem 8.3. *Let $W_1 = A_m \wr_{\Omega^{*[1]}} A_m$, where $\Omega^{*[1]} = \{1, 2, \ldots, m\}$, for some $m \geq 5$ and $m \neq 6$. Denote the base group $A_m^{(m)} =: B$ and the permuting top group $A_m =: T$. Therefore $W_1 = B \rtimes T$.*

Define

$$M_0(L) = B \rtimes L, \text{ where } L \text{ is a maximal subgroup of } A_m.$$

Define

$$M_1 = \{(x, x, \ldots, x) : x \in A_m\} \times T.$$

Define

$$M_2(L) = L^{(m)} \rtimes T, \text{ where } L \text{ is a maximal subgroup of } A_m.$$

Then the groups $M_0(L)$, M_1^g, where $g \in B$, and $M_2(L)^g$, where $g \in B$, are maximal subgroups of W_1 and every maximal subgroup of W_1 is one of these.

We count exactly one conjugacy class in W_1 of maximal subgroups of the form M_1^g, where $g \in B$ (see Remark 8.7, Section 8.2). We finalise Bhattacharjee's work and prove that the number of conjugacy classes in W_1 of maximal subgroups of the form $M_2(L)^g$, where $g \in B$, is the same as the number of conjugacy classes in A_m of maximal subgroups L of A_m (see Remark 8.8, Section 8.2).

Additionally, due to classifying all the maximal subgroups of W_1 by conjugation, we have been able to count them precisely.

Corollary 8.4. *Let W_1 be the group as defined in Theorem 8.3. Then the number of maximal subgroups of the form:*

- *M_1^g, where $g \in B$, is $|A_m|^{m-1}$;*
- *$M_2(L)^g$, where L is a maximal subgroup of A_m and $g \in B$, is*

$$\sum_{L \leq_{\max} A_m} |A_m : L|^{m-1},$$

where the summation runs over all maximal subgroups of A_m.

1.1.4 Finite generation for Wilson groups

Using M. Quick's work [25], we see that the Wilson groups $\varprojlim(G_n)_{n\geq 0}$ such that $|G_0| > 35!$ are positively finitely generated by two elements. Therefore any Wilson group is finitely generated provided $|G_0| > 35!$.

1.2 Thesis outline

We now set out to the reader how the material of this thesis is organised within the chapters. Notations, definitions and basic group theory results, required for the understanding of this thesis, are contained in Chapter 2.

Chapter 3 considers a motivating example of just infinite profinite groups, which are not hereditarily just infinite, that are not virtually pro-p. We denote these groups, which are infinite iterated wreath products of alternating groups, by W. Their construction is very similar to that of Wilson's construction.

The techniques used to characterise the normal subgroups of the groups W are the same techniques that are used to characterise the normal subgroups of an arbitrary Wilson group. In chapter 3, we completely characterise the normal subgroups of the groups W and in so doing show that these groups are just infinite. We give an explanation as to why the groups W are not hereditarily just infinite and are not virtually pro-p.

In chapter 4 we give a detailed description of Wilson's construction, as described by J. S. Wilson in his paper [32]. Chapter 4 then explains how an arbitrary Wilson group arises from such a construction. The proofs of Theorem 1.1 and Corollary 1.2, found in [32], are briefly discussed. In particular, it is reasoned why the Wilson groups are not virtually pro-p.

We look at the structure of Wilson groups by first finding their normal subgroups. Chapter 5 contains a complete characterisation of the closed normal subgroups of any arbitrary Wilson group.

Since a normal subgroup of a group is also a subnormal subgroup of that group, to continue investigating the structure of Wilson groups, it is natural to consider subnormal subgroups. Every open subgroup of a pro-p group is subnormal. Therefore it is also appropriate to study subnormal subgroups of Wilson groups because we are not in the pro-p setting, where studying all the open subgroups provides all the subnormal subgroups.

Work concerning the subnormal subgroups of Wilson groups is contained in Chapter 6. This is the prime chapter of the thesis. The chapter is formed in three parts, since describing subnormal subgroups of Wilson groups was found to be rather complicated.

The results of the section after the introduction, Section 6.2, only applies to particular Wilson groups. That is, Wilson groups where the unspecified permutation representations of the finite groups G_n, in Wilson's construction, are taken to be the action of the groups on themselves by right multiplication. This guarantees that subnormal subgroups of the groups G_n have all their orbits containing at least two elements. The closed subnormal subgroups of these particular Wilson groups are completely characterised in Section 6.2. In fact, this characterisation holds for all Wilson groups such that the actions of the subnormal subgroups of the groups G_n, in their construction, have all their orbits containing at least two elements.

In Section 6.3, to give an indication of the path to take for finding subnormal subgroups of a general Wilson group, we find the subnormal subgroups of the just infinite profinite groups W first described in Section 3.2. We do this because the actions of subnormal subgroups of the finite groups W_n, involved in the construction of groups W, can have orbits of one element. Section 6.3 completely classifies the subnormal subgroups of the groups W. In particular, a recursive formula is given to calculate subnormal length.

The main results of this thesis are contained in Section 6.4. Here the closed subnormal subgroups of any arbitrary Wilson group have been completely classified. For an arbitrary Wilson group, the actions of subnormal subgroups of the finite groups G_n involved in the construction may have orbits of one element. The characterisation has been achieved by using Corollary 6.9, which has been developed previously in Section 6.3.

The normal subgroup growth and the subnormal subgroup growth of a Wilson group have been worked on, in Section 7.1 of Chapter 7. Since the normal subgroups and the subnormal subgroups of the finite Wilson quotients G_n have been completely classified, it was natural to count the number of normal subgroups and subnormal subgroups of a Wilson group up to index at most $|G_n|$, for $n \geq 0$. We give upper and lower bounds for the size of G_n in order to make statements about the rate of types of growth.

In Section 7.2, the number of subnormal subgroups of the infinite iterated wreath products W constructed from the alternating group A_m, have been counted by using a correspondence to the number of subtrees of the infinite m-regular rooted tree.

Another type of subgroup of a group is a maximal subgroup. We would have liked to have investigated the structure of Wilson's groups further by finding their maximal subgroups. Chapter 8 looks at maximal subgroups of Wilson groups. Again, to gain ideas of how to proceed, we resort to examining the maximal subgroups of the easier example of the infinite iterated wreath products of alternating groups W first described in Section 3.2. In particular, Section 8.2 examines the maximal subgroups of the finite group W_1 used to construct W and Section 8.3 goes on to examine the maximal subgroups of the finite group W_2 used to construct W.

In Section 8.4, information from Section 8.2 and Section 8.3 is used to describe the maximal subgroups of the first Wilson quotients $G_1 = X_1 \wr_{L_1} (X_1 \wr_{\Omega_{d_1}} G_0)$ such that $G_0 = X_0$ and X_1 are taken to be the alternating groups, and the unspecified permutation representation of the group G_0 is chosen to be the natural action of the alternating group.

Chapter 9 concerns positive finite generation, and therefore finite generation of Wilson groups. As an analogy, the finite generation of the infinite iterated wreath products W is considered.

Open problems which have evolved from the work produced in this thesis are listed in Chapter 10. They are referred to within the body of the thesis when they come to light.

Chapter 2

Preliminaries

The purpose of this chapter is to set out the notations, definitions and basic group theory results required for the understanding of the thesis.

2.1 Wreath products

Both the just infinite groups in Chapter 3 and the Wilson groups are constructed from permutational wreath products, therefore it is beneficial to recall the definition.

Definition 2.1. Let U be a finite permutation group acting on a finite set Ω. Let X be a finite group. Define

$$V = \prod_{\omega \in \Omega} X_\omega,$$

where $X_\omega \cong X$ for all $\omega \in \Omega$.

The *wreath product* of X by U, denoted $X \wr_\Omega U$, is the semidirect product $V \rtimes U$. The group U acts on V by

$${(x_\omega)_{\omega \in \Omega}}^u = (x_{\omega \cdot u^{-1}})_{\omega \in \Omega},$$

where $u \in U$ and $(x_\omega)_{\omega \in \Omega} \in V$. The normal subgroup V is called the *base group* of the wreath product. The group U is sometimes referred to as the *top group* of the wreath product.

Let X and Y be permutation groups acting on the sets Ω_1 and Ω_2 respectively. The wreath product constructed from the permutation groups X and Y is again a permutation group and it acts on the set $\Omega_1 \times \Omega_2$. When we wish to view the wreath product as such, it is called the *permutational wreath product*.

2.1.1 Minimal normal subgroups of some wreath products

In this subsection, we consider wreath products only where the base group is a product of finite non-abelian simple groups. The fact (3.1) in [32] describes minimal normal subgroups of these wreath products when the action is transitive. Lemma 2.3, below, is a generalisation of this fact (3.1), since it does not assume that the action is transitive. It says that the minimal normal subgroups of such wreath products $X \wr_\Omega U$ are contained in V and each corresponds to a U-orbit. This lemma is applied in Proposition 6.8 for the classification of subnormal subgroups of Wilson groups and subnormal subgroups of the just infinite iterated wreath products W considered in Chapter 3.

First we need a preliminary lemma to help in the proof of Lemma 2.3. (Lemma 2.2 is also used later in the proof of Proposition 6.2.)

Lemma 2.2. *Let U be a finite permutation group acting on a finite set Ω with orbits $\Omega_1, \Omega_2, \dots, \Omega_r$. Let X be a finite non-abelian simple group. Define the permutational wreath product $G = X \wr_\Omega U$. Denote the base group of the wreath product as $V = \prod_{\omega\in\Omega} X_\omega$, where $X_\omega \cong X$ for all $\omega \in \Omega$.*

Define

$$N_i = \{(x_\omega)_{\omega\in\Omega} \in V : x_\omega = 1 \text{ if } \omega \notin \Omega_i\}\,,$$

for each $i = 1, 2, \dots, r$. Suppose that N is a normal subgroup of G.

To show that N contains N_i, for each $i = 1, 2, \dots, r$, it is sufficient to prove that N contains the coordinate subgroup

$$V_{\omega_1} = \{(y_\omega)_{\omega\in\Omega} \in V : y_\omega = 1 \text{ if } \omega \neq \omega_1\}$$

for at least one $\omega_1 \in \Omega_i$.

Proof. Since $N_i = \prod_{\omega_1\in\Omega_i} V_{\omega_1}$, it is enough to show that N contains the coordinate subgroups V_{ω_1}, for every $\omega_1 \in \Omega_i$. In fact, it is enough to show that N contains V_{ω_1} for at least one $\omega_1 \in \Omega_i$. This is because U acts transitively on the orbit Ω_i and, for any $u \in U$, we have $V_{\omega_1}^u = V_{\omega_1 \cdot u^{-1}}$. □

Lemma 2.3. *Let group $G = X \wr_\Omega U$ be the permutational wreath product as defined in Lemma 2.2.*

Then the minimal normal subgroups of G are contained in V and are precisely the groups

$$N_1 = \{(x_\omega)_{\omega\in\Omega} \in V : x_\omega = 1 \text{ if } \omega \notin \Omega_1\}\,,$$

$$N_2 = \{(x_\omega)_{\omega\in\Omega} \in V : x_\omega = 1 \text{ if } \omega \notin \Omega_2\}\,,$$

$$\vdots$$

$$N_r = \{(x_\omega)_{\omega\in\Omega} \in V : x_\omega = 1 \text{ if } \omega \notin \Omega_r\}\,.$$

Proof. We want to show that the minimal normal subgroups of G are precisely the groups $N_1, N_2, \ldots, N_r$. Obviously, this will imply that the minimal normal subgroups of G are contained in V.

Let $i \in \{1, 2, \ldots, r\}$. Since $G = VU$, to check that N_i is normal it is sufficient to show that $[N_i, V] \subseteq N_i$ and $[N_i, U] \subseteq N_i$.

To show $[N_i, V] \subseteq N_i$, let $\underline{x} = (x_\omega)_{\omega\in\Omega} \in N_i$ and $\underline{y} = (y_\omega)_{\omega\in\Omega} \in V$. Setting $\Delta := \Omega\backslash\Omega_i$ we have $\underline{x} = ((x_\omega)_{\omega\in\Omega_i}, (1)_{\omega\in\Delta})$ and $\underline{y} = ((y_\omega)_{\omega\in\Omega_i}, (y_\omega)_{\omega\in\Delta})$. So $[\underline{x}, \underline{y}] = ([x_\omega, y_\omega])_{\omega\in\Omega}$ can be written as

$$\big(([x_\omega, y_\omega])_{\omega\in\Omega_i}, ([1, y_\omega])_{\omega\in\Delta}\big) = \big(([x_\omega, y_\omega])_{\omega\in\Omega_i}, (1)_{\omega\in\Delta}\big) \in N_i.$$

To show $[N_i, U] \subseteq N_i$, let $\underline{x} = (x_\omega)_{\omega\in\Omega} \in N_i$ and $u \in U$. Setting $\Delta := \Omega\backslash\Omega_i$ we have $\underline{x} = ((x_\omega)_{\omega\in\Omega_i}, (1)_{\omega\in\Delta})$. Since Ω_i is a U-orbit, this allows us to write $[\underline{x}, u] = \underline{x}^{-1} \cdot \underline{x}^u$ as

$$\big((x_\omega^{-1})_{\omega\in\Omega_i}, (1^{-1})_{\omega\in\Delta}\big) \cdot \big((x_{\omega\cdot u^{-1}})_{\omega\in\Omega_i}, (1)_{\omega\in\Delta}\big) = \big((x_\omega^{-1} x_{\omega\cdot u^{-1}})_{\omega\in\Omega_i}, (1)_{\omega\in\Delta}\big) \in N_i.$$

Next we show that N_i is minimal normal in G. For this, we need that the normal closure in G of any non-trivial element $\underline{x} = (x_\omega)_{\omega\in\Omega} \in N_i$ is equal to N_i.

($*$) Choose $\omega_1 \in \Omega_i$ such that $x_{\omega_1} \neq 1$. Since X is non-abelian simple it has trivial centre and we can find $y \in X$ such that $[x_{\omega_1}, y] \neq 1$. Consider $\underline{y} = (y_\omega)_{\omega\in\Omega} \in V$ with $y_\omega = y$ if $\omega = \omega_1$ and $y_\omega = 1$ otherwise. Then $[\underline{x}, \underline{y}] \in \langle \underline{x} \rangle^G$ can be written as $(([x_\omega, y])_{\omega\in\{\omega_1\}}, (1)_{\omega\in\Omega\backslash\{\omega_1\}}) \neq 1$. As X is simple, the normal closure of $[\underline{x}, \underline{y}]$ in V is equal to V_{ω_1}. Therefore $V_{\omega_1} \subseteq \langle \underline{x} \rangle^G$ and Lemma 2.2 proves the claim.

It remains to prove that every minimal normal subgroup of G is one of $N_1, N_2, \ldots, N_r$. Let N be a minimal normal subgroup of G.

Suppose $N \subseteq V$. We can find $1 \neq \underline{x} = (x_\omega)_{\omega\in\Omega} \in N$. Replacing $\langle \underline{x} \rangle^G$ by N in argument ($*$) implies $N_i \subseteq N$, for one $i = 1, 2, \ldots, r$.

Now suppose $N \not\subseteq V$. Then we can find $u\underline{x} = u(x_\omega)_{\omega\in\Omega} \in N$ with $u \in U\backslash\{1\}$ and $\underline{x} \in V$. Since $u \neq 1$, we can obtain $\omega_1 \in \Omega$ such that $\omega_2 := \omega_1 \cdot u \neq \omega_1$. Choose $y \in X\backslash\{1\}$. Consider $\underline{y} = (y_\omega) \in V$ with $y_\omega = y$ if $\omega = \omega_1$ and $y_\omega = 1$ otherwise. Then

$[\underline{y}, u\underline{x}] = \underline{y}^{-1}(\underline{x}^{-1}\underline{y}^u\underline{x}) \in N$ can be written as

$$\big((y^{-1})_{\omega=\omega_1}, (1)_{\omega=\omega_2}, (1)_{\omega\in\Omega\setminus\{\omega_1,\omega_2\}}\big)\cdot \big((x_{\omega_1}^{-1}.1.x_{\omega_1})_{\omega=\omega_1}, (x_{\omega_2}^{-1}.y.x_{\omega_2})_{\omega=\omega_2}, (x_{\omega}^{-1}.1.x_{\omega})_{\omega\in\Omega\setminus\{\omega_1,\omega_2\}}\big)$$

$$= \big((y^{-1})_{\omega=\omega_1}, (y^{x_{\omega_2}})_{\omega=\omega_2}, (1)_{\omega\in\Omega\setminus\{\omega_1,\omega_2\}}\big). \tag{2.1}$$

Since $y \neq 1$, we have $((y^{-1})_{\omega=\omega_1}, (y^{x_{\omega_2}})_{\omega=\omega_2}, (1)_{\omega\in\Omega\setminus\{\omega_1,\omega_2\}}) \neq 1$. As X is simple, the normal closure of $[\underline{y}, u\underline{x}]$ in V contains V_{ω_1}. Therefore $V_{\omega_1} \subseteq N$ and Lemma 2.2 proves the claim. □

2.2 Subnormal subgroups

A subgroup T of a group G is *subnormal* in G if there exists subgroups

$$G = T_0 \geq T_1 \geq \ldots \geq T_k = T \text{ such that } T_i \trianglelefteq T_{i-1},$$

for each $i = 1, 2, \ldots, k$. When k is the smallest possible number with this feature one says that T is *subnormal of length k in G*.

2.2.1 Normal subgroups of some direct products

To describe subnormal subgroups in Chapter 6, we will need to know the normal subgroups of a direct product of finite non-abelian simple groups. Therefore we will frequently make use of the following fact.

Lemma 2.4. *A normal subgroup of a direct product of non-abelian simple groups is a direct product of some of its factors.*

2.3 Maximal subgroups

A proper subgroup M of a group G is *maximal* in G if there exists no proper subgroup L of G strictly containing M.

2.3.1 The alternating groups

In Chapter 8, the classification of the maximal subgroups of wreath products of alternating groups involves the maximal subgroups of alternating groups. The O'Nan-Scott Theorem can be used to classify all the maximal subgroups of A_m, the alternating group of degree m; see [33, Sec. 2.6] for a very readable version. It appeared as a classification of the maximal subgroups of the symmetric group at a conference in Santa Cruz on finite groups [28]. A maximal subgroup L of A_m is of one of the following six types.

(a) $(S_l \times S_k) \cap A_m$, with $m = l + k$ and $l \neq k$ (intransitive type);

(b) $(S_l \wr S_k) \cap A_m$, with $m = lk$, $l > 1$ and $k > 1$ (imprimitive type);

(c) $\mathrm{AGL}_k(p) \cap A_m$, with $m = p^k$ and p a prime (affine type);

(d) $(H^k.(\mathrm{Out}(H) \times S_k)) \cap A_m$, with H a non-abelian simple group, $k \geq 2$ and $m = |H|^{k-1}$ (diagonal type);

(e) $(S_l \wr S_k) \cap A_m$, with $m = l^k$, $l \geq 5$ and $k > 1$, excluding the case where L is imprimitive on $\Omega^{*[1]} = \{1, 2, \ldots, m\}$ (product action type);

(f) $H \lhd L \leq \mathrm{Aut}(H)$, with H a non-abelian simple group, $H \neq A_m$ and L acting primitively on $\Omega^{*[1]} = \{1, 2, \ldots, m\}$ (almost simple type).

Here, S_l denotes the symmetric group of degree l; $\mathrm{AGL}_k(p)$ denotes the affine general linear group over the field of order p; $\mathrm{Out}(H)$ denotes the outer automorphism group of H; and $\mathrm{Aut}(H)$ denotes the automorphism group of H.

However, not all the subgroups of type (a) to (f) may be maximal in A_m. The paper [16] by M. W. Liebeck, C. E. Praeger and J. Saxl says that such groups L are in general maximal and gives an explicit list of exceptions.

2.4 The normaliser of a direct product in a wreath product

The following result occurs in the proof of Theorem 8.3, in Section 8.2, where the maximal subgroups of the groups $W_1 = A_m \wr_{\Omega^{*[1]}} A_m$, where $\Omega^{*[1]} = \{1, 2, \ldots, m\}$ and $m \geq 5$, are characterised up to conjugation. The result shows that the normaliser in the wreath product $X \wr_\Omega U$ of the direct product H^Ω, where H is a subgroup of X, can be computed from the normaliser of H in X.

Lemma 2.5. *Let U be a finite permutation group acting on a finite set $\Omega = \{1, 2, \ldots, n\}$. Let X be a finite group. Define the permutational wreath product $G = X \wr_\Omega U$.*

Suppose H is a subgroup of X. Then

$$N_G(H^\Omega) = (N_X(H))^\Omega U = (N_X(H))^\Omega \rtimes U.$$

Proof. Now $(g_1, g_2, \ldots, g_n)s \in N_G(H^\Omega)$

if and only if $(H^\Omega)^{(g_1,g_2,\ldots,g_n)s} = (H^{g_1} \times H^{g_2} \times \ldots \times H^{g_n})^s = H^\Omega$,

if and only if $H^{g_i} = H$, for all $i \in \Omega$,

if and only if $g_i \in N_X(H)$, for all $i \in \Omega$,

if and only if $(g_1, g_2, \ldots, g_n) \in (N_X(H))^\Omega$.

Then $N_G(H^\Omega) = (N_X(H))^\Omega U$. Further, $(N_X(H))^\Omega U = (N_X(H))^\Omega \rtimes U$ as $(N_X(H))^\Omega$ is the intersection of $N_G(H^\Omega)$ with the base group X^Ω. □

2.5 Profinite groups

In this section, we define the concept of profinite groups and give some basic properties. There are many characterisations of a profinite group; see [15, Ch. I] for a readable overview of profinite theory that is more specific to profinite groups. However, the prevalent one of this thesis is that of a profinite group being an inverse limit of finite groups.

A *directed set* is a partially ordered set I with respect to $\preceq$ with the property that for all $i, j \in I$ there exists $k \in I$ such that $i \preceq k$ and $j \preceq k$. For our work any directed set is taken to be the set $\mathbb{N} \cup \{0\}$ with respect to the ordinary order-relation $\leq$.

Definition 2.6. An *inverse system* (G_i, φ_{ij}) of topological groups indexed by a directed set I consists of a collection G_i, for $i \in I$, of topological groups and a collection of continuous group homomorphisms $\varphi_{ij} : G_j \longrightarrow G_i$ defined whenever $i \preceq j$, for $i, j \in I$, satisfying

$$\varphi_{ii} = \mathrm{id}_{G_i} \text{ and } \varphi_{ij}\varphi_{jk} = \varphi_{ik}$$

whenever $i \preceq j \preceq k$, for $i, j, k \in I$.

Definition 2.7. An *inverse limit* of an inverse system (G_i, φ_{ij}) of topological groups is a topological group G with a collection of continuous group homomorphisms $\varphi_i : G \longrightarrow G_i$, for all $i \in I$, such that

$$\varphi_{ij}\varphi_j = \varphi_i$$

whenever $i \preceq j$, for $i, j \in I$.

In addition, the inverse limit has the following universal property: whenever H is a topological group and $\psi_i : H \longrightarrow G_i$, for all $i \in I$, is a collection of continuous group homomorphisms satisfying $\varphi_{ij}\psi_j = \psi_i$ whenever $i \preceq j$, for $i, j \in I$, then there is a unique continuous group homomorphism $\psi : H \longrightarrow G$ such that $\varphi_i\psi = \psi_i$ for each i.

The inverse limit is denoted by $\varprojlim(G_i)_{i\in I}$.

The maps $\varphi_i : G \longrightarrow G_i$ of the inverse limit are not necessarily surjective, however, without loss of generality, the maps φ_i can be defined as being surjective. Therefore the inverse limit $\varprojlim(G_i)_{i\in I}$ is the group

$$\left\{(g_i)_{i\in I} \in \prod_{i\in I} G_i : \varphi_{ij}(g_j) = g_i \text{ whenever } i \preceq j\right\},$$

which is a subgroup of the direct product $\prod_{i\in I} G_i$. We will see that for the profinite groups considered in our work the maps φ_i are always surjective.

Definition 2.8. A *profinite group* is the inverse limit of an inverse system of finite groups.

Finite groups G_i, for $i \in I$, are regarded as topological groups with the discrete topology. Then the direct product $\prod_{i\in I} G_i$ is a topological group when given the product topology. In this way, the inverse limit $\varprojlim(G_i)_{i\in I}$, with the induced topology, becomes a topological group. Hence profinite groups are topological groups.

Another characterisation is that a profinite group is a compact Hausdorff topological group such that every open neighbourhood of the identity element contains an open subgroup. Therefore the open subsets of a profinite group G are precisely those sets which can be written as unions of cosets gN of open normal subgroups $N \trianglelefteq_o G$.

Let G be any group. Define

$$I = \{N \trianglelefteq G : N \text{ has finite index in } G\}$$

with respect to reverse inclusion. That is, $N \preceq M$ if and only if $M \subseteq N$. Now I is a directed set because the intersection of two normal subgroups of finite index is a normal subgroup of finite index. Define the natural projections

$$\varphi_{NM} : \frac{G}{M} \longrightarrow \frac{G}{N}$$

whenever $N \preceq M$. The finite quotients G/N and maps φ_{NM} form a natural inverse system. The inverse limit $\widehat{G} := \varprojlim(G/N)$ of this inverse system is a profinite group. The group $\widehat{G}$ is called the *profinite completion* of G.

Let p be a fixed prime. The normal subgroups of $\mathbb{Z}$ whose index is a power of p are of the form $p^i\mathbb{Z}$, for $i \in \mathbb{N}$. The finite quotient groups $\mathbb{Z}/p^i\mathbb{Z}$ and natural projections

$$\varphi_{ij} : \frac{\mathbb{Z}}{p^j\mathbb{Z}} \longrightarrow \frac{\mathbb{Z}}{p^i\mathbb{Z}}$$

whenever $p^j\mathbb{Z} \subseteq p^i\mathbb{Z}$ form an inverse system. The inverse limit $\mathbb{Z}_p := \varprojlim \mathbb{Z}/p^i\mathbb{Z}$ of this inverse system is called the group of *p-adic integers*. Each element of $\mathbb{Z}_p$ has a unique p-adic expansion

$$a_0 + a_1 p + a_2 p^2 + \ldots = (\ldots a_2 a_1 a_0)_p,$$

where $a_i \in \{0, 1, \ldots, p-1\}$ are called *p-adic digits*.

Definition 2.9. Let p be a fixed prime. A *pro-p group* is a topological group that is isomorphic to the inverse limit of finite p-groups.

Lemma 2.10. *Let G be a profinite group.*

Then every open subgroup of G is closed.

The contrapositive of the following result is used in Section 3.3 to prove that the infinite iterated wreath products W, constructed from alternating groups, are not hereditarily just infinite. The result is also used in showing that the groups W (in Section 3.4) are just infinite.

Lemma 2.11. *Let G be a profinite group. Suppose H is a closed subgroup of G.*

Then H is an open subgroup of G if and only if H has finite index in G.

The following result, found in [30, Lem. 0.3.1 (h)], is used to describe normal subgroups and subnormal subgroups of the profinite groups present in the thesis.

Lemma 2.12. *Let G be a compact topological group. Suppose X_i, for $i \in I$, is a collection of closed subsets of G with the property that for all $i, j \in I$ there exists $k \in I$ such that $X_k \subseteq X_i \cap X_j$.*

If Y is a closed subset of G then

$$\left(\bigcap_{i\in I} X_i\right) Y = \bigcap_{i\in I} X_i Y.$$

Let X be a subset of a profinite group G. We say that X *generates* G (topologically) if the subgroup generated by X is dense in G. The profinite group G is *finitely generated* (topologically) if it contains a finite subset X that generates G (topologically). We usually refer to topological generating sets as generating sets because we mostly consider profinite groups as topological groups.

A profinite group has a property *virtually* if it has an open normal subgroup with that property.

A profinite group G is *just infinite* if it is infinite and every non-trivial closed normal subgroup of G is open. It is *hereditarily just infinite* if every open subgroup of G is just infinite.

2.6 Subgroup growth

Studying subgroup growth of a group G involves considering the growth rate of the function

$$n \to s_n(G),$$

where $s_n(G)$ denotes the *number of subgroups of index at most n in G*. Subgroup growth gives a rough classification of groups into growth types.

A group G has *polynomial subgroup growth of degree* c if there exists a constant c such that

$$s_n(G) \leq n^c \text{ for all } n.$$

In particular, we say that the growth type is linear if the constant $c = 1$. In this thesis, we are concerned with other subgroup counting functions, which are:

$s_n^{\triangleleft}(G)$ denotes the *number of normal subgroups of index at most n in G*;

$s_n^{\triangleleft\triangleleft}(G)$ denotes the *number of subnormal subgroups of index at most n in G*;

$m_n(G)$ denotes the *number of maximal subgroups of index n in G*.

The language of growth types is extended to these functions in a natural way.

2.7 Positive finite generation

A profinite group G has a natural compact topology, induced by the discrete topology on the finite groups in the inverse system. Therefore it has a finite Haar measure μ, which is determined uniquely by the algebraic structure of G. We normalise this measure so that $\mu(G) = 1$ and we can consider G as a probability space. Thus we can define

$$P(G,k) = \mu \left\{ (g_1, g_2, \ldots, g_k) \in G^{(k)} : g_1, g_2, \ldots, g_k \text{ topologically generate } G \right\},$$

for any positive integer k, where μ also denotes the product measure on $G^{(k)}$. The Haar measure on profinite groups is discussed in [8, Ch. 18].

A profinite group G is *positively finitely generated* (PFG) if, for some k, the probability $P(G, k)$ that k randomly chosen elements of G topologically generate G is positive. This term was formally introduced with A. Mann's paper [18] and PFG groups were surveyed in [17, Ch. 11].

Chapter 3

Infinite iterated wreath products $\ldots \wr A_m \wr A_m \wr \ldots \wr A_m$, where $m \geq 5$

3.1 Introduction

Until recently all known hereditarily just infinite profinite groups were virtually pro-p groups. However, if one looks at just infinite profinite groups then it is not difficult to construct some that are not virtually pro-p groups. This chapter briefly describes some such just infinite profinite groups, which have been studied, in particular, by M. Bhattacharjee [3]. This is useful because their construction has similarities with that of Wilson's construction. The same techniques used to show that these groups are just infinite will be used to show that the groups constructed by Wilson are just infinite.

The just infinite profinite groups in this chapter are constructed from inverse limits of iterated wreath products of alternating groups. The properties described below remain true whether the alternating groups involved in the construction are allowed to vary or not. However, for ease of reading, the alternating groups are taken to be the same. In fact, the properties described still hold if the alternating groups are generalised to any arbitrary finite non-abelian simple group. The actions of these non-abelian simple groups would be required to be faithful and transitive.

3.2 The construction

We now construct the just infinite profinite groups. Fix the alphabet $A = \{1, 2, \ldots, m\}$, where $m \geq 5$. We define the sets

$$\Omega^{*[j]} = \{i_1 i_2 \ldots i_j : i_1, i_2, \ldots, i_j \in A\},$$

for each $j = 1, 2, \ldots$. Here $i_1 i_2 \ldots i_j$ denotes a sequence of numbers and not a product of numbers. The symbol $*$ used for concatenation is written in order to remind the reader of this.

Set $W_0 = A_m$, the alternating group of degree m. The group A_m acts naturally on the set $\Omega^{*[1]} = \{1, 2, \ldots, m\}$. We form the permutational wreath product $A_m \wr_{\Omega^{*[1]}} A_m$, which we denote by W_1. This group is described as the semidirect product $W_1 = A_m^{(m)} \rtimes W_0$, where W_0 acts on $A_m^{(m)}$ by permuting the factors.

We observe that W_1 acts naturally on the finite m-regular rooted tree of length 2. This has been depicted in Figure 3.1 for $m = 5$. In this action the root vertex $\emptyset$ is fixed and the group W_1 acts by coordinate permutations on the bottom layer of 25 vertices.

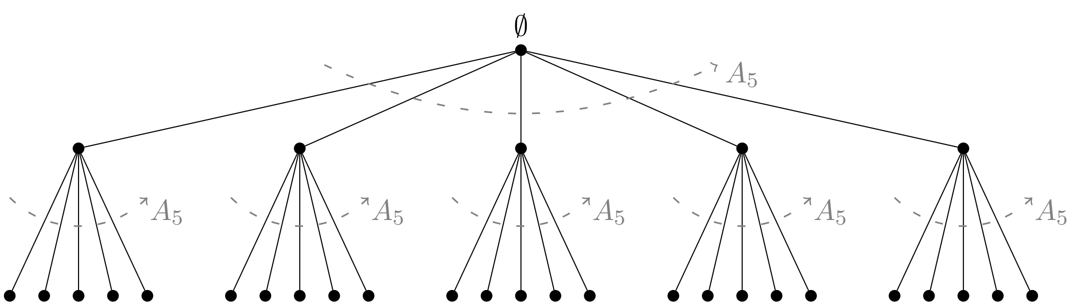

Figure 3.1: The wreath product $A_5 \wr A_5 = (A_5 \times A_5 \times A_5 \times A_5 \times A_5) \rtimes A_5$ acting naturally on the 5-regular rooted tree of length 2.

Now the group W_1 acts naturally on m^2 elements. We can then form the permutational wreath product $A_m \wr_{\Omega^{*[2]}} A_m \wr_{\Omega^{*[1]}} A_m$, which we denote by W_2. This is the semidirect product $W_2 = A_m^{(m^2)} \rtimes W_1$. The process can be continued to form the nth iterated wreath product

$$W_n = A_m \wr_{\Omega^{*[n]}} \ldots \wr_{\Omega^{*[2]}} A_m \wr_{\Omega^{*[1]}} A_m.$$

This is the same as the semidirect product $W_n = A_m^{(m^n)} \rtimes W_{n-1}$, for $n \geq 1$.

We construct a group W as the inverse limit of a sequence of finite groups $(W_n)_{n \geq 0}$

and the natural projections

$$\theta_n : W_n = A_m^{(m^n)} \rtimes W_{n-1} \longrightarrow W_{n-1},$$

for $n \geq 1$. The limit $W = \varprojlim (W_n)_{n \geq 0}$ has the natural projections $\phi_n : W \longrightarrow W_n$, for $n \geq 0$.

We give a pictorial description of the inverse limit W in Figure 3.2, below. The limit is indexed by the set $\mathbb{N} \cup \{0\}$ with respect to the ordinary order-relation $\leq$.

$$\begin{tikzcd}
W = \varprojlim W_n \arrow[dr, dashed, "\phi_n"] \arrow[drrr, dashed, "\phi_2"] \arrow[drrrr, dashed, "\phi_1"] \arrow[drrrrr, dashed, "\phi_0"] & & & & & \\
\cdots \arrow[r, "\theta_{n+1}"] & W_n = A_m^{(m^n)} \rtimes W_{n-1} \arrow[r, "\theta_n"] & \cdots \arrow[r, "\theta_3"] & W_2 = A_m^{(m^2)} \rtimes W_1 \arrow[r, "\theta_2"] & W_1 = A_m^{(m)} \rtimes W_0 \arrow[r, "\theta_1"] & W_0
\end{tikzcd}$$

Figure 3.2: A pictorial description of the inverse limit W.

3.3 Verifying not hereditarily just infinite and not virtually pro-p

We now verify that all such groups W, as defined above, are not hereditarily just infinite. Fix $m \geq 5$. Define

$$U = \ker(\phi_0 : W \longrightarrow W_0) \cong W^{(m)},$$

where $W^{(m)}$ denotes the direct product of m copies of W. Now U is an open subgroup of W because ϕ_0 is a continuous map. However, $N \cong W^{(m-1)}$ is a closed normal subgroup of U such that the index $U/N \cong W$ is infinite. The contrapositive of Lemma 2.11 implies that N cannot be open in U.

The fact that the groups W are not virtually pro-p, for some prime p, is because the groups W_n are constructed from wreath products of non-abelian simple group A_m.

3.4 Verifying just infinite

The rest of this chapter is concerned with determining the normal subgroups of the profinite groups W, as defined above. This is completed in Corollary 3.3. In particular, this shows that the profinite groups W are just infinite.

In Corollary 3.3, the non-trivial closed normal subgroups of a group W are denoted by V_j, for $j \geq 0$. Due to the definition of these subgroups V_j, their indices in W can easily be calculated. We have the indices $|W : V_j| = |W_{j-1}|$, for $j \geq 1$, and the index $|W : V_0| = 1$. All the indices are finite. The profinite group W is just infinite using Lemma 2.11.

Our work has been restricted in Corollary 3.3 to closed normal subgroups because we rely on Lemma 2.12, which only applies to normal subgroups that are closed. However, a result by N. Nikolov and D. Segal [22, Cor. 1.15] shows that all normal subgroups of a group W, since it is finitely generated (see Chapter 9), are automatically closed. Therefore the characterisation of normal subgroups, in Corollary 3.3, covers all the normal subgroups of the groups W.

3.4.1 The normal subgroups

Initially, we proceed in Theorem 3.2 by determining all the normal subgroups of the finite groups W_n. The construction of $W_n = A_m^{(m^n)} \rtimes W_{n-1}$ gives an indication of the outcome.

The proof of Theorem 3.2 uses the following lemma.

Lemma 3.1. *Let the finite groups W_n, for $n \geq 0$, be as defined above.*

The unique minimal normal subgroup of W_n is the group $A_m^{(m^n)}$.

This lemma comes directly from a standard fact about permutational wreath products, see [32, (3.1)] or Lemma 2.3. That is because W_{n-1} acts transitively on m^n elements and also the kernel of the action of W_n on $A_m^{(m^n)}$ is $A_m^{(m^n)}$.

Theorem 3.2. *Let W_n, for $n \geq 0$, be the finite groups as defined in Section 3.2. For $j \in \{1, 2, \ldots, n+1\}$, define*

$$V_j^n = \ker(W_n \longrightarrow W_{j-1}) = A_m^{(m^n)} \rtimes \ldots \rtimes (A_m^{(m^{j+1})} \rtimes A_m^{(m^j)}) \leq W_n,$$

and define

$$V_0^n = W_n.$$

Then the normal subgroups of W_n are precisely the groups V_j^n and V_0^n. In particular, *they form a complete chain*

$$\{1\} = V^n_{n+1} \subsetneq V^n_n \subsetneq \ldots \subsetneq V^n_1 \subsetneq V^n_0 = W_n.$$

Proof. We first prove that V^n_j are normal subgroups of W_n. The homomorphisms $W_n \longrightarrow W_{j-1}$ have kernels V^n_j, for $j \in \{1, 2, \ldots, n+1\}$.

We now prove, by induction on n, that V^n_j are the only normal subgroups of W_n. Suppose $N \trianglelefteq W_n$. For $n = 0$, all the normal subgroups of W_0 are $V^0_1 = \{1\}$ and $V^0_0 = W_0$ holds as W_0 is simple.

Now suppose $n \geq 1$. If $N = \{1\}$ then $N = V^n_{n+1}$. Assume $N \neq \{1\}$. We have $A_m^{(m^n)} \subseteq N$, since the group $A_m^{(m^n)}$ is the unique minimal normal subgroup of W_n, by Lemma 3.1. Then there are two possibilities: $A_m^{(m^n)} = N$ and $A_m^{(m^n)} \subsetneq N$.

For $A_m^{(m^n)} = N$ we are done, as $N = V^n_n$. We now look at the other possibility $A_m^{(m^n)} = V^n_n \subsetneq N$. The group V^n_n is the kernel of the homomorphism $\theta_n : W_n \longrightarrow W_{n-1}$. Then there is a one-to-one correspondence between the set of normal subgroups of W_n containing V^n_n and normal subgroups of W_{n-1}. By induction, we know that N is one of the groups V^n_j. $\square$

Corollary 3.3. *Let $W = \varprojlim (W_n)_{n \geq 0}$ be the inverse limit of the groups W_n as defined in Section 3.2. For $j \geq 0$, define*

$$V_j = \varprojlim (V^n_j)_{n \to \infty},$$

regarded as subgroups of W.

Then the non-trivial closed normal subgroups of W are precisely the groups V_j. In particular, they form a complete chain

$$\ldots \subsetneq V_{n+2} \subsetneq V_{n+1} \subsetneq V_n \subsetneq \ldots \subsetneq V_1 \subsetneq V_0 = W.$$

Proof. Theorem 3.2 showed that V^n_j, for $j \in \{0, 1, \ldots, n+1\}$, are all the normal subgroups of W_n and that they form the chain

$$\{1\} = V^n_{n+1} \subsetneq V^n_n \subsetneq \ldots \subsetneq V^n_1 \subsetneq V^n_0 = W_n.$$

We recall that there is an inverse system of surjective homomorphisms $\theta_n : W_n \longrightarrow W_{n-1}$, for $n \geq 1$, such that

$$\theta_n(V^n_j) = \begin{cases} V^{n-1}_j & \text{for } 0 \leq j \leq n, \\ \{1\} & \text{for } j = n+1. \end{cases} \tag{3.1}$$

Let $M \trianglelefteq W$ be a non-trivial closed normal subgroup of W. Since W is an inverse limit, we can find $n \geq 0$ such that the image of M in W_n under the natural projection $\phi_n : W \longrightarrow W_n$ is non-trivial. (This argument is used in the proof of (2.2) in [32].) Therefore $\phi_n(M) = V_j^n$, for some $j \in \{0, 1, \ldots, n\}$.

We claim that $M = V_j$. Since M is closed, it is enough to show that $\phi_m(M) = V_j^m$, for all $m \geq n$. Then $\phi_m(M) = \phi_m(V_j)$ implies $\ker \phi_m M = \ker \phi_m V_j$, for all $m \geq n$. Thus

$$\begin{aligned} M &= \left(\bigcap_{m \geq n} \ker \phi_m \right) M = \bigcap_{m \geq n} \big(\ker \phi_m M \big) \\ &= \bigcap_{m \geq n} \big(\ker \phi_m V_j \big) = \left(\bigcap_{m \geq n} \ker \phi_m \right) V_j = V_j, \end{aligned}$$

using Lemma 2.12.

Clearly $\phi_m(M) = V_j^m$ is true for $m = n$. Now suppose $m > n$. From

$$\{1\} \neq V_j^{m-1} = \phi_{m-1}(M) = \theta_m(\phi_m(M))$$

and mapping (3.1), we conclude $\phi_m(M) = V_j^m$. □

Remark. The non-trivial normal subgroups of W can be written as $V_{j+1} = \ker(\phi_j : W \longrightarrow W_j)$, for $j \geq 0$, and $V_0 = \ker(W \longrightarrow \{1\})$.

Chapter 4

Wilson groups

4.1 Wilson's construction

We now describe the hereditarily just infinite profinite groups constructed by Wilson in [32], which are not virtually pro-p. This construction provides numerous examples of groups with the properties described in Theorem 1.1.

Let $X_0, X_1, X_2, \ldots$ be any infinite sequence of finite non-abelian simple groups. Set $G_0 = X_0$. The group G_0 has a faithful transitive permutation representation of some degree d_1. For instance, when G_0 acts on itself by right multiplication, and then $d_1 = |G_0|$.

Suppose a group G_{n-1}, for $n \in \mathbb{N}$, with a faithful transitive permutation representation of degree d_n has been constructed. We construct the group G_n by two operations of taking permutational wreath products.

First let $L_n = X_n^{(d_n)}$, for $n \in \mathbb{N}$, the direct product of d_n copies of X_n. We form the first permutational wreath product

$$X_n \wr_{\Omega_{d_n}} G_{n-1}, \text{ where } \Omega_{d_n} = \{1, 2, \ldots, d_n\}.$$

This group is described as the semidirect product $L_n \rtimes G_{n-1}$, where G_{n-1} acts on L_n by permuting the factors.

Next we define a transitive permutation representation φ of $L_n G_{n-1}$ on the set L_n, with the subgroup L_n transitive. The ingredients are the action of L_n on itself by right multiplication and the action of G_{n-1} on L_n by conjugation. The action φ is

$$l\varphi(l'g) = (ll')^g, \text{ where } l \in L_n \text{ and } l'g \in L_n G_{n-1}. \tag{4.1}$$

Let $M_n = X_n^{(|L_n|)}$, for $n \in \mathbb{N}$, the direct product of $|L_n|$ copies of X_n. Now we form the second permutational wreath product

$$X_n \wr_{L_n} (L_n G_{n-1}),$$

which we denote by G_n. The group G_n is described as the semidirect product $M_n \rtimes (L_n G_{n-1})$, where the group $L_n G_{n-1}$ permutes the factors of M_n according to the permutation representation φ.

We now form the inverse limit G of the groups G_n as described above. The resulting group G is one of the groups having the properties stated in Theorem 1.1. We will refer to the groups arising from such a construction as Wilson groups. More specifically, a Wilson group G is the inverse limit of a sequence $(G_n)_{n\geq 0}$ of finite groups as defined above and the natural projections

$$\theta_n : G_n = (M_n L_n) \rtimes G_{n-1} \longrightarrow G_{n-1},$$

for $n \geq 1$. The limit $G = \varprojlim (G_n)_{n\geq 0}$ has the natural projections $\phi_n : G \longrightarrow G_n$, for $n \geq 0$.

The following, Figure 4.1, illustrates Wilson's construction in detail, passing from the finite group G_1 to the finite group G_0.

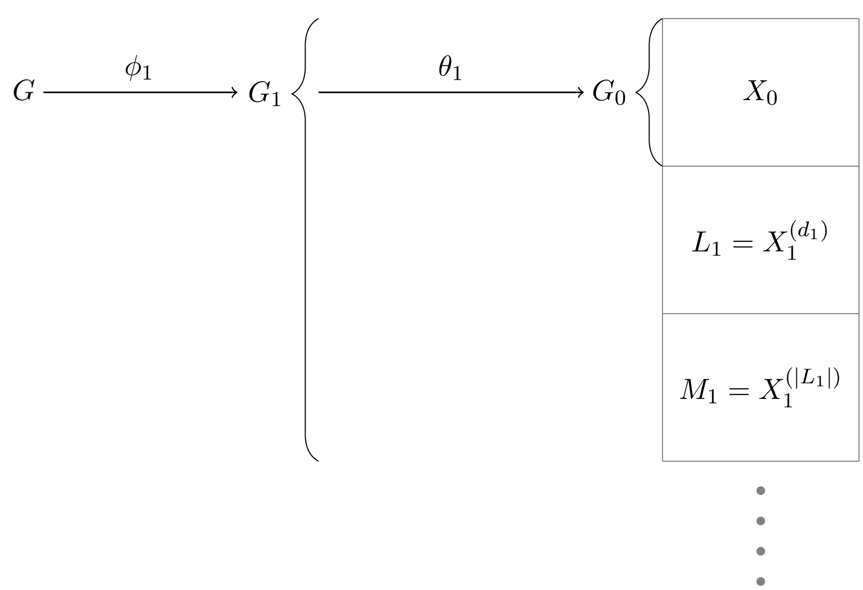

Figure 4.1: A pictorial description of Wilson's construction at level G_1.

We give an overview of the inverse limit G of a Wilson group in Figure 4.2, below.

The limit is indexed by the set $\mathbb{N} \cup \{0\}$ with respect to the ordinary order-relation $\leq$.

$$
\begin{tikzcd}
G = \varprojlim G_n \arrow[drr, dashed, "\phi_n"] \arrow[drrrrrr, dashed, "\phi_2"] \arrow[drrrrrrrr, dashed, "\phi_1"] \arrow[drrrrrrrrrr, dashed, "\phi_0"] & & & & & & & & & & \\
\cdots \arrow[rr, "\theta_{n+1}"] & & \begin{array}{c} G_n \\ = M_nL_n \rtimes G_{n-1} \end{array} \arrow[rr, "\theta_n"] & & \cdots \arrow[rr, "\theta_3"] & & \begin{array}{c} G_2 \\ = M_2L_2 \rtimes G_1 \end{array} \arrow[rr, "\theta_2"] & & \begin{array}{c} G_1 \\ = M_1L_1 \rtimes G_0 \end{array} \arrow[rr, "\theta_1"] & & G_0
\end{tikzcd}
$$

Figure 4.2: A pictorial description of the inverse limit G of a Wilson group.

4.2 Verifying hereditarily just infinite and not virtually pro-p

In this section, we briefly explain the proofs of Theorem 1.1 and Corollary 1.2, in Chapter 1. For further details the reader may refer to the original source of Wilson's paper [32].

Wilson develops the criterion (2.2) in [32], which says that the inverse limit of certain finite groups is either virtually abelian or hereditarily just infinite. Applying this criterion to the groups G_n as defined above, he shows that the Wilson groups are hereditarily just infinite by ruling out the possibility of them being virtually abelian. A Wilson group has all the composition factors of the finite continuous images non-abelian because the groups G_n are constructed from semidirect products of direct products of non-abelian simple groups $X_0, X_1, \ldots, X_n$.

We explain why the Wilson groups are not virtually pro-p, for some prime p. For a contradiction, suppose that a Wilson group G is virtually pro-p. We find a pro-p open normal subgroup N of G. Let K be an open normal subgroup of N. Then N/K is a finite p-group. Therefore all the composition factors of N/K are cyclic of order p. So G/K would have cyclic composition factors. That is G/K would have abelian composition factors. This contradicts the fact that all composition factors of finite continuous images of G are non-abelian.

It has been seen that every countably based profinite group can be embedded in the product

$$\prod_{n\geq 5} A_n = A_5 \times A_6 \times A_7 \times \dots$$

of alternating groups, refer to [30, (4.1.6)]. To prove Corollary 1.2, that every countably based profinite group can be embedded in a specific hereditarily just infinite profinite group, it suffices to embed the product $\prod_{n\geq 5} A_n$ in a specific Wilson group. For this embedding to take place, certain choices for X_n are required in the construction of this specific Wilson group. They are specified as $X_n = A_{n+5}$, for each $n \geq 0$.

The following technical result is used to prove that the Wilson groups are hereditarily just infinite, see [32, (3.2)].

Lemma 4.1 (Wilson [32])**.** *Let the finite groups L_n, M_n, for $n \geq 1$, and G_n, for $n \geq 0$, be as defined above.*

(a) *The unique minimal normal subgroup of G_n, for $n \geq 1$, is M_n.*

(b) *The unique minimal normal subgroup of $L_n G_{n-1}$, for $n \geq 1$, is L_n.*

The proof is elementary, but an important ingredient used from the construction of G_n is that the wreath product actions are transitive. Alternatively, the proof follows immediately from Lemma 2.3.

Lemma 4.1 is used in Chapter 5 to characterise the normal subgroups of the Wilson groups.

Chapter 5

Normal subgroups

5.1 General Wilson groups

In this chapter, we complete the characterisation of the closed normal subgroups of an arbitrary Wilson group. The characterisation holds for any choice of X_i, for $i \geq 0$, and for any choice of faithful transitive permutation representation of G_n, for $n \geq 1$, in the construction of a Wilson group.

Our work has been restricted in Lemma 5.2 to closed normal subgroups because we rely on Lemma 2.12, which only applies to normal subgroups that are closed. However, a result by N. Nikolov and D. Segal [22, Cor. 1.15] shows that all normal subgroups of a finitely generated Wilson group are automatically closed. Therefore the characterisation of normal subgroups, in Corollary 5.3, covers all the normal subgroups of any Wilson group provided the first group in Wilson's construction has size $|G_0| > 35!$ (see Chapter 9).

In finding the normal subgroups, we can see directly that all the Wilson groups are just infinite, which is implicit from [32, (3.3)]. Let G be a Wilson group arising as an inverse limit of finite groups G_n as defined in Section 4.1. In Corollary 5.3, the non-trivial closed normal subgroups of G are denoted by P_j and Q_j, for $j \geq 0$. Due to the definition of these subgroups P_j and Q_j, their indices in G can easily be calculated. We have the indices $|G : P_j| = |G_j|$, for $j \geq 0$, and $|G : Q_j| = |L_j G_{j-1}|$, for $j \geq 1$, and the index $|G : Q_0| = 1$. All the indices are finite. The profinite group G is just infinite using Lemma 2.11.

To describe normal subgroups of G, our strategy will be to first determine the normal subgroups of the finite groups G_n. As a motivation, the description of $G_n = M_n \rtimes (L_n \rtimes G_{n-1})$ implies $M_n \trianglelefteq G_n$ and $M_n L_n \trianglelefteq G_n$, for every $n \geq 1$. Therefore G_n has at least two types of normal subgroups.

For the purpose of what follows we define $M_0 = G_0$.

Theorem 5.1. *Let G_n, for $n \geq 0$, be the finite groups as defined in Section 4.1. For $j \in \{0, 1, \dots, n\}$, define*

$$P_j^n = M_n \rtimes \dots \rtimes (M_{j+1} \rtimes L_{j+1})$$

and define

$$Q_j^n = M_n \rtimes \dots \rtimes (M_{j+1} \rtimes (L_{j+1} \rtimes M_j)).$$

Then the normal subgroups of G_n are precisely the groups P_j^n and Q_j^n. In particular, they form a complete chain

$$\{1\} = P_n^n \subsetneq Q_n^n \subsetneq P_{n-1}^n \subsetneq \dots \subsetneq Q_1^n \subsetneq P_0^n \subsetneq Q_0^n = G_n.$$

Proof. We first prove that P_j^n and Q_j^n are normal subgroups of G_n. The homomorphisms $G_n \longrightarrow G_j$ have kernels P_j^n, for $j \in \{0, 1, \dots, n\}$. The homomorphisms $G_n \longrightarrow G_j/M_j$ have kernels Q_j^n, for $j \in \{0, 1, \dots, n\}$.

We now prove, by induction on n, that P_j^n and Q_j^n are the only normal subgroups of G_n. Suppose $N \trianglelefteq G_n$. For $n = 0$, all the normal subgroups of G_0 are $P_0^0 = \{1\}$ and $Q_0^0 = G_0$ holds as G_0 is simple.

Now suppose $n \geq 1$. If $N = \{1\}$ then $N = P_n^n$. Assume $N \neq \{1\}$. We have $M_n \subseteq N$, since the group M_n is the unique minimal normal subgroup of G_n, by Lemma 4.1 (a). Then there are two possibilities: $M_n = N$ and $M_n \subsetneq N$.

For $M_n = N$ we are done, as $N = Q_n^n$. For $M_n \subsetneq N$ we have $M_n L_n \subseteq N$ because L_n is the unique minimal normal subgroup of $L_n G_{n-1}$, by Lemma 4.1 (b). From $M_n L_n \subseteq N$ we have two cases. That is $M_n L_n = N$ implies $N = P_{n-1}^n$ and we are done. Alternatively $M_n L_n = P_{n-1}^n \subsetneq N$. Now P_{n-1}^n is the kernel of the homomorphism $\theta_n : G_n \longrightarrow G_{n-1}$. Then there is a one-to-one correspondence between the set of normal subgroups of G_n containing P_{n-1}^n and normal subgroups of G_{n-1}. By induction, we know that N is one of the groups P_j^n or Q_j^n. □

Figure 5.1, below, illustrates the chain of normal subgroups of the finite groups G_n, for $n \geq 0$.

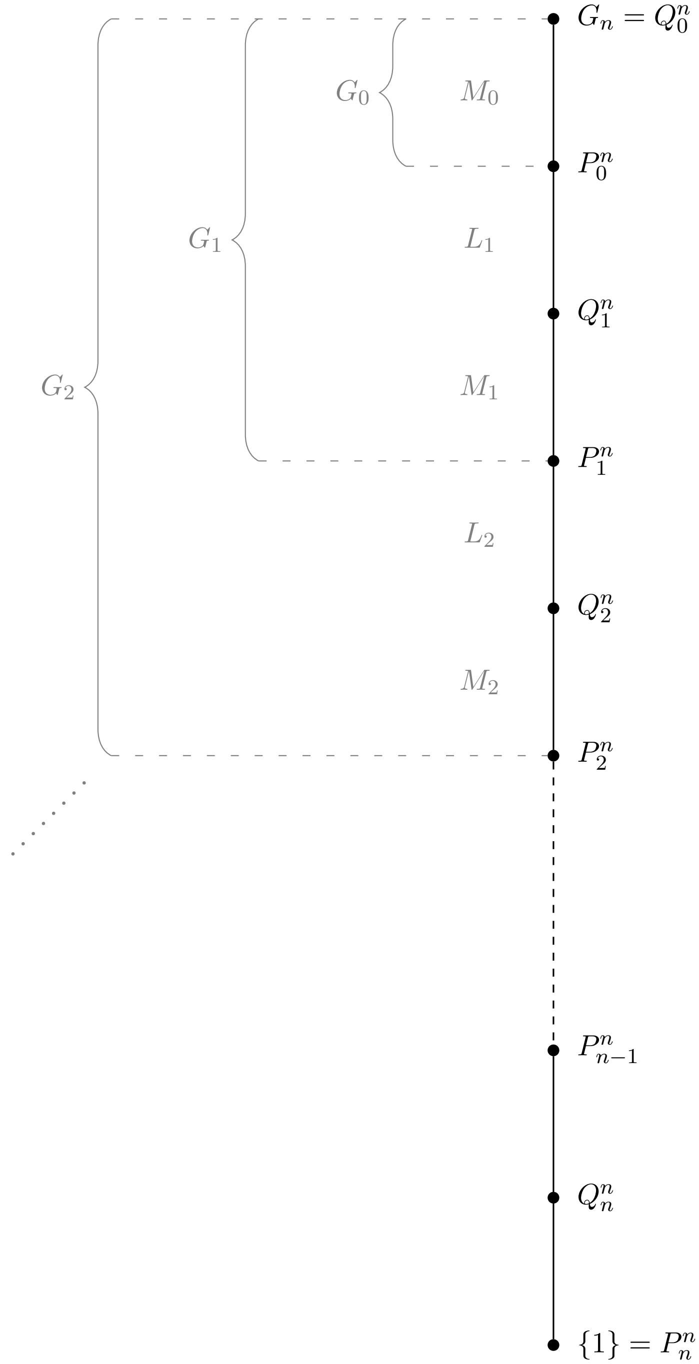

Figure 5.1: The chain of normal subgroups of the finite group G_n.

Lemma 5.2 is required due to the two different types of notation for the normal subgroups of G_n.

Lemma 5.2. *Given finite groups G_n, for $n \geq 0$, in which all the normal subgroups form a chain*

$$\{1\} = N^n_{2n+2} \subsetneq N^n_{2n+1} \subsetneq \ldots \subsetneq N^n_2 \subsetneq N^n_1 = G_n$$

and an inverse system of surjective homomorphisms $\theta_n : G_n \longrightarrow G_{n-1}$, for $n \geq 1$, such that

$$\theta_n(N^n_i) = \begin{cases} N^{n-1}_i & \text{for } 1 \leq i \leq 2n, \\ \{1\} & \text{for } i \in \{2n+1, 2n+2\}. \end{cases} \tag{5.1}$$

Then the inverse limit $G = \varprojlim(G_n)_{n\geq 0}$ has non-trivial closed normal subgroups precisely $N_i = \varprojlim(N^n_i)_{n\to\infty}$, for $i \geq 1$, regarded as subgroups of G.

Proof. Let $M \trianglelefteq G$ be a non-trivial closed normal subgroup of G. Since G is an inverse limit, we can find $n \geq 0$ such that the image of M in G_n under $\phi_n : G \longrightarrow G_n$ is non-trivial. Therefore $\phi_n(M) = N^n_i$, for some $i \in \{1, 2, \ldots, 2n+1\}$.

We claim that $M = N_i$. Since M is closed, it is enough to show that $\phi_m(M) = N^m_i$, for all $m \geq n$. Then $\phi_m(M) = \phi_m(N_i)$ implies $\ker \phi_m M = \ker \phi_m N_i$, for all $m \geq n$. Thus

$$\begin{aligned} M &= \left(\bigcap_{m\geq n} \ker \phi_m\right) M = \bigcap_{m\geq n} \left(\ker \phi_m M\right) \\ &= \bigcap_{m\geq n} \left(\ker \phi_m N_i\right) = \left(\bigcap_{m\geq n} \ker \phi_m\right) N_i = N_i, \end{aligned}$$

using Lemma 2.12.

Clearly $\phi_m(M) = N^m_i$ is true for $m = n$. Now suppose $m > n$. From

$$\{1\} \neq N^{m-1}_i = \phi_{m-1}(M) = \theta_m(\phi_m(M))$$

and mapping (5.1), we conclude $\phi_m(M) = N^m_i$. □

Corollary 5.3. *Let $G = \varprojlim(G_n)_{n\geq 0}$ be the inverse limit of the groups G_n as defined in Section 4.1. For $j \geq 0$, define*

$$P_j = \varprojlim(P^n_j)_{n\to\infty}$$

and define

$$Q_j = \varprojlim(Q^n_j)_{n\to\infty},$$

regarded as subgroups of G.

Then the non-trivial closed normal subgroups of G are precisely the groups P_j and Q_j. In particular, they form a complete chain

$$\ldots \subsetneq Q_{n+1} \subsetneq P_n \subsetneq Q_n \subsetneq P_{n-1} \subsetneq \ldots \subsetneq Q_1 \subsetneq P_0 \subsetneq Q_0 = G.$$

Proof. We apply Lemma 5.2 to the groups G_n, for $n \geq 0$, of Wilson's construction and their normal subgroups. Define

$$N_i^n = \begin{cases} Q^n_{\lfloor (i-1)/2 \rfloor} & \text{if } i \text{ is odd}, \\ P^n_{\lfloor (i-1)/2 \rfloor} & \text{if } i \text{ is even}, \end{cases}$$

where $i \in \{1, 2, \ldots, 2n+2\}$. For each n, these normal subgroups of G_n were defined in Theorem 5.1. It was shown that these are all the normal subgroups of G_n and they form a chain.

The definition of the groups N_i^n also shows that the second condition for Lemma 5.2 is satisfied. For $1 \leq i \leq 2n+2$,

$$\theta_n(N_i^n) = \begin{cases} \theta_n(Q^n_{\lfloor (i-1)/2 \rfloor}) = Q^{n-1}_{\lfloor (i-1)/2 \rfloor} = N_i^{n-1} & \text{if } i \text{ is odd}, \\ \theta_n(P^n_{\lfloor (i-1)/2 \rfloor}) = P^{n-1}_{\lfloor (i-1)/2 \rfloor} = N_i^{n-1} & \text{if } i \text{ is even}. \end{cases}$$

We take Q_n^{n-1}, P_n^{n-1}, N_{2n+1}^{n-1} and N_{2n+2}^{n-1} to be the trivial group $\{1\}$. □

Remark 5.4. Fix a prime number p and let K be a finite field of characteristic $p > 2$. The Nottingham group over K is the group $\mathcal{N}(K) := t + t^2K[[t]]$ of normalised formal power series over K under substitution. For every $i \in \mathbb{N}$, the sets $\mathcal{N}_i(K) := t + t^{i+1}K[[t]]$ are normal subgroups of $\mathcal{N}(K)$ and they form a chain

$$\ldots \subsetneq \mathcal{N}_3(K) \subsetneq \mathcal{N}_2(K) \subsetneq \mathcal{N}_1(K) = \mathcal{N}(K).$$

The Nottingham group is a pro-p group. This is because it is the inverse limit of the inverse system of finite p-groups $\mathcal{N}(K)/\mathcal{N}_i(K)$ and natural projections

$$\mathcal{N}(K)/\mathcal{N}_{i+1}(K) \longrightarrow \mathcal{N}(K)/\mathcal{N}_i(K),$$

for $i \in \mathbb{N}$, recall Section 2.5. The Nottingham group is a hereditarily just infinite group; see R. Camina [5].

For each $r \geq 0$, there are also $p+1$ non-trivial normal subgroups $\mathcal{H}$ of $\mathcal{N}(K)$ such that $\mathcal{N}_{p^r+3}(K) \subsetneq \mathcal{H} \subsetneq \mathcal{N}_{p^r+1}(K)$; referred to by B. Klopsch in [14]. Therefore the chain of normal subgroups of a Wilson group is more rigid than in the Nottingham group, where the normal subgroups almost form a chain.

An analogy with the Nottingham group poses many interesting questions for the Wilson groups and we include some of them in Question 2 of Chapter 10.

Chapter 6

Subnormal subgroups

6.1 Introduction

This chapter works on the task of characterising the subnormal subgroups of Wilson groups.

Let G be a Wilson group arising as an inverse limit of finite groups G_n as defined in Section 4.1. As before when describing the normal subgroups, our strategy will be to first determine the subnormal subgroups of the finite groups G_n. For ease of calculation, we will also consider a particular subgroup H_n of G_n such that there exists surjective homomorphisms $G_n \longrightarrow H_n \longrightarrow G_{n-1}$. That is, we define

$$H_n = L_n \rtimes G_{n-1},$$

for every $n \geq 1$.

Remark. Using the newly defined groups H_n, the non-trivial normal subgroups of G can be written as $P_j = \ker(\phi_j : G \longrightarrow G_j)$, for $j \geq 0$, $Q_j = \ker(G \longrightarrow H_j)$, for $j \geq 1$, and $Q_0 = \ker(G \longrightarrow \{1\})$.

The groups G_n, in the construction of a Wilson group, are formed from two types of transitive actions. One type are the unspecified actions of the groups G_{n-1}, for $n \geq 1$, on a set of d_n elements. The other type of transitive actions are the groups L_nG_{n-1}, for $n \geq 1$, acting on the sets L_n by the action defined in (4.1), from Section 4.1.

In the action (4.1), a subnormal subgroup of L_n acts on L_n by right multiplication and therefore the orbits of a non-trivial subnormal subgroup have at least two elements. However, the action of a subnormal subgroup of G_{n-1} on d_n elements may have orbits of one element, that is the action has fixed points. We illustrate this latter conclusion with the following example.

Example 6.1. Take a transitive and faithful action of X_{n-1} on a set Λ. Define $X_{n-1}^{[i]} = X_{n-1}$ and $\Lambda_i = \{(\lambda, i) : \lambda \in \Lambda\}$, for $i = 1, 2, \ldots, |L_{n-1}|$, noting that $\Lambda_i \cong \Lambda$ as X_{n-1}-spaces. Let $M_{n-1} = X_{n-1}^{(|L_{n-1}|)}$ act on the set $\bigcup_{i=1}^{|L_{n-1}|} \Lambda_i$, where $X_{n-1}^{[i]}$, for each $i = 1, 2, \ldots, |L_{n-1}|$, acts on Λ_i by the chosen action.

Recall $G_{n-1} = X_{n-1} \wr_{L_{n-1}} (L_{n-1}G_{n-2})$ is a wreath product and the part $L_{n-1}G_{n-2}$ acts transitively on the set L_{n-1}, according to the action (4.1). Set $\bigcup_{i=1}^{|L_{n-1}|} \Lambda_i = \Omega_{d_n}$. Consequently, the group G_{n-1} acts transitively on the set Ω_{d_n} by the natural permutational wreath product action, as explained in Section 2.1.

Non-trivial elements of $L_{n-1}G_{n-2}$ acting on the set L_{n-1}, according to the action (4.1), can have fixed points. However, these elements do move at least one other point and so this action is faithful. Consequently, the full action of the group G_{n-1} on the set Ω_{d_n} is faithful.

Now the action of the subnormal subgroup $X_{n-1} \times 1 \times \ldots \times 1 \subseteq M_{n-1}$ of G_{n-1} on Ω_{d_n} has many fixed points.

The above observation has an effect on the characterisation of the subnormal subgroups of Wilson groups which we now explain. Suppose K is a subnormal subgroup of $H_n = L_nG_{n-1}$ such that $K \not\subseteq L_n$. Then L_nK/L_n is isomorphic to a subnormal subgroup U of G_{n-1}. We consider the orbits of the action of U on d_n elements. As shown in Example 6.1, some orbits may have only one element. We see later in Corollary 6.9, within Section 6.3, to satisfy the condition of normality, the subnormal subgroup K must contain all the factors of $L_n = X_n^{(d_n)}$ which correspond to the U-orbits that contain at least two elements.

Section 6.2 characterises the subnormal subgroups of particular Wilson groups where a choice for the unspecified actions of G_{n-1}, for $n \geq 1$, on a set of d_n elements is made. This choice guarantees that subnormal subgroups of G_{n-1} have all their orbits containing at least two elements.

Later, in Section 6.4 we characterise the subnormal subgroups of an arbitrary Wilson group, that is, where the actions of the groups G_{n-1} remain unspecified. Consider the action of the groups G_j on the sets $\Omega_{d_{j+1}} = \{1, 2, \ldots, d_{j+1}\}$, for $j \geq 1$. The description of some subnormal subgroups of a general Wilson group (see Section 6.4) involves the set

$$\left\{\omega \in \Omega_{d_{j+1}} : \omega \cdot X_j^{I_{L_j}} \neq \{\omega\}\right\}, \text{ where } \emptyset \neq I_{L_j} \subseteq L_j;$$

the notation $\omega \cdot X_j^{I_{L_j}}$ denotes the orbit of ω under the action of the group $X_j^{I_{L_j}} \leq G_j$.

Let $g \in G_j$. The *support of* g is the set of points of $\Omega_{d_{j+1}}$ which are not fixed by g

and it is denoted by $\mathrm{supp}(g)$. So we write

$$\mathrm{supp}(g) = \left\{\omega \in \Omega_{d_{j+1}} : \omega \cdot g \neq \omega\right\}.$$

Therefore the reader could consider the set $\{\omega \in \Omega_{d_{j+1}} : \omega \cdot X_j^{I_{L_j}} \neq \{\omega\}\}$ as the support of the group $X_j^{I_{L_j}} \leq G_j$.

We make a short remark about notation, which occurs in the classification of subnormal subgroups of the Wilson groups (see Section 6.2 and Section 6.4) and subnormal subgroups of the infinite iterated wreath products W of alternating groups (see Section 6.3).

Let X be a group and let Ω be a set. Then

$$X^{\Omega} = \{f | f : \Omega \longrightarrow X\} \cong \{(x_\omega)_{\omega\in\Omega} : x_\omega \in X\}.$$

Let $I \subseteq \Omega$. Write $\Delta := \Omega \backslash I$. To define X^I we extend all functions f from I to X by setting $\tilde{f}(\Delta) = \{1\}$. Therefore

$$X^I \cong \{(x_\omega)_{\omega\in I} \times (1)_{\omega\in\Delta} : x_\omega \in X\}.$$

In so doing, it is acceptable to write $X^I \subseteq X^{\Omega}$.

6.2 Particular Wilson groups

In this section Wilson's construction is limited by specifying that the group G_{n-1}, for $n \geq 1$, acts on itself by right multiplication. Therefore $d_n = |G_{n-1}|$, for $n \geq 1$. This is a faithful and transitive action and so satisfying the conditions of Wilson's construction. Implicitly, the action of a non-trivial subnormal subgroup of G_{n-1} on d_n elements now has all its orbits containing at least two elements. Hence the characterisation of subnormal subgroups has been simplified.

Theorem 6.4 determines the subnormal subgroups of the finite groups G_n for this restricted construction. Then Corollary 6.6 completely classifies the closed subnormal subgroups of the Wilson groups that arise from this particular construction.

The inductive argument of Corollary 6.7, using the result by N. Nikolov and D. Segal [22, Cor. 1.15], shows that all subnormal subgroups of a Wilson group are automatically closed provided the first group in Wilson's construction has size $|G_0| > 35!$, and hence the Wilson group is finitely generated (see Chapter 9). Therefore the characterisation of subnormal subgroups, in Corollary 6.6, covers all the subnormal subgroups of our particular restricted Wilson groups provided $|G_0| > 35!$.

For the restriction, it is found that the subnormal subgroups of these Wilson groups are squeezed between consecutive normal subgroups. Therefore there are relatively few compared to the Nottingham group, which is a pro-p group and here every open subgroup is subnormal, refer to [27, 5.2.4]. At the end of this section, Figure 6.1 depicts the subnormal subgroups of these Wilson groups lying between the normal subgroups.

Also, it is found that the subnormal length for a Wilson group is at most 3 (found later in Corollary 6.17), and therefore is bounded. This is in contrast to the Nottingham group, where the subnormal length is unbounded, and is proved with the following short argument.

First note that there are finite p-groups G which have subnormal subgroups of arbitrarily large subnormal length in G. Now fix a finite p-group G with a subnormal subgroup H of subnormal length l in G. By a theorem of C. Leedham-Green and A. Weiss, see [4], G embeds as a subgroup into the Nottingham group $\mathcal{N}$. So we may assume that $G \leq \mathcal{N}$. There is an open normal subgroup N of $\mathcal{N}$ such that $G \cap N = \{1\}$, since G is finite and [30, Cor. 1.2.4 (iii)]. This implies $GN/N \cong G$ and, as $H \cap N = \{1\}$, also $HN/N \cong H$. Consider $S = HN$. Any chain

$$\mathcal{N} = T_0 \trianglerighteq T_1 \trianglerighteq \ldots \trianglerighteq T_{k-1} \trianglerighteq T_k = S$$

showing that S has subnormal length $\leq k$ in $\mathcal{N}$ intersects to a chain

$$GN = T_0 \cap GN \trianglerighteq T_1 \cap GN \trianglerighteq \ldots \trianglerighteq T_{k-1} \cap GN \trianglerighteq T_k \cap GN = S.$$

Therefore $HN/N \cong H$ has subnormal length $\leq k$ in $GN/N \cong G$. Thus $k \geq l$ and S has subnormal length $\geq l$ in $\mathcal{N}$.

To prove Theorem 6.4, determining the subnormal subgroups of the finite groups G_n of Wilson's restricted construction, we use Proposition 6.2 concerning subnormal subgroups of permutational wreath products as defined in Lemma 2.2 of Section 2.1. We recall the definition. Let U be a finite permutation group acting on a finite set Ω with orbits $\Omega_1, \Omega_2, \ldots, \Omega_r$. Let X be a finite non-abelian simple group. Define the permutational wreath product $G = X \wr_\Omega U$. Denote the base group of the wreath product as V.

In Proposition 6.2, the assumption is made that each of the U-orbits has at least two elements. Then the subnormal subgroups K of G such that $VK = G$ contain the base group V. Proposition 6.2 can be readily applied to the circumstance where U is taken to be a subnormal subgroup of G_{n-1} acting on G_{n-1} by right multiplication.

Proposition 6.2. *Let group $G = X \wr_\Omega U$ be the permutational wreath product as defined in Lemma 2.2. Assume that each of the U-orbits $\Omega_1, \Omega_2, \ldots, \Omega_r$ has at least two* *elements. Suppose K is a subnormal subgroup of G such that $VK = G$.*

Then $V \subseteq K$. In particular, this gives $K = G$.

Proof. We first show that we can assume $K \trianglelefteq G$. Since K is subnormal in G, we have $G = T_0 \trianglerighteq T_1 \trianglerighteq \ldots \trianglerighteq T_{k-1} \trianglerighteq T_k = K$. Without loss of generality, suppose this is a shortest chain. This means $G \neq T_1, T_1 \neq T_2, \ldots, T_{k-1} \neq K$.

Consider the beginning of the chain $G \trianglerighteq T_1$. Now $VK = G$ implies $VT_1 = G$. We apply the proposition, which we assume to be true for the special case where the subnormal subgroup is actually normal, to T_1. This gives $T_1 = G$. Therefore there is no such shortest chain involving the T_i, for $i = 1, 2, \ldots, k-1$, of subnormal length greater than 1. Thus $K \trianglelefteq G$.

To prove $V \subseteq K$, it is sufficient to show that each of the minimal normal subgroups of G is contained in K. Let $i \in \{1, 2, \ldots, r\}$. Let $\omega_1, \omega_2 \in \Omega_i$ such that ω_1 and ω_2 are distinct. We can find $u \in U$ such that $\omega_1 \cdot u = \omega_2$ because U acts transitively on Ω_i. As $VK = G$, we can obtain $\underline{x} \in V$ such that $u\underline{x} = u(x_\omega)_{\omega \in \Omega} \in K$. Choose $y \in X \backslash \{1\}$. Consider $\underline{y} = (y_\omega) \in V$ with $y_\omega = y$ if $\omega = \omega_1$ and $y_\omega = 1$ otherwise.

Then $[\underline{y}, u\underline{x}] \in K$ is similarly written as the element (2.1), in the final paragraph of the proof for Lemma 2.3. Continuing the argument, as written in the proof of Lemma 2.3 gives the required result. □

The proof of Theorem 6.4 (and later the proof of Theorem 6.15) also makes use of the following result.

Lemma 6.3. *Let G_n, for $n \geq 0$, be the finite groups as defined in Section 4.1. Recall that the group G_{n-1} has a faithful transitive action on Ω_{d_n}.*

Then each of the M_{n-1}-orbits has at least two elements.

Proof. Since G_{n-1} acts transitively on Ω_{d_n} and $M_{n-1} \trianglelefteq G_{n-1}$, all the M_{n-1}-orbits have the same size. This common size cannot be one because M_{n-1} is not trivial and G_{n-1} acts faithfully on Ω_{d_n}. Therefore each of the M_{n-1}-orbits has at least two elements. □

For the following, recall the normal subgroups P^n_j and Q^n_j, for $j \in \{0, 1, \ldots, n\}$, of G_n, defined in Theorem 5.1.

Theorem 6.4. *Let G_n, for $n \geq 0$, be the finite groups as defined in Section 4.1. In the Wilson construction, assume that the unspecified action of the group G_n, for $n \geq 0$, is taken to be right multiplication on itself.*

For $j \in \{0, 1, \ldots, n-1\}$, define

$$S^n_j(I_{d_{j+1}}) = Q^n_{j+1} \rtimes X_{j+1}^{I_{d_{j+1}}} \leq P^n_j, \text{ where } \emptyset \neq I_{d_{j+1}} \subseteq \Omega_{d_{j+1}},$$

and define

$$S_n^n = \{1\}.$$

For $j \in \{1, 2, \ldots, n\}$, define

$$T_j^n(I_{L_j}) = P_j^n \rtimes X_j^{I_{L_j}} \leq Q_j^n, \text{ where } \emptyset \neq I_{L_j} \subseteq L_j,$$

and define

$$T_0^n = G_n.$$

Then the subnormal subgroups of G_n are precisely the groups $S_j^n(I_{d_{j+1}})$, S_n^n, $T_j^n(I_{L_j})$ and T_0^n. In particular, for all I_{d_1}, I_{L_1}, $\ldots$, I_{d_n} and I_{L_n}, they form chains

$$\begin{aligned} S_n^n = P_n^n \subsetneq T_n^n(I_{L_n}) \subseteq Q_n^n \subsetneq S_{n-1}^n(I_{d_n}) \subseteq P_{n-1}^n \subsetneq \ldots \\ \subseteq P_1^n \subsetneq T_1^n(I_{L_1}) \subseteq Q_1^n \subsetneq S_0^n(I_{d_1}) \subseteq P_0^n. \end{aligned}$$

The subnormal length in G_n of the group $S_j^n(I_{d_{j+1}})$ is

$$\begin{cases} 1 & \text{if } I_{d_{j+1}} = \Omega_{d_{j+1}} \text{ (implying that } S_j^n(I_{d_{j+1}}) = P_j^n\text{)}, \\ 2 & \text{if } I_{d_{j+1}} \subsetneq \Omega_{d_{j+1}}. \end{cases}$$

The subnormal length in G_n of the group $T_j^n(I_{L_j})$ is

$$\begin{cases} 1 & \text{if } I_{L_j} = L_j \text{ (implying that } T_j^n(I_{L_j}) = Q_j^n\text{)}, \\ 2 & \text{if } I_{L_j} \subsetneq L_j. \end{cases}$$

Proof. We first check that the groups $S_j^n(I_{d_{j+1}})$, S_n^n, $T_j^n(I_{L_j})$ and T_0^n are all subnormal subgroups of G_n. Obviously $S_n^n = \{1\} \lhd G_n$ and $T_0^n = G_n \trianglelefteq G_n$. For any $\emptyset \neq I_{d_{j+1}} \subseteq \Omega_{d_{j+1}}$, we have

$$S_j^n(I_{d_{j+1}}) = Q_{j+1}^n \rtimes X_{j+1}^{I_{d_{j+1}}} \trianglelefteq Q_{j+1}^n \rtimes L_{j+1} = P_j^n \lhd G_n, \tag{6.1}$$

as $X_{j+1}^{I_{d_{j+1}}} \trianglelefteq L_{j+1}$. For any $\emptyset \neq I_{L_j} \subseteq L_j$, we have

$$T_j^n(I_{L_j}) = P_j^n \rtimes X_j^{I_{L_j}} \trianglelefteq P_j^n \rtimes M_j = Q_j^n \lhd G_n, \tag{6.2}$$

as $X_j^{I_{L_j}} \trianglelefteq M_j$.

If $I_{d_{j+1}} = \Omega_{d_{j+1}}$ then $S_j^n(I_{d_{j+1}}) = P_j^n$ and the subnormal series (6.1) reduces to a chain of length 1. Similarly, if $I_{L_j} = L_j$ then $T_j^n(I_{L_j}) = Q_j^n$ and the subnormal series

(6.2) reduces to a chain of length 1. For all other $S_j^n(I_{d_{j+1}})$ we have displayed the shortest length of a subnormal series (6.1) because P_j^n is the smallest normal subgroup of G_n containing $S_j^n(I_{d_{j+1}})$ and $S_j^n(I_{d_{j+1}})$ is not normal in G_n. A similar argument holds for all other $T_j^n(I_{L_j})$.

Recall the definition of the groups $H_n = L_n G_{n-1}$, for $n \geq 1$, as defined at the beginning of Section 6.1. Due to $H_n \cong G_n/M_n$, the theorem we are currently proving also implicitly makes a statement about the subnormal subgroups of H_n. We now prove, by induction on n, that every subnormal subgroup of G_n is one of the groups listed. Hence the subnormal subgroups of H_n are homomorphic images of the subnormal subgroups of G_n listed between Q_n^n and Q_0^n under the canonical map $G_n \longrightarrow H_n$.

For $n = 0$, all the subnormal subgroups of G_0 are $\{1\} = S_0^0$ and $G_0 = T_0^0$ holds as G_0 is simple. Although it will also follow from the general argument below, we now prove separately the implicit claim for H_1.

Suppose K is a subnormal subgroup of H_1. Then L_1K/L_1 is a subnormal subgroup of $H_1/L_1 \cong G_0$. Since G_0 is simple, we know

$$L_1K/L_1 \cong \{1\} \text{ or } L_1K/L_1 \cong G_0.$$

For the case $L_1K/L_1 \cong \{1\}$, we have $K \subseteq L_1$. Then K is subnormal in $L_1 = X_1^{(d_1)}$. There are two possibilities, either $K = \{1\} \cong M_1T_1^1(I_{L_1})/M_1$, for any $\emptyset \neq I_{L_1} \subseteq L_1$, or, since L_1 is a product of non-abelian simple groups X_1, using Theorem 2.4, we have $K = X_1^{I_{d_1}}$ is the image of $S_0^1(I_{d_1})$, for some $\emptyset \neq I_{d_1} \subseteq \Omega_{d_1}$, under the canonical map $G_1 \longrightarrow H_1$. Due to $H_1 \cong G_1/M_1$, there are subnormal subgroups of H_1 of this form.

For the case $L_1K/L_1 \cong G_0$, we have $L_1K = L_1 \rtimes G_0$. Since G_0 acts transitively on Ω_{d_1}, there is exactly one G_0-orbit Ω_{d_1}. Proposition 6.2 gives $L_1 \subseteq K$. Therefore $K = L_1 \rtimes G_0 \cong T_0^1/M_1$. For $n = 1$, the result holds for H_1.

Suppose that the result holds for G_{n-1}. Now we prove the result for H_n. Let K be a subnormal subgroup of H_n. Then there are two cases:

$$K \subseteq L_n \text{ (case 1), and } K \nsubseteq L_n \text{ (case 2).}$$

Case 1.

For $K \subseteq L_n$, we have K is subnormal in $L_n = X_n^{(d_n)}$. There are two possibilities, either $K = \{1\} \cong M_nT_n^n(I_{L_n})/M_n$, for any $\emptyset \neq I_{L_n} \subseteq L_n$, or, since L_n is a product of non-abelian simple groups X_n, using Theorem 2.4, we have $K = X_n^{I_{d_n}}$ is the image of $S_{n-1}^n(I_{d_n})$, for some $\emptyset \neq I_{d_n} \subseteq \Omega_{d_n}$, under the canonical map $G_n \longrightarrow H_n$.

Case 2.

Now suppose $K \nsubseteq L_n$. We know $\{1\} \ncong L_nK/L_n$ is a subnormal subgroup of $H_n/L_n \cong G_{n-1}$. Then there are two possibilities:

$$L_nK/L_n \subseteq L_nM_{n-1}/L_n \text{ (case 2a)},$$

and

$$L_nK/L_n \nsubseteq L_nM_{n-1}/L_n \text{ (case 2b)}.$$

Case 2a.

For $L_nK/L_n \subseteq L_nM_{n-1}/L_n$, we have $\{1\} \ncong L_nK/L_n$ is subnormal in $L_nM_{n-1}/L_n \cong M_{n-1}$. So

$$L_nK/L_n \cong X_{n-1}^{I_{L_{n-1}}} = T_{n-1}^{n-1}(I_{L_{n-1}}),$$

for some $\emptyset \neq I_{L_{n-1}} \subseteq L_{n-1}$. Put $T_{n-1}^{n-1}(I_{L_{n-1}}) =: T$. Then $L_nK = L_n \rtimes T$. Specifying that G_{n-1} acts on itself by right multiplication ensures that each of the T-orbits has at least two elements. Also $K \subseteq L_nT$ and so K is subnormal in L_nT. Proposition 6.2 gives $L_n \subseteq K$. Therefore $K = L_n \rtimes T$ is the image of $T_{n-1}^{n}(I_{L_{n-1}})$ under the canonical map $G_n \longrightarrow H_n$.

Case 2b.

For $L_nK/L_n \nsubseteq L_nM_{n-1}/L_n$, we have L_nK/L_n is subnormal in $H_n/L_n \cong G_{n-1}$ and is not contained in L_nM_{n-1}/L_n. By induction, we have $L_nK/L_n \cong S_j^{n-1}(I_{d_{j+1}})$, for some $j \in \{0, 1, \ldots, n-2\}$, or $L_nK/L_n \cong T_j^{n-1}(I_{L_j})$, for some $j \in \{1, 2, \ldots, n-2\}$, or $L_nK/L_n \cong T_0^{n-1}$.

We denote this isomorphic copy of L_nK/L_n in G_{n-1} by R. Then $L_nK = L_n \rtimes R$. Observe that $M_{n-1} \subseteq R$. Each of the orbits of M_{n-1} in its action upon Ω_{d_n}, and hence each of the orbits of R in its action upon Ω_{d_n}, has at least two elements (see Lemma 6.3). Therefore Proposition 6.2 can be applied irrespective of the chosen actions for the groups G_{n-1} on Ω_{d_n}.

Also $K \subseteq L_nR$ and so K is subnormal in L_nR. Proposition 6.2 gives $L_n \subseteq K$ and so $K = L_n \rtimes R$. Therefore K is the image of $S_j^n(I_{d_{j+1}})$ under the canonical map $G_n \longrightarrow H_n$, for some $j \in \{0, 1, \ldots, n-2\}$, or K is the image of $T_j^n(I_{L_j})$ under the canonical map $G_n \longrightarrow H_n$, for some $j \in \{1, 2, \ldots, n-2\}$, or $K \cong T_0^n/M_n$.

Suppose that the result holds for H_n. Now we prove the result for G_n. Let K be a subnormal subgroup of G_n. Then there are two cases:

$$K \subseteq M_n \text{ (case 1), and } K \nsubseteq M_n \text{ (case 2).}$$

Case 1.

For $K \subseteq M_n$, we have K is subnormal in $M_n = X_n^{(|L_n|)}$. There are two possibilities, either $K = \{1\} = S_n^n$, or, using Theorem 2.4, we have $K = X_n^{I_{L_n}} = T_n^n(I_{L_n})$, for some $\emptyset \neq I_{L_n} \subseteq L_n$.

Case 2.

Now suppose $K \nsubseteq M_n$. We know $\{1\} \ncong M_nK/M_n$ is a subnormal subgroup of $G_n/M_n \cong H_n$. Then there are two possibilities:

$$M_nK/M_n \subseteq M_nL_n/M_n \text{ (case 2a),}$$

and

$$M_nK/M_n \nsubseteq M_nL_n/M_n \text{ (case 2b).}$$

Case 2a.

For $M_nK/M_n \subseteq M_nL_n/M_n$, we have $\{1\} \ncong M_nK/M_n$ is subnormal in $M_nL_n/M_n \cong L_n$. So

$$M_nK/M_n \cong X_n^{I_{d_n}},$$

for some $\emptyset \neq I_{d_n} \subseteq \Omega_{d_n}$, which is the image of $S_{n-1}^n(I_{d_n})$ under the canonical map $G_n \longrightarrow H_n$. Put $S_{n-1}^n(I_{d_n}) =: S$. Then $M_nK = S$.

Right multiplication by L_n on itself in the action (4.1) implies that each of the orbits of $X_n^{I_{d_n}}$ in its action upon L_n has at least two elements. In the action of $X_n^{I_{d_n}}$ on L_n, each non-trivial element of $X_n^{I_{d_n}}$ acts fixed point freely. Therefore this action is faithful.

Also $K \subseteq S$ and so K is subnormal in S. Proposition 6.2 gives $M_n \subseteq K$. Therefore $K = S = S_{n-1}^n(I_{d_n})$.

Case 2b.

For $M_nK/M_n \nsubseteq M_nL_n/M_n$, we have M_nK/M_n is subnormal in $G_n/M_n \cong H_n$ and is not contained in M_nL_n/M_n. By induction, we have $M_nK/M_n = T_j^n(I_{L_j})/M_n$, for some $j \in \{1, 2, \dots, n-1\}$, or $M_nK/M_n = S_j^n(I_{d_{j+1}})/M_n$, for some $j \in \{0, 1, \dots, n-2\}$, or $M_nK/M_n = T_0^n/M_n$.

We denote this description of M_nK/M_n in H_n by R/M_n. Then $M_nK = R$. Again, right multiplication by L_n on itself in the action (4.1) implies that each of the orbits of R/M_n in its action upon L_n has at least two elements. In fact, since $L_n \subseteq R$, there is only one (R/M_n)-orbit, that is L_n.

In the action (4.1), non-trivial elements of R/M_n acting on L_n can have fixed points however these elements do move at least one other point. Therefore this action is faithful.

Also $K \subseteq R$ and so K is subnormal in R. Proposition 6.2 gives $M_n \subseteq K$ and so $K = R$. Therefore $K = T_j^n(I_{L_j})$, for some $j \in \{1, 2, \ldots, n-1\}$, or $K = S_j^n(I_{d_{j+1}})$, for some $j \in \{0, 1, \ldots, n-2\}$, or $K = T_0^n$.

□

Similarly as for the normal subgroups, our work has been restricted in Lemma 6.5 to closed subnormal subgroups because we rely on Lemma 2.12, which only applies to subnormal subgroups that are closed.

Lemma 6.5 is required due to the two different types of notation for the subnormal subgroups of G_n.

Lemma 6.5. *Given finite groups G_n, for $n \geq 0$, in which all the normal subgroups form a chain*

$$\{1\} = N_{2n+2}^n \subsetneq N_{2n+1}^n \subsetneq \ldots \subsetneq N_2^n \subsetneq N_1^n = G_n,$$

and an inverse system of surjective homomorphisms $\theta_n : G_n \longrightarrow G_{n-1}$, for $n \geq 1$, such that

$$\theta_n(N_i^n) = \begin{cases} N_i^{n-1} & \text{for } 1 \leq i \leq 2n, \\ \{1\} & \text{for } i \in \{2n+1, 2n+2\}. \end{cases}$$

Let $\mathcal{P}^i$, for $i \in \{1, 2, \ldots, 2n+1\}$, be finite disjoint index sets.

Suppose the non-trivial subnormal subgroups K_I^n of G_n are parameterised by I, where $\emptyset \neq I \in \mathcal{P}^i$, such that $N_{i+1}^n \subsetneq K_I^n \subseteq N_i^n$, and

$$\theta_n(K_I^n) = \begin{cases} K_I^{n-1} & \text{for } I \in \mathcal{P}^1, \mathcal{P}^2, \ldots, \mathcal{P}^{2n-1}, \\ \{1\} & \text{for } I \in \mathcal{P}^{2n}, \mathcal{P}^{2n+1}. \end{cases} \tag{6.3}$$

Then the inverse limit $G = \varprojlim (G_n)_{n\geq 0}$ has non-trivial closed subnormal subgroups precisely $K_I = \varprojlim (K_I^n)_{n\to\infty}$, where $\emptyset \neq I \in \mathcal{P}^i$ for $i \geq 1$, regarded as subgroups of G.

Proof. Let M be a non-trivial closed subnormal subgroup of G. Since G is an inverse limit, we can find $n \geq 0$ such that the image of M in G_n under $\phi_n : G \longrightarrow G_n$ is non-trivial. Therefore $\phi_n(M) = K_I^n$, where $\emptyset \neq I \in \mathcal{P}^i$, for some $i \in \{1, 2, \ldots, 2n+1\}$.

We claim that $M = K_I$. Since M is closed, it is enough to show that $\phi_m(M) = K_I^m$, for all $m \geq n$. Then $\phi_m(M) = \phi_m(K_I)$ implies $\ker \phi_m M = \ker \phi_m K_I$, for all $m \geq n$.

Thus

$$M = \Big(\bigcap_{m \geq n} \ker \phi_m \Big) M = \bigcap_{m \geq n} \big(\ker \phi_m M \big)$$
$$= \bigcap_{m \geq n} \big(\ker \phi_m K_I \big) = \Big(\bigcap_{m \geq n} \ker \phi_m \Big) K_I = K_I,$$

using Lemma 2.12.

Clearly $\phi_m(M) = K_I^m$ is true for $m = n$. Now suppose $m > n$. From

$$\{1\} \neq K_I^{m-1} = \phi_{m-1}(M) = \theta_m(\phi_m(M))$$

and mapping (6.3), we conclude $\phi_m(M) = K_I^m$. □

For the following, recall the normal subgroups P_j and Q_j, for $j \geq 0$, of a Wilson group G, defined in Corollary 5.3.

Corollary 6.6. *Let $G = \varprojlim (G_n)_{n \geq 0}$ be the inverse limit of the groups G_n as defined in Section 4.1. In the Wilson construction, assume that the unspecified action of the group G_n, for $n \geq 0$, is taken to be right multiplication on itself.*

For $j \geq 0$, define

$$S_j(I_{d_{j+1}}) = \varprojlim (S_j^n(I_{d_{j+1}}))_{n \to \infty}, \text{ where } \emptyset \neq I_{d_{j+1}} \subseteq \Omega_{d_{j+1}},$$

regarded as subgroups of G.

For $j \geq 1$, define

$$T_j(I_{L_j}) = \varprojlim (T_j^n(I_{L_j}))_{n \to \infty}, \text{ where } \emptyset \neq I_{L_j} \subseteq L_j,$$

and define

$$T_0 = \varprojlim (T_0^n)_{n \to \infty},$$

regarded as subgroups of G.

Then the non-trivial closed subnormal subgroups of G are precisely the groups $S_j(I_{d_{j+1}})$, $T_j(I_{L_j})$ and T_0. In particular, for all I_{d_1}, I_{L_1}, ... , I_{d_n}, I_{L_n}, $I_{d_{n+1}}$, ... , they form chains

$$\ldots \subsetneq S_n(I_{d_{n+1}}) \subseteq P_n \subsetneq T_n(I_{L_n}) \subseteq Q_n \subsetneq S_{n-1}(I_{d_n}) \subseteq P_{n-1} \subsetneq \ldots$$
$$\ldots \subseteq P_1 \subsetneq T_1(I_{L_1}) \subseteq Q_1 \subsetneq S_0(I_{d_1}) \subseteq P_0.$$

The subnormal length in G of the group $S_j(I_{d_{j+1}})$ is

$$\begin{cases} 1 & \text{if } I_{d_{j+1}} = \Omega_{d_{j+1}} \text{ (implying that } S_j(I_{d_{j+1}}) = P_j\text{)}, \\ 2 & \text{if } I_{d_{j+1}} \subsetneq \Omega_{d_{j+1}}. \end{cases}$$

The subnormal length in G of the group $T_j(I_{L_j})$ is

$$\begin{cases} 1 & \text{if } I_{L_j} = L_j \text{ (implying that } T_j(I_{L_j}) = Q_j\text{)}, \\ 2 & \text{if } I_{L_j} \subsetneq L_j. \end{cases}$$

Proof. We apply Lemma 6.5 to the groups G_n, for $n \geq 0$, of Wilson's construction with the specified actions, and to their subnormal subgroups.

For the finite index sets we take the power sets of Ω_{d_j} and L_j, for $1 \leq j \leq n$, and note that $\mathcal{P}^1 = \{1\}$. We remark that arbitrary sets A_1 and A_2 can be made disjoint when the elements $x \in A_1$ and $y \in A_2$ are labelled as $(1, x)$ and $(2, y)$.

Define

$$K_I^n = \begin{cases} S^n_{(i-2)/2}(I_{d_{(i-2)/2+1}}) \text{ if } i \text{ is even,} \\ T^n_{(i-1)/2}(I_{L_{(i-1)/2}}) \text{ if } i \text{ is odd,} \end{cases}$$

where $\emptyset \neq I \in \mathcal{P}^i$ for $i \in \{2, 3, \ldots, 2n+1\}$, and define $K_I^n = T_0^n$ for $\emptyset \neq I \in \mathcal{P}^1$. For each n, these subnormal subgroups of G_n were defined in Theorem 6.4. It was shown that these are all the non-trivial subnormal subgroups of G_n and they form chains.

The definition of the groups K_I^n also shows that the second condition for Lemma 6.5 is satisfied. For $2 \leq i \leq 2n+1$, where $\emptyset \neq I \in \mathcal{P}^i$,

$$\theta_n(K_I^n) = \begin{cases} \theta_n(S^n_{(i-2)/2}(I_{d_{(i-2)/2+1}})) = S^{n-1}_{(i-2)/2}(I_{d_{(i-2)/2+1}}) = K_I^{n-1} \text{ if } i \text{ is even,} \\ \theta_n(T^n_{(i-1)/2}(I_{L_{(i-1)/2}})) = T^{n-1}_{(i-1)/2}(I_{L_{(i-1)/2}}) = K_I^{n-1} \text{ if } i \text{ is odd.} \end{cases}$$

We take $S^{n-1}_{n-1}(I_{d_n})$, $T^{n-1}_n(I_{L_n})$, K_I^{n-1} for $\emptyset \neq I \in \mathcal{P}^{2n}$, and K_I^{n-1} for $\emptyset \neq I \in \mathcal{P}^{2n+1}$ to be the trivial group $\{1\}$. Also $\theta_n(K_I^n) = \theta_n(T_0^n) = T_0^{n-1} = K_I^{n-1}$ for $\emptyset \neq I \in \mathcal{P}^1$. $\square$

Below, Corollary 6.7 tells us which Wilson groups we know to have all their subnormal subgroups closed.

Corollary 6.7. *Let $G = \varprojlim(G_n)_{n \geq 0}$ be the inverse limit of the groups G_n, as defined in Section 4.1, such that $|G_0| > 35!$.*

Every subnormal subgroup of G is closed in G.

Proof. Let K be an abstract subnormal subgroup of G. We argue by induction on the subnormal length l of K in G. So

$$G = N_0 \trianglerighteq N_1 \trianglerighteq \ldots \trianglerighteq N_{l-1} \trianglerighteq N_l = K.$$

For $l = 1$ we have $K \trianglelefteq G$. Since $|G_0| > 35!$, we have that G is finitely generated (see Chapter 9). Applying the result by N. Nikolov and D. Segal [22, Cor. 1.15], the normal subgroup K is closed in G.

Suppose the result holds for $l > 1$. Note that N_{l-1} has subnormal length $l-1$ in G. By induction, the subnormal subgroup N_{l-1} is closed in G. From the classification Corollary 6.17, all the closed subnormal subgroups of a general Wilson group have finite index, therefore N_{l-1} is open in G. Then N_{l-1} is a hereditarily just infinite profinite group, since G is hereditarily just infinite, and also N_{l-1} is finitely generated, see [30, Prop. 4.3.1]. Applying again the result [22, Cor. 1.15], the subnormal subgroup K is closed in N_{l-1} and therefore K is closed in G. $\square$

The following diagram illustrates the chains of subnormal subgroups of Wilson groups constructed such that G_n, for $n \geq 0$, acts on itself by right multiplication. The diagram includes the chain of normal subgroups for any arbitrary Wilson group. Additionally, these chains of subnormal subgroups hold for any Wilson group constructed such that the actions of the non-trivial subnormal subgroups of the groups G_n, for $n \geq 1$, have all their orbits containing at least two elements.

Remark. The subnormal subgroup lattice in Figure 6.1 is very symmetric. However, there are no subnormal subgroups between the groups P_0 and G. This is because G_0 is a simple group. Wilson's construction can be slightly modified to make the lattice more symmetrical. Instead of starting the construction with $G_0 = X_0$, set G_0 to be a direct product of the finite non-abelian simple group X_0. That is $G_0 = X_0^{(d_0)}$. All the previous arguments hold while some extra normal subgroups are produced of the form $P_0 \rtimes X_0^{I_{d_0}}$, where $\emptyset \neq I_{d_0} \subsetneq \{1, 2, \ldots, d_0\}$.

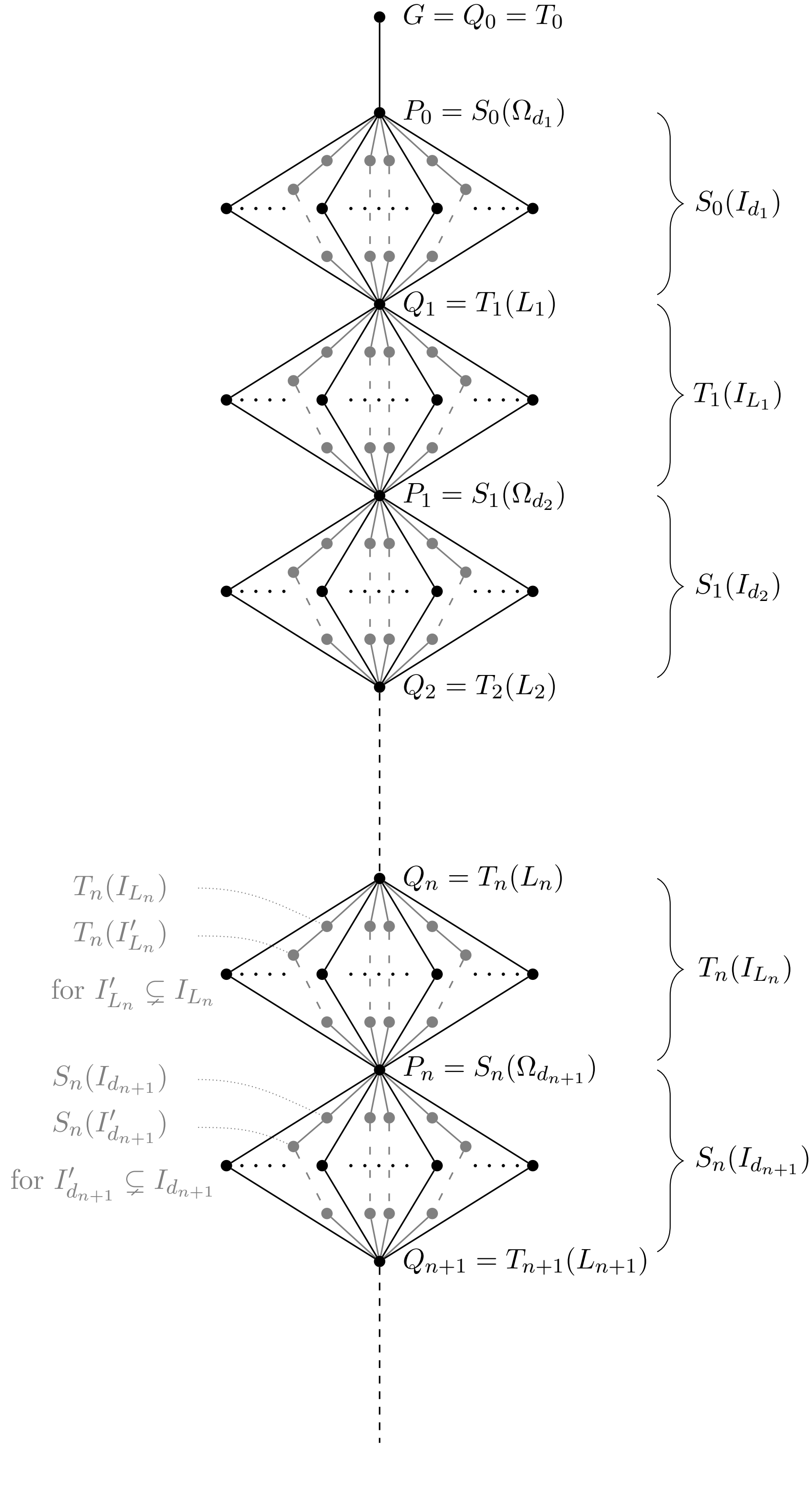

Figure 6.1: The subnormal subgroup lattice of particular Wilson groups.

6.3 Infinite iterated wreath products $\ldots \wr A_m \wr A_m \wr \ldots \wr A_m$, where $m \geq 5$

We recall the just infinite profinite groups W defined in Section 3.2. Fix the alphabet $A = \{1, 2, \ldots, m\}$, where $m \geq 5$. We define the sets

$$\Omega^{*[j]} = \{i_1 i_2 \ldots i_j : i_1, i_2, \ldots, i_j \in A\},$$

for each $j = 1, 2, \ldots$, where $i_1 i_2 \ldots i_j$ denotes a sequence of numbers and not a product. Set $W_0 = A_m$. We form the iterated wreath products

$$W_n = A_m \wr_{\Omega^{*[n]}} \ldots \wr_{\Omega^{*[2]}} A_m \wr_{\Omega^{*[1]}} A_m,$$

for $n \geq 1$. They are the same as the semidirect products $W_n = A_m^{(m^n)} \rtimes W_{n-1}$. A group $W = \varprojlim (W_n)_{n \geq 0}$ is constructed as the inverse limit of a sequence of finite groups $(W_n)_{n \geq 0}$.

The action of a subnormal subgroup of W_{n-1} on m^n elements may have orbits of one element, that is the action has fixed points. Similarly for a general Wilson group G, the action of a subnormal subgroup of G_{n-1} on d_n elements may have orbits of one element. To progress in the characterisation of subnormal subgroups of a general Wilson group, it would be most beneficial to describe the subnormal subgroups of the just infinite groups W.

We recall the permutational wreath products $X \wr_\Omega U$ as defined in Lemma 2.2 of Section 2.1. That is, where U is a finite permutation group acting on a finite set Ω with orbits $\Omega_1, \Omega_2, \ldots, \Omega_r$ and X is a finite non-abelian simple group. We need a generalisation of Proposition 6.2, found in the previous section, which makes no assumption as to the number of elements in each of the U-orbits. That is, a U-orbit can have one element. Proposition 6.8 says that the subnormal subgroups K of $X \wr_\Omega U$ such that $VK = X \wr_\Omega U$ contain all the minimal normal subgroups of $X \wr_\Omega U$ that correspond to orbits which have at least two elements.

Proposition 6.8. *Consider the permutational wreath product $X \wr_\Omega U$ as defined in Lemma 2.2. The base group is denoted $V = \prod_{\omega \in \Omega} X_\omega$, where $X_\omega \cong X$ for all $\omega \in \Omega$. Define*

$$Y = \{(x_\omega)_{\omega \in \Omega} \in V : x_\omega = 1 \text{ if } \omega \cdot U = \{\omega\}\};$$

the notation $\omega \cdot U$ denotes the orbit of ω under the action of the group U. Let G be a subgroup of $X \wr_\Omega U$ such that $Y \subseteq G$ and $VG = X \wr_\Omega U$.

Suppose K is a subnormal subgroup of G such that $VK = X \wr_\Omega U$. Then $Y \subseteq K$.

Proof. We first show that we can reduce to the case where the subnormal subgroup K is normal in G. Since K is subnormal in G, we have $G = T_0 \trianglerighteq T_1 \trianglerighteq \ldots \trianglerighteq T_{k-1} \trianglerighteq T_k = K$. Without loss of generality, suppose this is a shortest chain. This means $G \neq T_1, T_1 \neq T_2, \ldots, T_{k-1} \neq K$.

Consider the beginning of the chain $G \trianglerighteq T_1$. Now $VK = X \wr_\Omega U$ implies $VT_1 = X \wr_\Omega U$. We apply the proposition, which we assume to be true for the special case where the subnormal subgroup is actually normal, to T_1. This gives $Y \subseteq T_1$. Replacing G by T_1 satisfies the conditions of the proposition. We inductively have $Y \subseteq K$.

Let $\Omega_1, \Omega_2, \ldots, \Omega_r$ be the U-orbits which have at least two elements. By Lemma 2.3, we have the corresponding minimal normal subgroups $N_1, N_2, \ldots, N_r$ of $X \wr_\Omega U$. Obviously $Y = N_1 \times N_2 \times \ldots \times N_r$. We now show that $N_1, N_2, \ldots, N_r$ are also minimal normal subgroups of G. Let $i \in \{1, 2, \ldots, r\}$. Since $N_i \subseteq Y \subseteq G \subseteq X \wr_\Omega U$ and $N_i \trianglelefteq X \wr_\Omega U$, we have $N_i \trianglelefteq G$.

Next we show that N_i is minimal normal in G. For this we need that the normal closure in G of any non-trivial element $\underline{x} = (x_\omega)_{\omega \in \Omega} \in N_i$ is equal to N_i. Choose $\omega_1 \in \Omega_i$ such that $x_{\omega_1} \neq 1$. We follow argument $(*)$ in the proof of Lemma 2.3, which supplies an element $\underline{y}$ with certain properties. Noting that since Ω_i is a U-orbit which has at least two elements, we have $\underline{y} \in Y \subseteq G$. We take the normal closure of $[\underline{x}, \underline{y}]$ in Y to gain $V_{\omega_1} \subseteq \langle \underline{x} \rangle^G$. For all $\omega_2 \in \Omega_i$ with $\omega_1 \neq \omega_2$ we can find $u \in U$ such that $\omega_1 \cdot u = \omega_2$ because U acts transitively on Ω_i. As $VG = X \wr_\Omega U$, we can obtain $\underline{v} \in V$ such that $\underline{v}u \in G$. Then $V_{\omega_2} = V_{\omega_1}^u = V_{\omega_1}^{\underline{v}u} \subseteq \langle \underline{x} \rangle^G$.

To prove $Y \subseteq K$, it is sufficient to show that each of the minimal normal subgroups $N_1, N_2, \ldots, N_r$ of G is contained in K. Let $i \in \{1, 2, \ldots, r\}$. Let $\omega_1, \omega_2 \in \Omega_i$ such that ω_1 and ω_2 are distinct. We can find $u \in U$ such that $\omega_1 \cdot u = \omega_2$ because U acts transitively on Ω_i. As $VK = X \wr_\Omega U$, we can obtain $\underline{x} \in V$ such that $u\underline{x} = u(x_\omega)_{\omega \in \Omega} \in K$. Choose $y \in X \backslash \{1\}$ and consider $\underline{y} = (y_\omega) \in Y \subseteq G$ with $y_\omega = y$ if $\omega = \omega_1$ and $y_\omega = 1$ otherwise.

Then $[\underline{y}, u\underline{x}] \in K$ is similarly written as the element (2.1), in the final paragraph of the proof for Lemma 2.3. Now we know that K contains a non-trivial element from N_i. We have found that N_i is a minimal normal subgroup of G and since K is a normal subgroup of G, we have N_i is contained in K. □

The following corollary is a special case of Proposition 6.8, regarding subnormal subgroups for a particular group G.

Corollary 6.9. *Consider the permutational wreath product $X \wr_\Omega U$ as defined in Lemma 2.2. The base group is denoted $V = \prod_{\omega \in \Omega} X_\omega$, where $X_\omega \cong X$ for all $\omega \in \Omega$.*

Define

$$Y = \{(x_\omega)_{\omega \in \Omega} \in V : x_\omega = 1 \text{ if } \omega \cdot U = \{\omega\}\}\,;$$

the notation $\omega \cdot U$ denotes the orbit of ω under the action of the group U.

Suppose K is a subnormal subgroup of $X \wr_\Omega U$ such that $VK = X \wr_\Omega U$. Then $Y \subseteq K$.

Proof. Apply Proposition 6.8 where $G = X \wr_\Omega U$. □

Theorem 6.10 determines the subnormal subgroups of any arbitrary group W_n, for $n \geq 0$. In the proof, Corollary 6.9 is applied to the circumstance where U is taken to be a subnormal subgroup of W_{n-1} acting on m^n elements.

At the end of this section, Corollary 6.11 completely classifies the subnormal subgroups of the inverse limits W of the finite groups W_n. The characterisation covers all the subnormal subgroups of the groups W, as shown by Corollary 6.12. Then Figure 6.2, also at the end of this section, gives a pictorial description of one such subnormal subgroup.

For the description of subnormal subgroups, we now define some new notation which is required. The reader can refer to Figure 3.1, in Section 3.2, to visualise the geometric meaning of these concepts.

As before, fix the alphabet $A = \{1, 2, \ldots, m\}$. We have the set

$$\Omega^{*[j]} = \{i_1 i_2 \ldots i_j : i_1, i_2, \ldots, i_j \in A\}\,,$$

for each $j = 1, 2, \ldots$, which can be interpreted as the vertices on the jth layer of the m-regular rooted tree. In particular, this means that $\Omega^{*[0]} = \{\emptyset\}$.

For $j = 1, 2, \ldots$, denote the orbits of the base group $A_m^{(m^j)}$ of W_j, acting on the $(j+1)$th layer $\Omega^{*[j+1]}$, as

$$\Omega^{*[j+1]}_{i_1 i_2 \ldots i_j} = \{i_1 i_2 \ldots i_j i_{j+1} : i_{j+1} \in A\}\,,$$

where $i_1, i_2, \ldots, i_j \in A$ are fixed. In particular, this means that the orbit of $W_0 = A_m$ acting on $\Omega^{*[1]}$ is $\Omega^{*[1]}_\emptyset = \{1, 2, \ldots, m\}$.

For the following, recall the normal subgroups V_j^n, for $j \in \{1, 2, \ldots, n+1\}$, and V_0^n of W_n, defined in Theorem 3.2.

Theorem 6.10. *Let W_n, for $n \geq 0$, be the finite groups as defined in Section 3.2. For $j \in \{1, 2, \ldots, n\}$, define*

$$U_j^n(I_{*[j]}, I_{*[j+1]}, \ldots, I_{*[n]}) = A_m^{I_{*[n]} \cup \Delta_{*[n]}} \rtimes \ldots \rtimes (A_m^{I_{*[j+1]} \cup \Delta_{*[j+1]}} \rtimes A_m^{I_{*[j]}}) \leq W_n,$$

where

$$\begin{aligned}
& & \emptyset \neq I_{*[j]} &\subseteq \Omega^{*[j]},\\
\Delta_{*[j+1]} &= \bigcup_{i_1 i_2 \ldots i_j \in I_{*[j]}} \Omega^{*[j+1]}_{i_1 i_2 \ldots i_j}, & I_{*[j+1]} &\subseteq \Omega^{*[j+1]} \backslash \Delta_{*[j+1]},\\
\Delta_{*[j+2]} &= \bigcup_{i_1 i_2 \ldots i_{j+1} \in \Delta_{*[j+1]} \cup I_{*[j+1]}} \Omega^{*[j+2]}_{i_1 i_2 \ldots i_{j+1}}, & I_{*[j+2]} &\subseteq \Omega^{*[j+2]} \backslash \Delta_{*[j+2]},\\
&\vdots & &\vdots\\
\Delta_{*[n]} &= \bigcup_{i_1 i_2 \ldots i_{n-1} \in \Delta_{*[n-1]} \cup I_{*[n-1]}} \Omega^{*[n]}_{i_1 i_2 \ldots i_{n-1}}, & I_{*[n]} &\subseteq \Omega^{*[n]} \backslash \Delta_{*[n]},
\end{aligned}$$

and define

$$U^n_{n+1} = \{1\}$$

and

$$U^n_0 = W_n.$$

Then the subnormal subgroups of W_n are precisely the groups U^n_{n+1}, U^n_0 and $U^n_j(I_{*[j]}, I_{*[j+1]}, \ldots, I_{*[n]})$.

The subnormal length in W_n of the group $U^n_j(I_{*[j]}, I_{*[j+1]}, \ldots, I_{*[n]})$ is bounded above by $n - j + 2$. (See Theorem 6.14, later, which gives a recursive formula for the exact subnormal length.)

Proof. We first check that the groups $U^n_j(I_{*[j]}, I_{*[j+1]}, \ldots, I_{*[n]})$, U^n_{n+1} and U^n_0 are all subnormal subgroups of W_n. Obviously $U^n_{n+1} = \{1\} \lhd W_n$ and $U^n_0 = W_n \trianglelefteq W_n$.

We claim

$$\begin{aligned}
U^n_j(I_{*[j]}, I_{*[j+1]}, \ldots, I_{*[n]}) &\trianglelefteq A_m^{(m^n)} U^{n-1}_j(I_{*[j]}, I_{*[j+1]}, \ldots, I_{*[n-1]})\\
&\trianglelefteq A_m^{(m^n)} A_m^{(m^{n-1})} U^{n-2}_j(I_{*[j]}, I_{*[j+1]}, \ldots, I_{*[n-2]}) \trianglelefteq \ldots\\
&\trianglelefteq A_m^{(m^n)} A_m^{(m^{n-1})} \ldots A_m^{(m^{j+1})} U^j_j(I_{*[j]}) \trianglelefteq V^n_j \lhd W_n. \quad (6.4)
\end{aligned}$$

It is sufficient to show, for $k \in \{j+1, j+2, \ldots, n\}$, that

$$\begin{aligned}
&U^k_j(I_{*[j]}, I_{*[j+1]}, \ldots, I_{*[k]})\\
&\quad = A_m^{I_{*[k]} \cup \Delta_{*[k]}} \rtimes (A_m^{I_{*[k-1]} \cup \Delta_{*[k-1]}} \rtimes \ldots \rtimes (A_m^{I_{*[j+1]} \cup \Delta_{*[j+1]}} \rtimes A_m^{I_{*[j]}}))\\
&\quad \trianglelefteq A_m^{(m^k)} \rtimes (A_m^{I_{*[k-1]} \cup \Delta_{*[k-1]}} \rtimes \ldots \rtimes (A_m^{I_{*[j+1]} \cup \Delta_{*[j+1]}} \rtimes A_m^{I_{*[j]}}))\\
&\quad = A_m^{(m^k)} U^{k-1}_j(I_{*[j]}, I_{*[j+1]}, \ldots, I_{*[k-1]}).
\end{aligned}$$

Put

$$U_j^{k-1}(I_{*[j]}, I_{*[j+1]}, \ldots, I_{*[k-1]}) =: U.$$

From Lemma 2.3, we see that $A_m^{I_{*[k]} \cup \Delta_{*[k]}}$ is a product of some minimal normal subgroups of $A_m^{(m^k)} \rtimes U$ and so $A_m^{I_{*[k]} \cup \Delta_{*[k]}}$ is normal in $A_m^{(m^k)} \rtimes U$. It is left to show $[A_m^{(m^k)}, U] \subseteq A_m^{I_{*[k]} \cup \Delta_{*[k]}}$. This holds as U moves points in the set $\Delta_{*[k]}$ and fixes points in the sets $I_{*[k]}$ and $\Omega^{*[k]} \setminus (I_{*[k]} \cup \Delta_{*[k]})$.

The subnormal length in W_n of any group $U_j^n(I_{*[j]}, I_{*[j+1]}, \ldots, I_{*[n]})$ is $\leq n - j + 2$ because the subnormal series (6.4) has length $n - j + 2$.

We now prove, by induction on n, that every subnormal subgroup of W_n is one of the groups listed. For $n = 0$, all the subnormal subgroups of W_0 are $\{1\} = U_1^0$ and $W_0 = U_0^0$ holds as W_0 is simple.

Suppose that the result holds for W_{n-1}. Now we prove the result for W_n. Let K be a subnormal subgroup of W_n. Then there are two cases:

$$K \subseteq A_m^{(m^n)} \text{ (case 1), and } K \not\subseteq A_m^{(m^n)} \text{ (case 2).}$$

Case 1.

For $K \subseteq A_m^{(m^n)}$, we have K is subnormal in $A_m^{(m^n)}$. There are two possibilities, either $K = \{1\} = U_{n+1}^n$, or, using Theorem 2.4, we have $K = A_m^{I_{*[n]}} = U_n^n(I_{*[n]})$, for some $\emptyset \neq I_{*[n]} \subseteq \Omega^{*[n]}$.

Case 2.

Now suppose $K \not\subseteq A_m^{(m^n)}$. We know $\{1\} \ncong A_m^{(m^n)} K / A_m^{(m^n)}$ is a subnormal subgroup of $W_n / A_m^{(m^n)} \cong W_{n-1}$. Then, by induction, we have

$$A_m^{(m^n)} K / A_m^{(m^n)} \cong U_j^{n-1}(I_{*[j]}, I_{*[j+1]}, \ldots, I_{*[n-1]}),$$

for some $j \in \{1, 2, \ldots, n-1\}$, or $A_m^{(m^n)} K / A_m^{(m^n)} \cong U_0^{n-1}$.

We denote this isomorphic copy of $A_m^{(m^n)} K / A_m^{(m^n)}$ in W_{n-1} by U. Then $A_m^{(m^n)} K = A_m^{(m^n)} \rtimes U$. If $U = U_j^{n-1}(I_{*[j]}, I_{*[j+1]}, \ldots, I_{*[n-1]})$, for some $j \in \{1, 2, \ldots, n-1\}$, then $\Delta_{*[j+1]}, \Delta_{*[j+2]}, \ldots, \Delta_{*[n]}$ are all defined. If $U = U_0^{n-1}$ then we set $\Delta_{*[n]} = \Omega^{*[n]}$. The elements of the set $\Omega^{*[n]} \setminus \Delta_{*[n]}$ are fixed points for the action of U on $\Omega^{*[n]}$. Also $K \subseteq A_m^{(m^n)} U$ and so K is subnormal in $A_m^{(m^n)} U$. Corollary 6.9 gives

$$\Big\{ (x_\omega)_{\omega \in \Omega^{[n]}} \in A_m^{(m^n)} : x_\omega = 1 \text{ if } \omega \cdot U = \{\omega\} \Big\} = A_m^{\Delta_{*[n]}} \subseteq K.$$

We have found that

$$A_m^{\Omega^{*[n]}\backslash\Delta_{*[n]}}K = A_m^{\Omega^{*[n]}\backslash\Delta_{*[n]}}A_m^{\Delta_{*[n]}}U. \tag{6.5}$$

To finalise the characterisation of K, observe that $K \cap A_m^{\Omega^{*[n]}\backslash\Delta_{*[n]}}U$ is a subnormal subgroup of $A_m^{\Omega^{*[n]}\backslash\Delta_{*[n]}}U \cong A_m^{\Omega^{*[n]}\backslash\Delta_{*[n]}} \times U$ and that it projects onto the factor U. Using Lemma 2.4, there exists some subset $I_{*[n]} \subseteq \Omega^{*[n]}\backslash\Delta_{*[n]}$ such that $K \cap A_m^{\Omega^{*[n]}\backslash\Delta_{*[n]}}U = A_m^{I_{*[n]}}U \cong A_m^{I_{*[n]}} \times U$. From this and the fact that $A_m^{\Delta_{*[n]}} \leq K$, we establish $K = A_m^{I_{*[n]}}A_m^{\Delta_{*[n]}}U$. Thus $K = U_j^n(I_{*[j]}, I_{*[j+1]}, \ldots, I_{*[n]})$, for some $j \in \{1, 2, \ldots, n-1\}$, or $K = U_0^n$.

□

For the following, recall the normal subgroups V_j, for $j \geq 0$, of a group W, defined in Corollary 3.3.

Corollary 6.11. *Let $W = \varprojlim(W_n)_{n\geq 0}$ be the inverse limit of the groups W_n as defined in Section 3.2. For $j \geq 1$, define*

$$U_j(I_{*[j]}, I_{*[j+1]}, I_{*[j+2]}, \ldots) = \varprojlim(U_j^n(I_{*[j]}, I_{*[j+1]}, \ldots, I_{*[n]}))_{n\to\infty},$$

where

$$\begin{aligned}
& & \emptyset \neq I_{*[j]} &\subseteq \Omega^{*[j]},\\
\Delta_{*[j+1]} &= \bigcup_{i_1 i_2 \ldots i_j \in I_{*[j]}} \Omega^{*[j+1]}_{i_1 i_2 \ldots i_j}, & I_{*[j+1]} &\subseteq \Omega^{*[j+1]}\backslash\Delta_{*[j+1]},\\
\Delta_{*[j+2]} &= \bigcup_{i_1 i_2 \ldots i_{j+1} \in \Delta_{*[j+1]} \cup I_{*[j+1]}} \Omega^{*[j+2]}_{i_1 i_2 \ldots i_{j+1}}, & I_{*[j+2]} &\subseteq \Omega^{*[j+2]}\backslash\Delta_{*[j+2]},\\
&\vdots & &\vdots
\end{aligned}$$

and define

$$U_0 = \varprojlim(U_0^n)_{n\to\infty},$$

regarded as subgroups of W.

Then the non-trivial closed subnormal subgroups of W are precisely the groups $U_j(I_{*[j]}, I_{*[j+1]}, I_{*[j+2]}, \ldots)$ and U_0.

The subnormal length in W of the group $U_j(I_{*[j]}, I_{*[j+1]}, I_{*[j+2]}, \ldots)$ is bounded above by $n - j + 2$ for $I_{*[n]} \subsetneq \Omega^{*[n]}\backslash\Delta_{*[n]}$ and $I_{*[n+1]} = \Omega^{*[n+1]}\backslash\Delta_{*[n+1]}$.

Proof. Let K be a non-trivial closed subnormal subgroup of W. The profinite group W has the chain of open normal subgroups $\ldots \subsetneq V_2 \subsetneq V_1 \subsetneq V_0 = W$, see [26, Thm. 2.1.3]. These open normal subgroups form a base for the topology on W. Therefore, as K is a closed subgroup, we have $K = \varprojlim (V_i K/V_i)_{i \to \infty}$, refer to [30, Thm. 1.2.5 (a)]. From Theorem 6.10, we know that $V_i K/V_i$ is determined by a finite chain of sets $I_{*[j]}$, $I_{*[j+1]}$, $\ldots$, $I_{*[i-1]}$, for some $j \in \{0, 1, 2, \ldots, i\}$. Thus K is parametrised by the infinite chain of sets $I_{*[j]}$, $I_{*[j+1]}$, $I_{*[j+2]}$, $\ldots$. □

Remark (regarding the proof of Corollary 6.11). The infinite iterated wreath product W, constructed from alternating groups A_m, can be encoded differently using m-adic integers. (Refer to Section 2.5 for a description of the m-adic integers.)

The group W is viewed as acting naturally on the infinite m-regular rooted tree, that is where every vertex has m children (see P. de la Harpe [6, pg. 211-212]). However, each path of the tree corresponds uniquely to an element of the m-adic integers $\mathbb{Z}_m$. Therefore the collection of all these paths is $\mathbb{Z}_m$.

For $U_j(I_{*[j]}, I_{*[j+1]}, I_{*[j+2]}, \ldots)$, we can now think of each $I_{*[j]}$, $I_{*[j+1]}$, $I_{*[j+2]}$, $\ldots$ as prescribing a subset of the m-adic integers. In particular, each of these subsets of the m-adic integers is a union of cosets because everything from some point onwards is included. Unions of cosets are exactly the open subsets of $\mathbb{Z}_m$. Therefore $I_{*[j]}$, $I_{*[j+1]}$, $I_{*[j+2]}$, $\ldots$ can be interpreted as open subsets of $\mathbb{Z}_m$.

Corollary 6.11 can be proved from knowing that the m-adic integers has an infinite number of open subsets. Whether one subnormal subgroup is contained in another can be read off from the index sets $I_{*[j]}$, $I_{*[j+1]}$, $I_{*[j+2]}$, $\ldots$. In the new interpretation of W, one subnormal subgroup is contained in another when its open sets are contained in the others open sets.

Below, Corollary 6.12 tells us that all the subnormal subgroups of the groups W are closed. For this we note, a normal subgroup N of a profinite group G is *virtually dense* in G if the closure of N is open in G.

Corollary 6.12. *Let* $W = \varprojlim (W_n)_{n \geq 0}$ *be the inverse limit of the groups* W_n *as defined in Section 3.2.*

Every subnormal subgroup of W *is closed in* W*.*

Proof. Let K be an abstract subnormal subgroup of W. We argue by induction on the subnormal length l of K in W. So

$$W = N_0 \trianglerighteq N_1 \trianglerighteq \ldots \trianglerighteq N_{l-1} \trianglerighteq N_l = K.$$

For $l = 1$ we have $K \trianglelefteq W$. Applying the result by N. Nikolov and D. Segal [22, Cor. 1.15], the normal subgroup K is closed in W.

Suppose the result holds for $l > 1$. Note that N_{l-1} has subnormal length $l - 1$ in W. By induction, the subnormal subgroup N_{l-1} is closed in W. From the classification Corollary 6.11, all the closed subnormal subgroups of W have finite index, therefore N_{l-1} is open in W. Thus N_{l-1} is a finitely generated profinite group, see [30, Prop. 4.3.1].

Consider $K \trianglelefteq N_{l-1}$. The closure of K in N_{l-1} has finite index in N_{l-1}, by Corollary 6.11, and so K is a virtually dense normal subgroup of N_{l-1}.

Let U be an open subgroup of N_{l-1}. Then U is a finitely generated profinite group. So all its finite quotients are continuous quotients, refer to [21]. The normal subgroup $\mathrm{Core}_W(U) = \bigcap_{g \in W} U^g$ is open in W, as U is open in W and by Lemma 2.10. Therefore $\mathrm{Core}_W(U)$ has finite index in U, using Lemma 2.10.

If U had an infinite abelian quotient then $\mathrm{Core}_W(U)$ would have an infinite abelian quotient. But the only composition factors of finite quotients of $\mathrm{Core}_W(U)$ are isomorphic to A_m because all the composition factors of W are isomorphic to A_m.

If U had a quotient isomorphic to an infinite product of non-abelian finite simple groups then $\mathrm{Core}_W(U)$ would map onto an infinite product of non-abelian finite simple groups, using Lemma 2.4. But then $\mathrm{Core}_W(U)$ must map onto arbitrarily long products $A_m \times A_m \times \ldots \times A_m$ and so $\mathrm{Core}_W(U)$ cannot be finitely generated.

Finally, U cannot map onto any connected Lie groups because U is totally disconnected, see [30, Cor. 1.2.4 (iv)]. Thus the theorem of N. Nikolov and D. Segal [22, Thm. 1.14] implies that K has finite index in N_{l-1}. So K is open in N_{l-1}, refer to [21], and hence K is closed in W, using Lemma 2.10. $\square$

It is standard to view the group W as acting on the infinite m-regular rooted tree, where every vertex has m children. P. de la Harpe [6, pg. 211-212] gives an introduction to groups acting on these trees. Taking $m = 5$, we now use this tree to illustrate an example of a subnormal subgroup of W. The following diagram is a pictorial description of the subnormal subgroup

$$\begin{aligned}
\ldots \rtimes \Big(A_5^{(5^3)} \times A_5^{(5^3)} \times A_5^{(5^3)} \times A_5^{(5^3)} \times A_5^{(5^3)} \Big) \\
\rtimes \Big(A_5^{(5^2)} \times (\{1\} \times A_5 \times A_5 \times A_5 \times A_5) \times A_5^{(5)} \times A_5^{(5)} \\
\times (A_5 \times A_5 \times \{1\} \times A_5 \times A_5) \times A_5^{(5)} \times A_5^{(5^2)} \times A_5^{(5^2)} \times A_5^{(5^2)} \Big) \\
\rtimes \Big(A_5^{(5)} \times (\{1\} \times A_5 \times A_5 \times \{1\} \times \{1\}) \times A_5^{(5)} \times A_5^{(5)} \times A_5^{(5)} \Big) \\
\rtimes (A_5 \times \{1\} \times A_5 \times A_5 \times A_5)
\end{aligned}$$

of W. It is represented by the black squares being the index sets which select the factors A_5 of the subnormal subgroup.

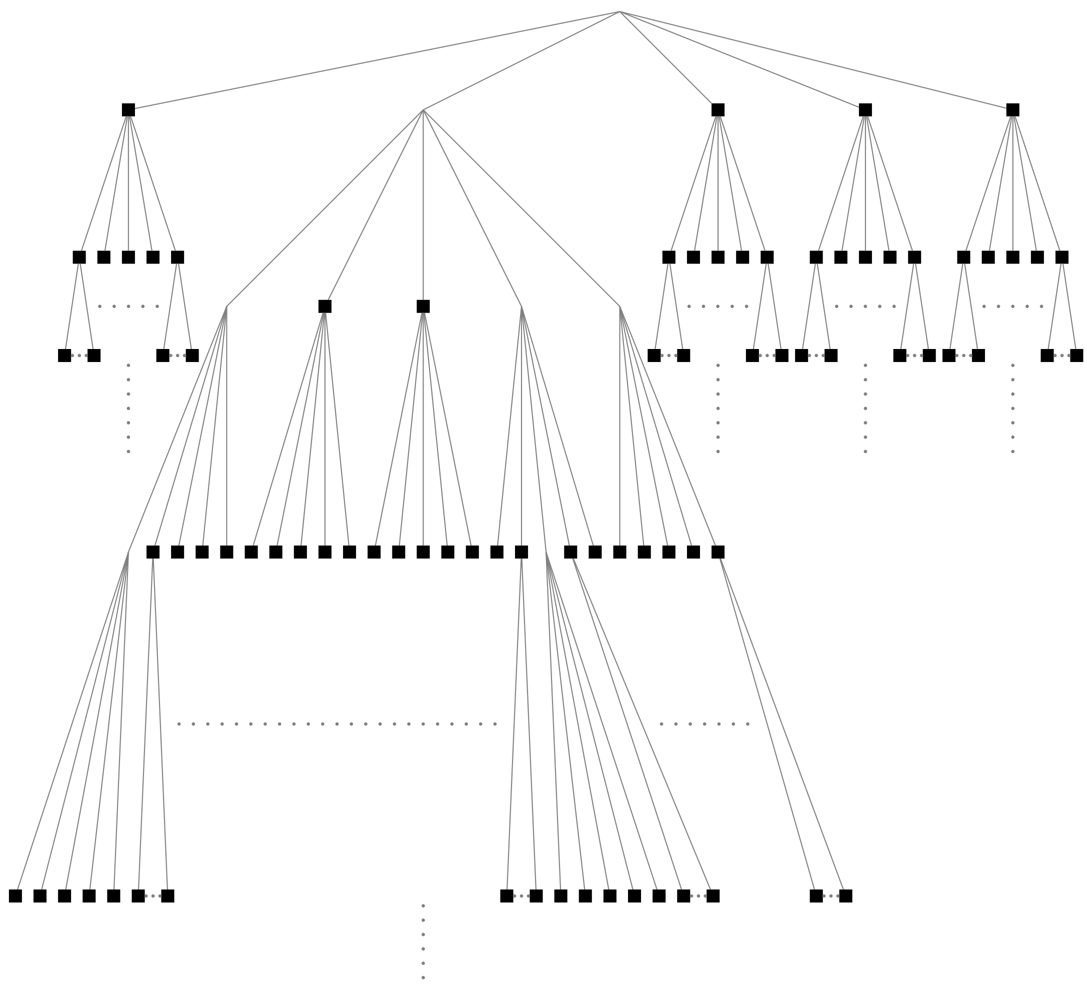

Figure 6.2: A subnormal subgroup of W represented on the infinite 5-regular rooted tree.

6.3.1 The subnormal length

We have only seen an upper bound for the subnormal length in the profinite groups W, refer to Theorem 6.10. The exact subnormal length of a subnormal subgroup of W_n, and hence of W, is given by the recursive formula in Theorem 6.14. Later, we see some examples, Figure 6.3 and Figure 6.4, to show how the formula works.

First, Lemma 6.13, below, is required. As a consequence of this lemma, the subnormal subgroups of a direct product of iterated wreath products of non-abelian simple groups are similarly direct products of the same form.

Lemma 6.13. *Let W_{n_i}, for $n_i \geq 0$, be the groups as defined in Section 3.2. Recall the normal subgroups $V_{j_i}^{n_i}$, for $j_i \in \{1, 2, \ldots, n_i + 1\}$, and $V_0^{n_i}$ of W_{n_i}, defined in Theorem 3.2.*

The normal subgroups of any direct product

$$W_{n_1} \times W_{n_2} \times \ldots \times W_{n_r}$$

are precisely the groups

$$V_{j_1}^{n_1} \times V_{j_2}^{n_2} \times \ldots \times V_{j_r}^{n_r}.$$

Proof. Let N be normal subgroup of $W_{n_1} \times W_{n_2} \times \ldots \times W_{n_r}$. The normal subgroup N projects onto a normal subgroup, say $V_{j_i}^{n_i}$, in the ith factor W_{n_i}. Clearly

$$N \subseteq V_{j_1}^{n_1} \times V_{j_2}^{n_2} \times \ldots \times V_{j_r}^{n_r}.$$

We claim

$$N \supseteq V_{j_1}^{n_1} \times V_{j_2}^{n_2} \times \ldots \times V_{j_r}^{n_r}.$$

It suffices to show, for all $i \in \{1, 2, \ldots, r\}$,

$$N \supseteq \{1\} \times \ldots \times \{1\} \times V_{j_i}^{n_i} \times \{1\} \times \ldots \times \{1\}.$$

Suppose $V_{j_i}^{n_i} \neq \{1\}$. Since N projects onto $V_{j_i}^{n_i}$ in the W_{n_i} factor, there is an

$$x = (x_1, x_2, \ldots, x_r) \in N$$

such that $x_i \in V_{j_i}^{n_i}$ but $x_i \notin V_{j_i+1}^{n_i}$.

By considering $[x, y]^z$ for elements $y = (1, \ldots, 1, y_i, 1, \ldots, 1)$ with $y_i \in W_{n_i}$ in the ith position and arbitrary z, we see that N contains the subgroup

$$\{1\} \times \ldots \times \{1\} \times \langle [x_i, W_{n_i}] \rangle^{W_{n_i}} \times \{1\} \times \ldots \times \{1\}.$$

By the classification of normal subgroups of W_{n_i}, Theorem 3.2, it is left to show that there exists $y_i \in W_{n_i}$ such that $[x_i, y_i] \notin V^{n_i}_{j_i+1}$.

Now $V^{n_i}_{j_i}/V^{n_i}_{j_i+1} \cong A_m^N$ for some N and $x_i \not\equiv 1 \bmod V^{n_i}_{j_i+1}$ gives a non-trivial element of the factor group. As $Z(V^{n_i}_{j_i}/V^{n_i}_{j_i+1}) \cong Z(A_m^N) \cong Z(A_m)^N = 1$, hence there exists $y_i \in V^{n_i}_{j_i}$ such that $[x_i, y_i] \not\equiv 1 \bmod V^{n_i}_{j_i+1}$ and this y_i will do. □

For the purpose of the following formula, set $I_{*[i]} = \emptyset$, for $i < j$, and $\Delta_{*[i]} = \emptyset$, for $i \le j$, and $I_{*[n+1]} \cup \Delta_{*[n+1]} = \Omega^{*[n+1]}$.

Theorem 6.14. *Let W_n, for $n \ge 0$, be the finite groups as defined in Section 3.2. For $j \in \{1, 2, \ldots, n\}$, recall the subnormal subgroups*

$$U^n_j(I_{*[j]}, I_{*[j+1]}, \ldots, I_{*[n]}) = A_m^{I_{*[n]} \cup \Delta_{*[n]}} \rtimes \ldots \rtimes (A_m^{I_{*[j+1]} \cup \Delta_{*[j+1]}} \rtimes A_m^{I_{*[j]}}),$$

where

$$\begin{aligned}
& & \emptyset \ne I_{*[j]} &\subseteq \Omega^{*[j]}, \\
\Delta_{*[j+1]} &= \bigcup_{i_1 i_2 \ldots i_j \in I_{*[j]}} \Omega^{*[j+1]}_{i_1 i_2 \ldots i_j}, & I_{*[j+1]} &\subseteq \Omega^{*[j+1]} \backslash \Delta_{*[j+1]}, \\
\Delta_{*[j+2]} &= \bigcup_{i_1 i_2 \ldots i_{j+1} \in \Delta_{*[j+1]} \cup I_{*[j+1]}} \Omega^{*[j+2]}_{i_1 i_2 \ldots i_{j+1}}, & I_{*[j+2]} &\subseteq \Omega^{*[j+2]} \backslash \Delta_{*[j+2]}, \\
&\vdots & &\vdots \\
\Delta_{*[n]} &= \bigcup_{i_1 i_2 \ldots i_{n-1} \in \Delta_{*[n-1]} \cup I_{*[n-1]}} \Omega^{*[n]}_{i_1 i_2 \ldots i_{n-1}}, & I_{*[n]} &\subseteq \Omega^{*[n]} \backslash \Delta_{*[n]},
\end{aligned}$$

of W_n, as defined in Theorem 6.10.

The subnormal length of $U^n_j(I_{*[j]}, I_{*[j+1]}, \ldots, I_{*[n]})$ in W_n is given by the formula

$$\max_{i_1 i_2 \ldots i_n} |\{l_0, l_1, \ldots, l_n\}|,$$

where $i_1 i_2 \ldots i_n$ runs through all paths in the rooted tree up to level n, and

$$l_0 = l(\emptyset),$$

$$l_{r+1} = l_r + l(i_1 i_2 \ldots i_{l_r}),\ \textit{for}\ 0 \le r < n,$$

with

$$l(i_1 i_2 \ldots i_k) = \min\bigl\{l \mid 0 \le l \le n+1-k \text{ such that } \\ \exists i'_{k+1}, i'_{k+2}, \ldots, i'_{k+l} : i_1 i_2 \ldots i_k i'_{k+1} i'_{k+2} \ldots i'_{k+l} \in I_{*[k+l]} \cup \Delta_{*[k+l]}\bigr\}.$$

Proof. We prove the formula for the subnormal length in W_n by induction on n. For $n = 1$, the subnormal subgroups $U_1^1(I_{*[1]})$ of W_1 have subnormal length

$$\begin{cases} 1 & \text{if } I_{*[1]} = \Omega^{*[1]} \text{ (implying that } U_1^1(I_{*[1]}) = V_1^1), \\ 2 & \text{if } I_{d_{j+1}} \subsetneq \Omega^{*[1]}, \end{cases}$$

which are the same lengths given by the formula.

Suppose the formula holds for W_m, for $m < n$. Now we prove the formula for W_n. Let $U_j^n(I_{*[j]}, I_{*[j+1]}, \ldots, I_{*[n]})$ be a subnormal subgroup of W_n and we denote $U_j^n(I_{*[j]}, I_{*[j+1]}, \ldots, I_{*[n]}) =: U$. The unique smallest normal subgroup of W_n that contains the subnormal subgroup U is V_j^n. The subnormal length of U in W_n is equal to

$$1 + \text{ the subnormal length of } U \text{ in } V_j^n.$$

Notice

$$V_j^n \cong \underbrace{W_{n-j} \times W_{n-j} \times \ldots \times W_{n-j}}_{m^j \text{ times}}.$$

By Lemma 6.13, there is a unique smallest normal subgroup N_1 of $W_{n-j} \times W_{n-j} \times \ldots \times W_{n-j}$ containing the subnormal subgroup U. Since N_1 is isomorphic to a direct product of groups of the form W_{n_i}, using Lemma 6.13, there is again a unique smallest normal subgroup N_2 of N_1 containing U and we descend so on. The formula for the subnormal length records how many steps this procedure requires until we reach U.

The subnormal length of U in $V_j^n \cong W_{n-j} \times W_{n-j} \times \ldots \times W_{n-j}$ is computed recursively as the maximum of the subnormal lengths of the intersection of U with each factor isomorphic to W_{n-j} in that factor isomorphic to W_{n-j}. The possible choices for descending to such factors are parameterized by the paths $i_1 i_2 \ldots i_n$. $\square$

We apply the formula of Theorem 6.14, below, to calculate the subnormal length for two examples of subnormal subgroups of W_5. The subnormal subgroups are illustrated using the simpler 2-regular rooted tree, since the formula does not depend on the degree of the alternating groups used to construct W_n.

The subnormal subgroups are represented by the black squares being the index sets which select the factors A_m of the subnormal subgroup. The black dots on the rooted trees remind the reader that for the purpose of the formula we take $I_{*[6]} \cup \Delta_{*[6]} = \Omega^{*[6]}$.

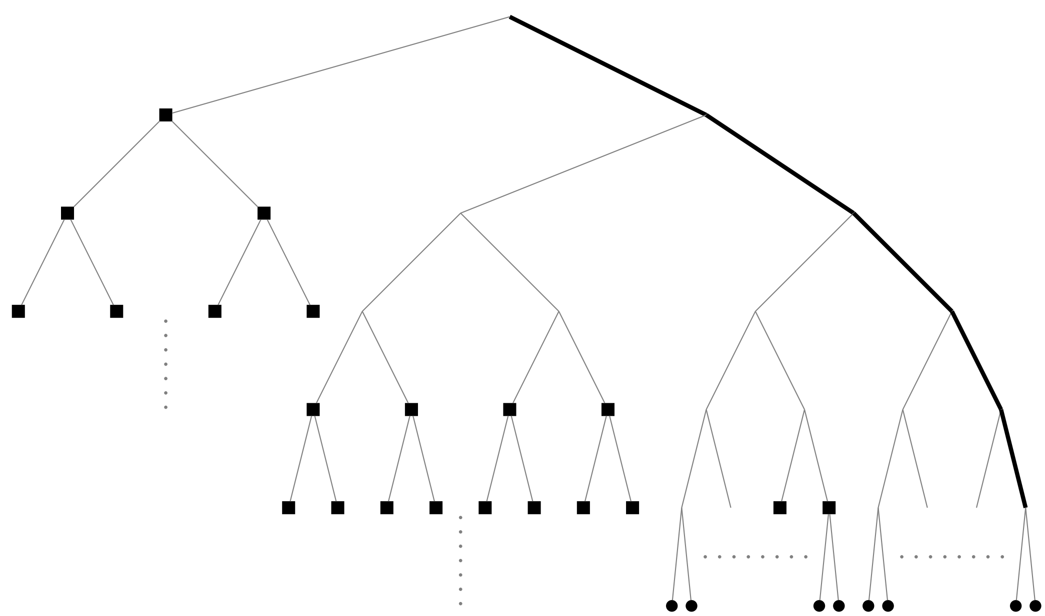

Figure 6.3: A subnormal subgroup $U_1^5(I_{*[1]}, I_{*[2]}, \ldots, I_{*[5]})$ of W_5 represented on the rooted tree of length 6.

Using the formula for the highlighted path $i_1 i_2 \ldots i_5$ on the far right of the tree, in Figure 6.3, gives:

$$\begin{aligned}
l_0 &= l(\emptyset) = 1, \\
l_1 &= l_0 + l(i_1) = 1 + 3 = 4, \\
l_2 &= l_1 + l(i_1 i_2 i_3 i_4) = 4 + 2 = 6, \\
l_3 &= l_2 + l(i_1 i_2 i_3 i_4 i_5 i_6) = 6 + 0 = 6.
\end{aligned}$$

This path produces the maximum $|\{l_0, l_1, \ldots, l_3\}| = 3$, and hence the subnormal length is 3.

The following example is to show how the subnormal length can grow with n. It shows the largest possible subnormal length in W_5.

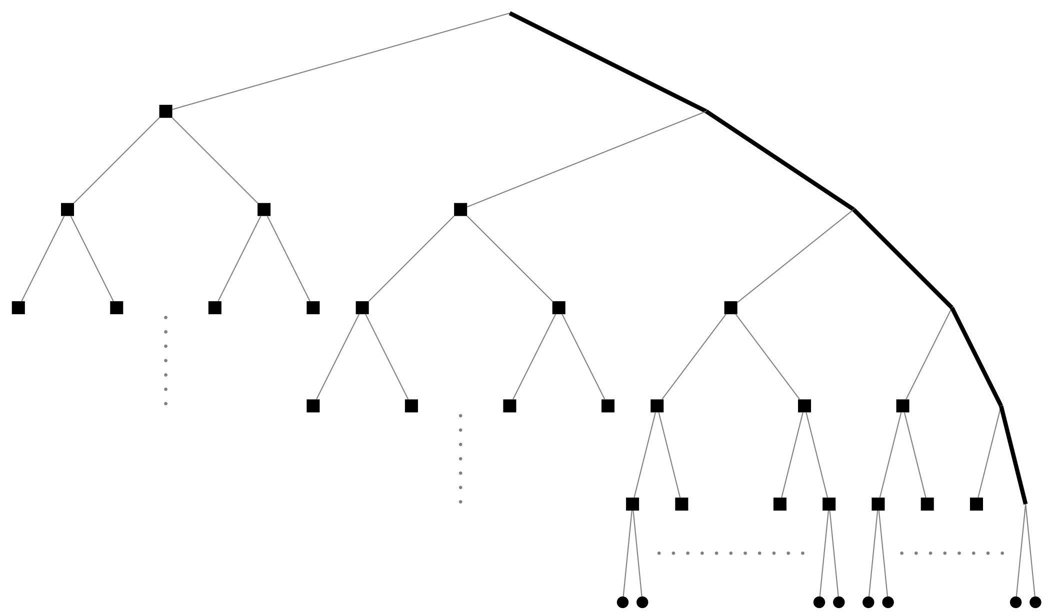

Figure 6.4: A subnormal subgroup of W_5 of subnormal length 6.

Using the formula for the highlighted path $i_1 i_2 \ldots i_5$ on the far right of the tree, in Figure 6.4, gives:

$$\begin{aligned}
l_0 &= l(\emptyset) = 1, \\
l_1 &= l_0 + l(i_1) = 1 + 1 = 2, \\
l_2 &= l_1 + l(i_1 i_2) = 2 + 1 = 3, \\
l_3 &= l_2 + l(i_1 i_2 i_3) = 3 + 1 = 4, \\
l_4 &= l_3 + l(i_1 i_2 i_3 i_4) = 4 + 1 = 5, \\
l_5 &= l_4 + l(i_1 i_2 i_3 i_4 i_5) = 5 + 1 = 6.
\end{aligned}$$

This path produces the maximum $|\{l_0, l_1, \ldots, l_5\}| = 6$, and hence the subnormal length is 6.

6.4 General Wilson groups

In this section, we complete the characterisation of the subnormal subgroups of an arbitrary Wilson group. The characterisation holds for any choice of X_i, for $i \geq 0$, and for any choice of faithful transitive permutation representation of G_n, for $n \geq 1$, in the construction of a Wilson group. Here we do not have the previously imposed restrictions, of Section 6.2, for the groups G_{n-1}, for $n \geq 1$, acting on themselves by right multiplication. Thus the action of a subnormal subgroup of G_{n-1} on d_n elements may have orbits of one element.

Theorem 6.15 determines the subnormal subgroups of the finite groups G_n for the general Wilson construction. To prove this theorem, we apply Corollary 6.9. Taking U to be a subnormal subgroup of G_{n-1} acting on d_n elements, the corollary holds when the subnormal subgroup has orbits of one element.

Corollary 6.17 completely classifies the closed subnormal subgroups of a general Wilson group. Then Corollary 6.7 shows that all subnormal subgroups of a Wilson group are automatically closed provided the first group in Wilson's construction has size $|G_0| > 35!$, and hence the Wilson group is finitely generated (see Chapter 9). Therefore the characterisation of subnormal subgroups, in Corollary 6.17, covers all the subnormal subgroups of any Wilson group provided $|G_0| > 35!$.

At the end of this section, Figure 6.5 gives a pictorial illustration of the subnormal subgroups of a general Wilson group. In comparison with the particular Wilson groups studied in Section 6.2, the subnormal subgroups are still squeezed between normal subgroups, however not consecutively; recall Figure 6.1.

For the following, recall the normal subgroups P_j^n and Q_j^n, for $j \in \{0, 1, \ldots, n\}$, of G_n, defined in Theorem 5.1. We define $L_0 = \{1\}$ for the working of the subsequent proof.

Theorem 6.15. *Let G_n, for $n \geq 0$, be the finite groups as defined in Section 4.1.*

For $j \in \{0, 1, \ldots, n-1\}$, define

$$S_j^n(I_{d_{j+1}}) = Q_{j+1}^n \rtimes X_{j+1}^{I_{d_{j+1}}} \leq P_j^n, \text{ where } \emptyset \neq I_{d_{j+1}} \subseteq \Omega_{d_{j+1}},$$

and define

$$S_n^n = \{1\}.$$

For $j \in \{1, 2, \ldots, n-1\}$, define

$$T_j^n(I_{L_j}, I_{d_{j+1}}) = Q_{j+1}^n \rtimes (X_{j+1}^{I_{d_{j+1}} \cup \Delta_{d_{j+1}}} \rtimes X_j^{I_{L_j}}) \leq Q_j^n, \text{ where } \emptyset \neq I_{L_j} \subseteq L_j,$$

$$\Delta_{d_{j+1}} = \{\omega \in \Omega_{d_{j+1}} : \omega \cdot X_j^{I_{L_j}} \neq \{\omega\}\} \text{ and } I_{d_{j+1}} \subseteq \Omega_{d_{j+1}} \backslash \Delta_{d_{j+1}}$$

(the notation $\omega \cdot X_j^{I_{L_j}}$ *denotes the orbit of* ω *under the action of the group* $X_j^{I_{L_j}} \leq G_j$*), and define*

$$T_n^n(I_{L_n}) = X_n^{I_{L_n}}, \text{ where } \emptyset \neq I_{L_n} \subseteq L_n,$$

and

$$T_0^n = G_n.$$

Then the subnormal subgroups of G_n *are precisely the groups* $S_j^n(I_{d_{j+1}})$*,* S_n^n*,* $T_j^n(I_{L_j}, I_{d_{j+1}})$*,* $T_n^n(I_{L_n})$ *and* T_0^n*.*

In particular, for all $j \in \{1, 2, \ldots, n-1\}, I_{d_j}, I_{L_j}$ *and* $I_{d_{j+1}}$*, they form chains*

$$Q_{j+1}^n \subsetneq T_j^n(I_{L_j}, I_{d_{j+1}}) \subseteq Q_j^n \subsetneq S_{j-1}^n(I_{d_j}) \subseteq P_{j-1}^n.$$

Also, for all I_{d_n} *and* I_{L_n}*, they form chains*

$$S_n^n = P_n^n \subsetneq T_n^n(I_{L_n}) \subseteq Q_n^n \subsetneq S_{n-1}^n(I_{d_n}) \subseteq P_{n-1}^n.$$

The subnormal length in G_n *of the group* $S_j^n(I_{d_{j+1}})$ *is*

$$\begin{cases} 1 & \text{if } I_{d_{j+1}} = \Omega_{d_{j+1}} \text{(implying that } S_j^n(I_{d_{j+1}}) = P_j^n\text{)}, \\ 2 & \text{if } I_{d_{j+1}} \subsetneq \Omega_{d_{j+1}}. \end{cases}$$

The subnormal length in G_n *of the group* $T_j^n(I_{L_j}, I_{d_{j+1}})$ *is*

$$\begin{cases} 1 & \text{if } I_{L_j} = L_j \text{(implying that } T_j^n(I_{L_j}, I_{d_{j+1}}) = Q_j^n\text{)}, \\ 2 & \text{if } I_{L_j} \subsetneq L_j \text{ and } I_{d_{j+1}} \cup \Delta_{d_{j+1}} \text{ is a union of } M_j\text{-orbits}, \\ 3 & \text{if } I_{L_j} \subsetneq L_j \text{ and } I_{d_{j+1}} \cup \Delta_{d_{j+1}} \text{ is not a union of } M_j\text{-orbits.} \end{cases}$$

The subnormal length in G_n *of the group* $T_n^n(I_{L_n})$ *is*

$$\begin{cases} 1 & \text{if } I_{L_n} = L_n \text{(implying that } T_n^n(I_{L_n}) = Q_n^n\text{)}, \\ 2 & \text{if } I_{L_n} \subsetneq L_n. \end{cases}$$

We remark in the above definition of $\Delta_{d_{j+1}}$ the dependency on I_{L_j} is implicit.

Proof. We first check that the groups $S_j^n(I_{d_{j+1}})$, S_n^n, $T_j^n(I_{L_j}, I_{d_{j+1}})$, $T_n^n(I_{L_n})$ and T_0^n are all subnormal subgroups of G_n. Obviously $S_n^n = \{1\} \lhd G_n$ and $T_0^n = G_n \trianglelefteq G_n$. For any $\emptyset \neq I_{L_n} \subseteq L_n$, we have

$$T_n^n(I_{L_n}) = X_n^{I_{L_n}} \trianglelefteq M_n = Q_n^n \lhd G_n, \tag{6.6}$$

using Theorem 2.4. For any $\emptyset \neq I_{d_{j+1}} \subseteq \Omega_{d_{j+1}}$, we have

$$S_j^n(I_{d_{j+1}}) = Q_{j+1}^n \rtimes X_{j+1}^{I_{d_{j+1}}} \trianglelefteq Q_{j+1}^n \rtimes L_{j+1} = P_j^n \lhd G_n, \tag{6.7}$$

as $X_{j+1}^{I_{d_{j+1}}} \trianglelefteq L_{j+1}$.

For any $\emptyset \neq I_{L_j} \subseteq L_j$ and $I_{d_{j+1}} \subseteq \Omega_{d_{j+1}} \backslash \Delta_{d_{j+1}}$, we show that

$$T_j^n(I_{L_j}, I_{d_{j+1}}) \trianglelefteq P_j^n \rtimes X_j^{I_{L_j}} \trianglelefteq Q_j^n \lhd G_n. \tag{6.8}$$

We have $P_j^n \rtimes X_j^{I_{L_j}} \trianglelefteq P_j^n \rtimes M_j = Q_j^n$, as $X_j^{I_{L_j}} \trianglelefteq M_j$. For

$$T_j^n(I_{L_j}, I_{d_{j+1}}) = Q_{j+1}^n \rtimes (X_{j+1}^{I_{d_{j+1}} \cup \Delta_{d_{j+1}}} \rtimes X_j^{I_{L_j}}) \trianglelefteq$$
$$Q_{j+1}^n \rtimes (L_{j+1} \rtimes X_j^{I_{L_j}}) = P_j^n \rtimes X_j^{I_{L_j}},$$

we need to show that $X_{j+1}^{I_{d_{j+1}} \cup \Delta_{d_{j+1}}} \rtimes X_j^{I_{L_j}} \trianglelefteq L_{j+1} \rtimes X_j^{I_{L_j}}$. From Lemma 2.3, we see that $X_{j+1}^{I_{d_{j+1}} \cup \Delta_{d_{j+1}}}$ is a product of some minimal normal subgroups of $L_{j+1} \rtimes X_j^{I_{L_j}}$ and so $X_{j+1}^{I_{d_{j+1}} \cup \Delta_{d_{j+1}}}$ is normal in $L_{j+1} \rtimes X_j^{I_{L_j}}$. It is now left to show that $[L_{j+1}, X_j^{I_{L_j}}] \subseteq X_{j+1}^{I_{d_{j+1}} \cup \Delta_{d_{j+1}}}$. This holds as $X_j^{I_{L_j}}$ moves points in the set $\Delta_{d_{j+1}}$ and fixes points in the sets $I_{d_{j+1}}$ and $\Omega_{d_{j+1}} \backslash (I_{d_{j+1}} \cup \Delta_{d_{j+1}})$.

We check that the subnormal lengths given in the statement of the theorem are correct for the groups $T_n^n(I_{L_n})$ and $S_j^n(I_{d_{j+1}})$. If $I_{L_n} = L_n$ then $T_n^n(I_{L_n}) = Q_n^n$ and the subnormal series (6.6) reduces to a chain of length 1. Similarly, if $I_{d_{j+1}} = \Omega_{d_{j+1}}$ then $S_j^n(I_{d_{j+1}}) = P_j^n$ and the subnormal series (6.7) reduces to chain of length 1. For all other $T_n^n(I_{L_n})$ we have displayed the shortest length of a subnormal series (6.6) because Q_n^n is the smallest normal subgroup of G_n containing $T_n^n(I_{L_n})$ and $T_n^n(I_{L_n})$ is not normal in G_n. A similar argument holds for all other $S_j^n(I_{d_{j+1}})$.

We check that the subnormal lengths given in the statement of the theorem are correct for the groups $T_j^n(I_{L_j}, I_{d_{j+1}})$. If $I_{L_j} = L_j$ then $\Delta_{d_{j+1}} = \Omega_{d_{j+1}}$ because Lemma 6.3 implies that the action of $X_j^{L_j} = M_j$ on $\Omega_{d_{j+1}}$ has no fixed points. So $T_j^n(I_{L_j}, I_{d_{j+1}}) = Q_j^n$ and the subnormal series (6.8) reduces to a chain of length 1.

If $I_{L_j} \subsetneq L_j$ and $I_{d_{j+1}} \cup \Delta_{d_{j+1}}$ is a union of M_j-orbits then the subnormal series (6.8) reduces to

$$T_j^n(I_{L_j}, I_{d_{j+1}}) \lhd Q_j^n \lhd G_n,$$

a chain of length 2, as M_j normalises $X_{j+1}^{I_{d_{j+1}} \cup \Delta_{d_{j+1}}}$. This is the shortest length of a subnormal series because Q_j^n is the smallest normal subgroup of G_n containing $T_j^n(I_{L_j}, I_{d_{j+1}})$ and $T_j^n(I_{L_j}, I_{d_{j+1}})$ is not normal in G_n.

If $I_{L_j} \subsetneq L_j$ and $I_{d_{j+1}} \cup \Delta_{d_{j+1}}$ is not a union of M_j-orbits we check that we have displayed the shortest length 3 of a subnormal series (6.8) for $T_j^n(I_{L_j}, I_{d_{j+1}})$. This is because Q_j^n is the smallest normal subgroup of G_n containing $T_j^n(I_{L_j}, I_{d_{j+1}})$. Also $T_j^n(I_{L_j}, I_{d_{j+1}})$ is not normal in Q_j^n since M_j does not normalise $X_{j+1}^{I_{d_{j+1}} \cup \Delta_{d_{j+1}}}$.

Recall the definition of the groups $H_n = L_n G_{n-1}$, for $n \geq 1$, as defined at the beginning of Section 6.1. Due to $H_n \cong G_n / M_n$, the theorem we are currently proving also implicitly makes a statement about the subnormal subgroups of H_n. We now prove, by induction on n, that every subnormal subgroup of G_n is one of the groups listed. Hence the subnormal subgroups of H_n are homomorphic images of the subnormal subgroups of G_n listed between Q_n^n and Q_0^n under the canonical map $G_n \longrightarrow H_n$.

For $n = 0$, all the subnormal subgroups of G_0 are $\{1\} = S_0^0$ and $G_0 = T_0^0 = T_0^0(I_{L_0})$, where $I_{L_0} = \{1\}$ (we have set $L_0 = \{1\}$), holds as G_0 is simple. Although it will also follow from the general argument below, we now prove separately the implicit claim for H_1.

Suppose K is a subnormal subgroup of H_1. Then $L_1 K / L_1$ is a subnormal subgroup of $H_1 / L_1 \cong G_0$. Since G_0 is simple, we know

$$L_1 K / L_1 \cong \{1\} \text{ or } L_1 K / L_1 \cong G_0.$$

For the case $L_1 K / L_1 \cong \{1\}$, we have $K \subseteq L_1$. Then K is subnormal in $L_1 = X_1^{(d_1)}$. There are two possibilities, either $K = \{1\} \cong M_1 T_1^1(I_{L_1}) / M_1$, for any $\emptyset \neq I_{L_1} \subseteq L_1$, or, using Theorem 2.4, we have $K = X_1^{I_{d_1}}$ is the image of $S_0^1(I_{d_1})$, for some $\emptyset \neq I_{d_1} \subseteq \Omega_{d_1}$, under the canonical map $G_1 \longrightarrow H_1$. Due to $H_1 \cong G_1 / M_1$, there are subnormal subgroups of H_1 of this form.

For the case $L_1 K / L_1 \cong G_0$, we have $L_1 K = L_1 \rtimes G_0$. Since G_0 acts faithfully and transitively on Ω_{d_1}, there is exactly one G_0-orbit of size at least two. Proposition 6.2 gives $L_1 \subseteq K$. Therefore $K = L_1 \rtimes G_0 \cong T_0^1 / M_1$. For $n = 1$, the result holds for H_1.

Suppose that the result holds for G_{n-1}. Now we prove the result for H_n. Let K be a subnormal subgroup of H_n. Then there are two cases:

$$K \subseteq L_n \text{ (case 1), and } K \nsubseteq L_n \text{ (case 2).}$$

Case 1.

For $K \subseteq L_n$, we have K is subnormal in $L_n = X_n^{(d_n)}$. There are two possibilities, either $K = \{1\} \cong M_n T_n^n(I_{L_n})/M_n$, for any $\emptyset \neq I_{L_n} \subseteq L_n$, or, using Theorem 2.4, we have $K = X_n^{I_{d_n}}$ is the image of $S_{n-1}^n(I_{d_n})$, for some $\emptyset \neq I_{d_n} \subseteq \Omega_{d_n}$, under the canonical map $G_n \longrightarrow H_n$.

Case 2.

Now suppose $K \nsubseteq L_n$. We know $\{1\} \ncong L_n K/L_n$ is a subnormal subgroup of $H_n/L_n \cong G_{n-1}$. Then there are two possibilities:

$$L_n K/L_n \subseteq L_n M_{n-1}/L_n \text{ (case 2a)},$$

and

$$L_n K/L_n \nsubseteq L_n M_{n-1}/L_n \text{ (case 2b)}.$$

Case 2a

For $L_n K/L_n \subseteq L_n M_{n-1}/L_n$, we have $\{1\} \ncong L_n K/L_n$ is subnormal in $L_n M_{n-1}/L_n \cong M_{n-1}$. So

$$L_n K/L_n \cong X_{n-1}^{I_{L_{n-1}}} = T_{n-1}^{n-1}(I_{L_{n-1}}),$$

for some $\emptyset \neq I_{L_{n-1}} \subseteq L_{n-1}$. Put

$$T_{n-1}^{n-1}(I_{L_{n-1}}) =: T.$$

Then $L_n K = L_n \rtimes T$.

The action of T on Ω_{d_n} may have fixed points. Also $K \subseteq L_n T$ and so K is subnormal in $L_n T$. Corollary 6.9 gives

$$\big\{(x_\omega)_{\omega \in \Omega_{d_n}} \in L_n : x_\omega = 1 \text{ if } \omega \cdot T = \{\omega\}\big\} = X_n^{\Delta_{d_n}} \subseteq K.$$

We have found that

$$X_n^{\Omega_{d_n} \setminus \Delta_{d_n}} K = X_n^{\Omega_{d_n} \setminus \Delta_{d_n}} X_n^{\Delta_{d_n}} T. \tag{6.9}$$

To finalise the characterisation of K, observe that $K \cap X_n^{\Omega_{d_n} \setminus \Delta_{d_n}} T$ is a subnormal subgroup of $X_n^{\Omega_{d_n} \setminus \Delta_{d_n}} T \cong X_n^{\Omega_{d_n} \setminus \Delta_{d_n}} \times T$ and that it projects onto the factor T. Using Lemma 2.4, there exists some subset $I_{d_n} \subseteq \Omega_{d_n} \setminus \Delta_{d_n}$ such that $K \cap X_n^{\Omega_{d_n} \setminus \Delta_{d_n}} T = X_n^{I_{d_n}} T \cong X_n^{I_{d_n}} \times T$. From this and the fact that $X_n^{\Delta_{d_n}} \leq K$, we establish $K = X_n^{I_{d_n}} X_n^{\Delta_{d_n}} T$. Therefore $K = X_n^{I_{d_n} \cup \Delta_{d_n}} \rtimes T$ is the image of $T^n_{n-1}(I_{L_{n-1}}, I_{d_n})$ under the canonical map $G_n \longrightarrow H_n$.

Case 2b.

For $L_nK/L_n \nsubseteq L_nM_{n-1}/L_n$, we have L_nK/L_n is subnormal in $H_n/L_n \cong G_{n-1}$ and is not contained in L_nM_{n-1}/L_n. By induction, we have $L_nK/L_n \cong S^{n-1}_j(I_{d_{j+1}})$, for some $j \in \{0, 1, \dots, n-2\}$, or $L_nK/L_n \cong T^{n-1}_j(I_{L_j}, I_{d_{j+1}})$, for some $j \in \{1, 2, \dots, n-2\}$, or $L_nK/L_n \cong T^{n-1}_0$.

We denote this isomorphic copy of L_nK/L_n in G_{n-1} by R. Then $L_nK = L_n \rtimes R$. Observe that $M_{n-1} \subseteq R$. Each of the orbits of M_{n-1} in its action upon Ω_{d_n}, and hence each of the orbits of R in its action upon Ω_{d_n}, has at least two elements (see Lemma 6.3). Also $K \subseteq L_nR$ and so K is subnormal in L_nR.

Proposition 6.2 gives $L_n \subseteq K$ and so $K = L_n \rtimes R$. Therefore K is the image of $S^n_j(I_{d_{j+1}})$ under the canonical map $G_n \longrightarrow H_n$, for some $j \in \{0, 1, \dots, n-2\}$, or K is the image of $T^n_j(I_{L_j}, I_{d_{j+1}})$ under the canonical map $G_n \longrightarrow H_n$, for some $j \in \{1, 2, \dots, n-2\}$, or $K \cong T^n_0/M_n$.

Suppose that the result holds for H_n. Now we prove the result for G_n. Let K be a subnormal subgroup of G_n. Then there are two cases:

$$K \subseteq M_n \text{ (case 1), and } K \nsubseteq M_n \text{ (case 2).}$$

Case 1.

For $K \subseteq M_n$, we have K is a subnormal subgroup of $M_n = X_n^{(|L_n|)}$. There are two possibilities, either $K = \{1\} = S^n_n$, or, using Theorem 2.4 we have $K = X_n^{I_{L_n}} = T^n_n(I_{L_n})$, for some $\emptyset \neq I_{L_n} \subseteq L_n$.

Case 2.

Now suppose $K \nsubseteq M_n$. We know $\{1\} \ncong M_nK/M_n$ is a subnormal subgroup of $G_n/M_n \cong H_n$. Then there are the two possibilities:

$$M_nK/M_n \subseteq M_nL_n/M_n \text{ (case 2a),}$$

and

$$M_nK/M_n \nsubseteq M_nL_n/M_n \text{ (case 2b).}$$

Case 2a.

For $M_nK/M_n \subseteq M_nL_n/M_n$, we have $\{1\} \ncong M_nK/M_n$ is subnormal in

$M_nL_n/M_n \cong L_n$. So
$$M_nK/M_n \cong X_n^{I_{d_n}},$$
for some $\emptyset \neq I_{d_n} \subseteq \Omega_{d_n}$, which is the image of $S_{n-1}^n(I_{d_n})$ under the canonical map $G_n \longrightarrow H_n$. Put $S_{n-1}^n(I_{d_n}) =: S$. Then $M_nK = S$.

As said in the proof of the analogue case for Theorem 6.4, right multiplication by L_n on itself in the action (4.1) implies that each of the orbits of $X_n^{I_{d_n}}$ in its action upon L_n has at least two elements. In the action of $X_n^{I_{d_n}}$ on L_n, each non-trivial element of $X_n^{I_{d_n}}$ acts fixed point freely. Therefore this action is faithful.

Also $K \subseteq S$ and so K is subnormal in S. Proposition 6.2 gives $M_n \subseteq K$. Therefore $K = S = S_{n-1}^n(I_{d_n})$.

Case 2b.

For $M_nK/M_n \not\subseteq M_nL_n/M_n$, we have M_nK/M_n is subnormal in $G_n/M_n \cong H_n$ and is not contained in M_nL_n/M_n. By induction, we have $M_nK/M_n = T_j^n(I_{L_j}, I_{d_{j+1}})/M_n$, for some $j \in \{1, 2, \ldots, n-1\}$, or $M_nK/M_n = S_j^n(I_{d_{j+1}})/M_n$, for some $j \in \{0, 1, \ldots, n-2\}$, or $M_nK/M_n = T_0^n/M_n$.

We denote this description of M_nK/M_n in H_n by R/M_n. Then $M_nK = R$. Again, right multiplication by L_n on itself in the action (4.1) implies that each of the orbits of R/M_n in its action upon L_n has at least two elements. We claim separately that each of the $(X_n^{I_{d_n}\cup\Delta_{d_n}} X_n^{I_{L_{n-1}}})$-orbits has at least two elements. Obviously $X_n^{I_{L_{n-1}}}$ is not the trivial group because $I_{L_{n-1}} \neq \emptyset$. The action of $1 \neq X_n^{I_{L_{n-1}}}$ on Ω_{d_n} is faithful and therefore at least one point is moved. So $\Delta_{d_n} \neq \emptyset$. Thus $X_n^{I_{d_n}\cup\Delta_{d_n}}$ is not the trivial group.

In the action (4.1), non-trivial elements of R/M_n acting on L_n can have fixed points however these elements do move at least one other point. Therefore this action is faithful.

Also $K \subseteq R$ and so K is subnormal in R. Proposition 6.2 gives $M_n \subseteq K$ and so $K = R$. Therefore $K = T_j^n(I_{L_j}, I_{d_{j+1}})$, for some $j \in \{1, 2, \ldots, n-1\}$, or $K = S_j^n(I_{d_{j+1}})$, for some $j \in \{0, 1, \ldots, n-2\}$, or $K = T_0^n$.

□

Again, our work has been restricted in Lemma 6.16 to closed subnormal subgroups because we rely on Lemma 2.12, which only applies to subnormal subgroups that are closed.

Lemma 6.16 is required due to the two different types of notation for the subnormal subgroups of G_n.

Lemma 6.16. *Given finite groups H_n, for $n \geq 1$, in which all the normal subgroups form a chain*

$$\{1\} = N^n_{2n+1} \subsetneq N^n_{2n} \subsetneq \ldots \subsetneq N^n_2 \subsetneq N^n_1 = H_n,$$

and an inverse system of surjective homomorphisms $\psi_n : H_n \longrightarrow H_{n-1}$, for $n \geq 2$, such that

$$\psi_n(N^n_i) = \begin{cases} N^{n-1}_i & \text{for } 1 \leq i \leq 2n-1, \\ \{1\} & \text{for } i \in \{2n, 2n+1\}. \end{cases}$$

Let $\mathcal{P}^i$, $\mathcal{Q}^i$ and $\mathcal{R}^i$, for $i \in \{1, 2, \ldots, n\}$, be finite disjoint index sets.

Suppose the non-trivial subnormal subgroups K^n_p and $K^n_{q,r}$ of H_n are parameterised by p, q and r, where $\emptyset \neq p \in \mathcal{P}^i$, $\emptyset \neq q \in \mathcal{Q}^i$ and $r \in \mathcal{R}^i$, such that $N^n_{2i+1} \subsetneq K^n_r, K^n_{p,q} \subseteq N^n_{2i-1}$, and

$$\psi_n(K^n_p) = \begin{cases} K^{n-1}_p & \text{for } p \in \mathcal{P}^1, \mathcal{P}^2, \ldots, \mathcal{P}^{n-1}, \\ \{1\} & \text{for } p \in \mathcal{P}^n, \end{cases} \tag{6.10}$$

and

$$\psi_n(K^n_{q,r}) = \begin{cases} K^{n-1}_{q,r} & \text{for } q \in \mathcal{Q}^1, \mathcal{Q}^2, \ldots, \mathcal{Q}^{n-1} \text{ and } r \in \mathcal{R}^1, \mathcal{R}^2, \ldots, \mathcal{R}^{n-1}, \\ \{1\} & \text{for } q \in \mathcal{Q}^n \text{ and } r \in \mathcal{R}^n. \end{cases} \tag{6.11}$$

Then the inverse limit $G = \varprojlim(H_n)_{n\geq 1}$ has non-trivial closed subnormal subgroups precisely $K_p = \varprojlim(K^n_p)_{n\to\infty}$ and $K_{q,r} = \varprojlim(K^n_{q,r})_{n\to\infty}$, where $\emptyset \neq p \in \mathcal{P}^i$, $\emptyset \neq q \in \mathcal{Q}^i$ and $r \in \mathcal{R}^i$ for $i \geq 1$, regarded as subgroups of G.

Proof. Let M be a non-trivial closed subnormal subgroup of G. Since G is an inverse limit, we can find $n \geq 1$ such that the image of M in H_n under $\pi_n : G \longrightarrow H_n$ is non-trivial. Therefore $\pi_n(M) = K^n_p$ or $\pi_n(M) = K^n_{q,r}$, where $\emptyset \neq p \in \mathcal{P}^i$, $\emptyset \neq q \in \mathcal{Q}^i$ and $r \in \mathcal{R}^i$, for some $i \in \{1, 2, \ldots, n\}$.

We claim that $M = K_p$ or $M = K_{q,r}$. Since M is closed, it is enough to show that $\pi_m(M) = K^m_p$, for all $m \geq n$, or $\pi_m(M) = K^m_{q,r}$, for all $m \geq n$. Then $\pi_m(M) = \pi_m(K_p)$ implies $\ker \pi_m M = \ker \pi_m K_p$, for all $m \geq n$, or $\pi_m(M) = \pi_m(K_{q,r})$ implies

$\ker \pi_m M = \ker \pi_m K_{q,r}$, for all $m \geq n$. Thus

$$M = \left(\bigcap_{m \geq n} \ker \pi_m \right) M = \bigcap_{m \geq n} \left(\ker \pi_m M \right)$$
$$= \bigcap_{m \geq n} \left(\ker \pi_m K_p \right) = \left(\bigcap_{m \geq n} \ker \pi_m \right) K_p = K_p$$

or similarly $M = K_{q,r}$, using Lemma 2.12.

Clearly $\pi_m(M) = K_p^m$ or $\pi_m(M) = K_{q,r}^m$ is true for $m = n$. Now suppose $m > n$. From

$$\{1\} \neq K_p^{m-1} = \pi_{m-1}(M) = \psi_m(\pi_m(M))$$

and mapping (6.10), we conclude $\pi_m(M) = K_p^m$. Or from

$$\{1\} \neq K_{q,r}^{m-1} = \pi_{m-1}(M) = \psi_m(\pi_m(M))$$

and mapping (6.11), we conclude that $\pi_m(M) = K_{q,r}^m$. □

For the following, recall the normal subgroups P_j and Q_j, for $j \geq 0$, of a Wilson group G, defined in Corollary 5.3.

Corollary 6.17. *Let $G = \varprojlim (G_n)_{n \geq 0}$ be the inverse limit of the groups G_n as defined in Section 4.1.*

For $j \geq 0$, define

$$S_j(I_{d_{j+1}}) = \varprojlim (S_j^n(I_{d_{j+1}}))_{n \to \infty}, \text{ where } \emptyset \neq I_{d_{j+1}} \subseteq \Omega_{d_{j+1}},$$

regarded as subgroups of G.

For $j \geq 1$, define

$$T_j(I_{L_j}, I_{d_{j+1}}) = \varprojlim (T_j^n(I_{L_j}, I_{d_{j+1}}))_{n \to \infty}, \text{ where } \emptyset \neq I_{L_j} \subseteq L_j,$$

$$\Delta_{d_{j+1}} = \{\omega \in \Omega_{d_{j+1}} : \omega \cdot X_j^{I_{L_j}} \neq \{\omega\}\} \text{ and } I_{d_{j+1}} \subseteq \Omega_{d_{j+1}} \backslash \Delta_{d_{j+1}},$$

and define

$$T_0 = \varprojlim (T_0^n)_{n \to \infty},$$

regarded as subgroups of G.

Then the non-trivial closed subnormal subgroups of G are precisely the groups $S_j(I_{d_{j+1}})$, $T_j(I_{L_j}, I_{d_{j+1}})$ and T_0. In particular, for all $j \geq 1$, I_{d_j}, I_{L_j} and $I_{d_{j+1}}$, they *form chains*

$$T_j(I_{L_j}, I_{d_{j+1}}) \subseteq Q_j \subsetneq S_{j-1}(I_{d_j}) \subseteq P_{j-1}.$$

The subnormal length in G of the group $S_j(I_{d_{j+1}})$ is

$$\begin{cases} 1 & \text{if } I_{d_{j+1}} = \Omega_{d_{j+1}} \text{ (implying that } S_j(I_{d_{j+1}}) = P_j\text{)}, \\ 2 & \text{if } I_{d_{j+1}} \subsetneq \Omega_{d_{j+1}}. \end{cases}$$

The subnormal length in G of the group $T_j(I_{L_j}, I_{d_{j+1}})$ is

$$\begin{cases} 1 & \text{if } I_{L_j} = L_j \text{ (implying that } T_j(I_{L_j}, I_{d_{j+1}}) = Q_j\text{)}, \\ 2 & \text{if } I_{L_j} \subsetneq L_j \text{ and } I_{d_{j+1}} \cup \Delta_{d_{j+1}} \text{ is a union of } M_j\text{-orbits}, \\ 3 & \text{if } I_{L_j} \subsetneq L_j \text{ and } I_{d_{j+1}} \cup \Delta_{d_{j+1}} \text{ is not a union of } M_j\text{-orbits}. \end{cases}$$

Proof. We apply Lemma 6.16 to the groups H_n, for $n \geq 1$, of Wilson's construction and their subnormal subgroups. For the finite index sets we take the power sets of Ω_{d_j}, L_j and $\Omega_{d_{j+1}} \backslash \Delta_{d_{j+1}}$, for $1 \leq j \leq n-1$, and Ω_{d_n}. Note that $\mathcal{Q}^1 = \{1\}$ and $\mathcal{R}^1 = \emptyset$. We remark that arbitrary sets A_1 and A_2 can be made disjoint when the elements $x \in A_1$ and $y \in A_2$ are labelled as $(1, x)$ and $(2, y)$.

Define $K_p^n = S_{i-1}^n(I_{d_i})/M_n$, where $\emptyset \neq p \in \mathcal{P}^i$ for $i \in \{1, 2, \ldots, n\}$, $K_{q,r}^n = T_{i-1}^n(I_{L_{i-1}}, I_{d_i})/M_n$, where $\emptyset \neq q \in \mathcal{Q}^i$ and $r \in \mathcal{R}^i$ for $i \in \{2, 3, \ldots, n\}$, and $K_{q,r}^n = T_0^n/M_n$, where $\emptyset \neq q \in \mathcal{Q}^1$ and $r \in \mathcal{R}^1$. For each n, these subnormal subgroups of H_n were defined in Theorem 6.15. It was shown that these are all the non-trivial subnormal subgroups of H_n and they form chains.

The definition of the groups K_p^n and $K_{q,r}^n$ also shows that the second condition for Lemma 6.16 is satisfied. For $1 \leq i \leq n$, where $\emptyset \neq p \in \mathcal{P}^i$, we have $\psi_n(K_p^n) = \psi_n(S_{i-1}^n(I_{d_i})/M_n) = S_{i-1}^{n-1}(I_{d_i})/M_{n-1} = K_p^{n-1}$. We take $S_{n-1}^{n-1}(I_{d_n})/M_{n-1}$ and K_p^{n-1} for $\emptyset \neq p \in \mathcal{P}^n$ to be the trivial group $\{1\}$. For $2 \leq i \leq n$, where $\emptyset \neq q \in \mathcal{Q}^i$ and $r \in \mathcal{R}^i$, we have

$$\psi_n(K_{q,r}^n) = \psi_n(T_{i-1}^n(I_{L_{i-1}}, I_{d_i})/M_n) = T_{i-1}^{n-1}(I_{L_{i-1}}, I_{d_i})/M_{n-1} = K_{q,r}^{n-1}.$$

We take $T_{n-1}^{n-1}(I_{L_{n-1}}, I_{d_n})/M_{n-1}$ and $K_{q,r}^{n-1}$ for $\emptyset \neq q \in \mathcal{Q}^n$ and $r \in \mathcal{R}^n$ to be the trivial group $\{1\}$. Also $\psi_n(K_{q,r}^n) = \psi_n(T_0^n/M_n) = T_0^{n-1}/M_{n-1} = K_{q,r}^{n-1}$ for $\emptyset \neq q \in \mathcal{Q}^1$ and $r \in \mathcal{R}^1$. $\square$

Remark. The indices of the closed subnormal subgroups of the Wilson groups are finite, due to the definition of the subnormal subgroups. Therefore the subnormal subgroups of the Wilson groups are open, using Lemma 2.11.

Remark. The results of this section, every closed subnormal subgroup of a Wilson group is of finite index, provide an alternative proof to the proof of [32, (3.3)], Wilson groups are hereditarily just infinite.

Let G be a Wilson group. Suppose H is an open subgroup of G and N is a closed normal subgroup of H. So $K = \mathrm{Core}_G(H)$ is an open normal subgroup of G, using Lemma 2.10. Then $N \cap K$ is a closed subnormal subgroup of G. From Corollary 6.17, we know that $N \cap K$ has finite index in G. Hence N has finite index in G and so H is just infinite.

The following diagram illustrates the chains of subnormal subgroups of a general Wilson group.

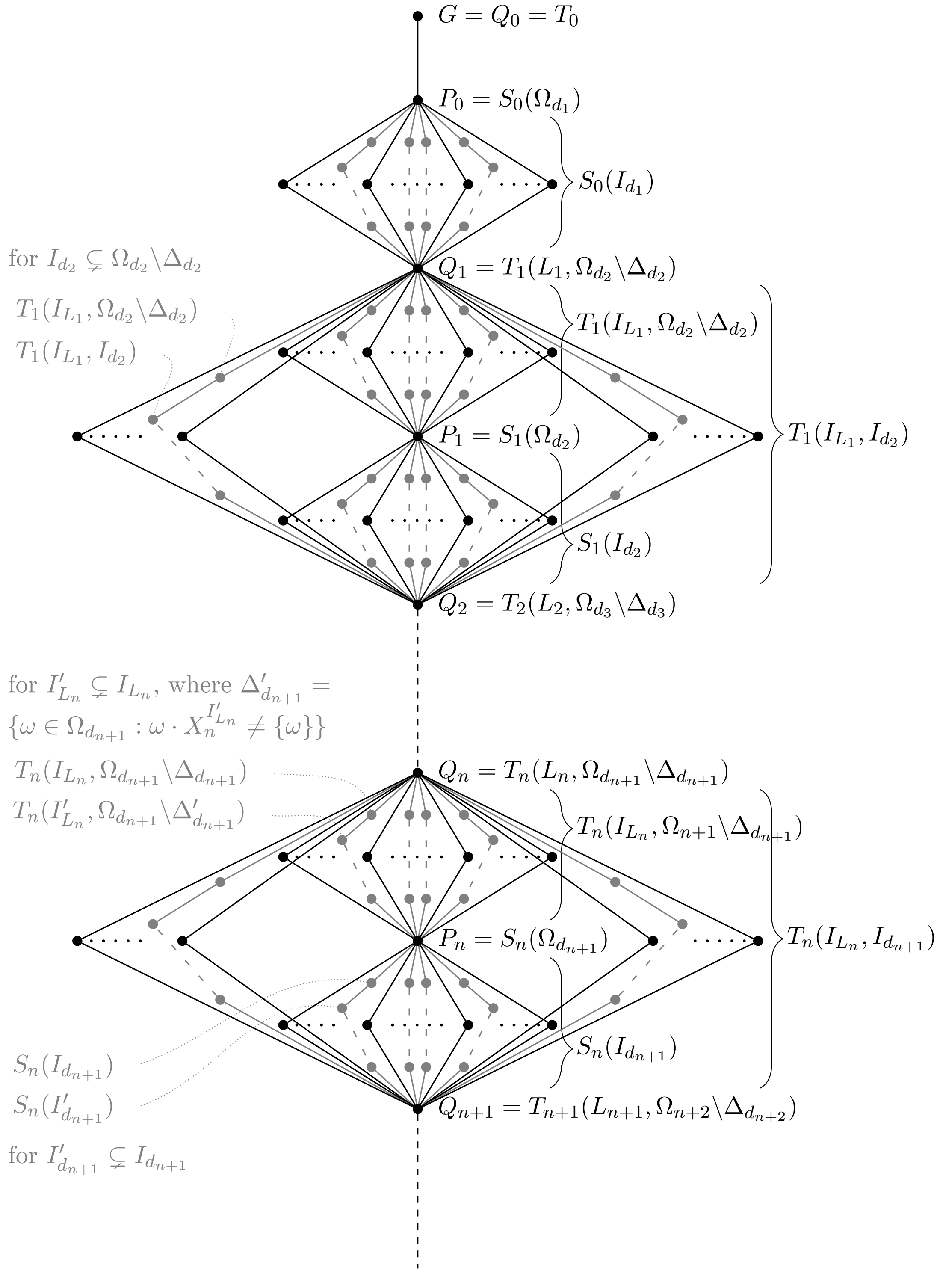

Figure 6.5: The subnormal subgroup lattice of an arbitrary Wilson group.

Chapter 7

Subnormal subgroup growth

7.1 General Wilson groups

Let G be a Wilson group arising as an inverse limit of finite groups G_n as defined in Section 4.1. The number of normal subgroups of the Wilson quotient G_n is $2n+2$, for $n \geq 0$. Therefore the Wilson group G has $2n+2$ normal subgroups of index at most $|G_n|$.

Define the number of normal subgroups of G of index at most $|G_n|$ as

$$S^{\lhd}_{|G_n|}(G) = 2n+2,$$

for $n \geq 0$, which is a step function. This normal subgroup growth is very slow because the number $S^{\lhd}_{|G_n|}(G)$ is much smaller than the number $|G_n| = |X_n|^{|X_n|^{d_n}} |X_n|^{d_n} |G_{n-1}|$. By choosing carefully the finite non-abelian simple groups X_i, for $i \geq 0$, namely X_i very large, we could make this growth function $S^{\lhd}_{|G_n|}(G)$ grow as slow as we like.

We give an alternative description of the normal subgroup growth of a Wilson group. Recall the normal subgroups P_n, for $n \geq 0$, of a Wilson group G, defined in Corollary 5.3. Define the number of normal subgroups of G of index at most $|G : P_n|$ by $S^{\lhd}_{|G:P_n|}(G)$. So the growth function $S^{\lhd}_{|G:P_n|}(G) = 2n+2$ is linear in n.

Theorem 7.1, below, gives an estimate for the size of the groups G_n, for $n \geq 0$. Using the lower bound for $|G_n|$ in this theorem, we can make a more precise statement regarding normal subgroup growth of a Wilson group G. Since

$$2n+2 \leq \underbrace{4^{4^{\cdot^{\cdot^{\cdot^{4}}}}}}_{n+2},$$

we have that $S^{\lhd}_{|G_n|}(G)$ grows very slowly, that is slower than the functions

$\underbrace{\log\log\ldots\log}_{r}|G_n|$ for any fixed r.

Theorem 7.1. *Let G_n, for $n \geq 0$, be the finite groups as defined in Section 4.1. Suppose there exists a constant c such that $|X_i| \leq c$, for all $i \geq 0$.*

Then

$$\underbrace{4^{4^{\cdot^{\cdot^{\cdot^{4}}}}}}_{n+2} \leq |G_n| \leq \underbrace{\tilde{c}^{\tilde{c}^{\cdot^{\cdot^{\cdot^{\tilde{c}}}}}}}_{2n+2},$$

where $\tilde{c} = 3c$.

Proof. First we confirm the lower bound

$$\underbrace{4^{4^{\cdot^{\cdot^{\cdot^{4}}}}}}_{n+2}$$

for $|G_n|$. We have $|X_n| \geq 60 \geq 2^5$ because X_n is a finite non-abelian simple group. Then

$$\begin{aligned}|G_n| &= |X_n|^{|X_n|^{d_n}}|X_n|^{d_n}|G_{n-1}| \\ &\geq (2^5)^{2^{5d_n}+d_n}|G_{n-1}| \\ &\geq (2^5)^{2^{5d_n}}. \end{aligned} \tag{7.1}$$

The degree d_1 of the faithful transitive action of $G_0 = X_0$ is such that $d_1 \geq 5 \geq 4$, as the minimal degree of a faithful transitive permutation representation of A_5 is 5. Therefore $d_n \geq 4$, for $n \geq 1$. Now

$$|G_{n-1}| \leq d_n! \leq {d_n}^{d_n} \tag{7.2}$$

because the permutation representation of G_{n-1} of degree d_n is faithful. Then

$$\begin{aligned}{d_n}^{3/2} &\geq \frac{d_n(\log_2 d_n)}{5} \geq \frac{\log_2|G_{n-1}|}{5}, \text{ using (7.2)}, \\ &\geq 2^{5d_{n-1}}, \text{ using (7.1)}.\end{aligned}$$

So $d_n \geq 2^{(10/3)d_{n-1}} \geq 4^{d_{n-1}}$. Therefore, by induction,

$$d_n \geq \underbrace{4^{4^{\cdot^{\cdot^{\cdot^{4}}}}}}_{n} \tag{7.3}$$

and

$$|G_n| \geq 60^{60^{d_n}} \geq \underbrace{4^{4^{\cdot^{\cdot^{\cdot^{4}}}}}}_{n+2}.$$

Now we confirm the upper bound

$$\underbrace{\tilde{c}^{\tilde{c}^{\cdot^{\cdot^{\cdot^{\tilde{c}}}}}}}_{2n+2}, \text{ where } \tilde{c} = 3c,$$

for $|G_n|$. Suppose there exists a constant c such that $|X_i| \leq c$, for all $i \geq 0$. Then

$$\begin{aligned}
|G_n| &= |X_n|^{|X_n|^{d_n}} |X_n|^{d_n} |G_{n-1}| \\
&\leq c^{(c^{d_n}+d_n)} |G_{n-1}| \\
&\leq c^{(c^{d_n}+d_n)} {d_n}^{d_n}, \text{ using (7.2)}, \\
&= c^{(c^{d_n}+d_n+d_n \log_c d_n)} \\
&\leq c^{3c^{d_n}} \\
&\leq (3c)^{(3c)^{d_n}}
\end{aligned}$$

Now

$$d_n \leq |G_{n-1}| \leq (3c)^{(3c)^{d_{n-1}}}$$

because the permutation representation of G_{n-1} of degree d_n is transitive. Therefore, putting $\tilde{c} = 3c$, by induction,

$$d_n \leq \underbrace{\tilde{c}^{\tilde{c}^{\cdot^{\cdot^{\cdot^{\tilde{c}}}}}}}_{2n}$$

and

$$|G_n| \leq \tilde{c}^{\tilde{c}^{d_n}} \leq \underbrace{\tilde{c}^{\tilde{c}^{\cdot^{\cdot^{\cdot^{\tilde{c}}}}}}}_{2n+2}.$$

□

We now consider subnormal subgroup growth of Wilson groups. The following theorem gives a formula for the number of subnormal subgroups of G_n in terms of d_j and X_j, for $1 \leq j \leq n$. The power set notation $\mathcal{P}(X)$ is used to denote the set of all subsets of the set X.

Theorem 7.2. *Let G_n, for $n \geq 0$, be the finite groups as defined in Section 4.1. Then* *the number of subnormal subgroups of G_n, for $n \geq 1$, is*

$$2^{|X_n|^{|\Omega_{d_n}|}} + \sum_{j=1}^{n} 2^{|\Omega_{d_j}|} + \sum_{j=2}^{n} \left(\sum_{I \in \mathcal{P}(X_{j-1}^{\Omega_{d_{j-1}}}) \backslash \{\emptyset, X_{j-1}^{\Omega_{d_{j-1}}}\}} 2^{|\Omega_{d_j} \backslash \Delta_{d_j}(I)|} \right), \tag{7.4}$$

where $\emptyset \neq I \subsetneq X_{j-1}^{\Omega_{d_{j-1}}}$ and $\Delta_{d_j}(I) = \{\omega \in \Omega_{d_j} : \omega \cdot {X_{j-1}}^{I} \neq \{\omega\}\}$.

Proof. We prove the result by induction on n. Recall the subnormal subgroups of G_n defined in Theorem 6.15. The subnormal subgroups of G_1 are:

$$T_1^1(I_{L_1}) = X_1^{I_{L_1}}, \text{ where } \emptyset \neq I_{L_1} \subseteq L_1;$$

$$S_0^1(I_{d_1}) = Q_1^1 \rtimes X_1^{I_{d_1}}, \text{ where } \emptyset \neq I_{d_1} \subseteq \Omega_{d_1};$$

$$S_1^1 = \{1\} \text{ and } T_0^1 = G_1.$$

The number of subnormal subgroups of G_1 is $|\mathcal{P}(L_1)\backslash\{\emptyset\}| + |\mathcal{P}(\Omega_{d_1})\backslash\{\emptyset\}| + 2$. When recalling that $L_1 = X_1^{\Omega_{d_1}}$, this number can be written as $|\mathcal{P}(X_1^{\Omega_{d_1}})\backslash\{\emptyset\}| + |\mathcal{P}(\Omega_{d_1})\backslash\{\emptyset\}| + 2$. Since X_1 and Ω_{d_1} are finite, the number of subnormal subgroups of G_1 becomes

$$\begin{aligned} &\left(2^{|X_1|^{|\Omega_{d_1}|}} - 1\right) + \left(2^{|\Omega_{d_1}|} - 1\right) + 2 \\ &= 2^{|X_1|^{|\Omega_{d_1}|}} + 2^{|\Omega_{d_1}|}. \end{aligned}$$

Now putting $n = 1$ into the formula (7.4) shows that the result holds for G_1.

Suppose the result is true for G_{n-1}. The subnormal subgroups of G_n are:

(a)
$$T_n^n(I_{L_n}) = X_n^{I_{L_n}}, \text{ where } \emptyset \neq I_{L_n} \subseteq L_n;$$

(b)
$$S_{n-1}^n(I_{d_n}) = Q_n^n \rtimes X_n^{I_{d_n}}, \text{ where } \emptyset \neq I_{d_n} \subseteq \Omega_{d_n};$$

(c)
$$\begin{aligned} T_{n-1}^n(I_{L_{n-1}}, I_{d_n}) = Q_n^n \rtimes (X_n^{I_{d_n} \cup \Delta_{d_n}} \rtimes X_{n-1}^{I_{L_{n-1}}}), \text{ where } \emptyset \neq I_{L_{n-1}} \subseteq L_{n-1}, \\ \Delta_{d_n} = \{\omega \in \Omega_{d_n} : \omega \cdot X_{n-1}^{I_{L_{n-1}}} \neq \{\omega\}\} \text{ and } I_{d_n} \subseteq \Omega_{d_n} \backslash \Delta_{d_n}; \end{aligned}$$

and, by induction,

(d)

$$M_n L_n S_j^{n-1}(I_{d_{j+1}}), \text{ where } \emptyset \neq I_{d_{j+1}} \subseteq \Omega_{d_{j+1}}, \qquad \text{for } j \in \{0, 1, \ldots, n-2\};$$

$$\begin{aligned} &M_n L_n T_j^{n-1}(I_{L_j}, I_{d_{j+1}}), \text{ where } \emptyset \neq I_{L_j} \subseteq L_j, \\ &\qquad \Delta_{d_{j+1}} = \{\omega \in \Omega_{d_{j+1}} : \omega \cdot X_j^{I_{L_j}} \neq \{\omega\}\} \text{ and } I_{d_{j+1}} \subseteq \Omega_{d_{j+1}} \backslash \Delta_{d_{j+1}}, \\ &\qquad\qquad\qquad\qquad \text{for } j \in \{1, 2, \ldots, n-2\}; \end{aligned}$$

$$M_n L_n T_0^{n-1} = G_n \text{ and } \frac{M_n L_n S_{n-1}^{n-1}}{M_n L_n} \cong \{1\}.$$

We count the number of subnormal subgroups of each type (a) to (d):

(a) $|\mathcal{P}(L_n)\backslash\{\emptyset\}|$;

(b) $|\mathcal{P}(\Omega_{d_n})\backslash\{\emptyset\}|$;

(c) $\displaystyle\sum_{I_{L_{n-1}} \in \mathcal{P}(L_{n-1})\backslash\{\emptyset, L_{n-1}\}} |\mathcal{P}(\Omega_{d_n} \backslash \Delta_{d_n}(I_{L_{n-1}}))| + 1$;

(d) the number of subnormal subgroups of $G_{n-1} - |\mathcal{P}(L_{n-1})\backslash\{\emptyset\}|$.

Recalling that $L_i = X_i^{\Omega_{d_i}}$, the number of subnormal subgroups of G_n is equal to

$$\begin{aligned} &|\mathcal{P}(X_n^{\Omega_{d_n}})\backslash\{\emptyset\}| \\ &+ |\mathcal{P}(\Omega_{d_n})\backslash\emptyset| \\ &+ \sum_{I \in \mathcal{P}(X_{n-1}^{\Omega_{d_{n-1}}})\backslash\{\emptyset, X_{n-1}^{\Omega_{d_{n-1}}}\}} |\mathcal{P}(\Omega_{d_n} \backslash \Delta_{d_n}(I))| + 1 \\ &+ \text{ the number of subnormal subgroups of } G_{n-1} - |\mathcal{P}(X_{n-1}^{\Omega_{d_{n-1}}})\backslash\{\emptyset\}|. \end{aligned}$$

Using that Ω_j and X_j, for $0 \leq j \leq n$, are finite, this number can be written as

$$\begin{aligned}
&\left(2^{|X_n|^{|\Omega_{d_n}|}} - 1\right) \\
&+ \left(2^{|\Omega_{d_n}|} - 1\right) \\
&+ \sum_{I \in \mathcal{P}(X_{n-1}^{\Omega_{d_{n-1}}}) \backslash \{\emptyset, X_{n-1}^{\Omega_{d_{n-1}}}\}} |\mathcal{P}(\Omega_{d_n} \backslash \Delta_{d_n}(I))| + 1 \\
&+ \left[2^{|X_{n-1}|^{|\Omega_{d_{n-1}}|}} + \sum_{j=1}^{n-1} 2^{|\Omega_{d_j}|} + \sum_{j=2}^{n-1} \left(\sum_{I \in \mathcal{P}(X_{j-1}^{\Omega_{d_{j-1}}}) \backslash \{\emptyset, X_{j-1}^{\Omega_{d_{j-1}}}\}} 2^{|\Omega_{d_j} \backslash \Delta_{d_j}(I)|} \right) \right] \\
&\qquad - \left(2^{|X_{n-1}|^{|\Omega_{d_{n-1}}|}} - 1\right)
\end{aligned}$$

$$\begin{aligned}
&= 2^{|X_n|^{|\Omega_{d_n}|}} + 2^{|\Omega_{d_n}|} \\
&\qquad + \sum_{I \in \mathcal{P}(X_{n-1}^{\Omega_{d_{n-1}}}) \backslash \{\emptyset, X_{n-1}^{\Omega_{d_{n-1}}}\}} |\mathcal{P}(\Omega_{d_n} \backslash \Delta_{d_n}(I))| \\
&\qquad + \sum_{j=1}^{n-1} 2^{|\Omega_{d_j}|} + \sum_{j=2}^{n-1} \left(\sum_{I \in \mathcal{P}(X_{j-1}^{\Omega_{d_{j-1}}}) \backslash \{\emptyset, X_{j-1}^{\Omega_{d_{j-1}}}\}} 2^{|\Omega_{d_j} \backslash \Delta_{d_j}(I)|} \right)
\end{aligned}$$

$$= 2^{|X_n|^{|\Omega_{d_n}|}} + \sum_{j=1}^{n} 2^{|\Omega_{d_j}|} + \sum_{j=2}^{n} \left(\sum_{I \in \mathcal{P}(X_{j-1}^{\Omega_{d_{j-1}}}) \backslash \{\emptyset, X_{j-1}^{\Omega_{d_{j-1}}}\}} 2^{|\Omega_{d_j} \backslash \Delta_{d_j}(I)|} \right).$$

□

We now give an upper bound to the number (7.4). We have that $X_{j-1}^{I_{L_{j-1}}}$, where $\emptyset \neq I_{L_{j-1}} \subseteq L_{j-1}$, acts faithfully on Ω_{d_j} because G_{j-1} acts faithfully on Ω_{d_j}. So $\Delta_{d_j} = \{\omega \in \Omega_{d_j} : \omega \cdot X_{j-1}^{I_{L_{j-1}}} \neq \{\omega\}\}$ contains at least two points. Therefore the maximal size of $\Omega_{d_j} \backslash \Delta_{d_j}$ is $|\Omega_{d_j}| - 2$. Thus the Wilson quotient G_n has less than or equal to

$$2^{|X_n|^{d_n}} + \sum_{j=1}^{n} 2^{d_j} + \sum_{j=2}^{n} 2^{d_j - 2}(2^{|X_{j-1}|^{d_{j-1}}} - 2) \tag{7.5}$$

subnormal subgroups.

Recall the definition of the groups $H_n = L_n G_{n-1}$, for $n \geq 1$, as defined at the beginning of Section 6.1. From the classification in Section 6.4, any subnormal subgroup of a Wilson group G that has index at most $|H_n|$ contains $Q_n = \ker(\phi_n : G \to H_n)$. Thus the number of subnormal subgroups of a Wilson group of index at most $|H_n|$, for $n \geq 1$, is less than or equal to the number (7.5).

In this expression (7.5), the term $\sum_{j=1}^{n} 2^{d_j}$ is very small in comparison with the other two terms. These two terms $2^{|X_n|^{d_n}}$ and $\sum_{j=2}^{n} 2^{d_j-2}(2^{|X_{j-1}|^{d_{j-1}}}-2)$ look similar in size. Since the d_j, for $j \geq 1$, increase in value (refer to (7.3) in the proof of Theorem 7.1), the term $2^{|X_n|^{d_n}}$ is the largest in the expression (7.5).

Define the number of subnormal subgroups of a Wilson group G of index at most $|H_n|$ as $S^{\lhd\lhd}_{|H_n|}(G)$. Using Theorem 7.1, we can conclude that $S^{\lhd\lhd}_{|H_n|}(G)$, which is less than the number (7.5), is roughly the size of the group G_n, although somewhat smaller. Therefore for some constant d we have $S^{\lhd\lhd}_{|H_n|}(G) \leq d|G_n|$, for $n \geq 1$.

7.2 Infinite iterated wreath products $\ldots \wr A_m \wr A_m \wr \ldots \wr A_m$, where $m \geq 5$

Recall the just infinite profinite groups $W = \varprojlim (W_n)_{n \geq 0}$, where

$$W_n = A_m \wr_{\Omega^{*[n]}} \ldots \wr_{\Omega^{*[2]}} A_m \wr_{\Omega^{*[1]}} A_m,$$

for $n \geq 1$, and where

$$\Omega^{*[j]} = \{i_1 i_2 \ldots i_j : i_1, i_2, \ldots, i_j \in \{1, 2, \ldots, m\}\},$$

for each $j = 1, 2, \ldots$, and $W_0 = A_m$, as defined in Section 3.2. It is standard to view the group W as acting on the infinite m-regular rooted tree, that is where every vertex has m children (see P. de la Harpe [6, pg. 211-212]). We denote this tree by T.

The subnormal subgroups of these groups are completely characterised in Section 6.3. Every non-trivial subnormal subgroup of W has index of the form $|A_m|^k$, for some $k \geq 1$. The number of subnormal subgroups of W with index $|A_m|^k$, for $k \geq 1$, is equal to the number of subtrees of T that have the same root and k vertices (or equivalently $k - 1$ edges). The following diagram is an example to illustrate this statement.

For $m = 5$, we consider the same subnormal subgroup of W that has been depicted previously in Figure 6.2, found towards the end of Section 6.3. Below, Figure 7.1

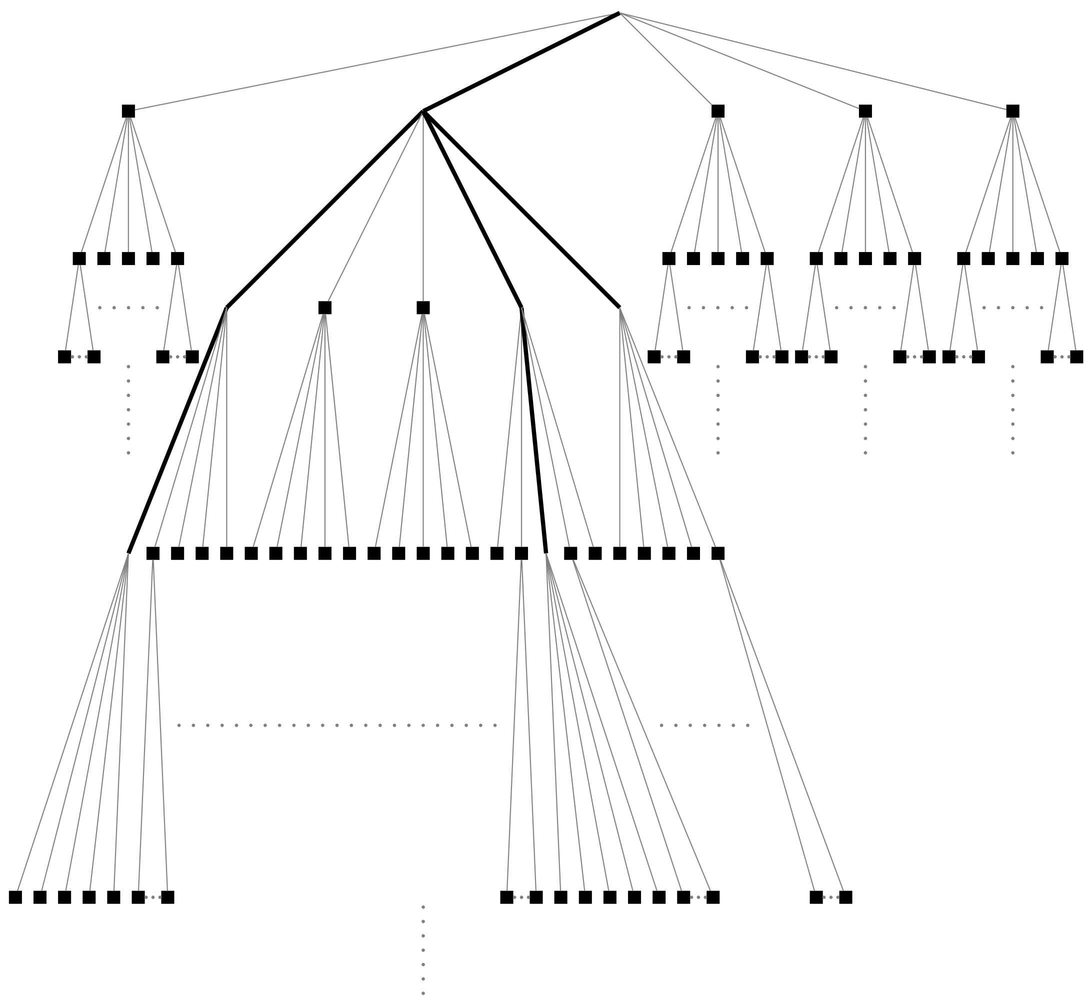

Figure 7.1: The index of a subnormal subgroup of W represented as a subtree of T.

represents the index of this subnormal subgroup as the highlighted subtree of the infinite tree T. The index in W of this subnormal subgroup is $|A_5|^7$.

We denote the number of subnormal subgroups of W with index $|A_m|^k$ by $\widetilde{a}_k^{\lhd\lhd}(W)$. The number of subtrees of T that have the same root and k vertices is the same as the Fuss-Catalan number

$$\frac{1}{(m-1)k+1}\binom{mk}{k},$$

refer to [1, Prop. 3.1]. Therefore the number of non-trivial subnormal subgroups of W with index at most $|A_m|^n$, for some n, is equal to the sum

$$\sum_{k=1}^{n} \widetilde{a}_k^{\lhd\lhd}(W) = \sum_{k=1}^{n} \frac{1}{(m-1)k+1}\binom{mk}{k}.$$

For further research concerning the subnormal subgroup growth of the groups W, see Chapter 10, Question 3.

Chapter 8

Maximal subgroups

8.1 Introduction

We now wish to investigate maximal subgroups of Wilson groups. Let G be a Wilson group arising as an inverse limit of finite groups G_n as defined in Section 4.1. We would first like to determine the maximal subgroups of the finite groups G_n.

Fix the alphabet $A = \{1, 2, \ldots, m\}$, where $m \geq 5$. For $n \geq 1$, recall the iterated wreath products

$$W_n = A_m \wr_{\Omega^{*[n]}} \ldots \wr_{\Omega^{*[2]}} A_m \wr_{\Omega^{*[1]}} A_m,$$

first defined in Section 3.2, where

$$\Omega^{*[j]} = \{i_1 i_2 \ldots i_j : i_1, i_2, \ldots, i_j \in A\},$$

for each $j = 1, 2, \ldots, n$. Here again $i_1 i_2 \ldots i_j$ denotes a sequence of numbers and not a product of numbers.

The groups G_n and W_n are both constructed from wreath products of finite non-abelian simple groups using transitive actions. Therefore determining maximal subgroups of the groups G_n is likely to involve the same techniques that are used to determine maximal subgroups of the groups W_n.

M. Bhattacharjee [3] has produced information on maximal subgroups of iterated wreath products that are constructed from alternating groups of degree at least 5. Her wreath products are a little different from our wreath products W_n, in that the alternating groups are allowed to vary giving $A_{m_k} \wr \ldots \wr A_{m_2} \wr A_{m_1}$, where $m_1, m_2, \ldots, m_k \geq 5$. The natural action of the alternating groups is used to form Bhattacharjee's wreath products and the natural action of the alternating groups is used to form the wreath products W_n.

Bhattacharjee's view point is that of finite generation of inverse limits of such wreath products. Her method requires her to analyse maximal subgroups, of the wreath products, which modulo the base group project onto the top group. She obtains upper bounds for the number of conjugacy classes of these maximal subgroups. Bhattacharjee's results fall short of a complete classification of such maximal subgroups.

8.2 Finite wreath products $A_m \wr A_m$, where $m \geq 5$

We now consider the maximal subgroups of the finite groups W_n, for $n \geq 1$. As we want to see how techniques can be applied to the groups G_n, the easiest step is to look at the first wreath product $W_1 = A_m \wr_{\Omega^{*[1]}} A_m$, where $\Omega^{*[1]} = \{1, 2, \ldots, m\}$ and $m \geq 5$.

Theorem 8.3 describes the maximal subgroups of W_1. The proof of this theorem is a special case of Bhattacharjee's work in [3, pg. 316 - 321]. This is because she works more generally applying to wreath products where the top group can also be an iterated wreath product. There are differences, some very subtle, between our work and Bhattacharjee's work, which we now go on to explain.

The proof of Theorem 8.3 separates the possibilities for the maximal subgroups of W_1 into types, referred to as Case 1, Case 2a, Case 2b and Case 2c. The Case 1 type found in Theorem 8.3 does not occur in Bhattacharjee's work because she is only concerned with maximal subgroups that modulo the base group project onto the top group.

In Theorem 8.3, the proof concerning the maximal subgroups of type Case 2a is new and different from Bhattacharjee's proof. It is also a little more self-contained than Bhattacharjee's, since it does not rely on Lemma 2.3 from [2] (alternatively, see the Appendix of our thesis for this lemma). Instead, because we can specify double-transpositions from A_m and work with them directly, we implicitly produce a proof that the action of A_m on $\Omega^{*[1]} = \{1, 2, \ldots, m\}$ is primitive, see Lemma 8.2. Later in this section we go further to provide accurate results for the counting of these types of maximal subgroups (see Remark 8.5) and the counting of conjugacy classes of these types of maximal subgroups (see Remark 8.7).

The proof of the maximal subgroups of type Case 2b in Theorem 8.3 is contained in Bhattacharjee's work and we have possibly written it in a more readable fashion. However, later in this section we do produce extra information regarding the counting of these maximal subgroups (see Remark 8.6) and the counting of conjugacy classes of these maximal subgroups (see Remark 8.8).

Our work on the maximal subgroups of type Case 2c in the proof of Theorem 8.3 is new and different from Bhattacharjee's proof. This is because we use the more recent results of C. Parker and M. Quick [23] to rule out the possibility of maximal subgroups of this type. Theorem A(i) of [23] gives a set of conditions for a wreath product to have a maximal subgroup which complements the base group. In Theorem 8.3, we will show that one of these conditions fails to hold for our wreath product W_1.

To help the readers understanding, we now state the theorem of Parker and Quick.

Theorem 8.1 (Parker and Quick [23]). *Let X and Y be groups with Y acting on the finite set Ω where $|\Omega| > 1$. Let $W = X \wr_\Omega Y$ be the wreath product of X by Y with respect to this action and let K be the base group of W.*

The wreath product W has a maximal subgroup which is a complement to K if and only if the following conditions hold:

(a) *X is a non-abelian simple group,*

(b) *Y acts transitively on Ω,*

(c) *there exists a surjective homomorphism $\phi : \mathrm{St}_Y(\omega) \longrightarrow X$ from the stabiliser of a point $\omega \in \Omega$ in Y to X, and*

(d) *if we view ϕ as a map $\mathrm{St}_Y(\omega) \longrightarrow \mathrm{Aut}(X)$, identifying X with its group of inner automorphisms, then ϕ is not the restriction of a homomorphism $H \longrightarrow \mathrm{Aut}(X)$ for any subgroup H of Y properly containing $\mathrm{St}_Y(\omega)$.*

On several occasions, the following lemma is applied in the proof of Theorem 8.3.

Lemma 8.2. *Let $W_1 = A_m \wr_{\Omega^{*[1]}} A_m$, where $\Omega^{*[1]} = \{1, 2, \ldots, m\}$, for some $m \geq 5$. Denote the base group $A_m^{(m)} =: B$ and the permuting top group $A_m =: T$.*

Suppose H is a subgroup of W_1 such that

(i) $HB = W_1$, and

(ii) $H \cap B$ is a proper subdirect product in B.

Then

$$H \cap B = \{(x, \varphi_2(x), \varphi_3(x), \ldots, \varphi_m(x)) : x \in A_m\},$$

where $\varphi_j \in \mathrm{Aut}(A_m)$, for $2 \leq j \leq m$.

We remark that the group $\{(x, \varphi_2(x), \varphi_3(x), \ldots, \varphi_m(x)) : x \in A_m\}$, where $\varphi_j \in \mathrm{Aut}(A_m)$, for $2 \leq j \leq m$, is referred to as a *diagonal subgroup* of the direct product $\prod_{i=1}^{m} A_m^{(i)}$ of alternating groups.

Proof. We claim that the first coordinate of an element of $H \cap B$ determines all the other coordinates of that element. For a contradiction, suppose

$$(x, y_1, *, \dots, *), (x, y_2, *, \dots, *) \in H \cap B$$

such that $y_1 \neq y_2$. Then

$$(x, y_1, *, \dots, *)(x, y_2, *, \dots, *)^{-1} = (1, y_1 y_2^{-1}, *, \dots, *) \in H \cap B$$

with $y_1 y_2^{-1} \neq 1$. Put $y_3 = y_1 y_2^{-1}$.

For $t = (13)(45) \in T$ we find $b = (b_1, b_2, \dots, b_m) \in B$ such that $tb \in H$ (using condition (i)). So

$$(1, y_3, *, \dots, *)^{tb} = (*, y_3^{b_2}, 1, *, \dots, *) \in H \cap B.$$

Put $\widetilde{y_3} = y_3^{b_2}$. If $[y_3^h, \widetilde{y_3}] = 1$, for all $h \in A_m$, then $[k, \widetilde{y_3}] = 1$, for all $k \in \langle y_3 \rangle^{A_m}$. Now $\langle y_3 \rangle^{A_m} = A_m$, since $y_3 \neq 1$ and A_m is simple. Therefore $\widetilde{y_3} \in Z(A_m) = \{1\}$. Contradicting $\widetilde{y_3} \neq 1$, as $y_3 \neq 1$. Thus there exists $h \in A_m$ such that $[y_3^h, \widetilde{y_3}] \neq 1$. We can find $(*, h, *, \dots, *) \in H \cap B$ because $H \cap B$ projects onto A_m in each coordinate (using condition (ii)). Then

$$(1, y_3, *, \dots, *)^{(*, h, *, \dots, *)} = (1, y_3^h, *, \dots, *) \in H \cap B$$

and

$$[(1, y_3^h, *, \dots, *), (*, \widetilde{y_3}, 1, *, \dots, *)] = (1, [y_3^h, \widetilde{y_3}], 1, *, \dots, *) \in H \cap B.$$

Put $y_4 = [y_3^h, \widetilde{y_3}]$.

For $t = (14)(35) \in T$ we find $b = (b_1, b_2, \dots, b_m) \in B$ such that $tb \in H$. So

$$(1, y_4, 1, *, \dots, *)^{tb} = (*, y_4^{b_2}, *, 1, 1, *, \dots, *) \in H \cap B.$$

Put $\widetilde{y_4} = y_4^{b_2}$. There exists $h \in A_m$ such that $[y_4^h, \widetilde{y_4}] \neq 1$ and we can find $(*, h, *, \dots, *) \in H \cap B$ because $H \cap B$ projects onto A_m in each coordinate. Then

$$(1, y_4, 1, *, \dots, *)^{(*, h, *, \dots, *)} = (1, y_4^h, 1, *, \dots, *) \in H \cap B$$

and

$$[(1, y_4^h, 1, *, \dots, *), (*, \widetilde{y_4}, *, 1, 1, *, \dots, *)] = (1, [y_4^h, \widetilde{y_4}], 1, 1, 1, *, \dots, *) \in H \cap B.$$

The process can be iterated $n-3$ times to obtain $(1, y_n, 1, 1, \ldots, 1) \in H \cap B$ with $y_n \neq 1$. Since $H \cap B$ projects onto A_m in each coordinate, we have $(*, g, *, \ldots, *) \in H \cap B$, for all $g \in A_m$. Then

$$(1, y_n, 1, 1, \ldots, 1)^{(*,g,*,\ldots,*)} = (1, y_n^g, 1, 1, \ldots, 1) \in H \cap B,$$

for all $g \in A_m$. Therefore

$$\begin{aligned}&\{1\} \times \langle y_n \rangle^{A_m} \times \{1\} \times \{1\} \times \ldots \times \{1\}\\ &= \{1\} \times A_m \times \{1\} \times \{1\} \times \ldots \times \{1\} \subseteq H \cap B,\end{aligned}$$

as $y_n \neq 1$ and A_m is simple. For all $t \in T$ we have $tb \in H$, for some $b \in B$, and conjugating by $tb \in H$ implies $A_m^{(m)} \subseteq H \cap B$. This contradicts $B \not\subseteq H$. Thus

$$H \cap B = \{(x, \varphi_2(x), \varphi_3(x), \ldots, \varphi_m(x)) : x \in A_m\},$$

where $\varphi_j : A_m \longrightarrow A_m$ are maps, for $2 \leq j \leq m$. That is $H \cap B \cong A_m$.

In fact, $H \cap B$ being a subdirect product in B implies that each of the maps φ_j is surjective. Since φ_j are surjective maps between the same finite set, they are injective. Now

$$\begin{aligned}&(x, \varphi_2(x), \varphi_3(x), \ldots, \varphi_m(x))(y, \varphi_2(y), \varphi_3(y), \ldots, \varphi_m(y))\\ &\qquad = (xy, \varphi_2(x)\varphi_2(y), \varphi_3(x)\varphi_3(y), \ldots, \varphi_m(x)\varphi_m(y)) \in H \cap B\end{aligned}$$

and, as the first coordinate of an element of $H \cap B$ determines all its other coordinates, this element is equal to $(xy, \varphi_2(xy), \varphi_3(xy), \ldots, \varphi_m(xy))$. Therefore $\varphi_j(x)\varphi_j(y) = \varphi_j(xy)$, for every $x, y \in A_m$, for $2 \leq j \leq m$, and the maps φ_j are homomorphisms. Hence

$$\varphi_j \in \mathrm{Aut}(A_m), \text{ for } 2 \leq j \leq m.$$

□

Theorem 8.3 classifies the maximal subgroups of W_1 up to conjugation. The maximal subgroups are conjugates of three types of subgroups and the theorem tells us that it is enough to conjugate by the elements of the base group B. The degree of the alternating groups has been restricted to $m \neq 6$ because the proof of the theorem makes use of the fact that $\mathrm{Aut}(A_m) \cong S_m$, for $m \geq 4$ and $m \neq 6$.

We now state Theorem 8.3 and we prove this theorem over the next several pages.

Theorem 8.3. *Let $W_1 = A_m \wr_{\Omega^{*[1]}} A_m$, where $\Omega^{*[1]} = \{1, 2, \ldots, m\}$, for some $m \geq 5$ and $m \neq 6$. Denote the base group $A_m^{(m)} =: B$ and the permuting top group $A_m =: T$. Therefore $W_1 = B \rtimes T$.*

Define

$$M_0(L) = B \rtimes L, \text{ where } L \text{ is a maximal subgroup of } A_m.$$

Define

$$M_1 = \{(x, x, \ldots, x) : x \in A_m\} \times T.$$

Define

$$M_2(L) = L^{(m)} \rtimes T, \text{ where } L \text{ is a maximal subgroup of } A_m.$$

Then the groups $M_0(L)$, M_1^g, where $g \in B$, and $M_2(L)^g$, where $g \in B$, are maximal subgroups of W_1 and every maximal subgroup of W_1 is one of these.

Proof. Let M be a maximal subgroup of W_1. Then there are two possibilities:

$$B \subseteq M \text{ (case 1), and } B \not\subseteq M \text{ (case 2).}$$

Case 1.

Suppose $B \subseteq M$. Since $B \trianglelefteq W_1$, we have the surjective group homomorphism $W_1 \longrightarrow W_1/B \cong T$. A group homomorphism preserves inclusion of subgroups. Then there is a one-to-one correspondence between the maximal subgroups of W_1 containing B and the maximal subgroups of T. Therefore $M = BL$, where L is a maximal subgroup of T. Now L normalising B and $B \cap L = \{1\}$ implies $M = B \rtimes L$. Hence $M = M_0(L)$.

Case 2.

Suppose $B \not\subseteq M$. Obviously $M \subseteq BM \subseteq W_1$. Since M is a maximal subgroup of W_1, we have $M = BM$ or $BM = W_1$. However, $M = BM$ contradicts $B \not\subseteq M$. Therefore

$$BM = W_1.$$

Then $B \trianglelefteq W_1$, by the 2nd isomorphism theorem, gives

$$M/(M \cap B) \cong BM/B = (B \rtimes T)/B \cong T.$$

So for all $t \in T$ there exists $b \in B$ such that $bt \in M$.

Let $i, j \in \Omega^{*[1]}$. We choose $t \in T$ such that $ti = j$, since T acts transitively on the set $\Omega^{*[1]}$. For this $t \in T$, we find $b \in B$ such that $tb \in M$. For $1 \leq i \leq m$, let π_i be the projection map from B onto the ith factor of B. Then $\pi_j((M \cap B)^{tb}) =$

$\pi_i(M \cap B)^{b_j}$, where $b = (b_1, b_2, \ldots, b_m) \in B$. Thus the projections of $M \cap B$ into the m factors of B are conjugate in A_m.

Define

$$K_i := \pi_i(M \cap B) \leq A_m, \text{ for } 1 \leq i \leq m.$$

Therefore

$$M \cap B \leq K_1 \times K_2 \times \ldots \times K_m.$$

Case 2 can be separated into three possibilities because the groups K_i are all conjugate subgroups of A_m.

(case 2a) Let the group $K_1 = A_m$. Then $K_j = A_m^{b_j} = A_m$, for all $j \in \Omega^{*[1]}$.

(case 2b) Let the group $K_1 \neq \{1\}$ and $K_1 \neq A_m$. Then $K_j = K_1^{b_j} \neq \{1\}$ and $K_j = K_1^{b_j} \neq A_m$, for all $j \in \Omega^{*[1]}$.

(case 2c) Let the group $K_1 = \{1\}$. Then $K_j = \{1\}^{b_j} = \{1\}$, for all $j \in \Omega^{*[1]}$.

Case 2a.

Assume the groups $K_i = A_m$, for all $i \in \Omega^{*[1]}$. Then $M \cap B$ is a proper subdirect product in B. Setting $H = M$, Lemma 8.2 tells us that

$$M \cap B = \{(x, \varphi_2(x), \varphi_3(x), \ldots, \varphi_m(x)) : x \in A_m\},$$

where $\varphi_j \in \mathrm{Aut}(A_m)$, for $2 \leq j \leq m$.

We first consider a special case where $\varphi_j = \mathrm{id}_{A_m}$, for all $2 \leq j \leq m$. That is

$$M \cap B = \{(x, x, \ldots, x) : x \in A_m\}.$$

We prove that if M is a maximal subgroup such that $M \cap B = \{(x, x, \ldots, x) : x \in A_m\}$ then T is contained in M.

Let $(x, x, \ldots, x) \in M \cap B$ and $bt \in M$, where $b = (b_1, b_2, \ldots, b_m) \in B$ and $t \in T$. Then

$$(x, x, \ldots, x)^{bt} = (x^{b_1}, x^{b_2}, \ldots, x^{b_m})^t \in M \cap B.$$

Therefore $x^{b_1} = x^{b_2} = \ldots = x^{b_m}$. So $x^{b_i b_j^{-1}} = x$, for all $i, j \in \Omega^{*[1]}$. Since this holds for all $x \in A_m$, we have $b_i b_j^{-1} \in Z(A_m) = \{1\}$, for all $i, j \in \Omega^{*[1]}$.

Therefore $b_i = b_j$, for all $i, j \in \Omega^{*[1]}$. Now $b = (b_1, b_1, \ldots, b_1) \in M$ and so $t = b^{-1}(bt) \in M$. Since this holds for all $t \in T$, we have $T \subseteq M$.

Therefore $M = (M \cap B)T$. Now $B \trianglelefteq W_1$ implies $M \cap B \trianglelefteq M$, and $B \cap T = \{1\}$ implies $(M \cap B) \cap T = \{1\}$. In fact, $T \leq M$ gives this particular maximal subgroup as the semidirect product $M = (M \cap B) \rtimes T$.

Furthermore, $T \trianglelefteq M$ because T acting by conjugation on the elements $(x, x, \ldots, x)$ permutes the coordinates and, since the coordinates are all the same, permuting them leaves the elements $(x, x, \ldots, x)$ unchanged. Therefore we actually have the direct product $M = (M \cap B) \times T$. Thus $M = M_1$, recalling that $M_1 = \{(x, x, \ldots, x) : x \in A_m\} \times T$.

We check that M_1 is a maximal subgroup of W_1. Clearly M_1 is a proper subgroup of W_1 because it does not contain all the elements of the base group B.

We now show, for all $g \in W_1 \backslash M_1$, that $\langle \{g\} \cup M_1 \rangle = W_1$. Take $g = bt \in W_1 \backslash M_1$, where $b \in B$ and $t \in T$. Then

$$\widetilde{g} = gt^{-1} = b \in B \backslash (B \cap M_1),$$

as $t \in M_1$. Therefore $\widetilde{g} = (x_1, x_2, \ldots, x_m)$ with $x_j \neq x_1$ for some $j \in \Omega^{*[1]}$. Since

$$\langle \{g\} \cup M_1 \rangle = \langle \{\widetilde{g}\} \cup M_1 \rangle,$$

we will consider the group $\langle \{\widetilde{g}\} \cup M_1 \rangle$. For a contradiction, suppose that $\langle \{\widetilde{g}\} \cup M_1 \rangle \subsetneq W_1$. We can apply Lemma 8.2 to the group $\langle \{\widetilde{g}\} \cup M_1 \rangle$, setting $H = \langle \{\widetilde{g}\} \cup M_1 \rangle$. Condition (i) holds because $T \subseteq M_1 \subseteq H$. Condition (ii) holds because if $H \cap B = B$ we would not have $H \subsetneq W_1$. Thus the first coordinate of an element of $H \cap B$ determines all the other coordinates of that element. This contradicts $\widetilde{g} \in H$ and $(x_1, x_1, \ldots, x_1) \in H$.

Now we look more generally at the maximal subgroups M such that

$$M \cap B = \{(x, \varphi_2(x), \varphi_3(x), \ldots, \varphi_m(x)) : x \in A_m\},$$

where $\varphi_j \in \mathrm{Aut}(A_m)$, for $2 \leq j \leq m$. For $m \geq 4$, with the exception of $m = 6$, it is known that $\mathrm{Aut}(A_m) \cong S_m$, where S_m acts on A_m by conjugation. Therefore

$$M \cap B = \{(x, x^{g_2}, x^{g_3}, \ldots, x^{g_m}) : x \in A_m\},$$

where $g_j \in S_m$, for $2 \leq j \leq m$.

For $t = (123) \in T$ we find $b = (b_1, b_2, \ldots, b_m) \in B$ such that $tb \in M$. Also

$$(x^{g_m b_m g_m^{-1}}, (x^{g_m b_m g_m^{-1}})^{g_2}, \ldots, (x^{g_m b_m g_m^{-1}})^{g_{m-1}}, x^{g_m b_m})$$

is an element in $M \cap B$. Multiplying the inverse of this element by $(x, x^{g_2}, \ldots, x^{g_m})^{tb}$ gives the element

$$(x^{g_m b_m g_m^{-1}}, x^{g_m b_m g_m^{-1} g_2}, \ldots, x^{g_m b_m g_m^{-1} g_{m-1}}, x^{g_m b_m})^{-1} \cdot (x^{g_3 b_1}, x^{b_2}, x^{g_2 b_3}, x^{g_4 b_4}, x^{g_5 b_5}, \ldots, x^{g_m b_m})$$

$$= ((x^{g_m b_m g_m^{-1}})^{-1} x^{g_3 b_1}, (x^{g_m b_m g_m^{-1} g_2})^{-1} x^{b_2}, (x^{g_m b_m g_m^{-1} g_3})^{-1} x^{g_2 b_3}, (x^{g_m b_m g_m^{-1} g_4})^{-1} x^{g_4 b_4}, (x^{g_m b_m g_m^{-1} g_5})^{-1} x^{g_5 b_5}, \ldots, (x^{g_m b_m g_m^{-1} g_{m-1}})^{-1} x^{g_{m-1} b_{m-1}}, 1),$$

which is in $M \cap B$, for all $x \in A_m$. Since the mth coordinate of this element is equal to 1, all the coordinates of this element are equal to 1. From the 1st coordinate, we deduce that $x^{g_m b_m g_m^{-1}} = x^{g_3 b_1}$, for all $x \in A_m$. Then $g_m b_m g_m^{-1} = g_3 b_1$ because $C_{S_m}(A_m) = \{1\}$. Considering this equation modulo A_m, we obtain that $1 \equiv g_3 \pmod{A_m}$, as $b_1, b_m \in A_m$. Working similarly, the 2nd coordinate gives $g_2 \equiv 1 \pmod{A_m}$ and the 3rd coordinate gives $g_3 \equiv g_2 \pmod{A_m}$. This argument can be applied repeatedly, taking in turn t as each of the 3-cycles in A_m. Therefore it is deduced that

$$1 \equiv g_2 \equiv \ldots \equiv g_m \pmod{A_m}.$$

So $g_2, g_3, \ldots, g_m \in A_m$ because $1 \in A_m$.

Now

$$M \cap B = \{(x, x, \ldots, x) : x \in A_m\}^g,$$

where $g = (1, g_2, g_3, \ldots, g_m) \in B$. Then

$$(M \cap B)^{g^{-1}} = M^{g^{-1}} \cap B^{g^{-1}} = M^{g^{-1}} \cap B = \{(x, x, \ldots, x) : x \in A_m\}.$$

So $M^{g^{-1}}$ is a maximal subgroup of W_1 such that $M^{g^{-1}} \cap B = \{(x, x, \ldots, x) : x \in A_m\}$. Therefore $T \subseteq M^{g^{-1}}$ and $M^{g^{-1}} = M_1$. Hence $M = M_1^g$, where $g = (1, g_2, g_3, \ldots, g_m) \in B$.

Case 2b.

Assume the groups $K_i \neq \{1\}$ and $K_i \neq A_m$, for all $i \in \Omega^{*[1]}$. We choose $g = (g_1, g_2, \ldots, g_m) \in B$ such that

$$K_1^{g_1} = K_2^{g_2} = \ldots = K_m^{g_m} = L \neq A_m,$$

as the groups K_i are all conjugate subgroups in A_m. Then $\pi_i((M \cap B)^g) = \pi_i(M^g \cap B) = L$, for $1 \leq i \leq m$, and so

$$M^g \cap B \leq L^{(m)}.$$

Instead, we now study the maximal subgroup M^g of W_1.

We claim that M^g is contained in the normaliser of $L^{(m)}$ in W_1. Let $(l_1, l_2, \ldots, l_m) \in L^{(m)}$ and $bt \in M^g$, where $b = (b_1, b_2, \ldots, b_m) \in B$ and $t \in T$. Then

$$(l_1, l_2, \ldots, l_m)^{bt} = (l_1^{b_1}, l_2^{b_2}, \ldots, l_m^{b_m})^t.$$

We need to show that $l_i^{b_i} \in L$, for each $i \in \Omega^{*[1]}$. Since $M^g \cap B$ projects onto L in each coordinate, in $M^g \cap B$ there will be elements $(*, \ldots, *, l_i, *, \ldots, *)$ where l_i is in the ith position, for each $i \in \Omega^{*[1]}$. Conjugating by the same element $bt \in M^g$ gives

$$(*, \ldots, *, l_i, *, \ldots, *)^{bt} = (*, \ldots, *, l_i^{b_i}, *, \ldots, *)^t \in M^g \cap B.$$

Again since $M^g \cap B$ projects onto L in each coordinate, we have proved that $l_i^{b_i} \in L$, for each $i \in \Omega^{*[1]}$.

Now

$$M^g \leq N_{W_1}(L^{(m)}) \leq W_1.$$

As M^g is a maximal subgroup of W_1, we have that $M^g = N_{W_1}(L^{(m)})$ or $N_{W_1}(L^{(m)}) = W_1$. If $N_{W_1}(L^{(m)}) = W_1$ then $(N_{A_m}(L))^{(m)} \rtimes T = A_m^{(m)} \rtimes T$, by Lemma 2.5. So $N_{A_m}(L) = A_m$ and $L \trianglelefteq N_{A_m}(L) = A_m$. Since A_m is simple, this implies the contradiction that $L = \{1\}$ or $L = A_m$. Therefore

$$M^g = N_{W_1}(L^{(m)}) = (N_{A_m}(L))^{(m)} \rtimes T.$$

Obviously $M^g \cap B = (N_{A_m}(L))^{(m)}$. So $M^g \cap B \leq L^{(m)}$ gives $(N_{A_m}(L))^{(m)} \leq$

$L^{(m)}$. As $L \leq N_{A_m}(L)$, we have

$$N_{A_m}(L) = L. \tag{8.1}$$

Therefore $M^g = L^{(m)} \rtimes T$.

Here L must be a maximal subgroup of A_m because if it was not then we can find a maximal subgroup L' lying between L and A_m. Then $(L')^{(m)} \rtimes T$ is a group properly containing M^g but is not W_1, and contradicting that M^g is maximal in W_1. So $M^g = M_2(L)$, recalling that $M_2(L) = L^{(m)} \rtimes T$, where L is a maximal subgroup of A_m. Hence $M = M_2(L)^{g^{-1}}$, where $g \in B$.

We check that any choice of maximal subgroup L of A_m leads to $M_2(L)$ being a maximal subgroup of W_1. Clearly $M_2(L)$ is a proper subgroup of W_1, since L is maximal in A_m we can find $x \in A_m \backslash L$ so that $(x, 1, 1, \ldots, 1) \notin M_2(L)$.

We now show, for all $g \in W_1 \backslash M_2(L)$, that $\langle \{g\} \cup M_2(L) \rangle = W_1$. Take $g = bt \in W_1 \backslash M_2(L)$, where $b \in B$ and $t \in T$. Then

$$\widetilde{g} = gt^{-1} = b \in B \backslash (B \cap M_2(L)),$$

as $t \in M_2(L)$. Therefore $\widetilde{g} = (y_1, y_2, \ldots, y_m)$ where without loss of generality $y_1 \notin L$. Since

$$\langle \{g\} \cup M_2(L) \rangle = \langle \{\widetilde{g}\} \cup M_2(L) \rangle,$$

we will consider the group $\langle \{\widetilde{g}\} \cup M_2(L) \rangle$. We have $\langle y_1, L \rangle = A_m$ because L is maximal in A_m. Therefore $\langle \{\widetilde{g}\} \cup M_2(L) \rangle$ contains elements $(h, *, \ldots, *)$ for any $h \in A_m$. Since $L \neq \{1\}$, there exists $(l, 1, \ldots, 1) \in L^{(m)} \subseteq \langle \{\widetilde{g}\} \cup M_2(L) \rangle$ with $l \neq 1$. Then

$$(l, 1, \ldots, 1)^{(h, *, \ldots, *)} = (l^h, 1, \ldots, 1) \in \langle \{\widetilde{g}\} \cup M_2(L) \rangle,$$

for all $h \in A_m$. Therefore

$$\langle l \rangle^{A_m} \times \{1\} \times \ldots \times \{1\} = A_m \times \{1\} \times \ldots \times \{1\} \subseteq \langle \{\widetilde{g}\} \cup M_2(L) \rangle,$$

as $l \neq 1$ and A_m is simple. Applying the action of T implies that $B \subseteq \langle \{\widetilde{g}\} \cup M_2(L) \rangle$. So $\langle \{\widetilde{g}\} \cup M_2(L) \rangle = W_1$ and this confirms that $M_2(L)$ is a maximal subgroup of W_1.

Case 2c.

Assume the groups $K_i = \{1\}$, for all $i \in \Omega^{*[1]}$. Therefore $M \cap B = \{1\}$. Also since $BM = W_1$, we have that in this case the maximal subgroup M is a complement for the base group B in W_1.

We show that condition (c) of Theorem 8.1 does not hold. In applying this theorem to our group W_1, we have that $X = A_m$ and $Y = A_m$. The stabiliser of any point $i \in \Omega^{*[1]}$ under the action of A_m is isomorphic to A_{m-1}. Thus there can be no surjective homomorphism from the stabiliser of a point $i \in \Omega^{*[1]}$ under the action of A_m to the group A_m. Hence W_1 has no maximal subgroups which complement the base group and Case 2c does not occur.

□

Remark. Theorem 8.3 implies that there are three types of maximal subgroups of W_1.

- Maximal subgroups M of the form $M_0(L)$ have the property that $M \cap B$ is equal to B (Case 1).

Maximal subgroups M that are conjugates of:

- M_1 have the property that $M \cap B$ is a proper subdirect product in B (Case 2a);
- $M_2(L)$ have the property that $M \cap B$ projects onto a maximal subgroup of A_m in each coordinate (Case 2b).

Remark. The groups $M_0(L)$ are semidirect products of $M_0(L) \cap B = B$ by L. The groups M_1^g, where $g \in B$, are semidirect products of $M_1^g \cap B$ by T^g. The groups $M_2(L)^g$, where $g \in B$, are semidirect products of $M_2(L)^g \cap B$ by T^g.

Therefore all the maximal subgroups M of W_1 are semidirect products of $M \cap B$ by a suitable non-trivial complement.

In [3], Bhattacharjee finds upper bounds for the number of conjugacy classes of maximal subgroups of the wreath products that she is considering. We are able to do a little more because our wreath products W_1 are a very specific subclass of Bhattacharjee's wreath products. Since we have classified the maximal subgroups of W_1 up to conjugation, we can count explicitly the number of them using the orbit-stabiliser theorem. These numbers are displayed below in Corollary 8.4.

Corollary 8.4. *Let W_1 be the group as defined in Theorem 8.3. Then the number of maximal subgroups M of W_1 with the property that $M \cap B$:*

- *is equal to B is precisely the number of maximal subgroups of A_m (Case 1);*
- *is a proper subdirect product in B is precisely $|A_m|^{m-1}$ (Case 2a);*

- *projects onto a maximal subgroup of A_m in each coordinate is precisely*

$$\sum_{L \leq_{\max} A_m} |A_m : L|^{m-1},$$

where the summation runs over all maximal subgroups of A_m (Case 2b).

Proof.

Case 2a.

Maximal subgroups of W_1 of the type in Case 2a are all of the form M_1^g, where $g \in B$. We calculate the number of distinct maximal subgroups of this type. The group B in W_1 acts on the orbit $\{M_1^g : g \in B\}$ by conjugation. The orbit-stabiliser theorem says that the length of this orbit is $|B : N_B(M_1)|$. Therefore we compute the normaliser of M_1 in B.

To simplify workings we notice that a conjugate of an element of M_1 is in B if and only if the element of M_1 is in B. We need to find elements $(g_1, g_2, \ldots, g_m) \in B$ where for all $x \in A_m$ there exists $y \in A_m$ such that $(x^{g_1}, x^{g_2}, \ldots, x^{g_m}) = (y, y, \ldots, y)$. That is $x^{g_i} = x^{g_j}$, for all $x \in A_m$ and for all $i, j \in \Omega^{*[1]}$. So $x^{g_i g_j^{-1}} = x$, for all $x \in A_m$, and $g_i g_j^{-1} \in Z(A_m) = \{1\}$, for all $i, j \in \Omega^{*[1]}$. Then $g_i = g_j$, for all $i, j \in \Omega^{*[1]}$.

We check that $T^{(g_1, g_1, \ldots, g_1)} \subseteq M_1$. In fact $T^{(g_1, g_1, \ldots, g_1)} = T$. Therefore

$$N_B(M_1) = \{(g_1, g_1, \ldots, g_1) : g_1 \in A_m\} \cong A_m.$$

The number of distinct conjugates M_1^g, where $g \in B$, is $|A_m|^m / |A_m| = |A_m|^{m-1}$.

Case 2b.

Maximal subgroups of W_1 of the type in Case 2b are all of the form $M_2(L)^g$, where L is a maximal subgroup of A_m and $g \in B$. We calculate the number of distinct maximal subgroups of this type. For fixed L, the group B in W_1 acts on the orbit $\{M_2(L)^g : g \in B\}$ by conjugation. The orbit-stabiliser theorem says that the length of this orbit is $|B : N_B(M_2(L))|$. Therefore we compute the normaliser of $M_2(L)$ in B.

Again, to simplify workings we use the fact that a conjugate of an element of $M_2(L)$ is in B if and only if the element of $M_2(L)$ is in B. We need to find elements $(g_1, g_2, \ldots, g_m) \in B$ where for all $(l_1, l_2, \ldots, l_m) \in L^{(m)}$ we have

$$(l_1, l_2, \ldots, l_m)^{(g_1, g_2, \ldots, g_m)} = (l_1^{g_1}, l_2^{g_2}, \ldots, l_m^{g_m}) \in L^{(m)}.$$

That is $l_i^{g_i} \in L$, for all $l_i \in L$ and for all $i \in \Omega^{*[1]}$. So $g_i \in N_{A_m}(L)$, for all $i \in \Omega^{*[1]}$.

From result (8.1), we know that $N_{A_m}(L) = L$. Then $(g_1, g_2, \dots, g_m) \in L^{(m)}$ gives $T^{(g_1, g_2, \dots, g_m)} \subseteq M_2(L)$. Therefore $N_B(M_2(L)) \subseteq (N_{A_m}(L))^{(m)} = L^{(m)}$.

Now $M_2(L) \subseteq N_{W_1}(M_2(L))$ implies

$$L^{(m)} = M_2(L) \cap B \subseteq N_{W_1}(M_2(L)) \cap B = N_B(M_2(L)).$$

Thus $N_B(M_2(L)) = L^{(m)}$. The number of distinct conjugates $M_2(L)^g$, where $g \in B$, is $|A_m : L|^m$.

The conjugacy class of $M_2(\widetilde{L})$ in B, for another maximal subgroup $\widetilde{L}$ of A_m, may be the same as the conjugacy class of $M_2(L)$ in B. This will occur when $\widetilde{L}$ is a conjugate of L in A_m. The number of conjugates of L in A_m is $|A_m : L|$, by result (8.1). Hence the total number of distinct maximal subgroups of W_1 of the type given in Case 2b is $\sum_{L \leq_{\max} A_m} |A_m : L|^{m-1}$.

□

Remark. As was seen in Case 2a, the maximal subgroups M_1^g are parametrised by the cosets $\{(x, x, \dots, x) : x \in A_m\}g$, where $g \in B$. Therefore we can describe them using the coset representatives $g_i \in B$, for $1 \leq i \leq |A_m|^{m-1}$.

Similarly, as was seen in Case 2b, the maximal subgroups $M_2(L)^g$ can be described using the coset representatives $g_i \in B$, for $1 \leq i \leq |A_m : L|^m$.

Remark. It would be interesting to know which of the three types of maximal subgroups of W_1 is the largest class.

The number

$$\sum_{L \leq_{\max} A_m} |A_m : L|^{m-1}$$

of maximal subgroups of type Case 2b is calculated by summing numbers that are at least 1 as we run through all the maximal subgroups of A_m. Therefore the number of maximal subgroups of type Case 2b is larger than the number of maximal subgroups of type Case 1.

It is left open as to whether the number $|A_m|^{m-1}$ of maximal subgroups of type Case 2a is larger than the number of maximal subgroups of type Case 2b.

Remark 8.5. We analyse Bhattacharjee's paper [3] with respect to counting the number of maximal subgroups M of W_1 with the property that $M \cap B$ is a proper subdirect product in B (Case 2a).

Since the only non-trivial T-congruence[1] on $\Omega^{*[1]}$ is $\Omega^{*[1]}$, Bhattacharjee describes these maximal subgroups as $N_{W_1}(D_1)$, where

$$D_1 = \{(x, \varphi_2(x), \varphi_3(x), \ldots, \varphi_m(x)) : x \in A_m\},$$

for some $\varphi_j \in \mathrm{Aut}(A_m)$, for $2 \leq j \leq m$. She estimates the number of conjugacy classes of these maximal subgroups by calculating the number of conjugacy classes of the groups D_1. Instead, we use Bhattacharjee's best description of $N_{W_1}(D_1)$ to count the number of possible maximal subgroups of this type.

The groups D_1 are uniquely determined by the maps $\varphi_2, \varphi_3, \ldots$ and φ_m. However, not all choices of $\varphi_j \in \mathrm{Aut}(A_m)$ may lead to $N_{W_1}(D_1)$ being maximal. Therefore Bhattacharjee's work only goes so far as to produce the overestimate of $|S_m|^{m-1}$, for $m \neq 6$, maximal subgroups of this type. Corollary 8.4 counts the exact number of these maximal subgroups as $|A_m|^{m-1}$. The difference of values occurs because Theorem 8.3 checks that the maximal subgroups are actually maximal and Bhattacharjee's work does not require such checking.

We comment further that subgroups of Bhattacharjee's description $N_{W_1}(D_1)$ which are not maximal must therefore be contained in maximal subgroups of the form $M_0(L) = B \rtimes L$. So $N_{W_1}(D_1)$ is maximal if $BN_{W_1}(D_1) = W_1$. The subgroups $N_{W_1}(D_1)$ that are not maximal are those which $D_1 = \{(x, \varphi_2(x), \varphi_3(x), \ldots, \varphi_m(x)) : x \in A_m\}$ for some $1 \neq \varphi_j \in \mathrm{Aut}(A_m)/\,\mathrm{Inn}(A_m) = \mathrm{Out}(A_m)$.

Remark 8.6. We analyse Bhattacharjee's paper [3] with respect to counting the number of maximal subgroups M of W_1 with the property that $M \cap B$ projects onto a maximal subgroup of A_m in each coordinate (Case 2b).

Bhattacharjee's method and therefore best description of these types of subgroups is the same as that of Theorem 8.3. She then estimates the number of conjugacy classes of these maximal subgroups. Bhattacharjee's usage does not necessitate her to conclude that she has enough information to proceed in the counting of these types of groups.

Since Theorem 8.3 has shown that any maximal subgroup L of A_m leads to these groups being maximal, Corollary 8.4 has counted the exact number of these types of maximal subgroups as $\sum_{L \leq_{\max} A_m} |A_m : L|^{m-1}$.

Remark 8.7. We use Theorem 8.3 to count the exact number of conjugacy classes of maximal subgroups M of W_1 with the property that $M \cap B$ is a proper subdirect product in B (Case 2a).

Since Bhattacharjee conjugates maximal subgroups by elements of the whole group

[1]A *T-congruence* on Ω is a T-invariant equivalence relation. That is, for $t \in T$ and Ω_i, there exists Ω_j such that $t\Omega_i = \Omega_j$; where Ω_i are the equivalence classes.

and not just the base group, in order to compare with Bhattacharjee we conjugate by elements of the whole group W_1. We show that the maximal subgroups M_1^g, where $g \in B$, as described in Theorem 8.3, form exactly one conjugacy class in W_1. For $bt \in W_1$, where $b \in B$ and $t \in T$, we have $(M_1^g)^{bt} = M_1^{t(t^{-1}gbt)} = M_1^{(gb)^t}$ and $(gb)^t \in B$.

For W_1, Bhattacharjee's work leads to the maximal subgroups of the type in Case 2a being $N_{W_1}(D_1)$, where

$$D_1 = \{(x, \varphi_2(x), \varphi_3(x), \ldots, \varphi_m(x)) : x \in A_m\},$$

for some $\varphi_j \in \mathrm{Aut}(A_m)$, for $2 \leq j \leq m$. The inner automorphisms of A_m give rise to a single conjugacy class of groups D_1 in B. Any $1 \neq \varphi_j \in \mathrm{Out}(A_m)$ leads to a single distinct conjugacy class. Therefore the number of conjugacy classses of subgroups of the form D_1 in B is $|\mathrm{Out}(A_m)|^{m-1}$. Since $|\mathrm{Out}(A_m)| = 2$, for $m \neq 6$, an upper bound for the number of distinct conjugacy classes of maximal subgroups of type Case 2a is 2^{m-1}. Therefore Bhattacharjee's work only goes so far as to produce this overestimate, whereas, our work calculates precisely one conjugacy class.

Remark 8.8. We use Theorem 8.3 to count the exact number of conjugacy classes of maximal subgroups M of W_1 with the property that $M \cap B$ projects onto a maximal subgroup of A_m in each coordinate (Case 2b).

Since Bhattacharjee conjugates maximal subgroups by elements of the whole group and not just the base group, in order to compare with Bhattacharjee we conjugate by elements of the whole group W_1. We claim that $M_2(L_1)$ is conjugate to $M_2(L_2)$ in W_1 if and only if the maximal subgroups L_1 and L_2 of A_m are conjugate in A_m.

Suppose $M_2(L_1)$ and $M_2(L_2)$ are conjugate in W_1. Then $M_2(L_1)^{bt} = M_2(L_2)$ for some $bt \in W_1$, where $b = (b_1, b_2, \ldots, b_m) \in B$ and $t \in T$. So

$$(L_1^{(m)} \rtimes T)^{bt} = (L_1^{(m)})^{bt} \rtimes T^{bt} = L_2^{(m)} \rtimes T.$$

Intersecting with B gives $(L_1^{(m)})^{bt} = L_2^{(m)}$. Therefore there exists some $b_i \in A_m$ such that $L_1^{b_i} = L_2$.

Suppose L_1 and L_2 are conjugate in A_m. Then $L_1^g = L_2$ for some $g \in A_m$. Therefore

$$\begin{aligned} M_2(L_1)^{(g,g,\ldots,g)} &= (L_1^{(m)})^{(g,g,\ldots,g)} \rtimes T^{(g,g,\ldots,g)} \\ &= (L_1^g)^{(m)} \rtimes T = (L_2)^{(m)} \rtimes T = M_2(L_2), \end{aligned}$$

where $(g, g, \ldots, g) \in B$.

Thus the number of conjugacy classes in W_1 of maximal subgroups of the form $M_2(L)^g$, where $g \in B$, is the same as the number of conjugacy classes in A_m of maximal subgroups L of A_m. The number of conjugacy classes of maximal subgroups of A_m can be worked out from the classification of maximal subgroups of A_m as set out in Section 2.3.

Bhattacharjee states that finding an upper bound for the number of distinct conjugacy classes in W_1 of these types of maximal subgroups reduces to finding an upper bound for the number of distinct conjugacy classes in A_m of maximal subgroups of A_m. She overestimates the number of conjugacy classes because she is not required to prove that her statement is necessary and sufficient, which we have done above.

8.3 Finite wreath products $A_m \wr A_m \wr A_m$, where $m \geq 5$

We continue the work of determining the maximal subgroups of the finite groups W_n, with a view to applying these techniques to the groups G_n of Wilson's construction. The next natural step is to look at the second wreath product

$$W_2 = A_m \wr_{\Omega^{*[2]}} (A_m \wr_{\Omega^{*[1]}} A_m),$$

where

$$\Omega^{*[1]} = \{1, 2, \dots, m\} \text{ and } \Omega^{*[2]} = \{i_1 i_2 : i_1, i_2 \in \{1, 2, \dots, m\}\},$$

and $m \geq 5$. The top group $A_m \wr_{\Omega^{*[1]}} A_m$ of this iterated wreath product is the group W_1. Therefore we can write

$$W_2 = A_m \wr_{\Omega^{*[2]}} W_1.$$

Theorem 8.10 describes the maximal subgroups of W_2. They are described by using the work of Bhattacharjee [3], and Parker and Quick [23], and our analysis for proving Theorem 8.3. Similarly, the proof of Theorem 8.10 separates the possibilities for the maximal subgroups of W_2 into types, referred to as Case 1, Case 2a, Case 2b and Case 2c. The proof concerning the maximal subgroups of type Case 2a is taken from Bhattacharjee's work in [3]. To obtain a self-contained analogue for Case 2a, as in W_1 of the previous section, was found to be too complicated and seemed unnecessary considering we have the work of Bhattacharjee. Therefore since we do not use the fact that $\mathrm{Aut}(A_m) \cong S_m$, for $m \geq 4$ and $m \neq 6$, Theorem 8.10 holds for $m \geq 5$.

In the previous section, for W_1 the maximal subgroups M of type Case 2a had $M \cap B$ equal to a single diagonal subgroup[2]. From paper [3, pg. 316], we see that this is because A_m acts primitively on $\Omega^{*[1]} = \{1, 2, \dots, m\}$ and so $\Omega^{*[1]}$ is the only

[2] The group $\{(x, \varphi_2(x), \varphi_3(x), \dots, \varphi_m(x)) : x \in A_m\}$, where $\varphi_j \in \mathrm{Aut}(A_m)$, for $2 \leq j \leq m$, is referred to as a *diagonal subgroup* of the direct product $\prod_{i=1}^{m} A_m^{(i)}$ of alternating groups.

non-trivial A_m-congruence[3] on $\Omega^{*[1]}$. However, for W_2 the subgroup $A_m^{(m)} \rtimes A_m$ acts imprimitively on the set $\Omega^{*[2]} = \{i_1 i_2 : i_1, i_2 \in \{1, 2, \ldots, m\}\}$; see Lemma 8.9 below. Therefore the maximal subgroups M, of W_2, of type Case 2a can have $M \cap B$ equal to a direct product of more than one diagonal subgroup.

Lemma 8.9. *Let $A_m^{(m)} \rtimes A_m$ act naturally on $\Omega^{*[2]} = \{i_1 i_2 : i_1, i_2 \in \{1, 2, \ldots, m\}\}$, for $m \geq 3$. Then*

$$\{1i_2 : i_2 \in \{1, 2, \ldots, m\}\}\ ,\ \{2i_2 : i_2 \in \{1, 2, \ldots, m\}\}\ ,\ \ldots\ ,\ \{mi_2 : i_2 \in \{1, 2, \ldots, m\}\}$$

is the only non-trivial system of blocks.

Proof. Fix $11 \in \Omega^{*[2]}$. Recall $W_1 = A_m^{(m)} \rtimes A_m$. Since the group W_1 acts transitively on $\Omega^{*[2]}$, there is a one-to-one correspondence between the non-trivial systems of blocks and the subgroups H such that $\mathrm{St}_{W_1}(11) \subsetneq H \subsetneq W_1$, where $\mathrm{St}_{W_1}(11)$ is the stabiliser of 11 in W_1. Now

$$\mathrm{St}_{W_1}(11) = A_{m-1} \times (A_m^{(m-1)} \rtimes A_{m-1}). \tag{8.2}$$

We claim $H = A_m^{(m)} \rtimes A_{m-1}$ is the only subgroup such that $\mathrm{St}_{W_1}(11) \subsetneq H \subsetneq W_1$. We write $B = A_m^{(m)}$ for the base group of W_1.

If $A_{m-1} \cong B\,\mathrm{St}_{W_1}(11)/B \subsetneq BH/B$ then $BH/B \cong A_m$, as A_{m-1} is a maximal subgroup of A_m, for $m \geq 3$. A short calculation, similar to that used in the proof of Lemma 8.2, shows that $B \subseteq H$. So we have the contradiction $H = W_1$.

Therefore $B\,\mathrm{St}_{W_1}(11)/B = BH/B$. Then $\mathrm{St}_{W_1}(11) \subsetneq H$ implies $A_{m-1} \times A_m^{(m-1)} = \mathrm{St}_{W_1}(11) \cap B \subsetneq H \cap B$. We have $B \subseteq H$, since $A_{m-1} \times A_m^{(m-1)}$ is a maximal subgroup of B, for $m \geq 3$. Hence the claim is proved. □

The proof of the maximal subgroups of type Case 2b in Theorem 8.10 is contained in Bhattacharjee's work. Our work on the maximal subgroups of type Case 2c in the proof of Theorem 8.10 is new and makes use of the Theorem 8.1 of Parker and Quick.

Theorem 8.10. *Let $W_2 = A_m \wr_{\Omega^{*[2]}} (A_m \wr_{\Omega^{*[1]}} A_m)$, where $\Omega^{*[1]} = \{1, 2, \ldots, m\}$ and $\Omega^{*[2]} = \{i_1 i_2 : i_1, i_2 \in \{1, 2, \ldots, m\}\}$, for some $m \geq 5$. Denote the base group $A_m^{(m^2)} =: B$ and the permuting top group $W_1 =: T$. Therefore $W_2 = B \rtimes T$.*

Define

$$M_0(K) = B \rtimes K, \text{ where } K \text{ is a maximal subgroup of } W_1.$$

[3] A *T-congruence* on Ω is a T-invariant equivalence relation. That is, for $t \in T$ and Ω_i, there exists Ω_j such that $t\Omega_i = \Omega_j$; where Ω_i are the equivalence classes.

Consider the normaliser

$$N_{W_2}(D_1),$$

where

$$D_1 = \{(x, \varphi_2(x), \varphi_3(x), \ldots, \varphi_{m^2}(x)) : x \in A_m\}$$

and

$$\varphi_j \in \mathrm{Aut}(A_m), \text{ for } 2 \leq j \leq m^2.$$

Consider the normaliser

$$N_{W_2}(D_1 \times D_2 \times \ldots \times D_m),$$

where

$$D_i = \{(x_i, \varphi_{(i-1)m+2}(x_i), \varphi_{(i-1)m+3}(x_i), \ldots, \varphi_{im}(x_i)) : x_i \in A_m\}, \text{ for } 1 \leq i \leq m,$$

and

$$\varphi_j \in \mathrm{Aut}(A_m), \text{ for } (i-1)m + 2 \leq j \leq im.$$

Define

$$M_2(L) = L^{(m)} \rtimes T, \text{ where } L \text{ is a maximal subgroup of } A_m.$$

Then the groups $M_0(K)$ and $M_2(L)^g$, where $g \in B$, are maximal subgroups of W_2 and every maximal subgroup of W_2 is one of the groups $M_0(K)$, $N_{W_2}(D_1)$, $N_{W_2}(D_1 \times D_2 \times \ldots \times D_m)$ or $M_2(L)^g$, where $g \in B$.

Proof. Let M be a maximal subgroup of W_2. Then there are two possibilities:

$$B \subseteq M \text{ (case 1), and } B \not\subseteq M \text{ (case 2).}$$

Case 1.

Suppose $B \subseteq M$. Using the same reasoning as Case 1 of the proof for Theorem 8.10 gives $M = B \rtimes K$, where K is a maximal subgroup of W_1. The maximal subgroups of W_1 have been classified in Theorem 8.3.

Case 2.

Suppose $B \not\subseteq M$. Since M is maximal, we have

$$BM = W_2.$$

Again using the facts $M/(M \cap B) \cong T$ and T acts transitively on the set $\Omega^{*[2]}$, we see that the projections of $M \cap B$ into the m^2 factors of B must be conjugate in A_m.

Denote K_i as the projection of $M \cap B$ into the ith factor of B, for $1 \leq i \leq m^2$. Case 2 can be separated into three possibilities because the groups K_i are all conjugate subgroups of A_m.

(case 2a) The groups $K_i = A_m$, for all $i \in \Omega^{*[2]}$.

(case 2b) The groups $K_i \neq \{1\}$ and $K_i \neq A_m$, for all $i \in \Omega^{*[2]}$.

(case 2c) The groups $K_i = \{1\}$, for all $i \in \Omega^{*[2]}$.

Case 2a.

We follow Bhattacharjee's work [3, pg. 316 - 317] to characterise the maximal subgroups M such that $M \cap B$ is a proper subdirect product in B.

Since $M \cap B$ is a subdirect product of a collection of non-abelian simple groups it can be written as

$$M \cap B = D_1 \times D_2 \times \ldots \times D_s,$$

where

$$\Omega^{*[2]} = \Omega_1 \cup \Omega_2 \cup \ldots \cup \Omega_s$$

is a partition of $\Omega^{*[2]}$ and each $D_i (\cong A_m)$ is a diagonal subgroup of the direct product $A_m^{\Omega_i}$ (see [2, Lem. 2.3] or the Appendix of our thesis). The partition of $\Omega^{*[2]}$ gives rise to a T-congruence on $\Omega^{*[2]}$.

Using Lemma 8.9, there are three possibilities for s: $s = 1$, $s = m$ or $s = m^2$. The possibility of $s = m^2$ is excluded because we would have the contradiction $M \cap B = B$. Therefore $s = 1$ or $s = m$.

If $s = 1$ then

$$M \cap B = D_1 = \{(x, \varphi_2(x), \varphi_3(x), \ldots, \varphi_{m^2}(x)) : x \in A_m\},$$

where

$$\varphi_j \in \mathrm{Aut}(A_m), \text{ for } 2 \leq j \leq m^2.$$

If $s = m$ then each diagonal subgroup D_i is of the form

$$D_i = \{(x_i, \varphi_{(i-1)m+2}(x_i), \varphi_{(i-1)m+3}(x_i), \ldots, \varphi_{im}(x_i)) : x_i \in A_m\},$$

where

$$\varphi_j \in \mathrm{Aut}(A_m), \text{ for } (i-1)m + 2 \leq j \leq im.$$

So

$$
\begin{aligned}
M \cap B = \{(x_1, \varphi_2(x_1), \ldots, \varphi_m(x_1), \\
x_2, \varphi_{m+2}(x_2), \ldots, \varphi_{2m}(x_2), \\
\ldots, \\
x_m, \varphi_{(m-1)m+2}(x_m), \ldots, \varphi_{m^2}(x_m)) : x_1, x_2, \ldots, x_m \in A_m\},
\end{aligned}
$$

where $\varphi_j \in \mathrm{Aut}(A_m)$.

Now $M \cap B \trianglelefteq M$ implies that M is contained in the normaliser of $M \cap B$ in W_2. As M is a maximal subgroup of W_2, we have $M = N_{W_2}(M \cap B)$ or $N_{W_2}(M \cap B) = W_2$. If the normaliser equals W_2 then

$$M \cap B \trianglelefteq N_{W_2}(M \cap B) = W_2.$$

Since T acts transitively on m^2 elements, there is only one T-orbit and Lemma 2.3 gives $M \cap B = B$. This contradicts $B \not\subseteq M$.

Thus if there is a maximal subgroup M such that $M \cap B = D_1$, we must have

$$M = N_{W_2}(D_1).$$

and if there is a maximal subgroup M such that $M \cap B = D_1 \times D_2 \times \ldots \times D_m$, we must have

$$M = N_{W_2}(D_1 \times D_2 \times \ldots \times D_m).$$

Case 2b.

Analogous methods of Bhattacharjee and of Case 2b of the proof for Theorem 8.3 can be used to describe the maximal subgroups M such that $K_i \neq \{1\}$ and $K_i \neq A_m$, for all $i \in \Omega^{*[2]}$.

For $B := A_m^{(m^2)}$ and $T := W_1$, the same methods of Theorem 8.3 give

$$M = N_{W_2}(L^{(m^2)})^{g^{-1}} = ((N_{A_m}(L))^{(m^2)} \rtimes T)^{g^{-1}} = (L^{(m^2)} \rtimes T)^{g^{-1}},$$

where L is a maximal subgroup of A_m and $g \in B$. Bhattacharjee's analysis [3, pg. 318] gives

$$M = N_{W_2}(K_1 \times K_2 \times \ldots \times K_{m^2}).$$

Choosing $g = (g_1, g_2, \ldots, g_{m^2}) \in B$ such that $K_1^{g_1} = K_2^{g_2} = \ldots = K_{m^2}^{g_{m^2}} = L$, we have

$$\begin{aligned} M &= N_{W_2}(K_1 \times K_2 \times \ldots \times K_{m^2}) \\ &= N_{W_2}(L^{g_1^{-1}} \times L^{g_2^{-1}} \times \ldots \times L^{g_{m^2}^{-1}}) \\ &= N_{W_2}(L^{(m^2)})^{g^{-1}} \\ &= (L^{(m^2)} \rtimes T)^{g^{-1}} \\ &= (L^{(m^2)})^{g^{-1}} \rtimes T^{g^{-1}} \\ &= (K_1 \times K_2 \times \ldots \times K_{m^2}) \rtimes T^{g^{-1}}. \end{aligned}$$

The same methods of Theorem 8.3 check that these groups are maximal.

Case 2c.

Assume the groups $K_i = \{1\}$, for all $i \in \Omega^{*[2]}$. Since $M \cap B = \{1\}$, the maximal subgroup M is a complement for the base group B in W_2. We show that condition (c) of Theorem 8.1 does not hold. In applying this theorem to our group W_2, we have that $X = A_m$ and $Y = W_1$.

The stabilisers of any two points in $\Omega^{*[2]}$ under the action of W_1 are conjugate, since the action is transitive. The stabiliser of any point $i \in \Omega^{*[2]}$ under the action of W_1 is conjugate to

$$A_{m-1} \times (A_m^{(m-1)} \rtimes A_{m-1}) = A_{m-1} \times (A_m \wr A_{m-1});$$

refer to (8.2).

We look at a potential surjective homomorphism ϕ from $A_{m-1} \times (A_m \wr A_{m-1})$ onto A_m. Now $A_{m-1} \times \{1\} \cong A_{m-1}$ is a normal subgroup of $A_{m-1} \times (A_m \wr A_{m-1})$. Since ϕ is surjective, the normal subgroup A_{m-1} must be mapped to a normal subgroup of A_m. The simple group A_m only has two normal subgroups and A_{m-1} cannot be mapped to A_m because it is too small. Therefore A_{m-1} maps to $\{1\}$ and it is in the kernel of ϕ.

It is now satisfactory to study the surjective homomorphism $A_m \wr A_{m-1} \longrightarrow A_m$. From the 1st isomorphism theorem, due to size, we see that this surjective homomorphism has to have a non-trivial kernel. From Lemma 2.3, since the natural action of A_{m-1} is transitive there is only one orbit, the unique minimal normal subgroup of $A_m \wr A_{m-1}$ is the direct product $A_m^{(m-1)}$. The kernel of this homomorphism, being a normal subgroup, must contain $A_m^{(m-1)}$. Therefore a subgroup of A_{m-1} would have to map onto A_m which is impossible.

Thus there can be no surjective homomorphism from the stabiliser of a point $i \in \Omega^{*[2]}$ under the action of W_1 to the group A_m. Hence W_2 has no maximal subgroups which complement the base group and Case 2c does not occur.

□

8.4 Particular first Wilson quotients G_1

Let G_n be the Wilson quotients as defined in Section 4.1. Recall that X_0 and X_1 are finite non-abelian simple groups. Also $G_0 = X_0$ has a faithful transitive action on the set $\Omega_{d_1} = \{1, 2, \ldots, d_1\}$ and $L_1 = X_1^{(d_1)}$. We would like to describe the maximal subgroups of the first Wilson quotients

$$G_1 = X_1 \wr_{L_1} (X_1 \wr_{\Omega_{d_1}} G_0),$$

where the top group $X_1 \wr_{\Omega_{d_1}} G_0 = L_1 G_0$ acts on the set L_1 according to the transitive action defined in (4.1), found in Section 4.1.

In order to apply the same techniques that are used to determine maximal subgroups of the groups W_1 and W_2, we take $X_0 = X_1 = A_m$, where $m \geq 5$. We also take the faithful transitive action of the group $G_0 = A_m$ to be the natural action. Therefore we now study the first Wilson quotients

$$G_1 = A_m \wr_{A_m^{(m)}} (A_m \wr_{\Omega^{*[1]}} A_m),$$

where the top group $A_m \wr_{\Omega^{*[1]}} A_m = A_m^{(m)} A_m$ acts on the set $A_m^{(m)}$ according to the transitive action (4.1). These groups are more specific than the groups of Section 6.2 because, in their construction, the groups X_0 and X_1 have been specified. Notice that the top groups of the Wilson quotients G_1 are the groups W_1 and therefore

$$G_1 = A_m \wr_{A_m^{(m)}} W_1.$$

Theorem 8.11 describes the maximal subgroups of these particular first Wilson quotients G_1. They are described by using the work of Bhattacharjee [3], and Parker and Quick [23], and our analysis for proving Theorem 8.3. Similarly, the proof of Theorem 8.11 separates the possibilities for the maximal subgroups of G_1 into types, referred to as Case 1, Case 2a, Case 2b and Case 2c. The proof concerning the maximal subgroups of type Case 2a is taken from Bhattacharjee's work in [3]. The proof of the maximal subgroups of type Case 2b in Theorem 8.11 is contained in Bhattacharjee's work. Our work on the maximal subgroups of type Case 2c in the proof of Theorem 8.11 is new and makes use of the Theorem 8.1 of Parker and Quick.

Theorem 8.11. *Let* $G_1 = A_m \wr_{A_m^{(m)}} (A_m \wr_{\Omega^{*[1]}} A_m)$*, where* $\Omega^{*[1]} = \{1, 2, \ldots, m\}$*, for some* $m \geq 5$*. Denote the base group* $A_m^{(|A_m|^m)} =: B$ *and the permuting top group* $W_1 =: T$*. The group* T *acts on the set* $A_m^{(m)}$ *according to the action defined in* (4.1)*. Therefore* $G_1 = B \rtimes T$*.*

Define

$$M_0(K) = B \rtimes K, \text{ where } K \text{ is a maximal subgroup of } W_1.$$

Consider the normaliser

$$N_{G_1}(D_1 \times D_2 \times \ldots \times D_s),$$

with the equivalence classes Ω_i*, for* $1 \leq i \leq s$ *and* $s \neq |A_m|^m$*, of a* T*-congruence on* $A_m^{(m)}$ *having* $|\Omega_i| = l$*, and where*

$$D_i = \{(x_i, \varphi_{(i-1)l+2}(x_i), \varphi_{(i-1)l+3}(x_i), \ldots, \varphi_{il}(x_i)) : x_i \in A_m\}, \text{ for } 1 \leq i \leq s,$$

and

$$\varphi_j \in \mathrm{Aut}(A_m), \text{ for } (i-1)l + 2 \leq j \leq il.$$

Define

$$M_2(L) = L^{(|A_m|^m)} \rtimes T, \text{ where } L \text{ is a maximal subgroup of } A_m.$$

Then the groups $M_0(K)$ *and* $M_2(L)^g$*, where* $g \in B$*, are maximal subgroups of* G_1 *and every maximal subgroup of* G_1 *is one of the groups* $M_0(K)$*,* $N_{G_1}(D_1 \times D_2 \times \ldots \times D_s)$ *or* $M_2(L)^g$*, where* $g \in B$*.*

Proof. Let M be a maximal subgroup of G_1. Then there are two possibilities:

$$B \subseteq M \text{ (case 1), and } B \not\subseteq M \text{ (case 2).}$$

Case 1.

Suppose $B \subseteq M$. Using the same reasoning as Case 1 of the proof for Theorem 8.3 gives $M = B \rtimes K$, where K is a maximal subgroup of W_1. The maximal subgroups of W_1 have been classified in Theorem 8.3.

Case 2.

Suppose $B \not\subseteq M$. Since M is maximal, we have

$$BM = G_1.$$

Again using the facts $M/(M \cap B) \cong T$ and T acts transitively on the set $A_m^{(m)}$, we see that the projections of $M \cap B$ into the $|A_m|^m$ factors of B must be conjugate in A_m.

Denote K_i as the projection of $M \cap B$ into the ith factor of B, for $1 \leq i \leq |A_m|^m$. Case 2 can be separated into three possibilities because the groups K_i are all conjugate subgroups of A_m.

(case 2a) The groups $K_i = A_m$, for all $i \in A_m^{(m)}$.

(case 2b) The groups $K_i \neq \{1\}$ and $K_i \neq A_m$, for all $i \in A_m^{(m)}$.

(case 2c) The groups $K_i = \{1\}$, for all $i \in A_m^{(m)}$.

Case 2a.

We follow Bhattacharjee's work [3, pg. 316 - 317] to characterise the maximal subgroups M such that $M \cap B$ is a proper subdirect product in B.

Since $M \cap B$ is a subdirect product of a collection of non-abelian simple groups it can be written as

$$M \cap B = D_1 \times D_2 \times \ldots \times D_s,$$

where the partition of

$$A_m^{(m)} = \Omega_1 \cup \Omega_2 \cup \ldots \cup \Omega_s$$

is a T-congruence on $A_m^{(m)}$ and each $D_i (\cong A_m)$ is a diagonal subgroup of the direct product $A_m^{\Omega_i}$. We have $s \neq |A_m|^m$ because $B \not\subseteq M$. Let $|\Omega_i| = l$, say, for all $1 \leq i \leq s$. Then

$$D_i = \{(x_i, \varphi_{(i-1)l+2}(x_i), \varphi_{(i-1)l+3}(x_i), \ldots, \varphi_{il}(x_i)) : x_i \in A_m\},$$

where

$$\varphi_j \in \mathrm{Aut}(A_m), \text{ for } (i-1)l + 2 \leq j \leq il.$$

Therefore

$$\begin{aligned} M \cap B = \{(&x_1, \varphi_2(x_1), \ldots, \varphi_l(x_1), \\ &x_2, \varphi_{l+2}(x_2), \ldots, \varphi_{2l}(x_2), \ldots, \\ &x_s, \varphi_{(s-1)l+2}(x_s), \ldots, \varphi_{sl}(x_s)) : x_1, x_2, \ldots, x_s \in A_m\}, \end{aligned}$$

where $\varphi_j \in \mathrm{Aut}(A_m)$.

Following the same argument in the proof for Case 2a of Theorem 8.10, if there is a maximal subgroup M such that $M \cap B = D_1 \times D_2 \times \ldots \times D_s$ then it is equal to the normaliser of $D_1 \times D_2 \times \ldots \times D_s$ in G_1. That is

$$M = N_{G_1}(D_1 \times D_2 \times \ldots \times D_s).$$

Case 2b.

Analogous methods of Bhattacharjee and of Case 2b of the proof for Theorem 8.3 can be used to describe the maximal subgroups M such that $K_i \neq \{1\}$ and $K_i \neq A_m$, for all $i \in A_m^{(m)}$.

For $B := A_m^{(|A_m|^m)}$ and $T := W_1$, the same methods of Theorem 8.3 give

$$M = N_{G_1}(L^{(|A_m|^m)})^{g^{-1}} = ((N_{A_m}(L))^{(|A_m|^m)} \rtimes T)^{g^{-1}} = (L^{(|A_m|^m)} \rtimes T)^{g^{-1}},$$

where L is a maximal subgroup of A_m and $g \in B$. Bhattacharjee's analysis [3, pg. 318] gives

$$M = N_{G_1}(K_1 \times K_2 \times \ldots \times K_{|A_m|^m}).$$

The same methods of Theorem 8.3 check that these groups are maximal.

Case 2c.

Assume the groups $K_i = \{1\}$, for all $i \in A_m^{(m)}$. Since $M \cap B = \{1\}$, the maximal subgroup M is a complement for the base group B in G_1. In this instance, we show that condition (c) of Theorem 8.1 does hold but condition (d) of Theorem 8.1 does not hold. Applying the theorem to the group G_1 gives $X = A_m$, $Y = W_1$ and $\Omega = A_m^{(m)}$.

The stabilisers of any two points in $A_m^{(m)}$ under the action of W_1 are conjugate, since the action is transitive. The stabiliser of the point $(1, \ldots, 1) \in A_m^{(m)}$ under the action of W_1 is the group of elements $(g_1, g_2, \ldots, g_m)t \in W_1$, where $(g_1, g_2, \ldots, g_m) \in A_m^{(m)}$ and $t \in A_m$, such that

$$(1, 1, \ldots, 1)(g_1, g_2, \ldots, g_m)t = (1, 1, \ldots, 1).$$

That is $(g_1, g_2, \ldots, g_m)^t = (1, 1, \ldots, 1)$ and so the stabiliser is the top group A_m of W_1. Therefore the stabiliser of any point of $A_m^{(m)}$ in W_1 is a conjugate of A_m. Thus there are surjective homomorphisms ϕ from these stabilisers to A_m and condition (c) holds.

However we now show that condition (d) of Theorem 8.1 does not hold. The stabiliser A_m^y, for some $y \in W_1$, satisfies $W_1 = A_m^{(m)} \rtimes (A_m^y)$. Any surjective homomorphism $\phi : A_m^y \longrightarrow A_m$ can be formed as the restriction of a homomorphism $A_m^{(m)} \rtimes A_m^y \longrightarrow A_m$, where the base group $A_m^{(m)}$ lies in the kernel. Hence condition d) is not satisfied and G_1 has no maximal subgroups which complement the base group. Therefore Case 2c does not occur.

$\square$

For further research concerning the maximal subgroups of these first Wilson quotients, refer to Chapter 10, Question 4.

Chapter 9

Finite generation and PMSG

In [3], M. Bhattacharjee has produced a result regarding finite generation of an inverse limit of iterated wreath products of finite alternating groups of degree at least 5 formed using the natural action. That is, the profinite groups $\varprojlim(A_{m_k} \wr \ldots \wr A_{m_2} \wr A_{m_1})$, where $m_i \geq 5$, are generated by two random elements with positive probability and the probability approaches 1 as the size of m_1 tends to infinity. Therefore the profinite groups W, constructed from iterated wreath products of the same alternating group, in Section 3.2 are positively finitely generated by two elements.

M. Quick [24] extends Bhattacharjee's work by first replacing the alternating groups in the wreath products with arbitrary finite non-abelian simple groups G_i, for $i \geq 0$. The standard action is used when forming each iterated wreath product, that is the top group of the wreath product acting on itself by right multiplication. Quick concludes that the profinite groups, which are the inverse limits $\varprojlim(G_k \wr \ldots \wr G_1 \wr G_0)$ of these iterated wreath products, are positively finitely generated. The probability of generating these profinite groups with two random elements is positive and approaches 1 as the order of G_0 tends to infinity.

In the paper [25], Quick generalises further to iterated wreath products of finite non-abelian simple groups G_i, for $i \geq 0$, each constructed from any faithful transitive actions. Similarly, the profinite groups $\varprojlim(G_k \wr \ldots \wr G_1 \wr G_0)$ constructed from these iterated wreath products are positively finitely generated by two random elements provided $|G_0| > 35!$. Again this probability approaches 1 as the order of G_0 tends to infinity.

Let G be a Wilson group arising as an inverse limit of finite groups G_n as defined in Section 4.1. The iterated wreath products $G_n = X_n \wr_{L_n} (L_n G_{n-1})$ are formed from the transitive actions (4.1), found in Section 4.1, of the groups $L_n G_{n-1}$ on L_n, for $n \geq 1$. Non-trivial elements of the group $L_n G_{n-1}$ acting by (4.1) on the set L_n can have fixed points however these elements do move at least one other point. Therefore the action (4.1) is faithful. Thus all the wreath products G_n are constructed with faithful transitive actions.

Hence Quick's result, in [25], can be applied to the Wilson groups. That is, the Wilson groups $\varprojlim(G_n)_{n\geq 0}$ such that $|G_0| > 35!$ are positively finitely generated by two elements.

Consequently, these particular Wilson groups are finitely generated because there must be at least one collection of two elements that generate them. For future research concerning finite generation of Wilson groups, refer to Question 1 and Question 5, Chapter 10.

Recall, from Section 2.6, that $m_n(G)$ denotes the number of closed maximal subgroups of a profinite group G with index n. A profinite group G has *polynomial maximal subgroup growth* (PMSG) if there exists a constant c such that

$$m_n(G) \leq n^c \text{ for all } n.$$

A result by A. Mann and A. Shalev [19] implies that the Wilson groups such that $|G_0| > 35!$, since they are positively finitely generated, have polynomial maximal subgroup growth. Question 6 of Chapter 10 gives an idea of further work on polynomial maximal subgroup growth of Wilson groups.

Chapter 10

Open problems

1) We know that the Wilson groups $\varprojlim(G_n)_{n\geq 0}$, as defined in Section 4.1, are finitely generated provided $|G_0| > 35!$; refer to Chapter 9. Is any arbitrary Wilson group finitely generated?

2) Remark 5.4, in Section 5.1, compares the Nottingham group to the Wilson groups with regard to chains of normal subgroups. There are many interesting questions that have been resolved for the Nottingham group and these could be investigated for the Wilson groups. We outline a few below.

Let G be a Wilson group arising as an inverse limit of finite groups G_n as defined in Section 4.1.

- The lower central series is an important filtration for the Nottingham group that gives a graded Lie ring, see [5]. Is there a similar chain of characteristic subgroups for G and a substitute for an associated Lie ring for G?
- The Nottingham group is finitely presented; refer to M. V. Ershov [7]. Is there a finite or countably recursive presentation for G?
- The automorphism group of the Nottingham group has been determined; refer to B. Klopsch [13]. What are the automorphisms of G?

3) Recall the just infinite profinite groups $W = \varprojlim(W_n)_{n\geq 0}$, where

$$W_n = A_m \wr_{\Omega^{*[n]}} \ldots \wr_{\Omega^{*[2]}} A_m \wr_{\Omega^{*[1]}} A_m,$$

for $n \geq 1$, and where

$$\Omega^{*[j]} = \{i_1 i_2 \ldots i_j : i_1, i_2, \ldots, i_j \in \{1, 2, \ldots, m\}\},$$

for each $j = 1, 2, \dots$, and $W_0 = A_m$, as defined in Section 3.2.

In Section 7.2, it was found that the number of non-trivial subnormal subgroups of W with index at most $|A_m|^n$, for some n, is equal to the sum

$$\sum_{k=1}^{n} \frac{1}{(m-1)k+1}\binom{mk}{k}.$$

What can be deduced about the subnormal subgroup growth of these groups?

4) In Section 8.4, we considered the first Wilson quotients

$$G_1 = A_m \wr_{A_m^{(m)}} (A_m \wr_{\Omega^*[1]} A_m) = A_m \wr_{A_m^{(m)}} W_1.$$

The top group $A_m \wr_{\Omega^*[1]} A_m = A_m^{(m)} A_m$ of G_1 acts on the set $A_m^{(m)}$ according to the transitive action (4.1).

The maximal subgroups of these Wilson quotients have been described in Theorem 8.11. There are maximal subgroups of the form

$$N_{G_1}(D_1 \times D_2 \times \ldots \times D_s),$$

with the equivalence classes Ω_i, for $1 \leq i \leq s$ and $s \neq |A_m|^m$, of a $(A_m^{(m)} A_m)$-congruence on $A_m^{(m)}$ having $|\Omega_i| = l$, and where

$$D_i = \{(x_i, \varphi_{(i-1)l+2}(x_i), \varphi_{(i-1)l+3}(x_i), \dots, \varphi_{il}(x_i)) : x_i \in A_m\},$$

for $1 \leq i \leq s$, and $\varphi_j \in \mathrm{Aut}(A_m)$, for $(i-1)l + 2 \leq j \leq il$.

Can we further describe these maximal subgroups by finding the $(A_m^{(m)} A_m)$-congruence on $A_m^{(m)}$?

5) The Wilson groups $\varprojlim (G_n)_{n \geq 0}$, as defined in Section 4.1, are positively finitely generated by two random elements provided $|G_0| > 35!$; refer to Chapter 9. Allowing for a larger number of generators, is a general Wilson group positively finitely generated?

6) Recall, from Section 2.6, that $m_n(G)$ denotes the number of closed maximal subgroups of a profinite group G with index n, and G has polynomial maximal subgroup growth if there exists a constant c such that

$$m_n(G) \leq n^c \text{ for all } n.$$

In Chapter 9, it was stated that the Wilson groups $\varprojlim(G_n)_{n\geq 0}$, as defined in Section 4.1, such that $|G_0| > 35!$, have polynomial maximal subgroup growth. What is the degree c of the polynomial maximal subgroup growth of these Wilson groups?

Bibliography

[1] J.-C. Aval. Multivariate Fuss-Catalan numbers. *Discrete Math.*, 308(20):4660–4669, 2008. 94

[2] M. Bhattacharjee. *Amalgamated free products and inverse limits of wreath products of groups*. D. Phil. Thesis, Oxford, 1992. 96, 114, 130

[3] M. Bhattacharjee. The probability of generating certain profinite groups by two elements. *Israel J. Math.*, 86(1 - 3):311–329, 1994. 14, 29, 95, 96, 106, 108, 109, 111, 114, 115, 117, 119, 120, 122

[4] R. Camina. Subgroups of the Nottingham group. *J. Algebra*, 196(1):101–113, 1997. 8, 48

[5] R. Camina. The Nottingham group. In *New Horizons in Pro-p Groups*, volume 184 of *Progr. Math.*, pages 205–221. Birkhäuser Boston, Boston, MA, 2000. 43, 124

[6] P. de la Harpe. *Topics in Geometric Group Theory*. Chicago Lectures in Mathematics. University of Chicago Press, 2000. 7, 65, 66, 92

[7] M. V. Ershov. The Nottingham group is finitely presented. *J. London Math. Soc. (2)*, 71(2):362–378, 2005. 124

[8] M. D. Fried and M. Jarden. *Field Arithmetic*, volume 11 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, third edition, 2008. 27

[9] R. I. Grigorchuk. On Burnside's problem on periodic groups. *Funktsional. Anal. i Prilozhen.*, 14(1):53–54, 1980. 7

[10] N. Gupta and S. Sidki. On the Burnside problem for periodic groups. *Math. Z.*, 182(3):385–388, 1983. 7

[11] S. A. Jennings. Substitution groups of formal power series. *Canadian J. Math.*, 6:325–340, 1954. 8

[12] D. L. Johnson. The group of formal power series under substitution. *J. Austral. Math. Soc. Ser. A*, 45(3):296–302, 1988. 8

[13] B. Klopsch. Automorphisms of the Nottingham group. *J. Algebra*, 223(1):37–56, 2000. 124

[14] B. Klopsch. Normal subgroups in substitution groups of formal power series. *J. Algebra*, 228(1):91–106, 2000. 43

[15] B. Klopsch, N. Nikolov, and C. Voll. *Lectures on Profinite Topics in Group Theory*, volume 77 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 2011. 24

[16] M. W. Liebeck, C. E. Praeger, and J. Saxl. A classification of the maximal subgroups of the finite alternating and symmetric groups. *J. Algebra*, 111(2):365–383, 1987. 23

[17] A. Lubotzky and D. Segal. *Subgroup Growth*, volume 212 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 2003. 28

[18] A. Mann. Positively finitely generated groups. *Forum Math.*, 8(4):429–459, 1996. 28

[19] A. Mann and A. Shalev. Simple groups, maximal subgroups, and probabilistic aspects of profinite groups. *Israel J. Math.*, 96(part B):449–468, 1996. 123

[20] J. L. Mennicke. Finite factor groups of the unimodular group. *Ann. of Math. (2)*, 81:31–37, 1965. 7

[21] N. Nikolov and D. Segal. On finitely generated profinite groups. I. Strong completeness and uniform bounds. *Ann. of Math. (2)*, 165(1):171–238, 2007. 66

[22] N. Nikolov and D. Segal. Generators and commutators in finite groups; abstract quotients of compact groups. *Invent. Math.*, 190(3):513–602, 2012. 32, 39, 47, 57, 66

[23] C. Parker and M. Quick. Maximal complements in wreath products. *J. Algebra*, 266(1):320–337, 2003. 15, 97, 111, 117

[24] M. Quick. Probabilistic generation of wreath products of non-abelian finite simple groups. *Comm. Algebra*, 32(12):4753–4768, 2004. 14, 122

[25] M. Quick. Probabilistic generation of wreath products of non-abelian finite simple groups. II. *Internat. J. Algebra Comput.*, 16(3):493–503, 2006. 14, 16, 122, 123

[26] L. Ribes and P. Zalesskii. *Profinite Groups*, volume 40 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics].* Springer-Verlag, Berlin, 2000. 7, 65

[27] D. J. S. Robinson. *A Course in the Theory of Groups*, volume 80 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 1982. 48

[28] L. L. Scott. Representations in characteristic p. In *The Santa Cruz Conference on Finite Groups (Univ. California, Santa Cruz, Calif., 1979)*, volume 37 of *Proc. Sympos. Pure Math.*, pages 319–331. Amer. Math. Soc., Providence, R.I., 1980. 23

[29] J. S. Wilson. Groups with every proper quotient finite. *Proc. Cambridge Philos. Soc.*, 69:373–391, 1971. 7

[30] J. S. Wilson. *Profinite Groups*, volume 19 of *London Mathematical Society Monographs. New Series.* The Clarendon Press Oxford University Press, New York, 1998. 7, 8, 26, 38, 48, 57, 65, 66

[31] J. S. Wilson. On just infinite abstract and profinite groups. In *New Horizons in Pro-p Groups*, volume 184 of *Progr. Math.*, pages 181–203. Birkhäuser Boston, Boston, MA, 2000. 7

[32] J. S. Wilson. Large hereditarily just infinite groups. *J. Algebra*, 324(2):248–255, 2010. 7, 8, 16, 20, 32, 34, 35, 37, 38, 39, 84

[33] R. A. Wilson. *The Finite Simple Groups*, volume 251 of *Graduate Texts in Mathematics.* Springer-Verlag London Ltd., London, 2009. 23

[34] I. O. York. The exponent of certain finite p-groups. *Proc. Edinburgh Math. Soc. (2)*, 33(3):483–490, 1990. 8

Appendix A

Bhattacharjee's Lemma

For the reader's understanding we include the Lemma 2.3 from Bhattacharjee's D. Phil. Thesis [2].

Lemma A.1 (Bhattacharjee [2])**.** *Let* $I := \{1, 2, \ldots, m\}$ *and for every* $i \in I$ *let* G_i *be a simple group. If* $H \leq G_1 \times G_2 \times \ldots \times G_m$ *is a subdirect product then*

$$H \cong D_1 \times D_2 \times \ldots \times D_k,$$

with $k \leq m$ *and where there exist distinct* $i_1, i_2, \ldots, i_k \in I$ *such that* $D_i \cong G_{i_j}$ *for each* $i = 1, 2, \ldots, k$*.*

Furthermore, if the groups G_i *are all non-abelian simple then there is a partition*

$$I = \bigcup_{j=1}^{k} I_j$$

of I *such that all* G_i *for* $i \in I_j$ *are isomorphic and such that* D_j *is the diagonal subgroup of* $\prod_{i \in I_j} G_i$*.*

Proof. Let us proceed by induction on m. It is trivially true for $m = 1$. Let us assume that the lemma is true for a family of less than m simple groups.

If $G_m \leq H$ then

$$H = (H \cap (G_1 \times G_2 \times \ldots \times G_{m-1})) \times G_m.$$

But the first term in this expression is itself a subdirect product involving $m-1$ simple groups and hence is a direct product by induction. (This is because $\prod_i (H \cap (G_1 \times G_2 \times \ldots \times G_{m-1})) = \prod_i H \cap G_i = G_i$, for $1 \leq i \leq m-1$.) So in this case the lemma holds.

Otherwise, $G_m \not\leq H$ so that $G_m \cap H = \{1\}$ as G_m is simple and $G_m \cap H \trianglelefteq G_m$.

Therefore, the projection

$$G_1 \times G_2 \times \ldots \times G_m \longrightarrow G_1 \times G_2 \times \ldots \times G_{m-1}$$

maps H injectively into a subdirect product with fewer factors, which, by inductive hypothesis, is a direct product. Hence the first part of the lemma is proved.

To prove the rest of the lemma, define

$$I_j = \{i \in I : D_j \text{ projects non-trivially onto } G_i\}.$$

We need to show that this defines a partition on I. Clearly, $I = \bigcup_{j=1}^{k} I_j$ as H is a subdirect product. If possible, let $i \in I_{j_1} \cap I_{j_2}$ for distinct $j_1, j_2 \in \{1, 2, \ldots, k\}$. Then the groups D_{j_1} and D_{j_2} both project non-trivially onto G_i. Let $y_1 \in D_{j_1}$ and $y_2 \in D_{j_2}$ be such that their projection x_1 and x_2 respectively in G_i do not commute. Such elements exist since G_i is non-abelian and simple. But y_1 and y_2 commute as they belong to distinct factors in a direct product. This contradiction proves the lemma. $\square$