

INFORMATION PROPAGATION THROUGH DISORDERED QUANTUM SPIN CHAINS

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Abstract

One of the central aims of quantum information theory is to exploit quantum mechanical phenomena such as superposition and entanglement to build a quantum computer — a device capable of efficiently performing computational tasks which are not feasible on a classical computer. In order to achieve this goal it will be necessary to connect together the different components of a quantum computer with channels which are capable of quickly and faithfully transmitting quantum states. It has been suggested that a good channel for this purpose is a chain of permanently coupled quantum spins and we begin by reviewing some of the protocols which have been previously developed to transfer quantum information along such a spin chain. We then examine the effects of noise on the propagation of quantum information along spin chains, using a toolkit of methods which include Lieb-Robinson bounds, the Jordan-Wigner transform and correlation functions. Several fundamentally different noise models are considered including a static (time-independent) on-site disorder model and two different models of fluctuating (time-dependent) on-site disorder. Each noise model has a different and distinctive effect on the propagation of information — static disorder leads to exponential localisation whilst dynamic disorder can lead to diffusive or even ballistic propagation of information. We finish by reviewing how a spin chain (whether noisy or noise-free) can be viewed as a depolarising channel and we make a detailed study of the geometrical structure of all possible depolarising channels with respect to various bases, concentrating on the Pauli, Gell-Mann and Heisenberg-Weyl bases. In particular, we show precisely when the set of all possible depolarising channels forms a simplex in compression space.

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Chapter 1

Introduction

The fields of quantum information theory and quantum computing were born in 1982 when Richard Feynman proposed that the vast Hilbert space of a large quantum mechanical system could be utilised to create a *quantum computer* capable of simulating quantum systems which cannot be simulated efficiently by classical methods [1]. Since then it has been one of the central aims of quantum information theory to actually build such a device by exploiting quantum mechanical phenomena such as superposition and entanglement. It is worth noting that whilst entanglement is often necessary, it can be argued that “it is nevertheless misleading to view entanglement as a key resource for quantum computational power” [2].

In order to achieve the goal of building a quantum computer it will be necessary to connect together the different components (such as quantum memories and quantum information processors) with channels which are capable of quickly and faithfully transmitting arbitrary quantum states. The ideal channel to use over long distances is almost undoubtedly an optical fibre due to its low decoherence rate; in this case the quantum information is stored and transmitted as the quantum states of photonic qubits. Over shorter distances one might dispense with the optical fibre and opt for free-space propagation of the photonic qubits. However, unless the entire quantum computer is comprised of optical devices, this is not such a practical solution for communicating over

shorter distances: at each end of every optical channel there must be some kind of interfacing device which transfers the quantum states of the photonic qubits onto the non-optical qubits and vice-versa. Such interfaces are tricky to engineer and inevitably introduce a source of noise or even information loss. It has therefore been suggested that when the components of the quantum computer are non-optical devices, a good channel for communicating over shorter distances is a chain of permanently coupled quantum spins [3].

The purpose of this thesis is to examine the effects of noise (or disorder) on the propagation of quantum information along such spin chains. Whilst the effects of noise have been examined before, we consider fundamentally different noise models to those previously studied and observe dramatically different effects. Previous studies have considered noise on the interaction strengths between the particles in the spin chain, whereas we study the effects of on-site disorder (caused, perhaps, by a noisy transverse electromagnetic field). In the first instance this on-site disorder is fixed in time (we call this *static disorder*) and later on we study models where the on-site disorder fluctuates with time in either strength or direction (we refer to these as *dynamic disorder* models).

We finish by reviewing how a spin chain (whether noisy or noise-free) can be viewed as a generalised depolarising channel and we make a detailed study of the geometrical structure of all possible generalised depolarising channels with respect to various bases. We focus our attention on the Pauli, Gell-Mann and Heisenberg-Weyl bases, but also show the effects of changing to other bases. A key result is the proof and generalisation of a conjecture by Dixit and Sudarshan [4], which allows us to determine precisely when the set of all possible depolarising channels forms a simplex in compression space.

1.1 Quantum information theory

This section reviews some of the basics of quantum information theory, concentrating on only those aspects which are used regularly throughout this thesis. More detail and further information can be found in [5].

A quantum mechanical system of dimension D (a *qudit*) has a state $|\psi\rangle$ which can be thought of as a complex vector in a Hilbert space of dimension D . (Recall that a Hilbert space is a complex inner product space.) The state $|\psi\rangle$ can be written as a complex linear combination of D basis states; perhaps the most common basis is the *computational basis* $\{|0\rangle, |1\rangle, \dots, |D-1\rangle\}$, and if we work in this basis then we can write

$$|\psi\rangle = \sum_{j=0}^{D-1} \alpha_j |j\rangle \quad (1.1)$$

where α_j are complex coefficients satisfying $\sum_j |\alpha_j|^2 = 1$, which ensures *normalisation* of the state: $\langle\psi|\psi\rangle = 1$ where $\langle\psi|$ is the dual to $|\psi\rangle$. It is worth mentioning that whilst a qudit can be in a superposition of several basis states, such as the state $|\psi\rangle$ above, when we measure it in the computational basis we obtain measurement outcome “ j ” with probability $|\alpha_j|^2$ in which case the state of the qudit *collapses* to $|j\rangle$.

So far all the states we have considered are *pure states*, but a more general class of states exists called *mixed states*. These are statistical ensembles of pure states and can be written as a complex matrix which corresponds to the pure states of the ensemble. A rank k mixed state can be written as a density matrix

$$\rho = \sum_{j=1}^k p_j |\psi_j\rangle \langle\psi_j| \quad (1.2)$$

where $|\psi_j\rangle$ are pure states and p_j are probabilities: $0 \leq p_j \leq 1$ and $\sum_j p_j = 1$. Like pure states, mixed states are normalised in the sense that the density matrix has unit trace: $\text{tr } \rho = 1$.

We have seen how to describe the quantum states of simple quantum systems, but how do we describe the states of larger systems comprising several smaller systems? The answer lies in *tensor products*: if system \mathcal{A} is in state $|\psi_{\mathcal{A}}\rangle$ and system \mathcal{B} is in state $|\psi_{\mathcal{B}}\rangle$ then the state of the joint system is $|\psi_{\mathcal{A}}\rangle \otimes |\psi_{\mathcal{B}}\rangle$ which is often abbreviated to $|\psi_{\mathcal{A}}, \psi_{\mathcal{B}}\rangle$; similarly for mixed states, if system \mathcal{A} is in state $\rho_{\mathcal{A}}$ and system \mathcal{B} is in state $\rho_{\mathcal{B}}$ then the state of the joint system is $\rho_{\mathcal{A}} \otimes \rho_{\mathcal{B}}$. We can write this because the Hilbert space of the joint system is the tensor product of the Hilbert spaces of the individual systems $\mathcal{H}_{\mathcal{AB}} = \mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\mathcal{B}}$.

Physical processes acting on a quantum system cause the state of that system to change. We describe these processes using *operators* which are essentially linear maps. Let us suppose that a certain physical process corresponds to the operator A , then a system whose initial state is $|\psi\rangle$ before this process occurs is in the (normalised) state $\frac{A|\psi\rangle}{\|A|\psi\rangle\|}$ after the process has occurred. Similarly, an initial mixed state ρ becomes $\frac{A\rho A^\dagger}{\text{tr}(A\rho A^\dagger)}$ where A^\dagger is the hermitian conjugate (the complex conjugate of the transpose) of A . Just as pure states can be thought of as vectors and mixed states as matrices, operators can also be thought of as matrices. We can calculate the (j, k) th entry of the matrix A by finding the *matrix element* $a_{jk} = \langle j| A |k\rangle$. This allows us to write

$$A = \sum_{j,k=0}^{D-1} a_{jk} |j\rangle \langle k| \quad (1.3)$$

It should be noted that we can find the matrix elements of density matrices in precisely the same way.

The formalism introduced above is sufficient to fully describe quantum spin chains and the way in which they can be used to transfer quantum states and hence transmit quantum information. Indeed, it is sufficient to describe quantum computing in general.

If a quantum system has a state $|\psi(t)\rangle$ which evolves in time (that is, there is some physical process which is continuously affecting the system) then the

system evolves in accordance with the *Schrödinger equation* [6]

$$\frac{\partial}{\partial t} |\psi(t)\rangle = -iH(t) |\psi(t)\rangle \quad (1.4)$$

where $H(t)$ is the *Hamiltonian* of the system and is, generally speaking, time dependent. When the Hamiltonian is independent of time (we simply write H in this case) then the Schrödinger equation can be solved exactly to reveal the dynamics of the system

$$|\psi(t)\rangle = e^{-iHt} |\psi(0)\rangle \quad (1.5)$$

If the Hamiltonian is time-dependent then in principle one can still solve the Schrödinger equation exactly, but in practice the solution is not so simple to calculate. In this case, the solution is

$$|\psi(t)\rangle = \mathcal{T} e^{-i \int_0^t H(s) ds} |\psi(0)\rangle \quad (1.6)$$

where \mathcal{T} denotes a time-ordered exponential.

Many Hamiltonians which describe systems consisting only of *qubits* (two-dimensional qudits) are defined in terms of the *Pauli matrices* which are defined as

$$\sigma^0 = \mathbb{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (1.7)$$

It is sometimes convenient to use the vector of Pauli matrices $\boldsymbol{\sigma} = (\sigma^x, \sigma^y, \sigma^z)$.

1.2 Quantum spin chains as quantum information channels

Quantum spin chains consist of numerous quantum spin particles arranged in a one-dimensional array where the particles interact with close neighbours via local interactions. Figure 1.1 shows a typical example of a spin chain consisting of N qubits with nearest-neighbour interactions h_j acting between qubits j and $j + 1$.

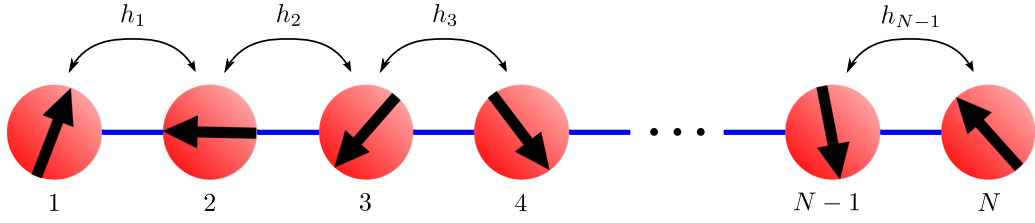


Figure 1.1: A typical quantum spin chain of length N : the red spheres represent qubits, the bold arrows their spin (or quantum state) and the blue lines between the qubits represent the local interactions h_j .

Spin chains are not just of theoretical interest: there are many physical manifestations which include arrays of Josephson junction qubits [7, 8, 9], quantum dots [10, 11, 12, 13], nuclear magnetic resonance [14, 15, 16, 17], atoms in optical lattices [18], coupled cavity arrays [19, 20, 21] and natural spin chains in compounds such as Sr_2CuO_3 and SrCuO_2 [22].

There are many ways in which one can use a quantum spin chain as a quantum channel and Bose [23] provides an excellent overview of these. Most protocols consider spin chains which exhibit nearest-neighbour interactions only, with no longer range interactions present between the spins. However, a few authors consider more general scenarios in which there might be next-nearest-neighbour interactions [24, 25, 26] or perhaps interactions whose effect decays algebraically with the distance between spins [27]. In this thesis we focus exclusively on nearest-neighbour interactions as they usually dominate over any longer-range interactions and are therefore a good approximation to many of the physical realisations of spin chains mentioned above.

We begin in section 1.2.1 by describing a basic protocol suggested by Bose [3] which uses a spin chain with isotropic Heisenberg nearest-neighbour interactions. This protocol is particularly simple and elegant but it suffers as a result: the maximum possible fidelity of quantum communication in this protocol decays with increasing chain length. There have been many successful attempts to improve on this and we briefly describe a small selection of them in section 1.2.2.

It should be noted that spin chain models are not just useful as quantum communication channels, but are also of interest for their entanglement transfer properties [28, 29], their ability to generate entanglement [30, 31, 32, 33, 34] and for quantum computing in general [35, 36].

1.2.1 Basic protocol

We now describe the basic protocol first proposed in [3] which enables two parties, Alice and Bob, to transmit quantum information along a quantum spin chain from one end to the other.

Alice and Bob take a chain of qubits where each qubit is coupled to its nearest-neighbours via an isotropic Heisenberg interaction. If this chain has length N then the Hamiltonian for the whole system is

$$H = - \sum_{j=1}^{N-1} J_j \boldsymbol{\sigma}_j \cdot \boldsymbol{\sigma}_{j+1} - \sum_{j=1}^N B_j \sigma_j^z \quad (1.8)$$

where the coupling strengths $J_j > 0$ and the magnetic field strengths $B_j > 0$ are all positive. The chain must now be *cooled* until it is in the ground state $|\mathbf{0}\rangle = |00\cdots 0\rangle$ (that is, all spins are pointing *down*). Alice now places the quantum state she wishes to send to Bob, $|\psi_{in}\rangle = \alpha |\mathbf{0}\rangle + \beta |\mathbf{1}\rangle$, on the first (left-hand) spin of the chain (see figure 1.2). We write the state of the whole spin chain as

$$|\psi(0)\rangle = \alpha |\mathbf{0}\rangle + \beta |\mathbf{1}\rangle \quad (1.9)$$

where we have introduced the notation $|\mathbf{j}\rangle = |00\cdots 010\cdots 00\rangle$ where the 1 is in the j th position ($j = 1, \dots, N$). That is, $|\mathbf{j}\rangle$ represents the spin chain in a state where all the spins are *down* except for the j th spin which is *up*. The spin chain is now allowed to evolve freely for a time t , after which its state has become

$$|\psi(t)\rangle = \alpha |\mathbf{0}\rangle + \beta \sum_{j=1}^N \langle \mathbf{j} | e^{-iHt} | \mathbf{1} \rangle |\mathbf{j}\rangle \quad (1.10)$$

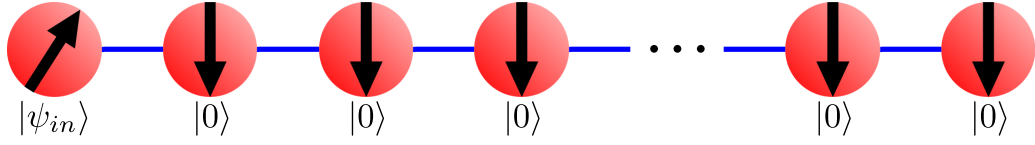


Figure 1.2: The initial setup for the basic spin chain communication protocol: all qubits are in the spin down ground state except Alice's (left-hand) qubit which is in the state $|\psi_{in}\rangle$.

At time t the reduced state of Bob's qubit (the right-hand qubit, labelled by N) is,

$$\rho_{out}(t) = p(t) |\psi_{out}\rangle \langle \psi_{out}| + (1 - p(t)) |0\rangle \langle 0| \quad (1.11)$$

where

$$|\psi_{out}\rangle = \frac{1}{\sqrt{p(t)}} (\alpha |0\rangle + \beta \langle \mathbf{N} | e^{-iHt} | \mathbf{1} \rangle |1\rangle) \quad (1.12)$$

and the probability $p(t) = |\alpha|^2 + |\beta|^2 |\langle \mathbf{N} | e^{-iHt} | \mathbf{1} \rangle|^2$ is, generally speaking, less than unity.

Bob has therefore received an approximation ρ_{out} to the state that Alice sent him and this state most closely approximates $|\psi_{in}\rangle \langle \psi_{in}|$ when $p(t)$ is at its closest to unity. The success or failure of this protocol can be measured by the *averaged fidelity* of the state transfer, $F = \frac{1}{4\pi} \int \langle \psi_{in} | \rho_{out} | \psi_{in} \rangle d\Omega$ where the integral is taken over all possible pure input states (*i.e.* the states which lie on the surface of the Bloch sphere). It has been shown [3] that the fidelity is better than that which could be achieved with a classical channel (namely $F = 2/3$) for all chain lengths up to around $N = 80$, provided Bob can calculate and choose the optimal time to extract his output ρ_{out} . Unfortunately for longer chains the fidelity drops below that which is achievable with classical channels, and it is to overcome this downfall that several advanced protocols have been designed. We briefly review a small selection of these protocols in the next section.

1.2.2 Advanced protocols

There have been various attempts at improving on the fidelity achieved by the basic protocol for transferring information along quantum spin chains [37, 38, 39, 40, 41, 42]. There are too many advancements to consider in detail here, so we restrict ourselves to reviewing just a few schemes to illustrate the kind of improvements that can be achieved.

Engineered spin chains

The first advanced protocol we consider is that of an engineered spin chain which was discovered independently by Christandl *et al.* [43] and Nikolopoulos *et al.* [44]; see also [45] for more information.

The engineered spin chain in question consists of XX -model interactions of the form $h_j = J_j(\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y)$ where we are at liberty to choose each of the interaction strengths J_j independently.

Now, the XX -model on a hypercube of dimension $N - 1$ with just two sites on each edge allows perfect state transfer between antipodal vertices [43]. The proof of this is based on some graph theoretic results and the fact that an XX -chain of length 2 allows perfect transfer of states from one particle to the other. One can map this hypercube onto an XX -model spin chain of length N , but to do so one must ensure that the coupling strengths are $J_j = \sqrt{j(N - j)}$; the Hamiltonian of the chain is therefore

$$H = - \sum_{j=1}^{N-1} \sqrt{j(N - j)} (\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y) \quad (1.13)$$

It is this engineered spin chain that allows perfect transfer of quantum states and, as in the basic protocol, we must prepare the chain in the state $|\mathbf{0}\rangle$ before Alice places the quantum state she wishes to transmit on the first spin at her end of the chain (see figure 1.3). The chain is allowed to evolve freely for a pre-determined length of time, after which Bob has received at his end of the chain precisely the state which was input by Alice, $\rho_{out} = |\psi_{in}\rangle \langle \psi_{in}|$.

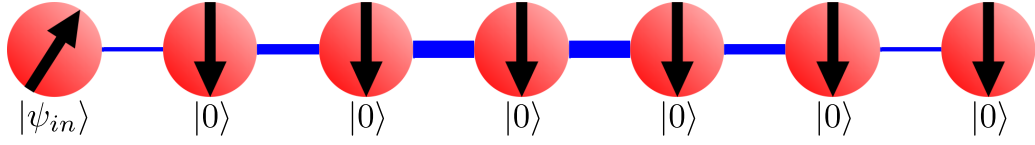


Figure 1.3: The initial setup for the spin chain with engineered nearest-neighbour couplings that achieves perfect transfer of quantum states: all qubits are in the spin down state except Alice’s (left-hand) qubit which is in the state $|\psi_{in}\rangle$. The thickness of the blue lines between the qubits represents the strength of the interactions (thicker lines represent stronger couplings).

Spin rings

The problem with engineered spin chains is that it is very difficult — if not impossible — to precisely engineer the interaction strengths as required. For this reason we now consider a different scheme which, although not providing perfect state transfer, nevertheless improves dramatically on what is possible with the basic protocol. This protocol [46] utilises a spin ring of N sites; see also [47] for more on spin rings. Alice and Bob both have access to a small number of spins at diametrically opposite sites on which they can perform unitary operations.

The first step of the protocol is to initialise the spin ring in the state $|0\rangle$. However, instead of simply placing her input state $|\psi_{in}\rangle = \alpha|0\rangle + \beta|1\rangle$ on a single site, Alice encodes her input state as a truncated Gaussian wave-packet $|\psi\rangle = \alpha|0\rangle + \beta|G\rangle$, by applying unitary operations on the spins \mathcal{A} to which she has access (see figure 1.4). Free evolution of the spin ring now takes place during which the wave-packet propagates around the ring, dispersing slightly as it goes. The wave-packet is chosen to be Gaussian as this minimises the dispersion, which equates to loss of information about the input state. Choosing the truncation width of the wave-packet (which is identical to the number of sites to which Alice has direct access) to be $\sqrt[3]{N}$ guarantees that the final width of the wave-packet is essentially independent of the size of the ring. Finally, Bob applies a set of decoding unitaries to the spins in his region

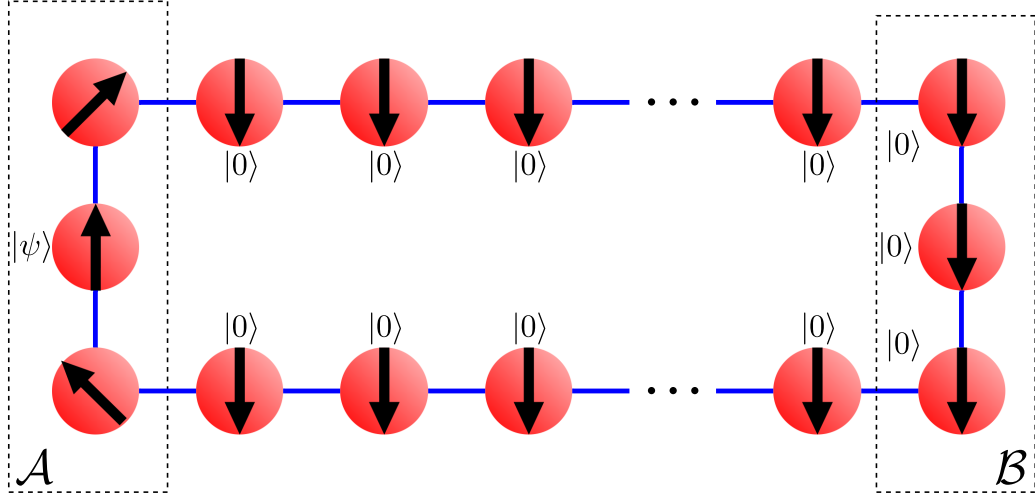


Figure 1.4: The initial setup for the spin ring communication protocol: all qubits are in the spin down state except for those in Alice's region \mathcal{A} which are in a truncated Gaussian wave-packet state $|\psi\rangle$ (which encodes Alice's input state $|\psi_{in}\rangle$). After a pre-determined time Bob will apply decoding unitaries on his region \mathcal{B} to extract his output ρ_{out} .

\mathcal{B} to recover a good approximation to $|\psi_{in}\rangle$.

Dual rail protocol

The final protocol we describe is the dual rail scheme which combines the advantages of both the engineered spin chain and the spin ring: it provides perfect state transfer but without the need to engineer specially designed chains.

The idea here is to use two identical parallel spin chains [48, 49]. Alice has access to the left-hand spin of each chain and Bob has access to the right-hand spins (see figure 1.5). If the Hamiltonian of each individual chain is H , then the Hamiltonian for the whole system is $H \otimes \mathbb{I} + \mathbb{I} \otimes H$. We begin by cooling both chains to their ground state $|0\rangle$, after which Alice encodes her input state $|\psi_{in}\rangle = \alpha |0\rangle + \beta |1\rangle$ as

$$|\psi_1\rangle = \alpha |0, 1\rangle + \beta |1, 0\rangle \quad (1.14)$$

where $|\mathbf{j}, \mathbf{k}\rangle$ represents the state of the system where the first spin chain is in state $|\mathbf{j}\rangle$ and the second chain in state $|\mathbf{k}\rangle$.

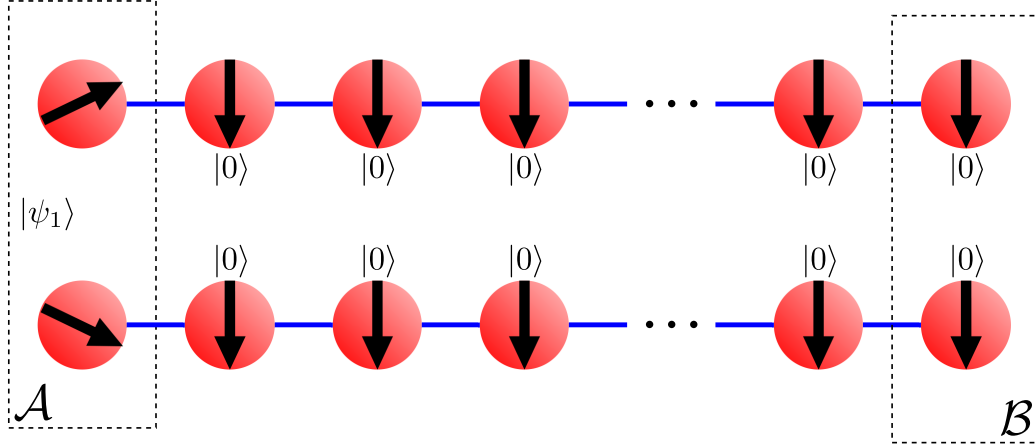


Figure 1.5: The initial setup for the dual rail communication protocol: all qubits are in the spin down ground state except for those at Alice's end of the chains \mathcal{A} which are in the state $|\psi_1\rangle$ encoding Alice's input state $|\psi_{in}\rangle$. Bob will repeatedly attempt to extract this state at his end of the chains \mathcal{B} until he is successful.

Allowing the chains to evolve freely for a time t results in the state

$$|\psi(t)\rangle = \sum_{j=1}^N \langle \mathbf{j} | e^{-iHt} | \mathbf{1} \rangle |\psi_j\rangle \quad (1.15)$$

where $|\psi_j\rangle = \alpha |\mathbf{0}, \mathbf{j}\rangle + \beta |\mathbf{j}, \mathbf{0}\rangle$. After this period of free evolution, Bob applies a CNOT gate on his two qubits which transforms the state $|\psi(t)\rangle$ into

$$|\psi_{CNOT}(t)\rangle = \langle \mathbf{N} | e^{-iHt} | \mathbf{1} \rangle (\alpha |\mathbf{0}\rangle + \beta |\mathbf{N}\rangle) \otimes |\mathbf{N}\rangle + \sum_{j=1}^{N-1} \langle \mathbf{j} | e^{-iHt} | \mathbf{1} \rangle |\psi_j\rangle \quad (1.16)$$

If Bob now measures his qubit on the second chain, he either gets measurement outcome 1 (which happens with probability $|\langle \mathbf{N} | e^{-iHt} | \mathbf{1} \rangle|^2$) and he is left with $|\psi_{in}\rangle$ on his qubit in the first chain, or he gets measurement outcome 0 (which happens with probability $1 - |\langle \mathbf{N} | e^{-iHt} | \mathbf{1} \rangle|^2$) and he is left with $|0\rangle$ on his qubit in the first chain — in which case the protocol has failed.

The advantage of this protocol is that when it fails, the information encoded in $|\psi_{in}\rangle$ remains in the spin chains — it has not been destroyed by the measurement but rather it has been localised to the first $N - 1$ spins. Whenever the protocol fails Bob simply waits for a short interval before repeating

his measurement; eventually he will obtain a positive measurement outcome and will be left with $|\psi_{in}\rangle$ on his qubit in the first spin chain. A rough approximation for the maximum length of time Bob will have to wait in order to receive $|\psi_{in}\rangle$ with probability $1 - \epsilon$ is $t \approx 0.55J^{-1}N^{5/3}|\ln \epsilon|$ [48, 49] (where J is the nearest-neighbour interaction coupling strength).

It is possible to generalise this protocol to the situation where Bob is only able to perform finite-strength continuous measurements at the receiving end of the chain [50].

1.2.3 Noisy chains

So far we have considered only *clean*, idealistic spin chains. Of course any physical realisation of these chains will deviate from the ideal as imperfections inevitably find their way into the setup; we call these *noisy spin chains*.

Several examples of noisy spin chains have been analysed before including a model similar to the basic protocol described above, but where each spin in the chain is coupled to a *spin bath*. It has been shown [49, 51] that if the mean square coupling to the environment (which consists of these spin baths) is G , then the fidelity of state transfer is modulated by a factor of $|\cos Gt|$. Other noisy models include chains where the interaction strengths J_j are random variables (so, for example, J_k might be stronger than J_j). In this situation there is plenty of evidence that perfect state transfer is still possible [49, 52, 53].

In this thesis we drive the study of noisy spin chains forward by considering noise models which are fundamentally different to those described above, namely models where the noise manifests itself as on-site disorder (see figure 1.6). In particular we will study the effects of this noise on the speed with which information can propagate along such spin chains. The results differ quite dramatically from what has been observed in the noise models described above.

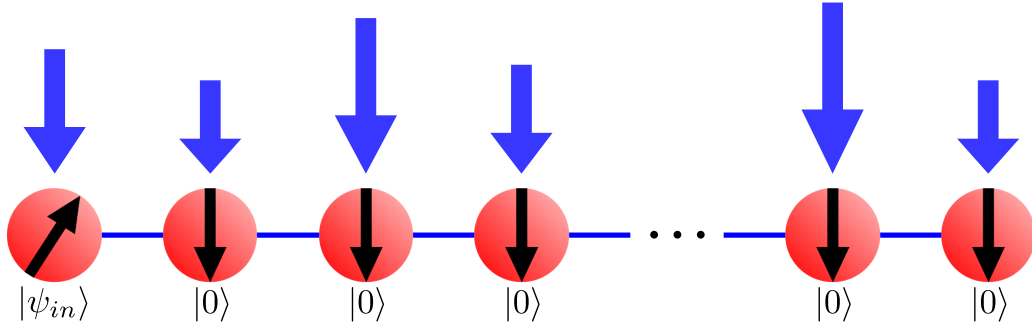


Figure 1.6: A typical initial setup of a noisy spin chain with on-site disorder (represented by the blue arrows). This on-site disorder might be caused, for example, by a non-uniform transverse electromagnetic field.

1.3 Bounds on the speed of propagation in quantum spin systems

Having reviewed some of the ways in which one may use quantum spin chains as quantum information channels, we now turn our attention to calculating the speed with which information can propagate along such spin chains. We begin by describing perhaps the most general tool for this, namely Lieb-Robinson bounds. We also describe alternative but equivalent bounds and briefly discuss the implications of both.

1.3.1 Lieb-Robinson bounds

Quantum lattice models (including quantum spin chains) obey a locality condition: once locally excited the excitation will travel through the lattice at a finite velocity. For spin models the speed at which information can propagate is limited by the Lieb-Robinson bound [54] which says that there is an effective *light cone* for correlations, with exponentially decaying tails, whose radius grows linearly with time [55]. This light cone is not of the kind we meet in special relativity: it is possible that information could propagate to regions outside the light cone, although the probability of this happening is exponentially small and the excitations are very likely to remain within the

cone.

There have been many studies focusing on Lieb-Robinson bounds and these have resulted in many simplified and alternative bounds [55, 56, 57, 58, 59] which are reviewed along with their implications in [60].

In its most general form, the Lieb-Robinson bound is a way of measuring the effect of an operator A applied to a region \mathcal{A} of the lattice at time 0, by computing its effect on another operator B applied to another region \mathcal{B} at time t . More formally, a Lieb-Robinson bound is an upper bound on the *Lieb-Robinson commutator*

$$C_B(t) := \sup_A \frac{\|[A, B(t)]\|}{\|A\|} \quad (1.17)$$

where the commutator $[X, Y]$ is defined to be $[X, Y] = XY - YX$ and $B(t)$ is the Heisenberg picture representation of the operator B . There are many different Lieb-Robinson bounds which apply to many different lattice systems, but they typically have the generic form

$$C_B(t) \leq \text{const} \times e^{\mu t - \nu d(\mathcal{A}, \mathcal{B})} \quad (1.18)$$

where μ and ν are constants and $d(\mathcal{A}, \mathcal{B})$ is the minimum lattice distance between the regions \mathcal{A} and \mathcal{B} . It is easy to see that this bound is exponentially growing in t but exponentially decaying in the distance between the regions; it is this feature that gives rise to the linear light cone and its exponentially decaying tails described above.

There are many consequences of Lieb-Robinson bounds. Apart from the aforementioned bounds on the velocity of information propagation, the Lieb-Robinson bound was exploited to establish the Lieb-Schultz-Mattis theorem in higher dimensions [56] (which relates periodicity in lattice systems to the excitation gap between the ground state and the first excited state). In generalising the proof of this theorem it was realised that the Lieb-Robinson bound can be used to provide a method to efficiently simulate the properties of low-dimensional spin networks [61, 62, 63, 64, 65, 66].

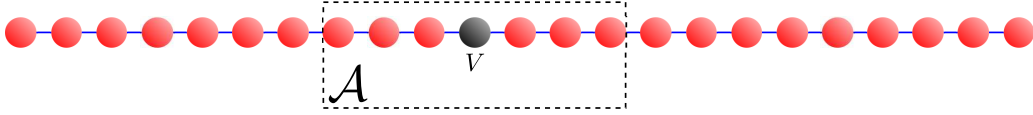


Figure 1.7: We approximate the Heisenberg picture time evolution of an operator V which acts initially on a small number of sites (represented here by a black sphere) by restricting evolution to a small block of sites \mathcal{A} . We obtain an approximation $V'(t) \approx V(t)$.

1.3.2 An alternative bound

Consider a quantum lattice system whose Hamiltonian is H and also consider a region \mathcal{A} within this lattice whose Hamiltonian is $H_{\mathcal{A}}$. Note that $H_{\mathcal{A}}$ can be found by restricting H to the region \mathcal{A} — that is, $H_{\mathcal{A}}$ consists of only those terms in H which correspond to interactions taking place within \mathcal{A} . Now consider any operator V which acts on a sub-region in the middle of \mathcal{A} . Assuming a time-independent Hamiltonian, we can approximate the time evolution of the operator $V(t) = e^{-itH} V e^{itH}$ by restricting to the region \mathcal{A} in the following manner: $V'(t) = e^{-itH_{\mathcal{A}}} V e^{itH_{\mathcal{A}}}$ (see figure 1.7).

Of course it is important to find out how good this approximation is by calculating an upper bound for $\|V(t) - V'(t)\|$. This alternative bound also describes the locality of the system and it has the same generic form [67] as the Lieb-Robinson bound, namely:

$$\|V(t) - V'(t)\| \leq \text{const} \times e^{\mu t - \nu |\mathcal{A}|} \quad (1.19)$$

where $|\mathcal{A}|$ is the number of spins in \mathcal{A} .

1.3.3 Improved bounds

Whilst we have not proved the Lieb-Robinson bound (or its alternative version), suffice it to say that the argument underlying its proof relies only on the *ultra-violet cut-off* imposed by the lattice structure and therefore the Lieb-Robinson bound is very general. There are, however, some situations where the Lieb-Robinson bound is not the best available. For example, when we

know more about the structure of the interactions it should be possible to construct tighter bounds.

Indeed, it is the pursuit of tighter bounds which occupies a large part of this thesis. We prove new tighter Lieb-Robinson and alternative bounds for one-dimensional quantum lattice systems, namely qubit spin chains, with various types of on-site disorder. Depending on the details of the disorder, we will find different bounds with different consequences for the propagation of excitations (and hence information) through the spin chains.

1.4 Generalised depolarising channels

Perhaps the simplest model of noise in a quantum system is that of the isotropic depolarising channel Φ_p where with probability p a quantum state ρ is left untouched while with probability $(1 - p)$ it is mapped to the completely mixed state \mathbb{I}/D

$$\Phi_p(\rho) = (1 - p)\frac{\mathbb{I}}{D} + p\rho \quad (1.20)$$

This channel results in the *Bloch “ball”* (the set of all quantum states) being compressed isotropically by a factor of p .

One can imagine a slightly more complicated noise model whereby the noise compresses the Bloch “ball” anisotropically along axes which are defined by the basis we choose to work in. The amounts by which we compress along each axis are called the *compression coefficients* and these form the components of the *compression vector*.

These *generalised depolarising channels* (also called *anisotropic depolarising channels*) form a broad class of quantum channels which can be realised experimentally (see for example [68]); in the single qubit case they include the bit-flip and phase-flip channels. It is worth noting that any quantum channel (including the use of a spin chain as a quantum channel) can be turned into a generalised depolarising channel by the method described in chapter 5.

Throughout this thesis we let $\{M_\alpha\}_{\alpha=0}^{D^2-1}$ be a basis for $D \times D$ complex matrices (for example, density matrices of a quantum system with dimension D) which satisfies the following conditions:

- $M_0 = \mathbb{I}$
- $\text{tr}(M_\alpha) = 0$ for all $\alpha \neq 0$
- $\text{tr}(M_\alpha^\dagger M_\beta) = 0$ for all $\alpha \neq \beta$

We call such a basis *trace-orthogonal* and, abusing terminology slightly, *trace-free* (note that $\text{tr}(M_0) \neq 0$). We do not need to restrict ourselves to trace-orthonormal bases as we can simply divide by $\sqrt{\text{tr}(M_\alpha^\dagger M_\alpha)}$ when necessary to normalise the basis.

If ρ is the density matrix of any D -dimensional quantum system, then we may write

$$\rho = \frac{1}{D} \left(\mathbb{I} + \sum_{\alpha=1}^{D^2-1} \sqrt{\frac{D(D-1)}{\text{tr}(M_\alpha^\dagger M_\alpha)}} a_\alpha M_\alpha \right) \quad (1.21)$$

We call $\mathbf{a} = (a_0, \dots, a_{D^2-1})$ the *polarisation vector* and a_α the *polarisation coefficients* of the state ρ with respect to the basis $\{M_\alpha\}$. For notational convenience we add an extra *zeroth* component, $a_0 \equiv 1$, which does not affect the value of $\|\mathbf{a}\|$. We have chosen the normalisation in equation (1.21) such that

- $\|\mathbf{a}\| = 1$ if and only if ρ is a pure state
- $\|\mathbf{a}\| < 1$ if and only if ρ is a mixed state

where the norm of \mathbf{a} is defined to be $\|\mathbf{a}\| \equiv \sum_{\alpha=1}^{D^2-1} |a_\alpha|^2$. (To see this, recall that $\text{tr}(\rho^2) = 1$ if and only if ρ is a pure state and $\text{tr}(\rho^2) < 1$ if and only if ρ is a mixed state.)

The *Bloch “ball”* is the set of all polarisation vectors corresponding to quantum states. It is important to note that — except for the single qubit case — the Bloch “ball” is not the ball of unit radius, but rather a convex

subset of this ball. This is because for $D > 2$ some vectors lying within the unit ball are not valid polarisation vectors as they do not correspond to positive states.

We can now define a (*generalised*) *depolarising channel with respect to the basis* $\{M_\alpha\}$ to be a map $\Phi_{\mathbf{v}}$ which satisfies the following properties:

- $\Phi_{\mathbf{v}}$ is a trace-preserving completely positive map
- $\Phi_{\mathbf{v}}$ compresses the Bloch “ball” in the following manner:

$$\Phi_{\mathbf{v}}(\rho) = \frac{1}{D} \left(\mathbb{I} + \sum_{\alpha=1}^{D^2-1} \sqrt{\frac{D(D-1)}{\text{tr}(M_\alpha^\dagger M_\alpha)}} v_\alpha a_\alpha M_\alpha \right) \quad (1.22)$$

We call $\mathbf{v} = (v_0, \dots, v_{D^2-1})$ the *compression vector* of the channel $\Phi_{\mathbf{v}}$ as it specifies the amount by which $\Phi_{\mathbf{v}}$ compresses the Bloch “ball” along each axis; the v_α are called *compression coefficients*. Again, for notational convenience, we have added a *zeroth* component $v_0 \equiv 1$ (v_0 is the compression coefficient for $M_0 = \mathbb{I}$, so $v_0 = 1$ ensures that $\Phi_{\mathbf{v}}$ is trace-preserving).

Note that $|v_\alpha| \leq 1$ for all α . To see this, let the largest possible magnitude of the polarisation coefficient a_α be $\tilde{a}_\alpha = \sup_\rho \{|a_\alpha|\}$ and let σ be a state with $|a_\alpha| = \tilde{a}_\alpha$ (so σ lies on the boundary of the Bloch “ball”). If $|v_\alpha| > 1$ then $\Phi_{\mathbf{v}}(\sigma)$ is not a state (it lies outside the “Bloch ball”), so $|v_\alpha| \leq 1$ as claimed.

It is important to note that the notion of generalised depolarising channels is highly basis dependent: we *must* define such channels with respect to a given basis.

If we work in a fixed basis it is clear that for each depolarising channel there is a unique compression vector. It is therefore natural to ask the question “which vectors \mathbf{v} are valid compression vectors corresponding to depolarising channels $\Phi_{\mathbf{v}}$?”. When we restrict to the case of a single qubit the answer to this question is already known (see for example [69]): the set of all compression vectors which lie in a specific tetrahedron (see figure 1.8) are the only possible compression vectors which correspond to generalised depolarising channels.

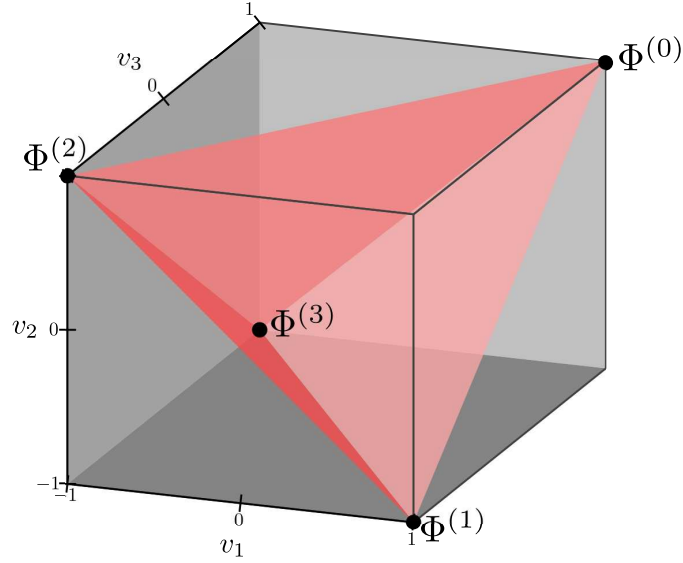


Figure 1.8: The tetrahedron in compression space representing all possible single qubit generalised depolarising channels: v_1 , v_2 and v_3 are the compression coefficients and the red tetrahedron encloses all compression vectors which correspond to valid generalised depolarising channels. $\Phi^{(0)}$, $\Phi^{(1)}$, $\Phi^{(2)}$ and $\Phi^{(3)}$ are the extremal channels whose compression vectors lie at the vertices of the tetrahedron (they are the identity, bit-flip, bit-phase-flip and phase-flip channels respectively).

Dixit and Sudarshan make a conjecture [4] which says that “when $D = 2^d$ ($d \in \mathbb{N}$) and we work in the Pauli basis, then the set of all compression vectors forms a simplex in compression space”. This conjecture is a generalisation of the single-qubit result discussed here; it is proved and generalised in chapter 6.

1.5 Structure of thesis

In this introduction we have reviewed the use of quantum spin chains as channels for quantum information. We have also reviewed Lieb-Robinson bounds and associated alternative bounds with a discussion of their implications. Finally, we have introduced generalised depolarising channels.

The remainder of this thesis (excepting parts of chapter 5 which contain review material) is devoted to original work.

We begin in chapter 2 by finding a new Lieb-Robinson bound for a qubit spin chain experiencing on-site disorder, the strength of which remains fixed in time. This new bound gives rise to a light cone whose radius grows *logarithmically* with time. Contrast this to the original Lieb-Robinson bound which exhibits a linear light cone. The work in this chapter is an expanded version of the material first published in [70].

Chapters 3 and 4 find bounds on the propagation of information through spin chains subjected to fluctuating on-site disorder. In chapter 3 we examine disorder which fluctuates in strength but is fixed in direction and discover bounds which imply that, on average, excitations (and hence information) are localised to a small region around where they started, despite the fact that individual excitations can propagate diffusively through the chain. Chapter 4 examines a model in which the on-site disorder can fluctuate in both strength and direction and we prove a Lieb-Robinson bound that identifies a potential noise threshold: above this threshold information is localised whilst below it information can potentially propagate ballistically through the chain. These chapters are an expanded version of work first published in [71].

Chapter 5 is a short review chapter which shows how any spin chain (whether noisy or clean) can act as a generalised depolarising channel, whilst chapter 6 studies these channels in detail, proving and generalising a conjecture of Dixit and Sudarshan [4]; this work was published in [72].

Finally we summarise and draw conclusions in chapter 7.

Chapter 2

Static disorder

In the previous chapter we saw how quantum spin chains can be used as quantum channels to faithfully transmit quantum states (and hence quantum information). In this chapter we study quantum spin chains which experience on-site disorder which is fixed in time and we examine the effects of this noise by calculating a Lieb-Robinson bound for this type of system.

This chapter is organised as follows. Once we have described the model in detail (§ 2.1) we apply the Jordan-Wigner transformation (§ 2.2) which allows us to calculate the dynamics (§ 2.3). Anderson localisation (§ 2.4) allows us to calculate bounds on a restricted class of Lieb-Robinson commutators (§ 2.4.1); coupling Anderson localisation with a decomposition of the propagator (§ 2.5) allows us to calculate the general Lieb-Robinson bound (§ 2.6). We finish with a short discussion (§ 2.7).

2.1 The static disorder model

Throughout this chapter we focus our attention on spin chains of length N with XX -model interactions between nearest-neighbouring spins and on-site disorder which is fixed in time but varies from site to site (see figure 2.1). Typically the nearest-neighbour interactions $h_j = J_j(\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y)$ have strengths which are all equal ($J_j = -J$ for all j), but this restriction is not

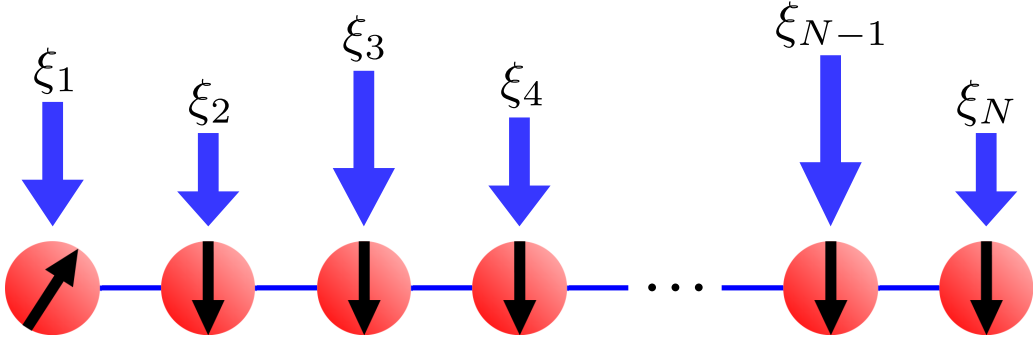


Figure 2.1: A spin chain experiencing static on-site disorder: the blue arrows represent the on-site disorder (which might be caused, for example, by a noisy electromagnetic field) and ξ_j is the strength of the disorder on site j .

necessary and we work with the more general case that they are all independently chosen. The on-site disorder is fixed in the z -direction and could arise experimentally from, for example, a transverse electromagnetic field whose field strength varies from site to site but is fixed in time. These field strengths ξ_j are drawn from independent identically distributed random variables \mathbb{P}_j , the only restriction being that the probability distributions have bounded probability density functions.

In summary, the spin chains we consider in this chapter have Hamiltonian

$$H = \sum_{j=1}^{N-1} J_j (\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y) + \sum_{j=1}^N \xi_j \sigma_j^z \quad (2.1)$$

Before we go any further it is important to note that this is a physical model which can be experimentally realised (see section 1.2). It is arguably the simplest of a class of models which preserve the number of excitations in the spin chain and it is amenable to precise analytical treatment.

2.2 Jordan-Wigner transformation

In order to calculate the dynamics of our model we will apply the Jordan-Wigner transformation [73], which maps systems of spin-half particles (for example qubit spin chains) into systems of spinless fermions hopping on a

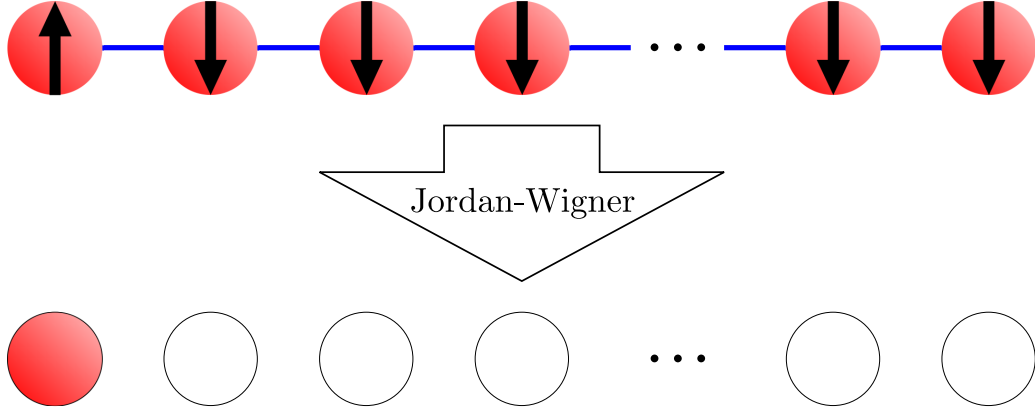


Figure 2.2: Schematic illustration of the Jordan-Wigner transform: spin-up qubits are mapped into spinless fermions (represented here by red spheres) whilst spin down qubits are mapped into empty lattice sites (represented by empty circles).

lattice. This is achieved by mapping an excited qubit (whose spin is up, corresponding to computational basis state $|1\rangle$) into a fermion on the corresponding lattice site; conversely an un-excited qubit (whose spin is down, corresponding to computational basis state $|0\rangle$) is mapped into an empty lattice site (see figure 2.2). Of course, any qubit which is in a superposition state $\alpha|0\rangle + \beta|1\rangle$ is mapped into a fermionic superposition state.

More precisely, the Jordan-Wigner transform is a relationship between the Pauli matrices and the fermionic creation and annihilation operators (a_j^\dagger and a_j respectively), which is defined by

$$a_j = (\sigma_1^z \otimes \cdots \otimes \sigma_{j-1}^z) \otimes \sigma_j^- \quad \text{where} \quad \sigma_j^- = \frac{1}{2} (\sigma_j^x + i\sigma_j^y) = |0\rangle\langle 1| \quad (2.2)$$

In this form, the fermionic creation and annihilation operators (which satisfy the anti-commutation relations $\{a_j, a_k^\dagger\} = \delta_{jk}$ and $\{a_j, a_k\} = \{a_j^\dagger, a_k^\dagger\} = 0$) are given in terms of the Pauli matrices. One can invert this to give the Pauli matrices in terms of fermionic creation and annihilation operators

$$\begin{aligned} \sigma_j^z &= a_j a_j^\dagger - a_j^\dagger a_j \\ \sigma_j^x &= \left(\prod_{k=1}^{j-1} \sigma_k^z \right) (a_j^\dagger + a_j) \\ \sigma_j^y &= i \left(\prod_{k=1}^{j-1} \sigma_k^z \right) (a_j^\dagger - a_j) \end{aligned} \quad (2.3)$$

It is this form of Jordan-Wigner that allows us to rewrite the Hamiltonian of our noisy spin chain as

$$H = 2 \sum_{j=1}^{N-1} J_j \left(a_j^\dagger a_{j+1} + a_{j+1}^\dagger a_j \right) - 2 \sum_{j=1}^N \xi_j a_j^\dagger a_j \quad (2.4)$$

Note that we have applied the fermionic anti-commutation relations to the on-site disorder terms and have ignored an irrelevant overall constant term $\sum_j \xi_j \mathbb{I}$ which simply shifts the energy eigenvalues by a constant $\sum_j \xi_j$.

We have seen how Jordan-Wigner maps the Pauli matrices into fermionic operators and we now discuss its effect on the states of the spin chain. The ground state $|\mathbf{0}\rangle$ with all spins down (that is, all qubits are in the computational basis state $|0\rangle$) is mapped into the empty fermionic lattice (sometimes referred to as the *vacuum*) which we denote by $|\Omega\rangle$; note that this state is an eigenstate of the Hamiltonian with eigenvalue 0. The states $|\mathbf{j}\rangle$ of the chain (which have all spins down except for spin j which is up) are mapped into single fermions on lattice site j which we denote by $|j\rangle := a_j^\dagger |\Omega\rangle$. For the remainder of this chapter we work exclusively in the single-excitation subspace and therefore the states described above are sufficient to describe the dynamics of our model.

2.3 Calculating the dynamics

Having applied the Jordan-Wigner transformation we are now in a position to calculate the dynamics of our spin chain. To do this we first note that the Hamiltonian preserves the number of excitations in the chain and these excitations are non-interacting; we can therefore work in the single excitation subspace. This allows us to write the Hamiltonian as a tri-diagonal $N \times N$ matrix, K , whose matrix elements are

$$\begin{aligned} \langle j | K | j \rangle &= -2\xi_j && \text{for } j \in \{1, \dots, N\} \\ \langle j | K | j+1 \rangle &= 2J_j && \text{for } j \in \{1, \dots, N-1\} \\ \langle j+1 | K | j \rangle &= 2J_j && \text{for } j \in \{1, \dots, N-1\} \\ \langle j | K | k \rangle &= 0 && \text{otherwise} \end{aligned} \quad (2.5)$$

The Hamiltonian may therefore be written as $H = \sum_{j,k=1}^N \langle j | K | k \rangle a_j^\dagger a_k$. We can now diagonalise the Hamiltonian by performing a Bogoliubov transformation, which defines a new set of fermionic operators $\{b_j\}$ which are related to the old set of fermionic operators $\{a_j\}$ by the relation

$$b_j = \sum_{k=1}^N u_{jk} a_k \quad (2.6)$$

where u_{jk} are complex numbers. By imposing the fermionic anti-commutation relations on the new set of fermionic operators $\{b_j\}$ (namely $\{b_j, b_k^\dagger\} = \delta_{jk}$ and $\{b_j, b_k\} = \{b_j^\dagger, b_k^\dagger\} = 0$) we force the following condition to hold

$$\sum_{k=1}^N u_{jk} u_{kl}^\dagger = \delta_{jl} \quad (2.7)$$

where $u_{kl}^\dagger = u_{lk}^*$. That is, the matrix $U := \sum_{jk} u_{jk} |j\rangle \langle k|$ is a unitary matrix. This allows us to invert the relationship of equation (2.6) to give the $\{a_j\}$ in terms of the $\{b_j\}$

$$a_j = \sum_{k=1}^N u_{jk}^\dagger b_k \quad (2.8)$$

Now, we are at liberty to choose the matrix U to diagonalise the Hamiltonian: $UKU^\dagger = \Lambda = \text{diag}\{\lambda_1, \dots, \lambda_N\}$, or in other words

$$H = \sum_{j=1}^N \lambda_j b_j^\dagger b_j \quad (2.9)$$

We are now in a position to calculate the time evolution of the $\{b_j\}$ in the Heisenberg picture: $b_j(t) := e^{itH} b_j e^{-itH}$. It is a simple matter to differentiate $b_j(t)$ with respect to time to obtain

$$\frac{d}{dt} b_j(t) = -i\lambda_j b_j(t) \quad (2.10)$$

which has solution

$$b_j(t) = e^{-i\lambda_j t} b_j \quad (2.11)$$

Since the Heisenberg picture time evolution of the $\{a_j\}$ can be written as $a_j(t) = \sum_{k=1}^N u_{jk}^\dagger b_k(t)$, equation (2.11) allows us to calculate the time evolution of the original fermionic annihilation operators

$$a_j(t) = \sum_{k,l=1}^N u_{jk}^\dagger e^{-i\lambda_k t} u_{kl} a_l =: \sum_{k=1}^N c_{jk}(t) a_k \quad (2.12)$$

The coefficients $c_{jk}(t) = \langle j | U^\dagger e^{-i\Lambda t} U | k \rangle = \langle j | e^{-iKt} | k \rangle$ can be called correlation functions — see section 3.3 of the next chapter, where we also demonstrate an alternative method of calculating the dynamics without using the Bogoliubov transformation.

2.4 Anderson localisation

Applying Jordan-Wigner maps our spin chain into the *tight-binding Anderson model* [74] with on-site disorder. This is a particularly well studied model of disorder and we can exploit some of its properties. In particular we make use of the phenomenon of *Anderson localisation*; this is a general phenomenon in sufficiently disordered systems and it is characterised by the absence of wave diffusion through such systems.

Anderson localisation is typically characterised by exponential localisation of the energy eigenstates, a type of localisation referred to as *exponential localisation* [75]. However, our particular model exhibits a stronger form of localisation called *strong sub-exponential HS-kernel decay (SSEHSD)* [75], which can be expressed as the following theorem

Theorem 1. *For any $0 < \zeta < 1$ and $j, k \in \{1, \dots, N\}$ then*

$$|\langle j | e^{-iKt} | k \rangle| \leq c_\zeta e^{-|j-k|^\zeta} \quad \text{for all } t \quad (2.13)$$

with probability $\geq 1 - e^{-L_k^\zeta}$ where $\frac{N-1}{2} \geq L_k \geq \left(\frac{N-1}{2}\right)^\frac{1}{\alpha}$ with $1 < \alpha < 2$

Proof. As the on-site disorder field strengths ξ_j are drawn from a probability distribution with bounded density we satisfy the assumptions of theorem 6.5 in [75], the proof of which establishes theorem 1. \square

This theorem is proved by a technique called *multiscale analysis*; see [76] for a good introduction to this topic. Multiscale analysis is an inductive proof technique which is used to prove localisation of the Anderson model on the infinite lattice. The inductive step assumes localisation of the model when restricted to a small box of size L_k and uses this to prove localisation on a larger box of size $L_{k+1} = L_k^\alpha$ with $1 < \alpha < 2$. (The initial scale satisfies $L_0 = e^{\text{const} \times \gamma}$ where γ is the second moment of the probability distribution \mathbb{P}_j .) Our disordered spin chain of length N is simply the one-dimensional infinite lattice Anderson model restricted to a finite sub-lattice with $2L_{k+1} + 1 \geq N \geq 2L_k + 1$ for some integer k .

Multiscale analysis proves what is observed in practice, namely that Anderson localisation is a general phenomenon which applies to a wide range of noise models [74, 77, 78, 79].

It is important to note that, due to the probabilistic nature of the on-site disorder, certain realisations of the on-site noise might lead to non-localised dynamics. However, the probability of this happening is exponentially small ($\leq e^{-L_k^\zeta}$). For the remainder of this chapter we assume that the on-site noise terms lead to localisation as characterised by theorem 1, as we know that this occurs with very high probability.

For any $0 < \zeta < 1$, Theorem 1 provides the following bound on the correlation functions of equation 2.12

$$|c_{jk}(t)| \leq c_\zeta e^{-|j-k|^\zeta} \quad (2.14)$$

It is now evident that the correlation functions are all exponentially small excepting those for which $|j - k|$ is small; this reflects the “almost diagonal” structure of the propagator e^{-iKt} for all times.

2.4.1 Direct calculation of a bound on a restricted class of Lieb-Robinson commutators

The purpose of this chapter is to calculate a Lieb-Robinson bound for our model of a spin chain experiencing static on-site disorder. Recall that a Lieb-Robinson bound is an upper bound on the Lieb-Robinson commutator $\|[A, B(t)]\|$ of equation (1.17). We are now in a position to calculate upper bounds on a restricted class of Lieb-Robinson commutators (namely those in which the operators A and B are linear combinations of the identity \mathbb{I} and the Pauli z matrix σ^z on single sites).

Unfortunately, due to the non-local nature of the Jordan-Wigner transformation (recall that $a_j = \left(\prod_{k=1}^{j-1} \sigma_k^z\right) (\sigma_j^x + i\sigma_j^y)$), direct calculation of an upper bound on arbitrary Lieb-Robinson commutators is futile: we obtain an extremely poor bound which contains a large constant and is therefore worse than the trivial bound $\|[A, B(t)]\| \leq 2\|A\|\|B\|$. Later on we use alternative methods to calculate an improved bound on arbitrary Lieb-Robinson commutators, beginning in section 2.5.

In order to calculate upper bounds on the restricted class of Lieb-Robinson commutators, we need only consider commutators such as $\|[\sigma_j^z, \sigma_k^z(t)]\|$ (as the identity \mathbb{I} commutes with everything). Working in the Heisenberg picture we apply Jordan-Wigner to the Pauli z matrix

$$\sigma_k^z(t) = a_k(t)a_k^\dagger(t) - a_k^\dagger(t)a_k(t) = \sum_{l,m=1}^N c_{kl}(t)c_{mk}^\dagger(t) (a_l a_m^\dagger - a_m^\dagger a_l) \quad (2.15)$$

Recalling that the $\{a_j\}$ satisfy the fermionic anti-commutation relations given in section 2.2 allows us to use the above expression to directly calculate our Lieb-Robinson commutators

$$[\sigma_j^z, \sigma_k^z(t)] = 2 \sum_{l=1}^N c_{kl}(t)c_{jk}^\dagger(t) (a_l a_j^\dagger - a_j^\dagger a_l) + 2 \sum_{m=1}^N c_{kj}(t)c_{mk}^\dagger(t) (a_j a_m^\dagger - a_m^\dagger a_j) \quad (2.16)$$

(This calculation is straight-forward but somewhat lengthy.) The Gersgorin

disk theorem [80] tells us that $\|a_j a_k^\dagger - a_k^\dagger a_j\| \leq 2$; using this and recalling the bound (2.14) on the correlation functions allows us to calculate an upper bound for the commutator of equation (2.16), namely

$$\|[\sigma_j^z, \sigma_k^z(t)]\| \leq 4 \sum_{l=1}^N \left(|c_{kl}(t)| |c_{jk}^\dagger(t)| + |c_{kj}(t)| |c_{lk}^\dagger(t)| \right) \leq 8\kappa_\zeta c_\zeta e^{-|j-k|^\zeta} \quad (2.17)$$

where we have employed the bound $\sum_{l=1}^N e^{-|k-l|^\zeta} \leq \kappa_\zeta$ where κ_ζ is a constant. This bound holds as only those terms where $|k-l| < \text{const}$ contribute significantly to the sum. As an illustrative example, for a chain of length $N = 101$ with $\zeta = 0.99$ and $k = 51$ (picking k to be in the middle of the chain maximises the sum), we numerically calculate that $\kappa \approx 2.18$. It is a simple matter to check that for all ζ the constant κ_ζ satisfies $\kappa_\zeta \leq \kappa_0 \leq \frac{N}{e}$

In summary we have directly calculated an upper bound for a class of Lieb-Robinson commutators with $A = \mu_A \mathbb{I} + \kappa_A \sigma^z$ and $B = \mu_B \mathbb{I} + \kappa_B \sigma^z$ (with μ_A, μ_B, κ_A and κ_B constants), namely

$$\|[A_j, B_k(t)]\| \leq 8\kappa_\zeta \kappa_A \kappa_B c_\zeta e^{-|j-k|^\zeta} \quad \text{for any } 0 < \zeta < 1 \quad (2.18)$$

Finally, we comment that this bound explains a phenomenon observed in tDMRG (time-dependent density matrix renormalisation group) numerical investigations reported in [81], where the two-point correlation functions $C(r, t) := 2^{-N} \text{tr}(\sigma_j^z(t) \sigma_{j+r}^z)$ exhibit a freezing in space after sufficiently large times. What is essentially happening is that at large times the above bound becomes saturated.

2.5 Calculation of the propagator

In the previous section we calculated upper bounds on a restricted class of Lieb-Robinson commutators. We now employ a different technique which allows us to calculate upper bounds on arbitrary Lieb-Robinson commutators, giving rise to a Lieb-Robinson bound which bounds the dynamics of state transfer (and hence information propagation) through our noisy spin chain.

We follow and adapt the scheme laid down by Osborne in [61] to approximate the propagator up to a known error. First we bipartition the chain into two parts, \mathcal{A} and \mathcal{B} . We let each partition evolve freely for a time t according to its local propagator, $e^{-itH_{\mathcal{A}}}$ or $e^{-itH_{\mathcal{B}}}$, before applying a *patching-up operator* $V(t)$ which is designed to exactly compensate for the local interaction that has been excluded due to the partitioning procedure

$$e^{-itH} = V(t)e^{-it(H_{\mathcal{A}}+H_{\mathcal{B}})} \quad (2.19)$$

The intuition to bear in mind is that although $V(t)$ acts on the whole chain, it acts most strongly on the sites around the partition boundary and acts in an increasingly weak manner as we move away from this boundary. For this reason it is possible to approximate $V(t)$ with another operator $V^{\mathcal{W}}(t)$ which acts only on a small number of sites close to the boundary between the two partitions (see figure 2.3). Before we formally define the operator $V^{\mathcal{W}}(t)$ we must first study $V(t)$ a little more closely. Noting that $V(t) = e^{-itH}e^{it(H_{\mathcal{A}}+H_{\mathcal{B}})}$ and choosing the partition boundary to lie between sites m and $m+1$ we can find a differential equation for $V(t)$

$$\frac{d}{dt}V(t) = -ih_m(t)V(t) \quad (2.20)$$

where we have defined $h_m(t) = e^{-itH}h_me^{itH}$. This differential equation has solution $V(t) = \mathcal{T}e^{-i\int_0^t h_m(s)ds}$ where \mathcal{T} denotes time-ordering.

We let \mathcal{W} denote a set of $|\mathcal{W}|$ spins centred on the partition boundary. By defining $h_m^{\mathcal{W}}(t) = h_m(t)|_{\mathcal{W}}$ to be the local interaction $h_m(t)$ truncated to the set \mathcal{W} (that is, $h_m^{\mathcal{W}}(t) = P_{\mathcal{W}}h_m(t)P_{\mathcal{W}}$ where $P_{\mathcal{W}}$ is the orthogonal projector onto \mathcal{W}), we may formally define $V^{\mathcal{W}}(t)$ via the differential equation

$$\frac{d}{dt}V^{\mathcal{W}}(t) = -ih_m^{\mathcal{W}}(t)V^{\mathcal{W}}(t) \quad (2.21)$$

We now wish to calculate the error introduced when we approximate $V(t)$ by $V^{\mathcal{W}}(t)$ which we do by finding an upper bound on the normed difference

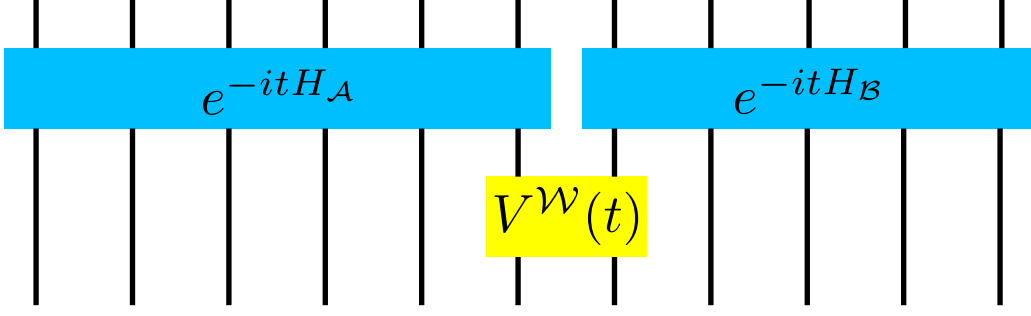


Figure 2.3: Bipartition of the propagator e^{-itH} into two halves (e^{-itH_A} and e^{-itH_B}) corrected by the patching-up operator $V^W(t)$ which bridges the gap between the two partitions (and compensates for the missing local interaction).

between these two operators, $\|V(t) - V^W(t)\|$ [82]. The Lie-Trotter expansion allows us to write

$$V(t) = \lim_{r \rightarrow \infty} \prod_{j=0}^{r-1} e^{-ih_m(\frac{jt}{r})\frac{j}{r}} \quad \text{and} \quad V^W(t) = \lim_{r \rightarrow \infty} \prod_{j=0}^{r-1} e^{-ih_m^W(\frac{jt}{r})\frac{j}{r}} \quad (2.22)$$

These Lie-Trotter expansions coupled with the differential equations (2.20) and (2.21) allow us [61] to bound

$$\|V(t) - V^W(t)\| \leq \int_0^t \|h_m(s) - h_m^W(s)\| ds \quad (2.23)$$

Calculating $\|h_m(t) - h_m^W(t)\|$ is a lengthy but simple task which we summarise below. We begin by noting that after application of the Jordan-Wigner transform the local interactions have the form $h_m = 2J_m(a_m^\dagger a_{m+1} + a_{m+1}^\dagger a_m)$ and so we must bound terms such as $\|a_m^\dagger(t)a_{m+1}(t) - (a_m^\dagger(t)a_{m+1}(t))|_W\|$. Recalling equation (2.12) which gives the Heisenberg picture fermionic annihilation operator in terms of the correlation functions, $a_m(t) = \sum_{k=1}^N c_{mk}(t)a_k$, allows us to conclude that

$$\left(a_j^\dagger(t)a_k(t)\right)|_W = a_j^\dagger(t)|_W a_k(t)|_W = a_j^{\dagger W}(t)a_k^W(t) \quad (2.24)$$

(where $a|_{\mathcal{W}}$ is the restriction of a onto \mathcal{W}) which in turn allows us to bound

$$\begin{aligned}
\|a_m^\dagger(t)a_{m+1}(t) - a_m^{\mathcal{W}\dagger}(t)a_{m+1}^{\mathcal{W}}(t)\| &= \|a_m^\dagger(t)a_{m+1}(t) - a_m^\dagger(t)a_{m+1}^{\mathcal{W}}(t) \\
&\quad + a_m^\dagger(t)a_{m+1}^{\mathcal{W}}(t) - a_m^{\mathcal{W}\dagger}(t)a_{m+1}^{\mathcal{W}}(t)\| \\
&\leq \|a_m^\dagger(t)\| \|a_{m+1}(t) - a_{m+1}^{\mathcal{W}}(t)\| \\
&\quad + \|a_m^\dagger(t) - a_m^{\mathcal{W}\dagger}(t)\| \|a_{m+1}^{\mathcal{W}}(t)\|
\end{aligned} \tag{2.25}$$

Since $a_m^{\mathcal{W}}(t)$ is a restriction of $a_m(t)$ we know that $\|a_m^{\mathcal{W}}(t)\| \leq \|a_m(t)\|$ (as $\|P_{\mathcal{W}}\| \leq 1$) and the Gersgorin disk theorem [80] coupled with the unitary equivalence of the operator norm allows us to calculate $\|a_m(t)\| \leq 1$.

To find an expression for the truncated operator $a_m^{\mathcal{W}}(t)$ we simply truncate the expression for $a_m(t)$ to the sites in \mathcal{W} to obtain: $a_m^{\mathcal{W}}(t) = \sum_{k \in \mathcal{W}} c_{mk}(t)a_k$. We may now conclude our argument by bounding the terms on the right hand side of equation (2.25), for example

$$\|a_{m+1}(t) - a_{m+1}^{\mathcal{W}}(t)\| \leq \left\| \sum_{k \notin \mathcal{W}} c_{mk}(t)a_k \right\| \leq \sum_{k \notin \mathcal{W}} c_\zeta e^{-|m-k|^\zeta} \tag{2.26}$$

Since \mathcal{W} is a set centred on the boundary between the partitions we know that $|m - k| \geq |\mathcal{W}|/2$ for all $k \notin \mathcal{W}$, which simplifies the above bound to

$$\|a_{m+1}(t) - a_{m+1}^{\mathcal{W}}(t)\| \leq c_\zeta(N - |\mathcal{W}|)e^{-\left(\frac{|\mathcal{W}|}{2}\right)^\zeta} \tag{2.27}$$

Finally, we may now bound the normed difference between the patching-up operator $V(t)$ and its approximation $V^{\mathcal{W}}(t)$ by substituting the bound $\|h_m(t) - h_m^{\mathcal{W}}(t)\| \leq 8|J_m|c_\zeta N e^{-\left(\frac{|\mathcal{W}|}{2}\right)^\zeta}$ into equation (2.23) to obtain

$$\|V(t) - V^{\mathcal{W}}(t)\| \leq c_\zeta \kappa t N e^{-\left(\frac{|\mathcal{W}|}{2}\right)^\zeta} \tag{2.28}$$

where $\kappa \leq \sup_j 8|J_j|$ is a constant. In particular, for any $\varepsilon \geq 0$ choosing \mathcal{W} to be a large enough set $\left(\frac{|\mathcal{W}|}{2}\right)^\zeta \geq \log(c_\zeta \kappa t N / \varepsilon)$ ensures that $V^{\mathcal{W}}(t)$ approximates $V(t)$ to within an error of at most ε

$$\|V(t) - V^{\mathcal{W}}(t)\| \leq \varepsilon \tag{2.29}$$

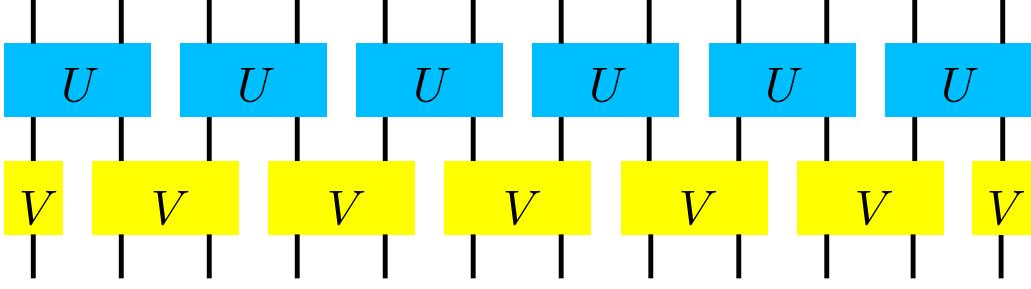


Figure 2.4: Complete decomposition of the propagator e^{-itH} into multiple blocks (each with its own propagator U) and the corresponding array of patching-up operators V , each of which bridges the gap between a pair of blocks and compensates for a missing local interaction.

We may therefore conclude that the propagator e^{-itH} may be approximated by the bipartitioning and patching-up procedure to within a known error

$$e^{-itH} = V^{\mathcal{W}}(t)e^{-it(H_A+H_B)} + \mathcal{O}\left(c_\zeta \kappa t N e^{-(\frac{|\mathcal{W}|}{2})^\zeta}\right) \quad (2.30)$$

Having discovered that the partitioning and patching-up procedure allows a good approximation of the propagator, we recursively apply this procedure to obtain a quantum cellular automata decomposition of the propagator [61] (see figure 2.4). Algebraically speaking, this decomposition can be written as

$$e^{-itH} = \left(\bigotimes_{k=0}^{N/|\mathcal{W}|} V^{\mathcal{W}'_k}(t) \right) \left(\bigotimes_{j=1}^{N/|\mathcal{W}|} e^{-itH_{\mathcal{W}_j}} \right) + \mathcal{O}\left(\frac{c_\zeta \kappa t N^2}{|\mathcal{W}|} e^{-(\frac{|\mathcal{W}|}{2})^\zeta}\right) \quad (2.31)$$

where $\mathcal{P}_1 = \{\mathcal{W}_j\}$ is a partition of the chain into $\frac{N}{|\mathcal{W}|}$ blocks (each of size $|\mathcal{W}|$) and $\mathcal{P}_2 = \{\mathcal{W}'_k\}$ is a partition of the chain into $\frac{N}{|\mathcal{W}|} + 1$ blocks obtained by shifting partition \mathcal{P}_1 along by $\frac{|\mathcal{W}|}{2}$ sites (each block is of size $|\mathcal{W}|$ excepting the blocks at each end which are each half this size). Note that we accumulate an error term for each partition we introduce and this is reflected in the increased size of the error term in equation (2.31).

2.6 A Lieb-Robinson bound for static disorder

In the previous section we saw how to repeatedly partition the chain in order to approximate the propagator (up to a known error, see equation (2.31))

and we now use this result to provide a new Lieb-Robinson bound for our noisy spin chain model which is far tighter than the original Lieb-Robinson bound. The original bound is extremely general and applies to a wide range of quantum lattice systems, in contrast with our new bound which applies only to the model studied in this chapter.

We define an operator $Q(t)$ to be the decomposition of the propagator we derived in the previous section, namely

$$Q(t) = \left(\bigotimes_{k=0}^{N/|\mathcal{W}|} V^{\mathcal{W}'_k}(t) \right) \left(\bigotimes_{j=1}^{N/|\mathcal{W}|} e^{-itH_{\mathcal{W}_j}} \right) \quad (2.32)$$

We begin by approximating the Heisenberg picture time evolution of the operator B by evolving according to the approximated propagator $Q(t)$ instead of the actual propagator e^{-itH}

$$B(t) = \tilde{B}(t) + \mathcal{O}(\epsilon) \quad (2.33)$$

where $\epsilon = \frac{c_\zeta \kappa t N^2}{|\mathcal{W}|} e^{-(\frac{|\mathcal{W}|}{2})^\zeta}$ and the approximation $\tilde{B}(t) = Q^\dagger(t) B Q(t)$ acts trivially on all sites which are a distance of more than $\frac{3}{2}|\mathcal{W}|$ away from those on which B acts. Therefore if the distance between the sites on which A and B act is large enough, $d(A, B) \geq \frac{3}{2}|\mathcal{W}|$, then A commutes with \tilde{B} (that is, $[A, \tilde{B}(t)] = 0$) and so

$$\|[A, B(t)]\| = \|[A, \mathcal{O}(\epsilon)]\| \leq 2\|A\| \|\mathcal{O}(\epsilon)\| \leq \frac{2\|A\| c_\zeta \kappa t N^2}{|\mathcal{W}|} e^{-(\frac{|\mathcal{W}|}{2})^\zeta} \quad (2.34)$$

It is now clear that the tightest bound on the Lieb-Robinson commutator is achieved by picking $|\mathcal{W}|$ to be as large as possible whilst still satisfying $d(A, B) \geq \frac{3}{2}|\mathcal{W}|$. Making the choice $|\mathcal{W}| = \frac{2}{3}d(A, B)$ and defining the constant $c'_\zeta = 3\|A\| c_\zeta \kappa$ gives us the Lieb-Robinson bound

$$\|[A, B(t)]\| \leq \frac{c'_\zeta t N^2}{|d(A, B)|} e^{-(\frac{d(A, B)}{3})^\zeta} \quad \text{for any } 0 < \zeta < 1 \quad (2.35)$$

This is the main result of this chapter.

2.7 Discussion

We have derived a new Lieb-Robinson bound for our noisy spin chain with static on-site disorder, but what are the consequences? Where the original Lieb-Robinson bound gave rise to a linear light cone, our new bound gives rise to a light cone whose radius grows logarithmically with time.

To derive the form of these light cones, we fix the value of the right-hand-side of the relevant bound and examine the possible values of t and $d(A, B)$ which are consistent with this value. Recalling the original Lieb-Robinson bound, $\|[A, B(t)]\| \leq \text{const} \times e^{k_1 t - k_2 d(A, B)}$ (with k_1 and k_2 constants), we see that $d(A, B)$ must grow in proportion to t for the right-hand side to remain constant (and so the radius of the light cone grows linearly with time). For our new bound, $\|[A, B(t)]\| \leq \text{const} \times t e^{-k d(A, B)^\zeta}$ (with k a constant), we see that $d(A, B)^\zeta$ must grow in proportion to the logarithm of t for the right-hand-side to remain constant (and so the radius of the light cone grows logarithmically with time).

The logarithmic light cone is problematic if we wish to use this spin chain as a quantum channel for transmitting quantum information between Alice and Bob (see chapter 1 for examples of how one might achieve this). Even if the protocol used guarantees perfect state transfer in the noise-free scenario, it will almost certainly fail in the presence of noise if we take into account the realistic situation where the spin chain is not perfectly isolated from the surrounding environment. Our bound shows that non-negligible amounts of information can reach Bob only when the time elapsed is exponential in the distance between Alice and Bob. In practice this exponentially long waiting time will lead to decoherence (due to interactions between the spin chain and the environment) which will corrupt the quantum information.

There is however partial mitigation of this bad news: in some situations it is possible to communicate beyond the localisation length [83] of our model.

This is done by performing error correction [5, 84, 85, 86, 87] at short intervals along the chain. Unfortunately there are some drawbacks to this solution; for example, it requires controllable access to the interior of the spin chain (which may not be possible in a practical situation).

It is worth noting that several other studies [88, 89, 90, 91, 92] have used tDMRG (time-dependent density matrix renormalisation group) to simulate the time evolution of our model and they numerically confirm our findings.

We have only examined one very specific model (the XX -model) of a noisy spin chain but multiscale analysis [76, 75] can be used to prove similar levels of localisation for many other models.

Chapter 3

Dynamic disorder: fluctuating field strengths

In chapter 2 we studied an XX -model spin chain with static on-site disorder. In this chapter we study the same model with a fluctuating disordered field oriented in the z -direction, a setting which is ubiquitous in the condensed matter context (see for example [93, 94, 95]). We abandon the Lieb-Robinson bound in favour of bounds on various correlation functions which are easier to calculate in this setting, discussing the reasons why a full Lieb-Robinson bound is unattainable in section 3.6.

This chapter is organised as follows. We describe the model in detail (§ 3.1) before deriving a master equation (§ 3.2) which governs the evolution of the system. We then define and calculate some ensemble averaged correlation functions (§ 3.3) and some ensemble variance correlation functions (§ 3.4) before analysing the results (§ 3.5). We finish with a short discussion (§ 3.6).

3.1 The dynamic disorder model with fixed field direction

In this chapter we study the XX -model on either a finite spin ring (with N sites) or an infinite spin chain. In either case, the model experiences on-site disorder which is fixed in the z -direction but whose field strengths $\xi_j(t)$

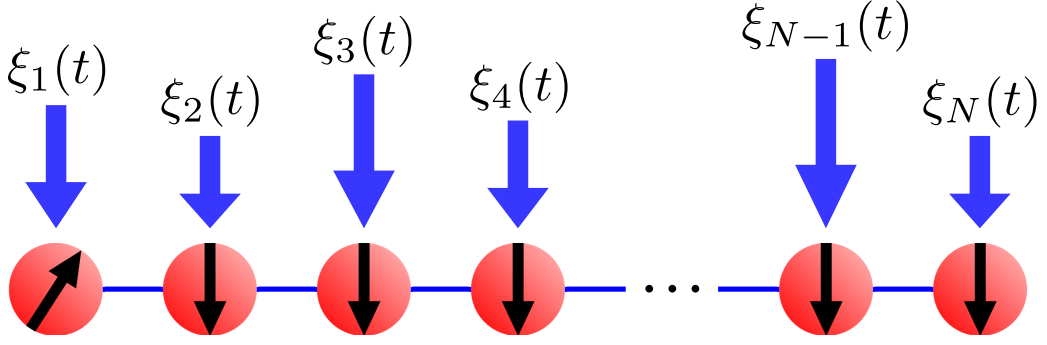


Figure 3.1: A spin chain with on-site disorder which fluctuates in the z -direction: $\xi_j(t)$ are the field strengths which fluctuate in time independently on each site.

fluctuate in time independently on each site (see figure 3.1). The Hamiltonian of the system is therefore

$$H(\xi, t) = \sum_j J_j (\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y) + \sum_j \xi_j(t) \sigma_j^z \quad (3.1)$$

Throughout this chapter and the next, we use ξ (with no subscript) to remind ourselves that the object in question (in this case the Hamiltonian $H(\xi, t)$) is dependent upon the disorder.

At each instant in time the field strengths $\xi_j(t)$ are independent, identically distributed random variables drawn from a probability distribution \mathbb{P}_ξ which has a finite second moment.

This well-studied model [96, 97, 98, 99] is a *physical* one: it is physically realisable as a continuously monitored dynamical process which can be simulated to an arbitrary level of precision by engineering a quantum system whereby we sequentially interact N quantum spins (which are evolving according to a particular spin-chain Hamiltonian — in our case the XX -model Hamiltonian) with a collection of harmonic oscillators, one after the other. This is the well-understood interaction of Caves and Milburn [100] and this forms the basis of the derivation of the master equation which follows in section 3.2. In particular, at each time step we interact each spin with its own harmonic oscillator with a strength that is proportional to the square root of the time step; this

process is well known to converge as the time step goes to zero to the so-called *continuous measurement* process.

In what follows we always intend the word *fluctuations* to mean fluctuations arising from measurement back-action, and do not mean fluctuations arising from interactions with a thermal bath at finite temperature. Indeed, we model the *non-equilibrium* noise induced by repeated measurement (an active process) and it should be noted that our spin chain is not in equilibrium with its environment.

3.2 A master equation for dynamic disorder

By solving the Schrödinger equation we see that the propagator is the time-ordered exponential

$$U(\xi, t) = \mathcal{T} \exp \left(-i \int_0^t H(\xi, s) ds \right) \quad (3.2)$$

In the static disorder model of chapter 2, the Hamiltonian was time-independent and so this time ordered exponential reduced to e^{-itH} and, courtesy of Anderson localisation, we were able to directly calculate bounds on the matrix elements of this propagator (which we called correlation functions). Unfortunately the model in this chapter has a time-dependent Hamiltonian and so $U(\xi, t)$ cannot be calculated in the same manner. We do not know the exact details of the noise (we only know the probability distribution from which the field strengths are drawn at each instant in time) and so we cannot calculate $U(\xi, t)$ exactly; neither can we directly calculate bounds on it. However, as we show below, it is possible to derive a master equation which governs the dynamics of our model and allows us to calculate the correlation functions we require to establish the localisation properties of this model.

After free evolution for a time t an initial state $\rho(0)$ of the spin chain becomes

$$\rho(\xi, t) = U(\xi, t) \rho(0) U^\dagger(\xi, t) \quad (3.3)$$

Clearly the state of the system at time t is dependent upon the path taken by the random field strengths $\xi_j(s)$ ($0 \leq s \leq t$) of which we have no specific knowledge except the probability distribution from which they were drawn. We are therefore motivated to take an ensemble average over all possible realisations of random field strengths, which we denote by \mathbb{E}_ξ .

By closely following and generalising the method laid down by Caves and Milburn [100], we derive a master equation for the ensemble averaged density operator

$$\rho(t) \equiv \mathbb{E}_\xi \rho(\xi, t) \quad (3.4)$$

By applying the Jordan-Wigner transformation we find that (up to an irrelevant constant) the Pauli spin operator σ_j^z is mapped to the fermionic *number operator*, $n_j = a_j^\dagger a_j$. Note that n_j is an operator which essentially performs a measurement of the excitation number of site j (it measures whether or not there is a fermion on lattice site j). As this measurement outcome is neither observed nor stored in any way, we see that the transverse field is effectively performing a continuous variable-strength non-selective measurement on the system. We model this process as a sequence of instantaneous measurements taking place at times $t_r = r\tau$ where τ is the time interval between measurements. Ultimately we will take the continuous measurement limit ($\tau \rightarrow 0$) in such a way as to avoid the infinite strength quantum Zeno effect. In the derivation that follows it does not matter whether we use σ_j^z or n_j , so we choose to work with the former as it simplifies the generalisation of this derivation to the model in chapter 4.

In our measurement model the system is coupled to a series of ancilla systems called *meters*. The meter for the r th measurement on site j has canonical variables \bar{x}_{jr} and \bar{p}_{jr} which satisfy the canonical commutation relations $[\bar{x}_{jr}, \bar{p}_{ks}] = i\delta_{jk}\delta_{rs}$. (Throughout this derivation we use the bar to denote an operator or measurement outcome on the meters.) We assume that the intrinsic Hamiltonian for each meter is the identity and can therefore be ignored.

The total Hamiltonian for our measurement model consisting of both the spin chain and the ancillary meters is therefore

$$H(\xi, t) = H_0 + \sum_j \sum_{r=1}^M \delta(t - r\tau) \sigma_j^z \xi_{jr} \bar{p}_{jr} \quad (3.5)$$

where $0 \leq t \leq M\tau$, H_0 is the intrinsic system Hamiltonian and $\xi_{jr} = \xi_j(t_r)$ is the field strength on site j at time t_r . We further assume that each of the measurements has an identical description and that each meter is prepared initially in a Gaussian state of width s

$$|G_{jr}\rangle = (\pi s)^{-1/4} \int e^{-\bar{x}^2/2s} |\bar{x}\rangle d\bar{x} \quad (3.6)$$

To simplify the notation in what follows we define

$$|G_r\rangle = \otimes_j |G_{jr}\rangle \quad (3.7)$$

That is, $|G_r\rangle$ is the joint state of all the meters involved in the r th set of measurements (which take place at time t_r).

If we let $\rho(\xi, t_r-)$ be the state of the system immediately prior to the r th set of measurements, then the joint state of the system and the r th set of meters immediately after the measurement interaction is

$$\tilde{\rho}(\xi, t_r+) = e^{-i\sum_j \sigma_j^z \xi_{jr} \bar{p}_{jr}} (|G_r\rangle \langle G_r| \otimes \rho(t_r-, \xi)) e^{i\sum_j \sigma_j^z \xi_{jr} \bar{p}_{jr}} \quad (3.8)$$

We define the following Gaussian operator on the chain

$$G(\xi, \bar{\mathbf{x}}_r) = \langle \bar{\mathbf{x}}_r | e^{-i\sum_j \sigma_j^z \xi_{jr} \bar{p}_{jr}} | G_r \rangle \quad (3.9)$$

If we assume that the measurement outcome on the r th set of meters is $\bar{\mathbf{x}}_r = \{\bar{x}_{jr}\}_{j \in I}$ (where $I = \{1, \dots, N\}$ for the finite ring and $I = \mathbb{Z}$ for the infinite chain) then we see that the state of the system immediately after the r th measurement is

$$\rho(\xi, t_r+, \bar{\mathbf{x}}_r) = \frac{\langle \bar{\mathbf{x}}_r | \tilde{\rho}(\xi, t_r+) | \bar{\mathbf{x}}_r \rangle}{\mathbb{P}(\bar{\mathbf{x}}_r | \xi)} = \frac{G(\xi, \bar{\mathbf{x}}_r) \rho(\xi, t_r-) G^\dagger(\xi, \bar{\mathbf{x}}_r)}{\mathbb{P}(\bar{\mathbf{x}}_r | \xi)} \quad (3.10)$$

where the probability of outcome $\bar{\mathbf{x}}_r$ from the r th set of measurements is

$$\mathbb{P}(\bar{\mathbf{x}}_r|\xi) = \text{tr}(|\bar{\mathbf{x}}_r\rangle\langle\bar{\mathbf{x}}_r|\tilde{\rho}(\xi, t_r+)) = \text{tr}(G^\dagger(\xi, \bar{\mathbf{x}}_r)G(\xi, \bar{\mathbf{x}}_r)\rho(\xi, t_r-)) \quad (3.11)$$

Now, between measurement interactions the system evolves freely under the intrinsic system Hamiltonian H_0 with corresponding propagator

$$U(\tau) = e^{-iH_0\tau} \quad (3.12)$$

If the measurement results from the first M sets of measurements form a sequence $\bar{\mathbf{x}} = \{\bar{\mathbf{x}}_r\}_{r=1}^M$ then the state of the system immediately after the M th measurement (at time $t_M = M\tau$) is

$$\rho(\xi, t_M+, \bar{\mathbf{x}}) = \frac{V(\xi, t_M, \bar{\mathbf{x}})\rho(0)V^\dagger(\xi, t_M, \bar{\mathbf{x}})}{\mathbb{P}(\bar{\mathbf{x}}|\xi)} \quad (3.13)$$

where $V(\xi, t_M, \bar{\mathbf{x}}) = \prod_{r=M}^1 (G(\xi, \bar{\mathbf{x}}_r)U(\tau))$.

Because the measurements are performed by the on-site disordered field we do not learn what their outcomes are (they are non-selective measurements) and so at time t the system is in the corresponding non-selective state, namely

$$\rho(\xi, t_M+) = \int \rho(\xi, t_M+, \bar{\mathbf{x}})\mathbb{P}(\bar{\mathbf{x}}|\xi)d\bar{\mathbf{x}} \quad (3.14)$$

Whilst this state is the final non-selective state of our system it nevertheless depends on the particular realisation of the disorder which has occurred throughout its evolution. We now take an ensemble average over all possible realisations of the disorder to obtain the ensemble averaged state, namely

$$\rho(t_M+) = \mathbb{E}_\xi \rho(\xi, t_M+) \quad (3.15)$$

We are now in a position to form a differential equation for $\rho(t)$ which we do by taking the continuous limit (*i.e.*, $M \rightarrow \infty$, $\tau \rightarrow 0$ and $D := s\tau = \text{const}$)

$$\frac{d}{dt}\rho(t) = \lim_{\tau \rightarrow 0} \frac{\rho(t_M+) - \rho(t_{M-1}+)}{\tau} \quad (3.16)$$

Note that we require D to be a finite constant: if this were not the case we would either have $D \rightarrow 0$ in which case we would encounter the infinite

strength quantum Zeno effect and the measurement model would inhibit any evolution of the system; or we would have $D \rightarrow \infty$ in which case we would be performing an infinitely weak measurement which is equivalent to turning off the on-site disorder term in the system Hamiltonian (3.1).

One should think of D as a physical parameter which is defined by the experimental setup. Technically speaking, we have modelled a continuous process by a series of very frequent measurements on the system (performed by the fluctuating field), which is approximated by a continuous measurement process. This approximation is valid provided all time scales of interest are larger than the typical time scale of the fluctuations in the on-site disorder (see [100] for further discussion). For example, if Alice and Bob were to use our system to attempt quantum communication, we would require the typical fluctuation time for the on-site disorder to be somewhat shorter than the time required to perform the quantum gates necessary for Alice to input and for Bob to extract the quantum information.

Utilising equations (3.13), (3.14) and (3.15) we may rewrite $\frac{d\rho}{dt}$ (equation (3.16)) as

$$\begin{aligned} \frac{d}{dt}\rho(t) = \mathbb{E}_\xi \lim_{\tau \rightarrow 0} \frac{1}{\tau} & \left(-\rho(\xi, t_{M-1}+) + \right. \\ & \left. \int G(\xi, \bar{\mathbf{x}}_M) U(\tau) \rho(\xi, t_{M-1}+) U^\dagger(\tau) G^\dagger(\xi, \bar{\mathbf{x}}_M) d\bar{\mathbf{x}}_M \right) \end{aligned} \quad (3.17)$$

We note the following Gaussian integral identity which is proved in section 3.2.1:

$$\int G(\xi, \bar{\mathbf{x}}_r) A G^\dagger(\xi, \bar{\mathbf{x}}_r) d\bar{\mathbf{x}}_r = A - \frac{1}{4s} \sum_j \xi_{jr}^2 [\sigma_j^z, [\sigma_j^z, A]] + \mathcal{O}\left(\frac{1}{s^2}\right) \quad (3.18)$$

Setting $A = U(\tau)\rho(t_{M-1}+, \xi)U^\dagger(\tau)$ in the Gaussian identity, and making use of the small τ limit $U(\tau) = 1 - iH_0\tau + \mathcal{O}(\tau^2)$, we derive a master equation for our spin chain with fluctuating noise

$$\frac{d}{dt}\rho(t) = -i[H_0, \rho(t)] - \gamma \sum_j [\sigma_j^z, [\sigma_j^z, \rho(t)]] \quad (3.19)$$

where $\gamma = \mathbb{E}(\xi_j^2)/4D$ characterises the strength of the disorder: $\mathbb{E}(\xi_j^2)$ is the second moment of the probability distribution \mathbb{P}_ξ from which the transverse field strengths are drawn; large γ corresponds to strong disorder whilst small γ corresponds to weak disorder.

3.2.1 Deriving the Gaussian identity

This brief technical section proves the Gaussian integral identity of equation (3.18) that was used in the derivation of the master equation (3.19).

We begin our proof by defining the Gaussian integral

$$I_r(A) = \int G(\xi, \bar{\mathbf{x}}_r) A G^\dagger(\xi, \bar{\mathbf{x}}_r) d\bar{\mathbf{x}}_r \quad (3.20)$$

The relation $e^{-i\lambda\bar{p}}|\bar{x}\rangle = |\bar{x} + \lambda\rangle$ allows us to rewrite $G(\xi, \bar{\mathbf{x}}_r)$ as

$$\begin{aligned} G(\xi, \bar{\mathbf{x}}_r) &= \langle \bar{\mathbf{x}}_r | e^{-i\sum_j \sigma_j^z \xi_{jr} \bar{p}_{jr}} | G_r \rangle \\ &= \prod_j \langle \bar{x}_{jr} | e^{-i\sigma_j^z \xi_{jr} \bar{p}_{jr}} | G_{jr} \rangle \\ &= \prod_j (\pi s)^{-1/4} \langle \bar{x}_{jr} | \int e^{-\bar{x}^2/2s} |\bar{x} + \sigma_j^z \xi_{jr}\rangle d\bar{x} \\ &= \prod_j (\pi s)^{-1/4} \langle \bar{x}_{jr} | \int e^{-(\bar{x} - \sigma_j^z \xi_{jr})^2/2s} |\bar{x}\rangle d\bar{x} \\ &= \prod_j (\pi s)^{-1/4} e^{-(\bar{x}_{jr} - \sigma_j^z \xi_{jr})^2/2s} \end{aligned} \quad (3.21)$$

which in turn allows us to rewrite the Gaussian integral as

$$I_r(A) = \int e^{-\sum_j (\bar{x}_{jr} - \sigma_j^z \xi_{jr})^2/2s} A e^{-\sum_j (\bar{x}_{jr} - \sigma_j^z \xi_{jr})^2/2s} \left(\prod_j \frac{d\bar{x}_{jr}}{\sqrt{\pi s}} \right) \quad (3.22)$$

We expand the exponentials as infinite sums to obtain

$$\begin{aligned} I_r(A) &= \sum_{\alpha, \beta=0}^{\infty} \int \left(\prod_j \frac{d\bar{x}_{jr}}{\sqrt{\pi s}} \right) \frac{e^{-\sum_j (\bar{x}_{jr}^2)/s}}{\alpha! \beta! (2s)^{\alpha+\beta}} \\ &\quad \times \left(\sum_j 2\bar{x}_{jr} \xi_{jr} \sigma_j^z - (\xi_{jr} \sigma_j^z)^2 \right)^\alpha A \\ &\quad \times \left(\sum_j 2\bar{x}_{jr} \xi_{jr} \sigma_j^z - (\xi_{jr} \sigma_j^z)^2 \right)^\beta \end{aligned} \quad (3.23)$$

Working through this integral term by term allows us to conclude that

$$I_r(A) = A - \frac{1}{4s} \sum_j \xi_{jr}^2 [\sigma_j^z, [\sigma_j^z, A]] + \mathcal{O}\left(\frac{1}{s^2}\right) \quad (3.24)$$

as claimed.

3.3 Correlation functions

In order to calculate the dynamics of our fluctuating disorder model we apply the Jordan-Wigner transform (as we did for our static disorder model); this transforms the system Hamiltonian into

$$H(\xi, t) = \sum_j \left(a_j^\dagger a_{j+1} + a_{j+1}^\dagger a_j + \xi_j(t) a_j^\dagger a_j \right) \quad (3.25)$$

where the sum is over a finite ring or an infinite chain. If the sum is over a finite ring consisting of N sites we impose periodic boundary conditions by identifying site $N + 1$ with site 1: $|N + 1\rangle \equiv |1\rangle$ and $\xi_{N+1}(t) \equiv \xi_1(t)$. Applying Jordan-Wigner to the master equation (3.19) results in

$$\frac{d}{dt} \rho(t) = -i[H_0, \rho(t)] - \gamma \sum_j [a_j^\dagger a_j, [a_j^\dagger a_j, \rho(t)]] \quad (3.26)$$

where $H_0 = \sum_j (a_j^\dagger a_{j+1} + a_{j+1}^\dagger a_j)$ is the intrinsic Hamiltonian and γ quantifies the strength of the disorder.

Now, we are interested in the propagation of information through this system and (as the particles are the mediators of information propagation) one way of studying this is to find the probability amplitude for a single particle to propagate from site k to site j when the system evolves freely for a time t

$$c_{jk}(\xi, t) = \langle \Omega | a_j U(\xi, t) a_k^\dagger | \Omega \rangle \quad (3.27)$$

We call such a quantity a *correlation function* and we also define an *ensemble averaged correlation function*

$$c_{jk}(t) = \mathbb{E}_\xi c_{jk}(\xi, t) \quad (3.28)$$

Noting that the vacuum $|\Omega\rangle$ is an eigenstate of the Hamiltonian (3.25) with eigenvalue zero (and it is therefore an eigenstate of the propagator $U(\xi, t)$ with eigenvalue one) enables us to rewrite the correlation functions as

$$c_{jk}(t) = \mathbb{E}_\xi \langle \Omega | a_j(\xi, t) a_k^\dagger | \Omega \rangle \quad (3.29)$$

where $a_j(\xi, t) = U^\dagger(\xi, t)a_jU(\xi, t)$ is the annihilation operator in the Heisenberg picture. We define the ensemble averaged annihilation operator in the Heisenberg picture to be $a_j(t) = \mathbb{E}_\xi a_j(\xi, t)$, which allows us to simplify the expression for the ensemble averaged correlation function: $c_{jk}(t) = \langle \Omega | a_j(t)a_k^\dagger | \Omega \rangle$.

In chapter 2 we used a Bogoliubov transformation to calculate $a(t)$ in terms of the correlation functions which, courtesy of Anderson localisation, we were able to bound. With the fluctuating disorder model studied here we have no way of bounding the correlation functions directly and so we follow an alternative route making use of the master equation (3.26). To do this we define

$$d_j^{(\rho)}(t) = \langle a_j(t) \rangle_\rho = \text{tr}(a_j \rho(t)) \quad (3.30)$$

Using the master equation we are able to form a differential equation for $d_j^{(\rho)}(t)$

$$\begin{aligned} \frac{d}{dt}d_j^{(\rho)}(t) = \text{tr}\left(a_j \frac{d}{dt}\rho(t)\right) &= -i \sum_r \text{tr}\left(\rho(t) \left[a_j, a_r^\dagger a_{r+1} + a_{r+1}^\dagger a_r\right]\right) \\ &\quad - \gamma \sum_r \text{tr}\left(\rho(t) \left[a_r^\dagger a_r, [a_r^\dagger a_r, a_j]\right]\right) \\ &= -i \left(d_{j+1}^{(\rho)}(t) + d_{j-1}^{(\rho)}(t)\right) - \gamma d_j^{(\rho)}(t) \end{aligned} \quad (3.31)$$

where we have used the cyclic rule of trace ($\text{tr} ABC = \text{tr} BCA$) several times. Since equation (3.31) holds for all choices of ρ we infer that a similar equation holds for the ensemble averaged annihilation operator

$$\frac{d}{dt}a_j(t) = -i(a_{j+1}(t) + a_{j-1}(t)) - \gamma a_j(t) \quad (3.32)$$

By defining the vector $\mathbf{a}(t) = (a_j(t))$ we rewrite the above equation as a vector differential equation

$$\frac{d}{dt}\mathbf{a}(t) = (-iR - \gamma I)\mathbf{a}(t) \quad (3.33)$$

where I is the identity matrix and $R = \sum_j (|j\rangle \langle j+1| + |j+1\rangle \langle j|)$ is the matrix corresponding to the intrinsic Hamiltonian H_0 after the Jordan-Wigner transformation has been applied. The vector differential equation (3.32) has solution

$$\mathbf{a}(t) = e^{(-iR - \gamma I)t} \mathbf{a}(0) = e^{-\gamma t} e^{-iRt} \mathbf{a}(0) \quad (3.34)$$

whose component form is

$$a_j(t) = e^{-\gamma t} \sum_k \langle j | e^{-iRt} | k \rangle a_k \quad (3.35)$$

Substituting this solution into equation (3.29) yields an expression for the ensemble averaged correlation function, namely

$$c_{jk}(t) = e^{-\gamma t} \sum_r \langle j | e^{-iRt} | r \rangle \langle \Omega | a_r a_k^\dagger | \Omega \rangle = e^{-\gamma t} \langle j | e^{-iRt} | k \rangle \quad (3.36)$$

We leave the analysis of this result to section 3.5.1.

3.4 Ensemble variances

The ensemble averaged correlation functions of the previous section define the average-case dynamics of our system, but one must be careful with their interpretation: it may be tempting to assume that the dynamics of our model are always close to the averaged dynamics, but this may not be the case. One must remember that $c_{jk}(t)$ is an ensemble average over the disorder-dependent correlation functions $c_{jk}(\xi, t)$. It is therefore perfectly possible that some realisations of the disorder will lead to dynamics which stray significantly from those represented by the ensemble averaged correlation functions and in section 3.5.2 we will see that this is indeed the case for our model.

In order to quantify how far the dynamics may stray from the average-case dynamics, we must look at the difference between the ensemble averaged correlation functions and the disorder-dependent correlation functions, $|c_{jk}(t) - c_{jk}(\xi, t)|$. We do this by studying ensemble variances

$$v_{jk}(t) = \text{Var}_\xi c_{jk}(\xi, t) = \mathbb{E}_\xi |c_{jk}(\xi, t)|^2 - |\mathbb{E}_\xi c_{jk}(\xi, t)|^2 \quad (3.37)$$

Clearly the second term on the right hand side of equation (3.37) is just the square of the ensemble averaged correlation function which we have already found, so we now move on to study the first term which we do by introducing

a new quantity

$$w_{ljk}(t) = \mathbb{E}_\xi c_{jk}(\xi, t) c_{lk}^*(\xi, t) \quad (3.38)$$

where c^* denotes the complex conjugate of c . Whilst we are only interested in the quantities $w_{jjk}(t)$, the dynamics of our system couples these quantities to quantities such as $w_{ljk}(t)$ with $l \neq j$. We note that $w_{ljk}(t)$ can be written as

$$w_{ljk}(t) = \mathbb{E}_\xi \text{tr}(\pi_{lj} \pi_{kk}(\xi, t)) = \text{tr}(\pi_{lj} \pi_{kk}(t)) \quad (3.39)$$

where $\pi_{jk} = a_j^\dagger |\Omega\rangle \langle \Omega| a_k$ is a projector, $\pi_{jk}(\xi, t) = U(\xi, t) \pi_{jk} U^\dagger(\xi, t)$ is the disorder-dependent time-evolved projector and $\pi_{jk}(t) = \mathbb{E}_\xi \pi_{jk}(\xi, t)$ is an ensemble averaged projector. We use the master equation (3.26) to form a differential equation for $w_{ljk}(t)$

$$\begin{aligned} \frac{d}{dt} w_{ljk}(t) = & -i(w_{l,j+1,k}(t) + w_{l,j-1,k}(t) - w_{l+1,jk}(t) - w_{l-1,jk}(t)) \\ & - 2\gamma(w_{ljk}(t) - \delta_{jl} w_{jjk}(t)) \end{aligned} \quad (3.40)$$

which we cast into a matrix equation by defining the matrices

$$W_k(t) = \sum_{j,l} w_{ljk}(t) |l\rangle \langle j| \quad (3.41)$$

Having defined these matrices we see that equation (3.40) becomes

$$\frac{d}{dt} W_k(t) = i[R, W_k(t)] - \gamma \mathcal{D}(W_k(t)) \quad (3.42)$$

where $\mathcal{D}(M) = 2(M - \text{diag}(M))$ is a dissipative operator which takes a general operator M and sets its diagonal entries to zero. The initial conditions of this equation are $W_k(0) = \pi_{kk}$ (that is, $W_k(0)$ is a matrix with all entries equal to zero excepting a 1 in the k th position on the diagonal).

Note that the matrix differential equation (3.42) is essentially the same as the master equation (3.26) for which there appears to be no analytic solution. (If there was an analytic solution we would solve for $\rho(t)$ directly and the correlation functions would be redundant.)

As we do not have an analytic solution for the master equation, we now present a method of bounding the diagonal elements of $W_k(t)$, and therefore the ensemble variance correlation functions $w_{jjk}(t)$ that we are trying to calculate. This method only works for the infinite chain (some steps are not valid for finite rings) and for this reason we restrict ourselves to working with an infinite chain for the remainder of this chapter.

We begin by defining the *squared position operator*

$$\hat{x}^2 = \sum_{j=-\infty}^{\infty} j^2 |j\rangle \langle j| \quad (3.43)$$

whose expectation is

$$\langle \hat{x}^2 \rangle(t) = \text{tr}(\hat{x}^2 \rho(t)) = \text{tr}(\hat{x}^2(t) \rho(0)) \quad (3.44)$$

where $\hat{x}^2(t) = \mathbb{E}_{\xi} U^{\dagger}(\xi, t) \hat{x}^2 U(\xi, t)$ is the ensemble average of the squared position operator in the Heisenberg picture. By repeating the steps used to derive the master equation, we find that $\hat{x}^2(t)$ obeys

$$\frac{d}{dt} \hat{x}^2(t) = i [R, \hat{x}^2(t)] - \gamma \mathcal{D}(\hat{x}^2(t)) \quad (3.45)$$

This is essentially the same as equation (3.42) which is satisfied by the variance matrices $W_k(t)$. In order to find bounds on the corresponding dynamics we must first define another set of matrices

$$\eta_{\alpha}^{\beta} = \sum_{j=-\infty}^{\infty} (j^{\alpha} |j\rangle \langle j + \beta| + (-1)^{\beta} j^{\alpha} |j + \beta\rangle \langle j|) \quad (3.46)$$

(Observe that $\eta_0^0 = 2I$ and $\eta_2^0 = 2\hat{x}^2$.) We now prove a small lemma which tells us that the dynamics we are interested in are spanned by just five of these η_{α}^{β} matrices.

Lemma 2. *The dynamics generated by equation (3.45) acting on the set $\mathcal{E} = \{\hat{x}^2, \mathbb{I}, \eta_0^1, \eta_1^1, \eta_0^2\}$ is closed. That is*

$$[R, \eta] \in \mathcal{E} \quad \text{and} \quad \mathcal{D}(\eta) \in \mathcal{E} \quad \text{for all } \eta \in \mathcal{E} \quad (3.47)$$

Proof. It is a simple matter to explicitly calculate the following

$$\begin{aligned}
[R, \hat{x}^2] &= 2\eta_1^1 + \eta_0^1 & \mathcal{D}(\hat{x}^2) &= 0 \\
[R, I] &= 0 & \mathcal{D}(I) &= 0 \\
[R, \eta_0^1] &= 0 & \mathcal{D}(\eta_0^1) &= 2\eta_0^1 \\
[R, \eta_1^1] &= \eta_0^2 - 2I & \mathcal{D}(\eta_1^1) &= 2\eta_1^1 \\
[R, \eta_0^2] &= 0 & \mathcal{D}(\eta_0^2) &= 2\eta_0^2
\end{aligned} \tag{3.48}$$

which completes the proof. \square

Note that this lemma does not hold for finite rings and it is for this reason that we are now restricted to working with infinite chains.

Since the dynamics we are interested in are spanned by the set \mathcal{E} , we define the following linear combination of the members of this set

$$\eta(t) = c_x(t)\hat{x}^2 + c_I(t)I + c_0^1(t)\eta_0^1 + c_1^1(t)\eta_1^1 + c_0^2(t)\eta_0^2 \tag{3.49}$$

Noting that $\eta(t)$ obeys a master equation similar to (3.45) we find that the coefficients obey the simultaneous differential equations

$$\frac{d}{dt} \begin{pmatrix} c_x(t) \\ c_I(t) \\ c_0^1(t) \\ c_1^1(t) \\ c_0^2(t) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2i & 0 \\ i & 0 & -2\gamma & 0 & 0 \\ 2i & 0 & 0 & -2\gamma & 0 \\ 0 & 0 & 0 & i & -2\gamma \end{pmatrix} \begin{pmatrix} c_x(t) \\ c_I(t) \\ c_0^1(t) \\ c_1^1(t) \\ c_0^2(t) \end{pmatrix} \tag{3.50}$$

We are interested in finding $\hat{x}^2(t)$ so we use the initial conditions $\eta(0) = \hat{x}^2$ as this allows us to write $\eta(t) = \hat{x}^2(t)$. Before solving this system of coupled differential equations we recall that (in order to calculate the ensemble variance correlation functions) our aim is to calculate $\langle \hat{x}^2 \rangle(t)$ with initial condition $\rho(0) = |k\rangle \langle k|$, which allows us to write

$$\begin{aligned}
\langle \hat{x}^2 \rangle(t) &= \text{tr}(\hat{x}^2(t) |k\rangle \langle k|) \\
&= c_x(t) \langle k | \hat{x}^2 | k \rangle + c_I(t) \langle k | I | k \rangle \\
&= k^2 c_x(t) + c_I(t)
\end{aligned} \tag{3.51}$$

We see, therefore, that we only need the partial solution of equation (3.50) which calculates $c_x(t)$ and $c_I(t)$; this is given in the following lemma.

Lemma 3. *The partial solution to equation (3.50) that we require is*

$$c_x(t) = 1 \quad c_I(t) = \frac{2t}{\gamma} + \frac{e^{-2\gamma t} - 1}{\gamma^2} \quad (3.52)$$

Proof. From equation (3.50) we see that the coefficient $c_x(t)$ does not evolve in time ($\frac{d}{dt}c_x(t) = 0$) and so it is constant: $c_x(t) = c_x(0) = 1$. We also see that the coefficients $c_1^1(t)$ and $c_I(t)$ evolve in time according to the differential equations

$$\frac{d}{dt}c_1^1(t) = 2ic_x(t) - 2\gamma c_1^1(t) \quad \text{and} \quad \frac{d}{dt}c_I(t) = -2ic_1^1(t) \quad (3.53)$$

whose solutions are

$$c_1^1(t) = \frac{i}{\gamma}(1 - e^{-2\gamma t}) \quad \text{and} \quad c_I(t) = \frac{2t}{\gamma} + \frac{1}{\gamma^2}(e^{-2\gamma t} - 1) \quad (3.54)$$

□

Combining equation (3.51) and the result of lemma 3 allows us to explicitly calculate $\langle \hat{x}^2 \rangle(t)$ according to the initial condition $\rho(0) = |0\rangle \langle 0|$

$$\langle \hat{x}^2 \rangle(t) = \frac{2t}{\gamma} + \frac{e^{-2\gamma t} - 1}{\gamma^2} =: f(t) \quad (3.55)$$

We are now in a position to bound the diagonal elements of $\rho(t)$ (with initial condition $\rho(0) = |0\rangle \langle 0|$) and to do so we rewrite $\langle \hat{x}^2 \rangle(t)$ as

$$\langle \hat{x}^2 \rangle(t) = \text{tr}(\rho(t)\hat{x}^2) = \sum_j j^2 \langle j | \rho(t) | j \rangle \quad (3.56)$$

This allows us to bound the diagonal elements of $\rho(t)$ by $\langle j | \rho(t) | j \rangle \leq \frac{f(t)}{j^2}$ for $j \neq 0$. Combining this with the trivial bound $\langle j | \rho(t) | j \rangle \leq 1$ we have $\langle j | \rho(t) | j \rangle \leq \min \left\{ 1, \frac{f(t)}{j^2} \right\}$ for $j \neq 0$ and $\rho(0) = |0\rangle \langle 0|$. Translational invariance of the infinite chain allows us to claim a similar bound for all initial conditions of the form $\rho(0) = |k\rangle \langle k|$

$$\langle j | \rho(t) | j \rangle \leq \min \left\{ 1, \frac{f(t)}{|j - k|^2} \right\} \quad \text{for } \rho(0) = |k\rangle \langle k| \text{ and } j \neq k \quad (3.57)$$

We are now in a position to state a bound on the ensemble variances of the disorder-dependent correlation functions for the infinite chain. Recall that

$$v_{jk}(t) = \text{Var}_{\xi} c_{jk}(\xi, t) = w_{jjk}(t) - (c_{jk}(t))^2 \leq w_{jjk}(t) \quad (3.58)$$

Furthermore, $c_{jk}(t) \rightarrow 0$ as $t \rightarrow \infty$ and so this bound becomes tight in the large time limit. Since $w_{jjk}(t)$ is the j th diagonal element of the matrix $W_k(t)$ which has initial condition $W_k(0) = |k\rangle\langle k|$ we can apply the result (3.57) to bound the ensemble variances of the correlation functions

$$v_{jk}(t) \leq \min \left\{ 1, \frac{f(t)}{|j-k|^2} \right\} \quad (3.59)$$

where $f(t)$ (defined in (3.55)) is asymptotically equivalent to $2t/\gamma$ for large t . Furthermore, $f(t) \leq 2t/\gamma$ for all t .

This is the main result of this section: it tells us how far the dynamics of our system can stray from the average-case dynamics found in the previous section. We analyse this bound and its consequences in section 3.5.2.

3.5 Analysis of results

In this section we analyse the consequences of both the ensemble averaged correlation functions (section 3.5.1) and the ensemble variance correlation functions (section 3.5.2).

3.5.1 Analysis of ensemble averaged correlation functions

Recall that in section 3.3 we calculated the ensemble averaged correlation functions to be

$$c_{jk}(t) = e^{-\gamma t} \langle j | e^{-iRt} | k \rangle \quad (3.60)$$

(see equation (3.36)). The unitary matrix e^{-iRt} corresponds to free evolution of the system when no disorder is present and this would allow $c_{jk}(t)$ to become significantly non-zero after a time proportional to the distance $|j-k|$ were it

not for the exponentially decaying term $e^{-\gamma t}$ which exponentially suppresses all correlations. In this section we will show that the only correlation functions which are appreciably non-zero are those for which both $|j - k|$ and t are small which essentially means that, on average, fermions can hop (and information can propagate) by at most a small number of sites along the chain.

We now analyse the expression (3.60) for the ensemble averaged correlation functions in detail and make the above statements precise.

Since e^{-iRt} is a unitary matrix there is the trivial bound $|\langle j | e^{-iRt} | k \rangle| \leq 1$ (for all j, k and t) which implies that $|c_{jk}(t)|$ is exponentially small for large t . More precisely, for $t \geq t_\epsilon = \log(\frac{1}{\epsilon})/\gamma$ we may bound $|c_{jk}(t)| \leq \epsilon$ and so by picking ϵ to be suitably small we may say that the ensemble averaged probability amplitude for a fermion to hop from site j to site k is negligible for $t \geq t_\epsilon$.

Having shown that the ensemble averaged correlation functions $c_{jk}(t)$ are negligible for large times $t \geq t_\epsilon$ we now analyse the behaviour at small times $t < t_\epsilon$. To do this we bound

$$|\langle j | e^{-iRt} | k \rangle| = \left| \sum_{l=0}^{\infty} \frac{\langle j | (-iRt)^l | k \rangle}{l!} \right| \leq \sum_{l=0}^{\infty} \frac{\|R\|^l |t|^l}{l!} \quad (3.61)$$

Stirling's approximation $n! = \sqrt{2n\pi} \left(\frac{n}{e}\right)^n (1 + \mathcal{O}(\frac{1}{n}))$ gives rise to a lower bound on factorials

$$n! \geq \sqrt{2n\pi} \left(\frac{n}{e}\right)^n \quad \text{for } n \geq 1 \quad (3.62)$$

which in turn gives rise to the upper bound

$$\frac{\|R\|^l |t|^l}{l!} \leq \frac{1}{\sqrt{2l\pi}} \left(\frac{e\|R\||t|}{l} \right)^l \quad (3.63)$$

The right hand side of this inequality is small for large l and furthermore the semi-infinite sum (from $l = m$ to ∞) over such terms is negligible (smaller than ϵ) for $m > t\kappa_\epsilon$ for some constant κ_ϵ .

We now define

$$Q_m(t) = e^{-iRt} - \sum_{l=0}^{m-1} \frac{(-iRt)^l}{l!} = \sum_{l=m}^{\infty} \frac{(-iRt)^l}{l!} \quad (3.64)$$

and we see (courtesy of equation 3.63) that $\|Q_m(t)\| \leq \epsilon$ is negligible when $m \geq t\kappa_\epsilon$. By noting the tri-diagonal form of R (which has 1s on the sub-diagonals and 0s elsewhere) we see that R^m is m -banded (that is, $\langle j | R^m | k \rangle = 0$ for $|j - k| > m$) and so

$$\langle j | Q_m(t) | k \rangle = \langle j | e^{-iRt} | k \rangle \quad \text{for } |j - k| \geq m \quad (3.65)$$

Hence we see that if $|j - k| \geq t\kappa_\epsilon$ then $|c_{jk}(t)|$ is negligible.

We have now shown that $|c_{jk}(t)|$ is negligible if either

- $t \geq t_\epsilon = \log(\frac{1}{\epsilon})/\gamma$
- $|j - k| \geq t\kappa_\epsilon$ for some constant κ_ϵ

Equivalently $|c_{jk}(t)|$ is non-negligible only when $|j - k| < \kappa_\epsilon t_\epsilon$ and so a fermion can, on average, hop by at most $\kappa_\epsilon t_\epsilon$ sites. This can be restated in an information theoretic manner as “non-negligible amounts of information can propagate, on average, by at most $\kappa_\epsilon t_\epsilon$ sites”.

Note that the second bullet point is essentially the original Lieb-Robinson bound: information can propagate by a distance proportional to the time elapsed. Our contribution is to add in the first bullet point for this noise model, which is in essence a cut-off point beyond which no non-negligible amount of information can, on average, propagate.

3.5.2 Analysis of ensemble variance correlation functions

Recall that in section 3.4 we calculated an upper bound (equation (3.59)) on the ensemble variances of the disorder-dependent correlation functions

$$v_{jk}(t) \leq \min \left\{ 1, \frac{f(t)}{|j - k|^2} \right\} \quad \text{where } f(t) = \frac{2t}{\gamma} + \frac{e^{-2\gamma t} - 1}{\gamma^2} \quad (3.66)$$

This bound tells us that for small times $t \ll \gamma|j - k|^2/2$ the ensemble variances $v_{jk}(t)$ are small and so no realisation of disorder is likely to lead to dynamics which are significantly different from the average-case dynamics represented by $c_{jk}(t)$. However for large times it is possible that the variances $v_{jk}(t)$ are large and so the actual dynamics may be very different from the averaged dynamics.

Recalling that $\langle j | \rho(t) | j \rangle$ (with initial condition $\rho(0) = |k\rangle \langle k|$) satisfies the same bound as $v_{jk}(t)$, we see that a particle placed on site k at time $t = 0$ has a low probability of diffusing through the lattice to a site j with $|j - k| \gg \sqrt{2t/\gamma}$. In other words we have a probabilistic light cone whose radius grows in proportion to the square root of the time elapsed, which means that one must wait for a time proportional to the square of the distance $|j - k|$ before there can be a high probability of a particle diffusing from site k to j .

To make the above statement about a light cone more precise we employ Chebyshev's inequality which states that for a random variable X with mean μ and finite variance σ^2 and for any positive real number κ

$$\mathbb{P}(|X - \mu| \geq \kappa\sigma) \leq \frac{1}{\kappa^2} \quad (3.67)$$

Choosing κ to be the constant of proportionality for the radius of the square-root light cone and identifying $X = c_{jk}(\xi, t)$, $\mu = c_{jk}(t)$ and $\sigma^2 = v_{jk}(t)$ we are able to say that a particle placed initially on site k remains within this light cone with probability greater than $(1 - \frac{1}{\kappa^2})$.

Recall that for a one-dimensional classical random walk with equal probabilities of stepping left and right, the expected distance from the starting point after N steps of the random walk is of order \sqrt{N} . That is, with high probability the particle is positioned within a light cone whose radius grows in proportion to the square root of the number of steps taken (or the time elapsed). This is exactly what we have observed for the fluctuating disorder model in this chapter and we may therefore conclude that in some sense the

noise in our system reduces the quantum mechanical behaviour of our model into classical random walk dynamics.

3.6 Discussion

To summarise the results of this chapter, we have found that information can propagate (on average) by at most a constant number of sites before it is lost in the noise of the system. However this is not so much a real physical effect but rather an effect of the ensemble averaging process we used. In reality, for a typical realisation of the disorder, information can propagate within a light cone whose radius grows in proportion to the square root of the time elapsed. There is a certain amount of similarity here with a classical random walk.

In contrast to the static disorder model of chapter 2 where the localisation was due to Anderson localisation, the localisation in this model is due to the finite strength quantum Zeno effect [101, 102, 103] caused by the on-site disorder which effectively performs continuous non-selective quantum measurements on the system.

Note that the correlation function $c_{jk}(t)$ in chapter 2 is really a disorder-dependent correlation function which we denote by $c_{jk}(\xi, t)$ in this chapter. In chapter 2 we did not average over the noise but simply said that with very high probability (exponentially close to 1) Anderson localisation occurs, which gives the bound $|c_{jk}(t)| < c_\zeta e^{-|j-k|^\zeta}$. Comparing this with the ensemble averaged correlation function found in this chapter, namely $c_{jk}(t) = e^{-\gamma t} \langle j | e^{-iRt} | k \rangle$, reveals that the dynamics of the two models differ significantly: for static noise the correlations are exponentially localised for all time, whereas for dynamic disorder the correlations are (on average) exponentially suppressed with time (although it is important to remember that this is a factor of the averaging process and not a real phenomenon).

In chapter 2 we calculated a Lieb-Robinson bound for the static disorder

model. A Lieb-Robinson bound is a general tool which describes the evolution of arbitrary operators in the Heisenberg picture, in contrast to correlation functions which calculate the dynamics of single excitations in the system. It would therefore be nice to calculate a Lieb-Robinson bound for the model of dynamic disorder studied in this chapter, but unfortunately if we attempt to calculate one using the same technique as that used in chapter 2 we run into problems: the derivation requires us to find the the ensemble average of a product of two disorder-dependent operators (for example, $a_m^\dagger(\xi, t)a_{m+1}(\xi, t)$). Unfortunately we are unable to do this as we would need to solve an equation similar to the variance equation (which we don't know how to do). The solution used in this chapter (to bound the ensemble variances $v_{jk}(t)$) would be of no help as it only bounds the diagonal elements whereas we would require bounds on the off-diagonal elements too.

Neither can we calculate a Lieb-Robinson bound directly using the method in chapter 4 as the calculations there rely on a symmetry (namely that the on-site disorder can point in any direction) which is not present in this model (where the on-site disorder is fixed in direction).

In conclusion, although a Lieb-Robinson bound for this model is very desirable, we have been unable to provide one and must instead rely on ensemble averages and ensemble variances of disorder-dependent correlation functions to describe the dynamics of the model.

Chapter 4

Dynamic disorder: fluctuating field directions

In the previous chapter we studied a noisy spin chain model where the on-site disorder fluctuated in strength but was fixed in the z -direction. In this chapter we study a similar model where both the strength and direction of the on-site disorder are free to fluctuate. The intrinsic Hamiltonian is an arbitrary nearest-neighbour Hamiltonian, in contrast with the previous two chapters where it was restricted to being the XX -model Hamiltonian. The additional symmetry present in this model (which was lacking in the previous dynamic disorder model due to the fixed field direction) allows us to once again calculate a Lieb-Robinson bound for the system.

This chapter is organised as follows. After describing the model (§ 4.1) we explain how to generalise the derivation of the master equation in chapter 3 to the model studied in this chapter, which allows us to form a differential equation for the Lieb-Robinson commutators (§ 4.2). We solve this differential equation to give a Lieb-Robinson bound (§ 4.3) which we analyse in detail (§ 4.4) and we finish with a short discussion (§ 4.5).

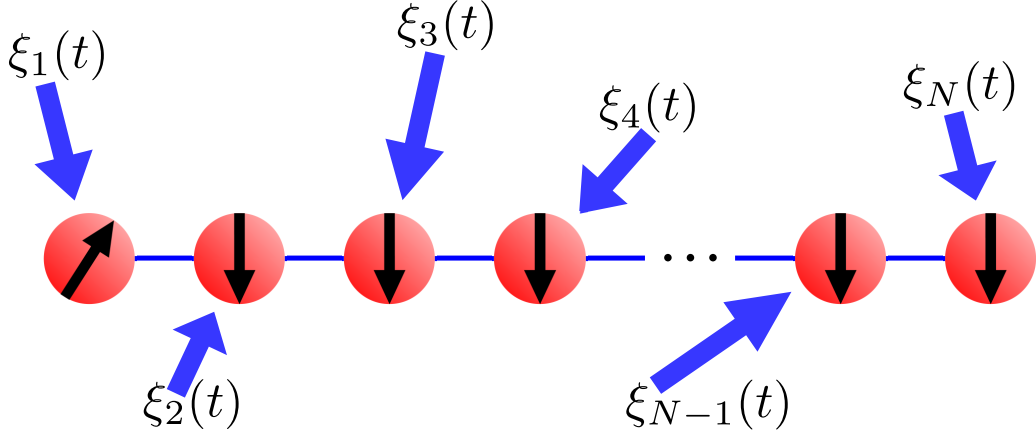


Figure 4.1: A spin chain with on-site disorder which fluctuates in both strength and direction: the disorder strengths $\xi_j(t) = (\xi_j^x(t), \xi_j^y(t), \xi_j^z(t))$ have components $\xi_j^\alpha(t)$ which fluctuate in time independently in each direction on each site.

4.1 The dynamic disorder model with fluctuating field direction

The model studied in this chapter is that of a qubit spin chain of length N whose intrinsic Hamiltonian H_0 consists of arbitrary nearest-neighbour interactions. Whilst it could be the XX -model Hamiltonian studied in chapters 2 and 3 it could also be any other nearest-neighbour Hamiltonian with finite norm $\|H_0\| < \infty$. The on-site disorder is free to fluctuate in both strength and direction independently on each site (see figure 4.1). In particular we model this noise by allowing the on-site disorder strengths to fluctuate independently in each of the three cardinal directions (x , y and z) with corresponding components $\xi_j^\alpha(t)$ ($\alpha \in \{x, y, z\}$). As with the fluctuating noise of chapter 3, at each instant in time the field strengths $\xi_j^\alpha(t)$ are independent, identically distributed random variables which are drawn from a probability distribution \mathbb{P}_ξ which has a finite second moment.

The system Hamiltonian for this model is therefore

$$H(\xi, t) = H_0 + \sum_{j=1}^N \sum_{\alpha \in \{x, y, z\}} \xi_j^\alpha(t) \sigma_j^\alpha \quad (4.1)$$

4.2 A differential equation for the Lieb-Robinson commutator

In this chapter we will calculate a Lieb-Robinson bound, which is an upper bound on the Lieb-Robinson commutator (see equation (1.17))

$$C_B(t) = \sup_A \frac{\|[A, B(t)]\|}{\|A\|} \quad (4.2)$$

where $B(t) = \mathbb{E}_\xi U^\dagger(\xi, t) B U(\xi, t)$ is an ensemble average over all possible realisations of the operator B in the Heisenberg picture and the norm $\|\cdot\|$ is the operator norm. We begin by calculating a Lieb-Robinson bound with the restriction that A has support only on site n and for this reason we relabel A to A_n and define a corresponding Lieb-Robinson commutator

$$C_B(n, t) = \sup_{A_n} C_{A,B}(n, t) \quad \text{where} \quad C_{A,B}(n, t) = \frac{\|[A_n, B(t)]\|}{\|A_n\|} \quad (4.3)$$

Once we have proved a bound on these Lieb-Robinson commutators, we will extend the result to a general Lieb-Robinson bound for arbitrary operators A (see section 4.3).

We begin by generalising the derivation of the master equation (3.19) of chapter 3, which we do by increasing the number of meters for each measurement site j at each time interval t_r from one to three: one for each of the three cardinal directions (x , y and z). These meters now have canonical variables \bar{x}_{jr}^α and \bar{p}_{jr}^α which satisfy the canonical commutation relations $[\bar{x}_{jr}^\alpha, \bar{p}_{ks}^\beta] = i\delta_{jk}\delta_{rs}\delta_{\alpha\beta}$. Following precisely the same steps as in chapter 3 leads to a master equation for the model we are studying in this chapter

$$\frac{d}{dt}\rho(t) = -i[H_0, \rho(t)] - \gamma \sum_{j=1}^N \sum_{\alpha \in \{x,y,z\}} [\sigma_j^\alpha, [\sigma_j^\alpha, \rho(t)]] \quad (4.4)$$

The remainder of this section shows how to use this master equation to form a differential equation for the Lieb-Robinson commutators of equation (4.3). In the next section we solve this differential equation to obtain a bound on the commutators, which is then generalised to give a full Lieb-Robinson bound.

As $B(t)$ is the ensemble average of the operator B in the Heisenberg picture, it obeys an equation similar to the master equation, namely

$$\frac{d}{dt}B(t) = i[H_0, B(t)] - \gamma\mathcal{F}(B(t)) \quad (4.5)$$

where we have written the dissipation term as

$$\mathcal{F}(B) = \sum_{j=1}^N \mathcal{F}_j(B) \quad \text{where} \quad \mathcal{F}_j(B) = \sum_{\alpha \in \{x,y,z\}} [\sigma_j^\alpha, [\sigma_j^\alpha, B]] \quad (4.6)$$

In order to form a differential equation for $C_B(n, t)$ we first use a Taylor expansion to calculate

$$\begin{aligned} \|A_n\|C_{A,B}(n, t + \epsilon) &= \|[A_n, B(t) + i\epsilon[H_0, B(t)] - \epsilon\gamma\mathcal{F}(B(t))] + \mathcal{O}(\epsilon^2)\| \\ &\leq (1 - 8\epsilon\gamma N)\|[A_n, B(t) + i\epsilon[H_0, B(t)]]\| \\ &\quad + \epsilon\gamma\|[A_n, 8NB(t) - \mathcal{F}(B(t))]\| + \mathcal{O}(\epsilon^2) \end{aligned} \quad (4.7)$$

where ϵ is chosen to be small enough so that $1 - 8\epsilon\gamma N \geq 0$ (later on we will take the limit $\epsilon \rightarrow 0$ and so this condition will be automatically satisfied).

We can rewrite the first term on the right-hand-side of equation (4.7) as

$$\begin{aligned} \|[A_n, B(t) + i\epsilon[H_0, B(t)]]\| &= \|[A_n, e^{i\epsilon H_0} B(t) e^{-i\epsilon H_0}] + \mathcal{O}(\epsilon^2)\| \\ &= \|[e^{-i\epsilon H_0} A_n e^{i\epsilon H_0}, B(t)] + \mathcal{O}(\epsilon^2)\| \\ &= \|[A_n - i\epsilon[H_0, A_n], B(t)] + \mathcal{O}(\epsilon^2)\| \\ &\leq \|A_n\|C_{A,B}(n, t) + \epsilon\|[H_0, A_n], B(t)\| + \mathcal{O}(\epsilon^2) \end{aligned} \quad (4.8)$$

where we have used unitary invariance of the operator norm and made repeated use of Taylor expansions. To deal with the double commutator term in equation (4.8) we make use of the trivial bound $\|[H_0, A_n]\| \leq 2\|H_0\|\|A_n\|$ which allows us to rewrite

$$[H_0, A_n] = 2\|H_0\|\|A_n\|V \quad \text{with} \quad \|V\| \leq 1 \quad (4.9)$$

Since H_0 consists only of nearest-neighbour interactions we may write V as

$$V = \sum_{\alpha, \beta \in \{x,y,z,0\}} \left(v_{n-1}^{\alpha\beta} \sigma_{n-1}^\alpha \sigma_n^\beta + v_n^{\alpha\beta} \sigma_n^\alpha \sigma_{n+1}^\beta \right) \quad (4.10)$$

where $|v_m^{\alpha\beta}| \leq \|V\| \leq 1$, which allows us to bound

$$\|[V, B(t)]\| \leq \sum_{\alpha, \beta \in \{x, y, z, 0\}} \left(|v_{n-1}^{\alpha\beta}| \|[\sigma_{n-1}^\alpha \sigma_n^\beta, B(t)]\| + |v_n^{\alpha\beta}| \|[\sigma_n^\alpha \sigma_{n+1}^\beta, B(t)]\| \right) \quad (4.11)$$

Using this and the commutator identity $[AB, C] = A[B, C] + [A, C]B$ allows us to calculate a bound on the double commutator term, namely

$$\begin{aligned} \frac{\|[H_0, A_n], B(t)\|}{\|A_n\|} &= 2\|H_0\| \|[V, B(t)]\| \\ &\leq 32\|H_0\| (2C_B(n, t) + C_B(n-1, t) + C_B(n+1, t)) \end{aligned} \quad (4.12)$$

Having expressed the first term on the right hand side of equation (4.7) in terms of the Lieb-Robinson commutators $C_{A,B}(n, t)$ and $C_B(n, t)$ we now do the same for the second term. We begin by noting the effect of the dissipation operators \mathcal{F}_k on the Pauli matrices

$$\begin{aligned} \mathcal{F}_k(\sigma_j^\alpha) &= 8\delta_{jk}\sigma_k^\alpha \quad \text{for } \alpha \in \{x, y, z\} \\ \mathcal{F}_k(\sigma_j^0) &= 0 \quad \text{where } \sigma^0 = \mathbb{I} \end{aligned} \quad (4.13)$$

We now restrict $B(t)$ to be of a tensor product form

$$B_\alpha = \sigma_1^{\alpha_1} \otimes \cdots \otimes \sigma_N^{\alpha_N} \quad (4.14)$$

and later on we will extend to general $B(t)$ by linearity. It is straightforward to use equation (4.13) to find the effect of the dissipation operators on B_α , which in turn allows us to calculate the following

$$\begin{aligned} 8B_\alpha - \mathcal{F}_k(B_\alpha) &= 8B_\alpha - \left(\bigotimes_{j \neq k} \sigma_j^{\alpha_j} \right) \otimes \mathcal{F}_k(\sigma_k^{\alpha_k}) \\ &= \begin{cases} 0 & \text{if } \sigma_k^{\alpha_k} \neq \mathbb{I}_k \\ 8B_\alpha & \text{if } \sigma_k^{\alpha_k} = \mathbb{I}_k \end{cases} \\ &= 4\text{tr}_k(B_\alpha) \otimes \mathbb{I}_k \end{aligned} \quad (4.15)$$

It is a simple matter to verify the following Pauli matrix identities

$$\begin{aligned} \sum_{\alpha \in \{0, x, y, z\}} \sigma_k^\alpha \sigma_k^\beta \sigma_k^\alpha &= 0 \quad \text{for } \beta \neq 0 \\ \sum_{\alpha \in \{0, x, y, z\}} \sigma_k^\alpha \sigma_k^0 \sigma_k^\alpha &= 4\mathbb{I}_k \end{aligned} \quad (4.16)$$

which allow us to re-express

$$8B_\alpha - \mathcal{F}_k(B_\alpha) = \sum_{\beta \in \{0, x, y, z\}} 2\sigma_k^\beta B_\alpha \sigma_k^\beta \quad (4.17)$$

Finally we observe that

$$[A_n, \sum_{\beta \in \{0,x,y,z\}} \sigma_n^\beta B_\alpha \sigma_n^\beta] = 0 \quad (4.18)$$

which allows us to calculate an upper bound on the second term of the right-hand-side of equation (4.7) (with the restriction that $B(t) = B_\alpha$ has tensor product form)

$$\begin{aligned} \epsilon\gamma \|[A_n, 8NB_\alpha - \mathcal{F}(B_\alpha)]\| &\leq \epsilon\gamma \sum_k \|[A_n, 8B_\alpha - \mathcal{F}_k(B_\alpha)]\| \\ &\leq 2\epsilon\gamma \sum_{k \neq n} \sum_{\beta \in \{0,x,y,z\}} \|[A_n, \sigma_k^\beta B_\alpha \sigma_k^\beta]\| \\ &= 8\epsilon\gamma(N-1)\|A_n\|C_{A,B_\alpha}(n,t) \end{aligned} \quad (4.19)$$

(we have used: $\|[A_n, \sigma_k^\beta B_\alpha \sigma_k^\beta]\| = \|[\sigma_k^\beta A_n \sigma_k^\beta, B_\alpha]\| = \|[A_n, B_\alpha]\|$ when $k \neq n$).

We now extend this result to general $B(t)$ by linearity

$$B(t) = \sum_{\alpha} c_\alpha(t) B_\alpha \quad (4.20)$$

It is straightforward to check that

$$8B - \mathcal{F}_k(B) = \dots = \sum_{\alpha} c_\alpha(t) \sum_{\beta \in \{0,x,y,z\}} 2\sigma_k^\beta B_\alpha \sigma_k^\beta \quad (4.21)$$

which allows us to bound the second term on the right-hand-side of equation (4.7)

$$\epsilon\gamma \|[A_n, 8NB(t) - \mathcal{F}(B(t))]\| \leq 8\epsilon\gamma(N-1)\|A_n\|C_{A,B}(n,t) \quad (4.22)$$

In summary, we have now bounded both terms on the right-hand-side of equation (4.7), which can therefore be rewritten as

$$\begin{aligned} C_{A,B}(n, t + \epsilon) &\leq 32\|H_0\|\epsilon(2C_B(n, t) + C_B(n-1, t) + C_B(n+1, t)) \\ &\quad + (1 - 8\epsilon\gamma)C_{A,B}(n, t) + \mathcal{O}(\epsilon^2) \end{aligned} \quad (4.23)$$

This expression holds for all A and in particular it holds for \tilde{A} , which is defined by the equality $C_B(n, t) = C_{\tilde{A}, B}(n, t)$. This allows us to bound the Lieb-Robinson commutator

$$\begin{aligned} C_B(n, t + \epsilon) &\leq (1 - 8\epsilon(\gamma - 8\|H_0\|))C_B(n, t) \\ &\quad + 32\|H_0\|(C_B(n-1, t) + C_B(n+1, t)) + \mathcal{O}(\epsilon^2) \end{aligned} \quad (4.24)$$

Finally we arrive at a bound on the time derivative of the Lieb-Robinson commutator $C_B(n, t)$

$$\begin{aligned} \frac{d}{dt} C_B(n, t) &= \lim_{\epsilon \rightarrow 0} \frac{C_B(n, t+\epsilon) - C_B(n, t)}{\epsilon} \\ &\leq -8(\gamma - 8\|H_0\|)C_B(n, t) \\ &\quad + 32\|H_0\|(C_B(n-1, t) + C_B(n+1, t)) \end{aligned} \tag{4.25}$$

This is the main result of this section.

4.3 A Lieb-Robinson bound for dynamic disorder

In this section we use the bound on the time derivative of the Lieb-Robinson commutator (equation (4.25) of the previous section) to find a Lieb-Robinson bound for single-site operators, A_n . We then generalise this result to arbitrary operators A , giving a fully general Lieb-Robinson bound for this noise model.

We begin by recasting equation (4.25) into a vector inequality by defining

$$\mathbf{C}_B(t) = \begin{pmatrix} C_B(1, t) \\ \vdots \\ C_B(N, t) \end{pmatrix} \tag{4.26}$$

which allows us to rewrite equation (4.25) as

$$\frac{d}{dt} \mathbf{C}_B(t) \leq (-8(\gamma - 8\|H_0\|)I + 32\|H_0\|R) \mathbf{C}_B(t) \tag{4.27}$$

where the inequality is to be understood component-wise, I is the $N \times N$ identity matrix and $R = \sum_{j=1}^{N-1} (|j\rangle \langle j+1| + |j+1\rangle \langle j|)$ as in chapter 3.

This differential inequality has solution

$$\mathbf{C}_B(t) \leq e^{-8(\gamma - 8\|H_0\|)t} e^{32\|H_0\|Rt} \mathbf{C}_B(0) \tag{4.28}$$

which is the vector form of the Lieb-Robinson bound (for single-site operators A_n) applicable to the fluctuating disorder model studied in this chapter.

To extract the component form of this bound, we must analyse the structure of the matrix exponential $e^{32\|H_0\|Rt}$. Noting that each matrix element of

the matrix e^{Dt} is upper bounded by $e^{\|D\|t}$, and defining E to be the matrix with every entry equal to unity (that is, $\langle j|E|k\rangle = 1$ for all j and k) we re-write the above Lieb-Robinson bound as

$$\mathbf{C}_B(t) \leq e^{-8(\gamma-8\|H_0\|)t} e^{32\|H_0\|\|R\|t} E \mathbf{C}_B(0) \quad (4.29)$$

whose components are

$$C_B(n, t) \leq e^{-8(\gamma-8\|H_0\|)t} e^{32\|H_0\|\|R\|t} C \quad (4.30)$$

where C is the constant

$$C = \sum_j C_B(j, 0) \quad (4.31)$$

Whilst a more careful analysis of the matrix $e^{32\|H_0\|Rt}$ might lead to a smaller value of C this would only improve the bound by a constant factor and would not affect the generic form of the bound. For this reason we do not perform a more detailed analysis of $e^{32\|H_0\|Rt}$ here.

We now show how to generalise this result for single-site operators A_n to arbitrary operators A with support on M contiguous sites n_1, \dots, n_M . To do this, we expand A in the Pauli basis

$$A = \sum_{\substack{\alpha_1, \dots, \alpha_M \\ \in \{0, x, y, z\}}} a_{\alpha_1, \dots, \alpha_M} \sigma_{n_1}^{\alpha_1} \otimes \dots \otimes \sigma_{n_M}^{\alpha_M} \quad (4.32)$$

which allows us to bound the commutator

$$\|[A, B(t)]\| \leq \sum_{\substack{\alpha_1, \dots, \alpha_M \\ \in \{0, x, y, z\}}} |a_{\alpha_1, \dots, \alpha_M}| \|\sigma_{n_1}^{\alpha_1} \otimes \dots \otimes \sigma_{n_M}^{\alpha_M}, B(t)\| \quad (4.33)$$

In section 4.2 we used the commutator identity $[AB, C] = A[B, C] + [A, C]B$; this can be applied iteratively to give a more general version, namely

$$[A_1 \dots A_M, B] = \sum_{j=1}^M \left(\prod_{k=0}^{j-1} A_k \right) [A_j, B] \left(\prod_{k=j+1}^{M+1} A_k \right) \quad (4.34)$$

where (for notational convenience) we have introduced $A_0 = A_{M+1} = \mathbb{I}$.

Noting that $|a_{\alpha_1, \dots, \alpha_M}| \leq \|A\|$, applying the above commutator identity and recalling that $\|\sigma_{n_j}^{\alpha_j}\| = 1$ allows us to calculate

$$\|[A, B(t)]\| \leq \sum_{\substack{\alpha_1, \dots, \alpha_M \\ \in \{0, x, y, z\}}} \|A\| \sum_{j=1}^M \left\| [\sigma_{n_j}^{\alpha_j}, B(t)] \right\| \leq 4^M \|A\| \sum_{j=1}^M C_B(n_j, t) \quad (4.35)$$

Finally we are in a position to calculate a general Lieb-Robinson bound for the noise model of this chapter with arbitrary operators A and B , namely

$$C_B(t) \leq e^{-8(\gamma - 8\|H_0\|)t} e^{32\|H_0\|\|R\|t} C' \quad (4.36)$$

where

$$C' = M 4^M \sum_j C_B(j, 0) \quad (4.37)$$

Note that this has the same form as the Lieb-Robinson bound (4.30) for single-site operators A_n excepting a constant multiplicative term which is exponential in the size of the support of the operator A . It is worth noting that if we have specific knowledge about the structure of the operator A , we may be able to dramatically reduce the number of terms in the expansion (4.32) and consequently reduce the size of the constant $M 4^M$ in the Lieb-Robinson bound.

4.4 Analysis of the new Lieb-Robinson bound

We now perform a detailed analysis of the general Lieb-Robinson bound (4.36) for the fluctuating disorder model studied in this chapter.

We see that the bound is decaying in time if $\gamma > (8 + 4\|R\|)\|H_0\|$; conversely, if $\gamma < (8 + 4\|R\|)\|H_0\|$ our bound is exponentially growing in time and is a useless bound: the original Lieb-Robinson bound is far superior to ours in this regime. We have therefore identified a potential noise threshold: if the noise level γ is below the threshold $(8 + 4\|R\|)\|H_0\|$ then our bound diverges and we are limited only by the ballistic transport of quantum states (and hence quantum information) identified in the original Lieb-Robinson bound;

however if the noise level is above the threshold then the dynamics are more localised.

We now analyse the localisation of the dynamics in the noisy regime ($\gamma > (8 + 4\|R\|)\|H_0\|$). Assuming the support of A is initially disjoint with the support of B then $C_B(0) = 0$. For small times (when the exponentially decaying term is still fairly close to unity) the matrix $e^{32\|H_0\|Rt}$ causes the support of $B(t)$ to grow at a rate limited by the linear light cone of the original Lieb-Robinson bound. In other words, $C_B(0)$ is negligibly small if the distance between the supports of A and B is greater than $\mathcal{O}(t)$. However at larger times the decaying term $e^{-8(\gamma - 8\|H_0\|)t}$ exponentially suppresses $C_B(t)$, irrespective of the distance between the supports of A and B . We note that information can propagate from the region supporting A to the region supporting B in time t only if the Lieb-Robinson commutator $C_B(t)$ is non-negligible. This means that information can (on average) spread by at most a constant number of sites along the chain before $C_B(t)$ is exponentially suppressed and the information is lost in the noise of the system.

We now rigorously analyse the Lieb-Robinson bound to make the above statements more precise. We begin by restating the Lieb-Robinson bound as

$$C_B(t) \leq e^{-\tilde{\gamma}t} e^{\tilde{R}t} C' \quad (4.38)$$

where $\tilde{\gamma} = 8(\gamma - 8\|H_0\|)$ and $\tilde{R} = 32\|H_0\|R$. This expression highlights the similarity of this Lieb-Robinson bound to the expression (3.60) for the correlation functions of chapter 3; for this reason the analysis below is very similar to that performed in section 3.5.1.

Defining a new constant $\Gamma = 8\gamma - 8(8 + 4\|R\|)\|H_0\|$ allows us to rewrite the Lieb-Robinson bound as

$$C_B(t) \leq e^{-\Gamma t} C' \quad (4.39)$$

(Note that as $\gamma > (8 + 4\|R\|)\|H_0\|$ is above the noise threshold then $\Gamma > 0$.)

From this version of the bound it is clear that $C_B(t)$ is negligible (that is $C_B(t) \leq \epsilon$) for large times $t \geq \tilde{t}_\epsilon$ where $\tilde{t}_\epsilon = \log(\frac{C'}{\epsilon})/\Gamma$.

Having analysed the large time behaviour of the Lieb-Robinson bound we now turn to the small time $t < \tilde{t}_\epsilon$ behaviour. We define

$$\tilde{Q}_m(t) = \sum_{l=m}^{\infty} \frac{(\tilde{R}t)^l}{l!} \quad (4.40)$$

and use the same argument as in section 3.5.1 to see that $\|\tilde{Q}_m(t)\| \leq \epsilon$ when $m \geq t\tilde{\kappa}_\epsilon$ where $\tilde{\kappa}_\epsilon$ is a constant. Noting that \tilde{R} is tri-diagonal (as was R in section 3.5.1) we see that

$$\langle j | \tilde{Q}_m(t) | k \rangle = \langle j | e^{\tilde{R}t} | k \rangle \quad \text{for} \quad |j - k| > m \quad (4.41)$$

which tells us that $e^{\tilde{R}t}$ (and hence the right-hand-side of the Lieb-Robinson bound) is negligible if $|j - k| \geq t\tilde{\kappa}_\epsilon$.

In summary, we have shown that the right-hand-side of our Lieb-Robinson bound is negligible if either

- $t \geq \tilde{t}_\epsilon = \log(\frac{C'}{\epsilon})/\Gamma$
- $|j - k| \geq t\tilde{\kappa}_\epsilon$ for some constant $\tilde{\kappa}_\epsilon$

Equivalently, the right-hand-side of the Lieb-Robinson bound is non-negligible only when $|j - k| < \tilde{\kappa}_\epsilon \tilde{t}_\epsilon$ or in other words the support of $B(t)$ can, on average, grow by at most $\tilde{\kappa}_\epsilon \tilde{t}_\epsilon$ sites. (Strictly speaking the support of B can grow more than this, however outside this range the support of B is exponentially suppressed.) This can be restated in an information theoretic manner as “non-negligible amounts of information can propagate, on average, by at most $\tilde{\kappa}_\epsilon \tilde{t}_\epsilon$ sites”.

4.5 Discussion

To summarise the results of this chapter, we have identified a potential noise threshold for our model of a spin chain experiencing on-site disorder which

fluctuates in both strength and direction. If this threshold is real (and not just a function of the way we calculated the bound), then it is similar to a quantum-to-classical phase transition, where low noise levels allow us to perform quantum computation but high noise levels prohibit this [104]. In the high noise regime, we have found that information can propagate (on average) by at most a constant number of sites before the information is lost in the noise of the system.

There is a fundamental difference between the dynamic disorder model studied in this chapter and that of chapter 3. In chapter 3 the noise model preserves the number of excitations in the spin chain (allowing, for example, the dual rail protocol discussed in section 1.2.2 to be used). However in the model studied in this chapter the on-site disorder components in the x and y directions create and destroy excitations. Therefore we can argue that any initial excitation really is lost in the noise and (unlike in chapter 3) this is not just a consequence of the averaging process but rather a real phenomenon. Consequently any attempt by Alice and Bob to use this spin chain for communication is going to run into problems: since excitations in the chain are indistinguishable, if Bob detects an excitation he cannot know if it is the excitation transmitted by Alice or an excitation created by the noise.

Note that (as with ensemble averaged correlation functions in chapter 3) this bound does not provide the full picture about the dynamics due to the ensemble averaging process we used to average over the noise. In order to better understand the dynamics we need to estimate the difference between the actual dynamics of B and the ensemble averaged dynamics. This would require a bound on $\|B(t) - B(\xi, t)\|$ but we cannot calculate this directly as we do not know the exact details of the noise (we only know the distribution of the field strengths at each instant in time).

Note that due to the fundamental differences between the two dynamic disorder models, we cannot simply assume that a square-root light cone exists

as it did for the noise model of chapter 3. We conjecture that (due to the creation and destruction of excitations in this model) this ensemble averaged Lieb-Robinson bound represents a fairly truthful picture of the dynamics for all realisations of the disorder: for high levels of noise ($\gamma > (8 + 4\|R\|)\|H_0\|$) the initial excitation will become completely lost in the noise of the system. If this is true, then communication of a fidelity high enough to allow quantum error correction is only possible for short times and consequently over short distances.

To confirm or disprove this conjecture one could attempt a variance calculation approach to finding $\|B(t) - B(\xi, t)\|$, but it is not entirely clear how one should do this. One might also be tempted to try and investigate further by using correlation functions (as we did in chapter 3) but unfortunately this does not work: $|\mathbf{0}\rangle$ (or $|\Omega\rangle$ if we apply the Jordan-Wigner transformation) is not an eigenstate of the Hamiltonian and so the method used to calculate the correlation functions in chapter 3 will not work here.

We do not provide a solution to this problem, but merely comment that further investigation is required if we are to fully understand the dynamics of this model.

Chapter 5

Spin chains as generalised depolarising channels

Generalised depolarising channels are an interesting area of study in their own right; furthermore, they are realistic models of noisy quantum channels and they are experimentally realisable [68]. However they are particularly important because *any* quantum channel Φ can be transformed into a generalised depolarising channel in the following manner.

The channel Φ is used to transmit one half of a bipartite maximally entangled state. The resultant state (which, generally speaking, will no longer be maximally entangled) is then used by Alice and Bob to attempt quantum teleportation [105] of a quantum state $\rho_{in} = |\psi\rangle\langle\psi|$ with resultant output state ρ_{out} . The map $\rho_{in} \mapsto \rho_{out}$ is a generalised depolarising channel [106].

This is particularly relevant to us if we use a spin chain communication protocol as the quantum channel Φ in the protocol described above; it is with this focus on spin chains that we explain the protocol in more detail.

It is worth noting that antiferromagnetic qubit spin chains at finite temperature can also be used to teleport or transfer the state of a qubit in a way which results in depolarising channels [107], although we do not study this here.

In this chapter, we first review the definition of a generalised depolarising channel (§ 5.1). We then describe how arbitrary chains consisting of spins of

dimension D can be used as generalised depolarising channels (§ 5.2); we focus on excitation-preserving spin chains (§ 5.2.1) before giving a worked example and some numerical simulations of qubit spin chains (§ 5.2.2). We finish by showing that a generalisation of the noise model where the on-site disorder fluctuates in both strength and direction can act as a generalised depolarising channel (§ 5.3).

5.1 Recap of generalised depolarising channels

We briefly recall the definition of generalised depolarising channels introduced in section 1.4, but first we recall that the state of a qudit (a D -dimensional quantum system) can be written as

$$\rho(t) = \frac{1}{D} \left(\mathbb{I} + \sum_{\alpha=1}^{D^2-1} \sqrt{\frac{D(D-1)}{\text{tr}(M_\alpha^\dagger M_\alpha)}} a_\alpha(t) M_\alpha \right) \quad (5.1)$$

where $M_0 = \mathbb{I}$ and $\{M_\alpha\}_{\alpha=0}^{D^2-1}$ is a basis which is trace-free ($\text{tr}(M_\alpha) = 0$ for all $\alpha \neq 0$) and trace-orthogonal ($\text{tr}(M_\alpha^\dagger M_\beta) = 0$ for all $\alpha \neq \beta$). The vector $\mathbf{a} = (a_0, \dots, a_{D^2-1})$ is called the polarisation vector and its components a_α the polarisation coefficients. The normalisation in equation (5.1) is chosen so that pure states have a polarisation vector of unit norm $\|\mathbf{a}\| := \sum_{\alpha=1}^{D^2-1} |a_\alpha|^2 = 1$ and mixed states have a polarisation vector of norm strictly less than unity $\|\mathbf{a}\| < 1$. Recall that $a_0 = 1$ to ensure that ρ is normalised with unit trace.

A generalised depolarising channel $\Phi_{\mathbf{v}}$ (with respect to the basis $\{M_\alpha\}$) is defined to be a completely positive, trace-preserving map which maps a quantum state ρ into

$$\Phi_{\mathbf{v}}(\rho) = \frac{1}{D} \left(\mathbb{I} + \sum_{\alpha=1}^{D^2-1} \sqrt{\frac{D(D-1)}{\text{tr}(M_\alpha^\dagger M_\alpha)}} v_\alpha a_\alpha M_\alpha \right) \quad (5.2)$$

where the vector $\mathbf{v} = (v_0, \dots, v_{D^2-1})$ is the compression vector and its components v_α are the compression coefficients which satisfy $|v_\alpha| \leq 1$ for all α . Recall that $v_0 = 1$ to ensure that $\Phi_{\mathbf{v}}$ is trace-preserving.

5.2 Arbitrary spin chains as generalised depolarising channels

In this section we show in detail how arbitrary spin chains (whether noisy or noise-free) consisting of D -dimensional qudits (spin $\frac{D-1}{2}$ particles) can be used as generalised depolarising channels. In particular, this applies to the noisy chains of qubits we studied in chapters 2, 3 and 4. By restricting to permutation Hamiltonians we are able to give an explicit formula for the compression coefficients and we present some numerical simulations to illustrate this in the case of qubits.

We begin by defining the Heisenberg-Weyl basis to be $\{M_{jk}\}_{j,k=0}^{D-1}$ where $M_{jk} := X^j Z^k$ and where

$$X := \sum_{j=0}^{D-1} |j\rangle \langle j+1 \pmod{D}| \quad \text{and} \quad Z := \sum_{j=0}^{D-1} \omega^j |j\rangle \langle j| \quad (5.3)$$

($\omega = e^{2\pi i/D}$ is the primitive D th root of unity.)

The bipartite maximally entangled states of two qudits can be conveniently expressed in terms of the Heisenberg-Weyl basis

$$|\Phi_{rs}\rangle = (M_{rs} \otimes \mathbb{I}) \frac{1}{\sqrt{D}} \sum_{j=0}^{D-1} |j, j\rangle = \frac{1}{\sqrt{D}} \sum_{j=0}^{D-1} \omega^{js} |j-r, j\rangle \quad (5.4)$$

where $|j, k\rangle = |j\rangle \otimes |k\rangle$.

If Alice has a maximally entangled bipartite state $|\Phi_{rs}\rangle$, she can retain one half of this state (on a qudit labelled by 0) and place the other half of it on the first site of a qudit spin chain of length N which has been initialised in the state $|\mathbf{0}\rangle = |0\rangle_1 \otimes \cdots \otimes |0\rangle_N$. If she does this then the state of the whole system is

$$|\psi(0)\rangle = \frac{1}{\sqrt{D}} \sum_{j=0}^{D-1} \omega^{js} |j-r\rangle_0 \otimes |j\rangle_1 \otimes |0\rangle_{\widehat{0,1}} \quad (5.5)$$

where $|0\rangle_{\widehat{0,k}} = |0\rangle_1 \otimes \cdots \otimes |0\rangle_{k-1} \otimes |0\rangle_{k+1} \otimes \cdots \otimes |0\rangle_N$ (and $\widehat{0,k}$ represents all sites except sites 0 and k).

If the Hamiltonian for the spin chain is $H(t)$ then solving the Schrödinger equation reveals that the propagator is $U(t) = \mathcal{T}e^{-i \int_0^t H(s) ds}$ and so after free evolution for a time t the spin chain is in the state

$$|\psi(t)\rangle = \frac{1}{\sqrt{D}} \sum_{j=0}^{D-1} \omega^{js} |j-r\rangle_0 \otimes \left(U(t) |j\rangle_1 \otimes |0\rangle_{\widehat{0,1}} \right) \quad (5.6)$$

Tracing out all sites of the spin chain except for site N (Bob's qudit) reveals that Alice and Bob now share the entangled state

$$\chi_{rs}(t) = \text{tr}_{\widehat{0,N}} |\psi(t)\rangle \langle \psi(t)| \quad (5.7)$$

which, when expanded in full, can be written as

$$\chi_{rs}(t) = \frac{1}{D} \sum_{j,k=0}^{D-1} \omega^{(j-k)s} |j-r\rangle_0 \langle k-r|_0 \otimes \text{tr}_{\widehat{0,N}} \left(U(t) |j\rangle_1 |0\rangle_{\widehat{0,1}} \langle k|_1 \langle 0|_{\widehat{0,1}} U^\dagger(t) \right) \quad (5.8)$$

It is known [106] that if we use $\chi_{rs}(t)$ to teleport the qudit ρ_{in} from Alice to Bob then Bob obtains output state

$$\tilde{\rho}_{rs} = \sum_{p,q=0}^{D-1} \text{tr} (\chi_{rs}(t) |\Phi_{pq}\rangle \langle \Phi_{pq}|) (X^{-p} Z^q) \rho_{in} (X^{-p} Z^q)^\dagger \quad (5.9)$$

which, due to the relation $X^a Z^b = \omega^{ab} Z^b X^a$, is a generalised depolarising channel with compression coefficients

$$v_{lm}^{(rs)}(t) = \sum_{p,q=0}^{D-1} p_{pq}^{(rs)}(t) \omega^{-mp-lq} \quad \text{where} \quad p_{pq}^{(rs)}(t) = \langle \Phi_{pq} | \chi_{rs}(t) | \Phi_{pq} \rangle \quad (5.10)$$

(One can check that $p_{pq}^{(rs)}$ are probabilities: $0 \leq p_{pq}^{(rs)} \leq 1$ and $\sum_{p,q=0}^{D-1} p_{pq}^{(rs)} = 1$.)

5.2.1 Excitation-preserving spin chains

We now apply the above results to spin chains whose Hamiltonians are *permutation Hamiltonians* (which preserve the number and type of excitations); this allows us to explicitly calculate the probabilities $p_{pq}^{(rs)}$.

A simple permutation Hamiltonian

We first consider a simple nearest-neighbour Hamiltonian for a qudit spin chain of length N

$$H = \sum_{j=1}^{N-1} P_{j,j+1} \quad \text{where} \quad P_{j,j+1} = \sum_{r,s=0}^{D-1} (X^r Z^s)_j \otimes (\omega^{-rs} X^{-r} Z^{-s})_{j+1} \quad (5.11)$$

Observe that the local interaction $P_{j,j+1}$ acts on states as follows

$$P_{j,j+1} |\alpha\rangle_j \otimes |\beta\rangle_{j+1} = N |\beta\rangle_j \otimes |\alpha\rangle_{j+1} \quad \text{where} \quad \alpha, \beta \in \{0, \dots, D-1\} \quad (5.12)$$

This shows us that our Hamiltonian preserves the number and type of excitations and is therefore a permutation Hamiltonian.

Generalised XX -Hamiltonian

We define the *generalised XX -Hamiltonian* to be

$$H = \sum_{j=1}^{N-1} h_{j,j+1} \quad \text{where} \quad h_{j,j+1} = \sum_{r=1}^{D-1} \sum_{s=0}^{D-1} (X^r Z^s)_j \otimes (\omega^{-rs} X^{-r} Z^{-s})_{j+1} \quad (5.13)$$

This time the local interaction $h_{j,j+1}$ acts on states as

$$h_{j,j+1} |\alpha\rangle_j \otimes |\beta\rangle_{j+1} = \begin{cases} N |\beta\rangle_j \otimes |\alpha\rangle_{j+1} & \alpha \neq \beta \\ 0 & \alpha = \beta \end{cases} \quad \alpha, \beta \in \{0, \dots, D-1\} \quad (5.14)$$

and we see that this Hamiltonian is also a permutation Hamiltonian.

General permutation Hamiltonians

If H is a general permutation Hamiltonian then an initial state of the spin chain $|j\rangle_1 \otimes |0\rangle_{\widehat{1}}$ evolves in time to become

$$U(t) |j\rangle_1 \otimes |0\rangle_{\widehat{1}} = \sum_{k=1}^N f_{1,k}^{(j)}(t) |j\rangle_k \otimes |0\rangle_{\widehat{k}} \quad (5.15)$$

where $f_{1,k}^{(j)}(t)$ are matrix elements of the propagator $f_{1,k}^{(j)}(t) = \langle j|_k U(t) |j\rangle_1$ which, due to the unitarity of the propagator, satisfy $\sum_{k=1}^N |f_{1,k}^{(j)}(t)| = 1$.

This allows us to find an explicit expression for the entangled state shared by Alice and Bob

$$\begin{aligned} \chi_{rs} = & \frac{1}{D} \left(|-r, 0\rangle + \sum_{j=1}^{D-1} \omega^{js} f_{1,N}^{(j)} |j-r, j\rangle \right) \left(\langle -r, 0| + \sum_{k=1}^{D-1} \omega^{-ks} f_{1,N}^{(k)*} \langle k-r, k| \right) \\ & + \frac{1}{D} \sum_{j=1}^{D-1} \left(1 - |f_{1,N}^{(j)}|^2 \right) |j-r, 0\rangle \langle j-r, 0| \end{aligned} \quad (5.16)$$

(This calculation is straight forward but somewhat lengthy.) Observe that if $f_{1,N}^{(j)} = 1$ for all j (which corresponds to perfect state transfer along the spin chain) then $\chi_{rs} = |\Phi_{rs}\rangle \langle \Phi_{rs}|$ is a maximally entangled state as we would expect.

The expression (5.16) allows us to calculate the probabilities $p_{pq}^{(rs)}$ explicitly

$$p_{pq}^{(rs)} = \frac{1}{D^2} \left((1 - \delta_{rp}) \left(1 - |f_{1,N}^{(r-p)}|^2 \right) + \delta_{rp} \left| 1 + \sum_{j=1}^{D-1} \omega^{j(s-q)} f_{1,N}^{(j)} \right|^2 \right) \quad (5.17)$$

Substituting this result into equation (5.10) completes an explicit expression for the compression coefficients and demonstrates the exact manner in which D -dimensional quantum spin chains can be used as D -dimensional generalised depolarising channels.

5.2.2 Qubit spin chains

The expressions (5.16) and (5.17) are somewhat long and unwieldy, so in an attempt to demonstrate the essence of what they show we consider an arbitrary qubit spin chain and use this as a generalised qubit depolarising channel.

Suppose that Alice and Bob share the maximally entangled (Bell) state $|\Phi_{00}\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$. Then equation (5.16) reduces to

$$\chi_{00} = \left(\frac{|00\rangle + f|11\rangle}{\sqrt{2}} \right) \left(\frac{\langle 00| + f^* \langle 11|}{\sqrt{2}} \right) + \left(\frac{1 - |f|^2}{2} \right) |10\rangle \langle 10| \quad (5.18)$$

where, for notational convenience, we have written $f := f_{1,N}^{(1)}$. Using equation (5.17) we find that the probabilities for this spin chain are

$$p_{00}^{(00)} = \frac{|1+f|^2}{4} \quad p_{01}^{(00)} = \frac{|1-f|^2}{4} \quad p_{10}^{(00)} = \frac{1-|f|^2}{4} \quad p_{11}^{(00)} = \frac{1-|f|^2}{4} \quad (5.19)$$

which in turn allows us to use equation (5.10) to calculate the compression coefficients of the generalised depolarising channel

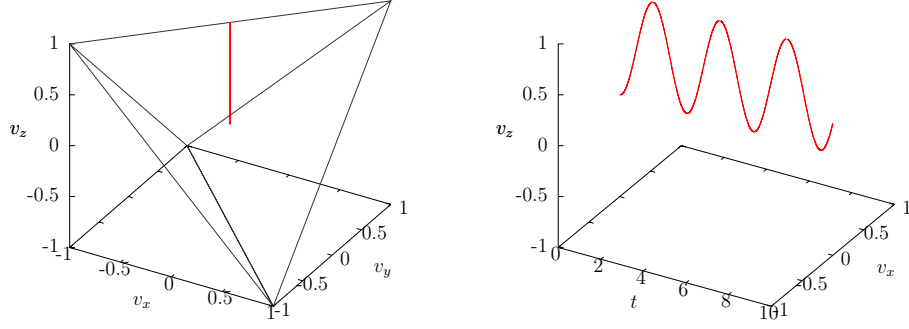
$$\begin{aligned} v_{00}^{(00)} &= p_{00}^{(00)} + p_{01}^{(00)} + p_{10}^{(00)} + p_{11}^{(00)} = 1 \\ v_{01}^{(00)} &= p_{00}^{(00)} + p_{01}^{(00)} - p_{10}^{(00)} - p_{11}^{(00)} = |f|^2 \\ v_{10}^{(00)} &= p_{00}^{(00)} - p_{01}^{(00)} + p_{10}^{(00)} - p_{11}^{(00)} = \frac{f+f^*}{2} = \text{Re}(f) \\ v_{11}^{(00)} &= p_{00}^{(00)} - p_{01}^{(00)} - p_{10}^{(00)} + p_{11}^{(00)} = \frac{f-f^*}{2} = \text{Im}(f) \end{aligned} \quad (5.20)$$

We know (see section 1.4) that all single qubit generalised depolarising channels lie within a certain tetrahedron in compression space. Numerical simulations of XX -model qubit spin chains give simple demonstrations of how the resultant generalised depolarising channels evolve (and how their compression vector wanders around the tetrahedron) as time progresses.

Figure 5.1 presents numerical simulations of the generalised depolarising channels (whose compression coefficients are given by equation (5.20)) which arise when Alice and Bob share the Bell state $|\Phi_{00}\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$ along a clean (noise-free) XX -model qubit spin chain. The top row shows the evolution of the generalised depolarising channel which arises from a chain of length $N = 2$ (which is known to allow perfect state transfer at certain times); the bottom row depicts a similar scenario but with a chain of length $N = 16$.

Figure 5.2 presents numerical simulations of the generalised depolarising channels which arise when Alice and Bob share the Bell state $|\Phi_{00}\rangle$ along a noisy XX -model qubit spin chain with static on-site disorder distributed according to a Cauchy distribution of width δ (see chapter 2). The top row shows the evolution of channels arising from low levels of disorder (that is, the Cauchy distribution is narrow, $\delta = 0.01$); the middle row depicts moderate

$$N = 2$$



$$N = 16$$

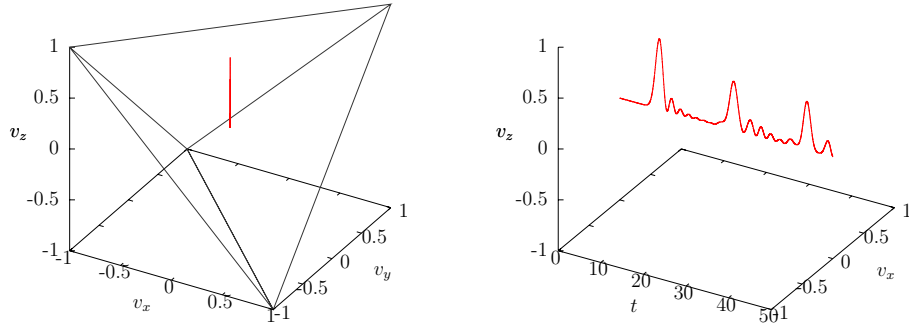
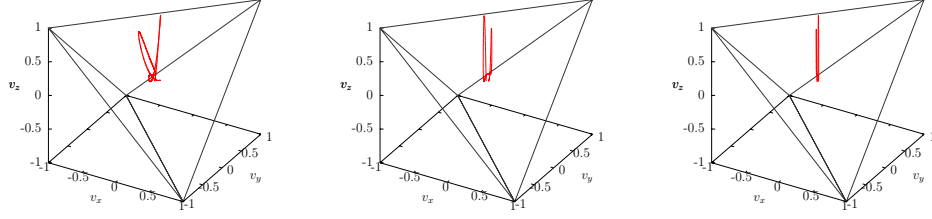
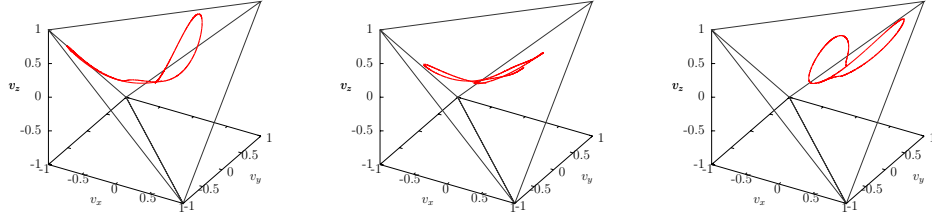


Figure 5.1: The generalised depolarising channels arising when Alice and Bob attempt to share the Bell state $|\Phi_{00}\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$ through clean (noise-free) XX -model spin chains before using this resource to attempt quantum teleportation. The left-hand diagrams show the line in the tetrahedron along which the generalised depolarising channels lie and the right-hand diagrams parameterise the evolution of the channels along this line. TOP ROW: a chain of length $N = 2$ evolving for times $0 \leq t \leq 10$; BOTTOM ROW: a chain of length $N = 16$ evolving for times $0 \leq t \leq 50$.

$$\delta = 0.01$$



$$\delta = 0.1$$



$$\delta = 1.0$$

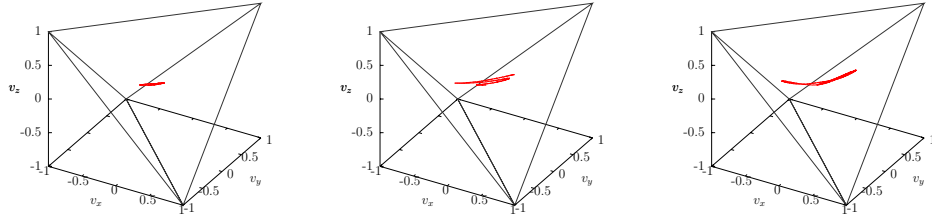


Figure 5.2: The generalised depolarising channels arising when Alice and Bob attempt to share the Bell state $|\Phi_{00}\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$ through noisy XX -model spin chains (with Cauchy-distributed static on-site disorder) before using this resource to attempt quantum teleportation. All chains depicted are of length $N = 4$ and evolution is shown for times $0 \leq t \leq 10$. TOP ROW: small disorder ($\delta = 0.01$); MIDDLE ROW: moderate disorder ($\delta = 0.1$); BOTTOM ROW: large disorder ($\delta = 1.0$).

levels of disorder ($\delta = 0.1$); and the bottom row depicts high levels of disorder ($\delta = 1.0$).

As one might expect, low levels of disorder give rise to channels which are close to the ones we obtain when the chain is noise-free, whilst moderate disorder levels produce significant variation from the noise-free scenario. High levels of disorder mean that Anderson localisation sets in (the localisation length of the energy eigenstates is shorter than the length of the chain) which results in Alice and Bob sharing minimal amounts of entanglement. This gives rise to channels which are close to the completely depolarising channel (which maps every input state to the completely mixed state). Note that for longer chains Anderson localisation sets in at lower levels of disorder, δ ; this is because the localisation length of the energy eigenstates (which scales as $1/\delta$) is shorter than the chain length N for all disorder levels greater than $\mathcal{O}(1/N)$.

5.3 Fluctuating noise as a generalised depolarising channel

In this section we show how a generalised version of the fluctuating on-site disorder model acting on a single qudit acts on the state of that qudit as a generalised depolarising channel.

Consider a single qudit experiencing on-site disorder which has one fluctuating component for each direction $\alpha \neq 0$. The field strengths $\xi^\alpha(t)$ are drawn at each instant in time from probability distributions \mathbb{P}_α with finite second moments. This is very similar to the noise model in chapter 4, but this time we allow the field strengths to have a different distribution in each direction.

Working with this noise model, we can generalise the derivation of the master equation given in chapter 3 by increasing the number of meters for each time interval t_r from one to $(D^2 - 1)$: one meter for each of the fluctuating fields $\xi^\alpha(t)$ with $\alpha \neq 0$. Following the same derivation as in chapter 3 (and

remembering that the M_α may not be hermitian) we arrive at the following master equation which governs the average case evolution of the state $\rho(t)$ of our qudit

$$\frac{d}{dt}\rho(t) = -i[H_0, \rho(t)] - \sum_{\alpha=1}^{D^2-1} \gamma_\alpha (M_\alpha M_\alpha \rho(t) + \rho(t) M_\alpha^\dagger M_\alpha^\dagger - 2M_\alpha \rho(t) M_\alpha^\dagger) \quad (5.21)$$

where H_0 is the intrinsic Hamiltonian of the qudit and $\gamma_\alpha \propto \mathbb{E}(\xi^\alpha)^2$ is proportional to the second moment of the probability distribution \mathbb{P}_α . If the basis $\{M_\alpha\}$ is hermitian then the dissipative term in this master equation reduces to the familiar double commutator of chapters 3 and 4.

We now restrict ourselves to working in a basis which is unitary, hermitian and which satisfies

$$M_\alpha M_\beta = e^{i\theta_{\alpha\beta}} M_\beta M_\alpha \quad \text{where} \quad \theta_{\alpha\beta} \in \mathbb{R} \quad (5.22)$$

This allows us to rewrite the dissipative term of the master equation as $\sum_{\alpha=0}^{D^2-1} 2\gamma_\alpha (\rho(t) - M_\alpha \rho(t) M_\alpha)$.

Assuming that the intrinsic Hamiltonian of the qudit is the identity $H_0 = \mathbb{I}$, we can use the master equation (5.21) and the expressions (5.1) and (5.22) to write the time derivative of the state $\rho(t)$ as

$$\frac{d}{dt}\rho(t) = -\frac{1}{D} \left(\mathbb{I} + \sum_{\alpha,\beta=1}^{D^2-1} \sqrt{\frac{D(D-1)}{\text{tr}(M_\beta^\dagger M_\beta)}} 2\gamma_\alpha (1 - e^{i\theta_{\alpha\beta}}) a_\beta(t) M_\beta \right) \quad (5.23)$$

From this differential equation, we are able to read off the differential equations satisfied by the polarisation coefficients $a_\beta(t)$, namely

$$\frac{d}{dt}a_\beta(t) = - \sum_{\alpha=1}^{D^2-1} 2\gamma_\alpha (1 - e^{i\theta_{\alpha\beta}}) a_\beta(t) \quad (5.24)$$

which have solution

$$a_\beta(t) = \exp \left(-2 \sum_{\alpha \neq 0} \gamma_\alpha (1 - e^{i\theta_{\alpha\beta}}) t \right) a_\beta(0) \quad (5.25)$$

This allows us to conclude that the fluctuating on-site disorder acts as a generalised depolarising channel on the state $\rho(t)$ of the qudit, with compression coefficients

$$v_\beta = \exp \left(-2 \sum_{\alpha \neq 0} \gamma_\alpha (1 - e^{i\theta_{\alpha\beta}}) t \right) \quad (5.26)$$

Chapter 6

Depolarising channels

In chapter 5 we showed how any quantum channel (and in particular a qudit spin chain) can be used as a generalised depolarising channel. In this chapter we study these generalised depolarising channels in detail, focusing on the structure (in compression space) of the set of all such channels with respect to a given basis.

Recall that the state of an arbitrary qudit (a D -dimensional quantum particle) can be written as

$$\rho = \frac{1}{D} \left(\mathbb{I} + \sum_{\alpha=1}^{D^2-1} \sqrt{\frac{D(D-1)}{\text{tr}(M_\alpha^\dagger M_\alpha)}} a_\alpha M_\alpha \right) \quad (6.1)$$

where the basis $\{M_\alpha\}_{\alpha=0}^{D^2-1}$ is trace-free and trace-orthogonal (except for $M_0 = \mathbb{I}$ which has non-zero trace). Also recall that a generalised depolarising channel $\Phi_{\mathbf{v}}$ (with respect to the basis $\{M_\alpha\}$) is a trace-preserving completely positive map which maps ρ into

$$\Phi_{\mathbf{v}}(\rho) = \frac{1}{D} \left(\mathbb{I} + \sum_{\alpha=1}^{D^2-1} \sqrt{\frac{D(D-1)}{\text{tr}(M_\alpha^\dagger M_\alpha)}} v_\alpha a_\alpha M_\alpha \right) \quad (6.2)$$

The polarisation coefficients a_α are the components of the polarisation vector $\mathbf{a} = (a_0, \dots, a_{D^2-1})$ and the compression coefficients v_α are the components of the compression vector $\mathbf{v} = (v_0, \dots, v_{D^2-1})$. (Recall that $a_0 \equiv 1$ so that ρ is normalised and $v_0 \equiv 1$ so that $\Phi_{\mathbf{v}}$ is trace-preserving.)

We make the following observation regarding the polarisation and compression coefficients

- If $M_\alpha = \gamma M_\beta^\dagger$ then the hermiticity of ρ tells us that $a_\alpha \gamma = a_\beta^*$ (where M^\dagger is the hermitian conjugate of M and a^* the complex conjugate of a). Furthermore, since $\Phi_{\mathbf{v}}(\rho)$ must be a quantum state (and must therefore be hermitian), $v_\alpha = v_\beta^*$.

We also note some special cases of the above observation, namely

- If $M_\alpha = \gamma M_\alpha^\dagger$ then $a_\alpha \gamma = a_\alpha^*$ and $v_\alpha = v_\alpha^* \in \mathbb{R}$.
- If M_α is hermitian ($\gamma = 1$) then $a_\alpha \in \mathbb{R}$ and $v_\alpha \in \mathbb{R}$.
- If M_α is skew-hermitian ($\gamma = -1$) then a_α is pure imaginary and $v_\alpha \in \mathbb{R}$.

If we work in a fixed basis $\{M_\alpha\}$ then it is clear that for each generalised depolarising channel there is a unique compression vector. It is therefore natural to ask the question “which vectors \mathbf{v} are valid compression vectors corresponding to depolarising channels $\Phi_{\mathbf{v}}$?” From the definition of a depolarising channel given above it is clear that the answer to this question is “a vector \mathbf{v} is a valid compression vector when the induced map $\Phi_{\mathbf{v}}$ is completely positive and $v_0 = 1$ ($\Phi_{\mathbf{v}}$ is trace-preserving)”. The purpose of this chapter is to identify which vectors \mathbf{v} meet this criterion.

It is clear from the observation $|v_\alpha| \leq 1$ (made in section 1.4) that all compression vectors \mathbf{v} which induce depolarising channels $\Phi_{\mathbf{v}}$ lie within the finite region $\mathbb{V} = \{\mathbf{v} \text{ such that } |v_\alpha| \leq 1 \text{ for all } \alpha\}$. However we will see that in general the converse is not true: not all vectors in \mathbb{V} induce depolarising channels. Indeed, we have already seen this for qubits: the set \mathbb{V} (which is the unit cube) is larger than the tetrahedron which encloses all possible compression vectors (see sections 1.4 and 6.2) and so there are vectors $\mathbf{v} \in \mathbb{V}$ which lie outside the tetrahedron and are therefore not valid compression vectors.

This chapter is organised as follows. We begin by making use of the Choi-Jamiolkowski representation (§ 6.1) to reformulate the problem of complete positivity into an equivalent problem concerning the positivity of large matrices. Having done this we discover the geometrical structure of the set of all possible compression vectors when we work in three specific bases: the Pauli basis (§ 6.2), the Gell-Mann basis (§ 6.3) and the Heisenberg-Weyl basis (§ 6.4). We also consider arbitrary bases (§ 6.5) and consider the effects of changing basis (§ 6.6) before showing how our formalism can be used to study more general channels (§ 6.7). Finally we conjecture that one can average any quantum channel with a generalised depolarising channel to obtain a mixed unitary channel (§ 6.8). We draw the chapter to a close with a brief summary of the results (§ 6.9).

6.1 Choi-Jamiolkowski representation

To help us decide which compression vectors \mathbf{v} induce completely positive maps $\Phi_{\mathbf{v}}$ (which are therefore generalised depolarising channels) we employ the *Choi-Jamiolkowski representation* of a map Φ

$$J(\Phi) = \sum_{j,k=0}^{D-1} \Phi(|j\rangle\langle k|) \otimes |j\rangle\langle k| \quad (6.3)$$

It is known [108, 109] that a map Φ is completely positive if and only if its Choi-Jamiolkowski representation $J(\Phi)$ is positive, which allows us to reformulate the problem of identifying completely positive maps into the problem of identifying positive matrices (which is a considerably easier task). In particular if

$$\Phi(\rho) = \text{tr}(A^\dagger \rho) B \quad (6.4)$$

where $A = \sum_{l,m=0}^{D-1} a_{lm} |l\rangle\langle m|$ and $B = \sum_{l,m=0}^{D-1} b_{lm} |l\rangle\langle m|$ are $D \times D$ matrices

then

$$\begin{aligned}
J(\Phi) &= \sum_{j,k=0}^{D-1} \text{tr} \left(\sum_{l,m=0}^{D-1} a_{ml}^* |l\rangle \langle m| j \rangle \langle k| \right) B \otimes |j\rangle \langle k| \\
&= \sum_{j,k=0}^{D-1} a_{jk}^* B \otimes |j\rangle \langle k| = B \otimes A^*
\end{aligned} \tag{6.5}$$

Since the basis $\{M_\alpha\}$ is trace-orthogonal, we may write

$$\rho = \sum_{\alpha=0}^{D^2-1} \frac{\text{tr}(M_\alpha^\dagger \rho)}{\text{tr}(M_\alpha^\dagger M_\alpha)} M_\alpha \quad \text{and} \quad \Phi_{\mathbf{v}}(\rho) = \sum_{\alpha=0}^{D^2-1} \frac{\text{tr}(M_\alpha^\dagger \rho)}{\text{tr}(M_\alpha^\dagger M_\alpha)} v_\alpha M_\alpha \tag{6.6}$$

which allows us to employ equation (6.5) to calculate the Choi-Jamiolkowski representation of an arbitrary map $\Phi_{\mathbf{v}}$

$$J(\Phi_{\mathbf{v}}) = \sum_{\alpha=0}^{D^2-1} v_\alpha \frac{M_\alpha \otimes M_\alpha^*}{\text{tr}(M_\alpha^\dagger M_\alpha)} \tag{6.7}$$

Depending upon the compression vector \mathbf{v} , the map $\Phi_{\mathbf{v}}$ may or may not be a generalised depolarising channel: $\Phi_{\mathbf{v}}$ is completely positive only when the eigenvalues of $J(\Phi)$ are all non-negative. In the following sections we work in different bases to determine precisely which vectors \mathbf{v} induce generalised depolarising channels $\Phi_{\mathbf{v}}$.

6.2 Pauli basis

In this section we prove a conjecture by Dixit and Sudarshan [4], which states that when $D = 2^d$ ($d \in \mathbb{N}$) and we work in the Pauli basis then the set of all valid compression vectors forms a simplex in compression space. We also illustrate our techniques with a worked example for single qubit generalised depolarising channels.

We restrict ourselves to a multiple qubit setting ($D = 2^d$) and choose the basis $\{M_\alpha\}$ to be formed of tensor products of the single qubit Pauli matrices. Let α be the number whose base-4 representation is $\alpha_d \alpha_{d-1} \cdots \alpha_1$

$$\alpha = \sum_{j=1}^d 4^{j-1} \alpha_j \tag{6.8}$$

Then we define the basis matrices M_α by

$$M_\alpha = \sigma^{\alpha_1} \otimes \cdots \otimes \sigma^{\alpha_d} \quad (6.9)$$

where $\sigma^0, \sigma^1, \sigma^2$ and σ^3 are the Pauli matrices $\mathbb{I}, \sigma^x, \sigma^y$ and σ^z . Note that this basis is trace-free and trace-orthogonal; furthermore each M_α is hermitian and so by the observation at the beginning of this chapter all a_α and v_α are real.

In this case it is a simple matter to find the eigenvectors and eigenvalues of $J(\Phi_{\mathbf{v}})$ and hence to determine which maps $J(\Phi_{\mathbf{v}})$ are depolarising channels. Since $J(\Phi)$ is a $D^2 \times D^2$ matrix (it is a *superoperator*) we can think of it as being an operator on $2d$ qubits (as $D^2 = 2^{2d}$). We define

$$|\Psi_{nm}\rangle_j = \sum_{r=0,1} (-1)^{rm} |r\rangle_j \otimes |r + n(\bmod 2)\rangle_{j+d} \quad \text{where } n, m \in \{0, 1\} \quad (6.10)$$

to be the maximally entangled Bell states on qubits j and $j + d$ (in more usual notation we have $|\Psi_{00}\rangle = |\Phi^+\rangle$, $|\Psi_{01}\rangle = |\Phi^-\rangle$, $|\Psi_{10}\rangle = |\Psi^+\rangle$ and $|\Psi_{11}\rangle = |\Psi^-\rangle$).

Observe the following relations

$$\begin{aligned} \sigma_j^0 \otimes (\sigma_{j+d}^0)^* |\Psi_{nm}\rangle_j &= |\Psi_{nm}\rangle_j \\ \sigma_j^1 \otimes (\sigma_{j+d}^1)^* |\Psi_{nm}\rangle_j &= (-1)^m |\Psi_{nm}\rangle_j \\ \sigma_j^2 \otimes (\sigma_{j+d}^2)^* |\Psi_{nm}\rangle_j &= (-1)^{n+m} |\Psi_{nm}\rangle_j \\ \sigma_j^3 \otimes (\sigma_{j+d}^3)^* |\Psi_{nm}\rangle_j &= (-1)^n |\Psi_{nm}\rangle_j \end{aligned} \quad (6.11)$$

which may be summarised as

$$\sigma_j^\alpha \otimes (\sigma_{j+d}^\alpha)^* |\Psi_{nm}\rangle_j = (-1)^{f(\alpha, n, m)} |\Psi_{nm}\rangle_j \quad (6.12)$$

where

$$f(\alpha, n, m) := \left\lfloor \frac{\alpha}{2} \right\rfloor n + \left\lfloor \frac{\alpha+1}{2} \right\rfloor m \quad (6.13)$$

It is now a straight forward matter to check that the eigenvectors $|J_{\mathbf{nm}}\rangle$ of the Choi-Jamiolkowski representation of an induced map $\Phi_{\mathbf{v}}$ are tensor products of these Bell states

$$|J_{\mathbf{nm}}\rangle = \bigotimes_{j=1}^d |\Psi_{n_j m_j}\rangle_j \quad \begin{aligned} \mathbf{n} &= (n_1, \dots, n_d) \\ \mathbf{m} &= (m_1, \dots, m_d) \end{aligned} \quad (6.14)$$

with corresponding eigenvalues

$$\lambda_{\mathbf{nm}} = \sum_{\alpha=0}^{D^2-1} \frac{v_{\alpha}}{D} (-1)^{\sum_{j=1}^d f(\alpha_j, n_j, m_j)} \quad (6.15)$$

Since there are $2^{2d} = D^2$ different eigenvectors $|J_{\mathbf{nm}}\rangle$ and eigenvalues $\lambda_{\mathbf{nm}}$, these are all the eigenvectors and eigenvalues of $J(\Phi)$. The important thing to notice here is that each $\lambda_{\mathbf{nm}}$ is a *linear* combination of the compression coefficients v_{α} .

We have therefore shown which vectors \mathbf{v} induce depolarising channels $\Phi_{\mathbf{v}}$ — namely those for which all eigenvalues of the Choi-Jamiolkowski representation of $\Phi_{\mathbf{v}}$ are non-negative: $\lambda_{\mathbf{nm}} \geq 0$ for all \mathbf{n} and \mathbf{m} . We are now in a position to prove a conjecture of Dixit and Sudarshan [4], which we present as the following theorem.

Theorem 4. *When $D = 2^d$ and we work in the Pauli basis, the set of all compression vectors forms a simplex in compression space.*

Proof. Since each $\lambda_{\mathbf{nm}}$ is linear in the compression coefficients v_{α} then the equation $\lambda_{\mathbf{nm}} = 0$ defines a hyperplane in compression space (which is a real Euclidean vector space of dimension $D^2 - 1$, with one dimension for each component of the compression vector excepting the zeroth component which we suppress as it is identically equal to 1).

Since $\Phi_{\mathbf{v}}$ is completely positive if and only if all eigenvalues $\lambda_{\mathbf{nm}}$ are non-negative, the hyperplanes $\lambda_{\mathbf{nm}} = 0$ must enclose precisely the set of vectors which induce completely positive depolarising channels $\Phi_{\mathbf{v}}$. In particular the hyperplanes enclose a finite region of compression space and the shape of this enclosed region must therefore be a simplex. \square

We now prove a small lemma before finding the *extremal channels* of the simplex (which are the depolarising channels whose compression vectors form the vertices of the simplex).

Lemma 5. *The eigenvalues of the Choi-Jamiołkowski representation of Φ_v sum to the system dimension, D .*

Proof. First recall that the sum of the eigenvalues of a matrix is simply the trace of that matrix. Then

$$\begin{aligned}
\sum_{\mathbf{n}, \mathbf{m}} \lambda_{\mathbf{n}\mathbf{m}} &= \text{tr}(J(\Phi_v)) \\
&= \text{tr} \left(\sum_{j,k=0}^{D-1} \Phi_v(|j\rangle \langle k|) \otimes |j\rangle \langle k| \right) \\
&= \text{tr} \left(\sum_{j=0}^{D-1} \Phi_v(|j\rangle \langle j|) \right) \\
&= D \text{tr} \Phi_v \left(\frac{\mathbb{I}}{D} \right) = D \text{tr} \left(\frac{\mathbb{I}}{D} \right) = D
\end{aligned} \tag{6.16}$$

□

Theorem 6. *The extremal channels are conjugations by the basis matrices*

$$\Phi^{(\alpha)}(\rho) := M_\alpha^\dagger \rho M_\alpha \quad \alpha \in \{0, \dots, D^2 - 1\} \tag{6.17}$$

Proof. First note the identity

$$\sigma^\beta \sigma^\alpha \sigma^\beta = (-1)^{g(\alpha, \beta)} \sigma^\alpha \quad \alpha, \beta \in \{0, 1, 2, 3\} \tag{6.18}$$

where

$$\begin{aligned}
g(\alpha, \beta) &\equiv \begin{cases} 1 \pmod{2} & \text{if } (\alpha, \beta) = (1, 2), (1, 3), (2, 3), \\ & (2, 1), (3, 1), (3, 2) \\ 0 \pmod{2} & \text{otherwise} \end{cases} \\
&\equiv \alpha\beta(\alpha - \beta)/2 \pmod{2}
\end{aligned} \tag{6.19}$$

We now fix β and prove that $\Phi^{(\beta)}$ is an extremal channel. First note that $\Phi^{(\beta)}$ is completely positive (see for example [5]) and apply the above identity to see that

$$\begin{aligned}
\Phi^{(\beta)}(\rho) &= \sum_{\alpha=0}^{D^2-1} \frac{\text{tr}(M_\beta^\dagger \rho)}{\text{tr}(M_\alpha^\dagger M_\alpha)} M_\beta^\dagger M_\alpha M_\beta \\
&= \sum_{\alpha=0}^{D^2-1} \frac{\text{tr}(M_\beta^\dagger \rho)}{\text{tr}(M_\alpha^\dagger M_\alpha)} (-1)^{\sum_{j=1}^d g(\alpha_j, \beta_j)} M_\alpha
\end{aligned} \tag{6.20}$$

It is then clear that $\Phi^{(\beta)}$ is a depolarising channel whose compression vector has components

$$v_\alpha = (-1)^{\sum_{j=1}^d g(\alpha_j, \beta_j)} \tag{6.21}$$

By combining expressions for the eigenvalue $\lambda_{\mathbf{pq}}$ (6.15) and the compression coefficients v_α (6.21) we see that

$$\lambda_{\mathbf{pq}} = \sum_{\alpha=0}^{D^2-1} \frac{1}{D} (-1)^{\sum_{j=1}^d g(\alpha_j, \beta_j) + f(\alpha_j, p_j, q_j)} \quad (6.22)$$

For fixed β and for all α , it is clear that if

$$s_{\alpha\beta\mathbf{pq}} := \sum_{j=1}^d g(\alpha_j, \beta_j) + f(\alpha_j, p_j, q_j) \equiv 0 \pmod{2} \quad (6.23)$$

then $\lambda_{\mathbf{pq}} = D$. We now show that there exist \mathbf{p} and \mathbf{q} such that (6.23) holds:

- When $\alpha = 4^{r-1}$ (i.e. $\alpha_r = 1$ and $\alpha_j = 0$ for all $j \neq r$) then

$$s_{\alpha\beta\mathbf{pq}} \equiv \frac{\beta_r(1 - \beta_r)}{2} + q_r \pmod{2} \quad (6.24)$$

and so (6.23) holds when

$$q_r \equiv \frac{\beta_r(1 - \beta_r)}{2} \pmod{2} \quad (6.25)$$

- When $\alpha = 3 \times 4^{r-1}$ (i.e. $\alpha_r = 3$ and $\alpha_j = 0$ for all $j \neq r$) then

$$s_{\alpha\beta\mathbf{pq}} \equiv \frac{3\beta_r(3 - \beta_r)}{2} + p_r \pmod{2} \quad (6.26)$$

and so (6.23) holds when

$$p_r \equiv \frac{3\beta_r(3 - \beta_r)}{2} \pmod{2} \quad (6.27)$$

- When $\alpha = 2 \times 4^{r-1}$ (i.e. $\alpha_r = 2$ and $\alpha_j = 0$ for all $j \neq r$) then

$$s_{\alpha\beta\mathbf{pq}} \equiv \frac{2\beta_r(2 - \beta_r)}{2} + p_r + q_r \pmod{2} \quad (6.28)$$

and so (6.23) holds when p_r and q_r are picked as in (6.25) and (6.27) above.

We have now shown that there exist \mathbf{p} and \mathbf{q} with $\lambda_{\mathbf{pq}} = D$. Since $\Phi^{(\beta)}$ is completely positive then all the eigenvalues of $J(\Phi^{(\beta)})$ are non-negative; they sum to D and we have found one which is equal to D ; therefore all other eigenvalues are zero

$$\lambda_{\mathbf{pq}} = D \quad \text{and} \quad \lambda_{\mathbf{nm}} = 0 \quad \text{for all } (\mathbf{n}, \mathbf{m}) \neq (\mathbf{p}, \mathbf{q}) \quad (6.29)$$

Clearly then the compression vector \mathbf{v} of the map $\Phi^{(\beta)}$ lies on all the hyperplanes $\lambda_{\mathbf{nm}} = 0$ except $\lambda_{\mathbf{pq}} = 0$, and so $\Phi^{(\beta)}$ must be an extremal channel.

To finish the proof, note that the simplex has D^2 vertices and there are D^2 channels of the form $\Phi^{(\beta)}$ so we have found all the extremal channels. \square

It is worth pointing out that any compression vector in the simplex can be written as a convex linear combination of the *extremal compression vectors* (which are the compression vectors forming the vertices of the simplex). This implies that any depolarising channel can be written as a convex linear combination of the extremal channels

$$\Phi_{\mathbf{v}} = \sum_{\alpha=0}^{D^2-1} p_{\alpha} \Phi^{(\alpha)}(\rho) \quad \text{where } 0 \leq p_{\alpha} \leq 1 \quad \text{and} \quad \sum_{\alpha=0}^{D^2-1} p_{\alpha} = 1 \quad (6.30)$$

Conversely, noting the relationship

$$v_{\alpha} = \sum_{\beta=0}^{D^2-1} p_{\beta} (-1)^{\sum_{j=1}^d g(\alpha_j, \beta_j)} \quad (6.31)$$

allows us to conclude that any channel of the form (6.30) is a depolarising channel.

Example 7. For a single qubit ($D = 2$) the compression space has dimension 3 and we see that for a general channel $\Phi_{\mathbf{v}}$

$$\begin{aligned} v_0 &= p_0 + p_1 + p_2 + p_3 = 1 \\ v_1 &= p_0 + p_1 - p_2 - p_3 \\ v_2 &= p_0 - p_1 + p_2 - p_3 \\ v_3 &= p_0 - p_1 - p_2 + p_3 \end{aligned} \quad (6.32)$$

and so we have the following correspondence between extremal channels and compression vectors (we suppress the zeroth component of v which is always equal to 1)

$$\begin{aligned}
\Phi^{(0)}(\rho) &= \mathbb{I}\rho\mathbb{I} \longleftrightarrow v = (1, 1, 1) \\
\Phi^{(1)}(\rho) &= X\rho X \longleftrightarrow v = (1, -1, -1) \\
\Phi^{(2)}(\rho) &= Y\rho Y \longleftrightarrow v = (-1, 1, -1) \\
\Phi^{(3)}(\rho) &= Z\rho Z \longleftrightarrow v = (-1, -1, 1)
\end{aligned} \tag{6.33}$$

6.3 Gell-Mann basis

In the previous section we restricted the dimension of the quantum system to be $D = 2^d$ so that we could employ the Pauli basis for multiple qubits. In this section we choose one possible generalisation of the Pauli basis, namely the Gell-Mann basis, which allows us to study systems of arbitrary dimension $D \in \mathbb{N}$. We will find that, in contrast to what was observed when we worked in the Pauli basis, the set of all generalised depolarising channels with respect to the Gell-Mann basis does not form a simplex in compression space.

The *Gell-Mann* basis is defined to be

$$\begin{aligned}
X_{jk} &:= |j\rangle\langle k| + |k\rangle\langle j| \\
Y_{jk} &:= -i(|j\rangle\langle k| - |k\rangle\langle j|) \\
Z_l &:= \sqrt{\frac{2}{l(l+1)}} \left(\sum_{r=0}^{l-1} |r\rangle\langle r| - l|l\rangle\langle l| \right)
\end{aligned} \quad \text{where } \begin{cases} j \in \{0, \dots, D-2\} \\ k, l \in \{1, \dots, D-1\} \\ j < k \end{cases} \tag{6.34}$$

For notational consistency with the rest of the chapter we identify

$$\begin{aligned}
M_0 &= \mathbb{I} \\
M_\alpha &= Z_\alpha \quad 1 \leq \alpha \leq D-1 \\
M_\alpha &= X_{jk} \quad \alpha = D(1+2j) + 2k - (j+1)(j+2) \\
M_\alpha &= Y_{jk} \quad \alpha = D(1+2j) + 2k - (j+1)(j+2) + 1
\end{aligned} \tag{6.35}$$

(i.e. $\{M_0, \dots, M_{D^2-1}\} = \{\mathbb{I}, Z_1, \dots, Z_{D-1}, X_{01}, Y_{01}, \dots, X_{D-2,D-1}, Y_{D-2,D-1}\}$)

This basis is trace-free and trace-orthogonal (and reduces to the Pauli basis when $D = 2$). Furthermore each M_α is hermitian and so all a_α and v_α are real.

We now attempt to find the eigenvectors and eigenvalues of $J(\Phi_v)$ and begin by defining

$$|J_{jk}^\pm\rangle = \frac{1}{\sqrt{2}} (|j, k\rangle \pm |k, j\rangle) \tag{6.36}$$

where $j, k \in \{0, \dots, D-1\}$ and $j < k$. It is a simple matter to check that

$$\begin{aligned}
\mathbb{I} \otimes \mathbb{I}^* |J_{jk}^\pm\rangle &= |J_{jk}^\pm\rangle \\
Z_l \otimes Z_l^* |J_{jk}^\pm\rangle &= \begin{cases} 0 |J_{jk}^\pm\rangle & l < k \\ \frac{-2}{l+1} |J_{jk}^\pm\rangle & l = k \\ \frac{2}{l(l+1)} |J_{jk}^\pm\rangle & l > k \end{cases} \\
X_{lm} \otimes X_{lm}^* |J_{jk}^\pm\rangle &= \begin{cases} \pm |J_{jk}^\pm\rangle & l = j \text{ and } m = k \\ 0 |J_{jk}^\pm\rangle & \text{else} \end{cases} \\
Y_{lm} \otimes Y_{lm}^* |J_{jk}^\pm\rangle &= \begin{cases} \mp |J_{jk}^\pm\rangle & l = j \text{ and } m = k \\ 0 |J_{jk}^\pm\rangle & \text{else} \end{cases}
\end{aligned} \tag{6.37}$$

and so it is clear that $|J_{jk}^\pm\rangle$ are eigenvectors of $J(\Phi_v)$ and that the corresponding eigenvalues λ_{jk}^\pm are linear in the compression coefficients v_α .

Now, $J(\Phi_v)$ has D^2 eigenvalues and we have found $D^2 - D$ of them. We let the remaining D eigenvalues be λ_j ($j \in \{0, \dots, D-1\}$). In order to establish which vectors in compression space induce depolarising channels we must now find when these remaining D eigenvalues of $J(\Phi_v)$ are non-negative.

Since $M_\alpha \otimes M_\alpha^*$ is symmetric for all α , $J(\Phi_v)$ is also symmetric and consequently has real eigenvalues. By considering the matrix elements of $J(\Phi_v)$ carefully we see that the only non-zero entries in the two columns indexed by $\langle j, k|$ and $\langle k, j|$ (with $j < k$) are in the two rows indexed by $|j, k\rangle$ and $|k, j\rangle$. We may conjugate $J(\Phi_v)$ by a permutation matrix P (to permute the rows and columns) to form a matrix $J'(\Phi_v)$ which has identical eigenvalues to $J(\Phi_v)$. By repeatedly conjugating by permutation matrices we can block-diagonalise $J(\Phi_v)$ to obtain a matrix $\tilde{J}(\Phi_v)$ which has the following structure

$$\tilde{J}(\Phi_v) = \begin{pmatrix} [2 \times 2] & & & \\ & \ddots & & \\ & & [2 \times 2] & \\ & & & [K(\Phi_v)] \end{pmatrix} \tag{6.38}$$

There are $D(D-1)/2$ blocks of size 2×2 (each of which has two eigenvalues, λ_{jk}^+ and λ_{jk}^- , for some j and k) and a large block of size $D \times D$ called $K(\Phi_v)$ (which has eigenvalues $\lambda_0, \dots, \lambda_{D-1}$).

By considering the characteristic equation of the matrix $K(\Phi_v)$ we see that the eigenvalues $\lambda_0, \dots, \lambda_{D-1}$ satisfy

$$0 = (t - \lambda_0) \cdots (t - \lambda_{D-1}) =: \sum_{j=0}^D (-1)^j t^{D-j} S_j \quad (6.39)$$

where we have defined the *eigenvalue sums* S_j to be

$$S_0 := 1 \quad \text{and (for } j \in \{1, \dots, D\}) \quad S_j := \sum_{0 \leq k_1 < \dots < k_j \leq D-1} \lambda_{k_1} \cdots \lambda_{k_j} \quad (6.40)$$

The following lemma proves that the eigenvalues $\lambda_0, \dots, \lambda_{D-1}$ are all non-negative precisely when the eigenvalue sums S_0, \dots, S_D are all non-negative.

Lemma 8. $\lambda_j \geq 0$ for all $j \in \{0, \dots, D-1\}$ if and only if $S_k \geq 0$ for all $k \in \{0, \dots, D\}$.

Proof. One way is trivial: if $\lambda_j \geq 0$ for all j then $S_k \geq 0$ for all k . We prove the converse by contradiction.

Assume that $S_k \geq 0$ for all $k \in \{1, \dots, D\}$, recall that $S_0 = 1$ and note that for any $j \in \{0, \dots, D-1\}$

$$\begin{aligned} S_D &= \lambda_j (S_{D-1} - \lambda_j (S_{D-2} - \cdots - \lambda_j (S_1 - \lambda_j) \cdots)) \\ &= \sum_{k=1}^D (-1)^{k-1} \lambda_j^k S_{D-k} \end{aligned} \quad (6.41)$$

(this is essentially an inclusion-exclusion argument). But then we may split this sum up into two terms

$$S_D = \lambda_j \sum_{k \text{ odd}} \lambda_j^{k-1} S_{D-k} - \sum_{k \text{ even}} \lambda_j^k S_{D-k} \quad (6.42)$$

It is clear that these sums over k odd and k even are both non-negative since each S_j is non-negative and λ_j appears to an even power in each term. But then $\lambda_j < 0$ implies that $S_D < 0$ also, which contradicts the assumption that $S_D \geq 0$; we therefore conclude that the eigenvalues $\lambda_j \geq 0$ are all non-negative. Since this argument holds for all $j \in \{0, \dots, D-1\}$ the proof is complete. \square

Returning to $J(\Phi_{\mathbf{v}})$ we see from equation (6.7) that each matrix element consists of a linear combination of the compression coefficients v_{α} , a property which is inherited by $K(\Phi_{\mathbf{v}})$. Careful consideration reveals that S_j is a j th-order polynomial in the compression coefficients

$$S_j = \sum_{0 \leq \alpha_1 \leq \dots \leq \alpha_j \leq D^2-1} c_{\alpha_1 \dots \alpha_j} v_{\alpha_1} \dots v_{\alpha_j} \quad (6.43)$$

It is tempting to conclude that $S_j = 0$ is a j th-order surface in compression space, but we must be careful: it is possible that all j th-order coefficients $c_{\alpha_1 \dots \alpha_j}$ (*i.e.* those with $\alpha_k \neq 0$ for all k) are equal to zero, in which case the degree of the surface $S_j = 0$ is strictly less than j . For example, when $D = 2$ the Gell-Mann basis reduces to the Pauli basis and we have already seen in the previous section that all eigenvalues are linear in the compression coefficients; in this case the surface $S_2 = 0$ is a plane and not a quadratic surface. In general we may only conclude that $S_j = 0$ is a surface of order at most j .

We have now proved that the set of all vectors \mathbf{v} which induce depolarising channels $\Phi_{\mathbf{v}}$ (with respect to the Gell-Mann basis) form a finite region in compression space which is bounded by

- $D^2 - D + 1$ hyperplanes
($D^2 - D$ are given by $\lambda_{jk}^{\pm} = 0$ and the remaining one is given by $S_1 = 0$)
- A surface with order at most 2 ($S_2 = 0$)
- \vdots
- A surface with order at most D ($S_D = 0$)

Note that the above does not tell us whether the eigenvalues $\lambda_0, \dots, \lambda_{D-1}$ are linear in the compression coefficients. It turns out that when $D > 2$ at least one of the λ_j must be non-linear in the compression coefficients (a fact which is proved in lemma 13 in section 6.5).

Whilst it is a slight disappointment that we have been unable to explicitly give expressions for all the eigenvalues of $J(\Phi_{\mathbf{v}})$, we can draw some solace from the fact that we have simplified the problem somewhat. In order to see if a vector \mathbf{v} induces a depolarising channel we must check to see if all the eigenvalues of the Choi-Jamiołkowski representation $J(\Phi_{\mathbf{v}})$ of the induced map $\Phi_{\mathbf{v}}$ are non-negative. Without the above results we would have to use a “brute-force” algorithm to calculate the D^2 eigenvalues of $J(\Phi_{\mathbf{v}})$ directly but the above results enable us to calculate $D^2 - D$ of these quickly and efficiently, leaving only D to be calculated by the brute-force algorithm (which is a substantial improvement and computational speed-up).

6.4 Heisenberg-Weyl basis

In this section we work in the Heisenberg-Weyl basis which is an alternative generalisation of the single-qubit Pauli basis to arbitrary dimension D . It has certain advantages over the Gell-Mann basis (for example all the Heisenberg-Weyl basis matrices are invertible whilst most of the Gell-Mann basis matrices are singular). However there is a price to pay for such convenience: the Heisenberg-Weyl basis is not hermitian and so the compression space is a complex vector space. We will show that when we work in the Heisenberg-Weyl basis, the set of all generalised depolarising channels forms a simplex in compression space.

Let us first recall the definition of the *Heisenberg-Weyl* basis

$$M_{\alpha} = M_{jk} := X^j Z^k \quad \begin{array}{l} \text{where } j, k \in \{0, \dots, D-1\} \\ \text{and } \alpha = jD + k \end{array} \quad (6.44)$$

where X and Z are defined to be

$$X := \sum_{j=0}^{D-1} |j\rangle \langle j+1 \pmod{D}| \quad \text{and} \quad Z := \sum_{j=0}^{D-1} \omega^j |j\rangle \langle j| \quad (6.45)$$

and where $\omega = e^{2\pi i/D}$ is the primitive D th root of unity. Note that this basis is trace-free and trace-orthogonal.

The r th powers ($r \in \{0, \dots, D-1\}$) of X and Z are $X^r = \sum_{j=0}^{D-1} |j\rangle \langle j+r|$ and $Z^r = \sum_{j=0}^{D-1} \omega^{jr} |j\rangle \langle j|$ and their inverses are X^{-r} and Z^{-r} .

We can now find the eigenvectors and eigenvalues of $J(\Phi_{\mathbf{v}})$ by defining

$$|J_{jk}\rangle = \sum_{r=0}^{D-1} \omega^{kr} |r, r+j\rangle \quad (6.46)$$

Then it is a simple matter to check that

$$M_{jk} \otimes M_{jk}^* |J_{lm}\rangle = \omega^{mj-kl} |J_{lm}\rangle \quad (6.47)$$

which shows us that $|J_{lm}\rangle$ are eigenvectors of $J(\Phi_{\mathbf{v}})$. Furthermore, there are D^2 distinct $|J_{lm}\rangle$ and we have therefore found all the eigenvectors of $J(\Phi_{\mathbf{v}})$. Relabelling the compression coefficients v_{α} as v_{jk} (with $\alpha = Dj + k$) we see that the eigenvalues of $J(\Phi_{\mathbf{v}})$ are

$$\lambda_{lm} = \sum_{j,k=0}^{D-1} \frac{v_{jk} \omega^{mj-kl}}{D} \quad (6.48)$$

The identity $X^r Z^s = \omega^{rs} Z^s X^r$ allows us to show that $M_{jk}^{\dagger} = \omega^{-jk} M_{-j, -k}$. Therefore, by the observation at the start of this chapter, $v_{jk} = v_{-j, -k}^*$.

In sections 6.2 and 6.3 we had one real compression coefficient for each basis matrix. In this section we have seen that most of the compression coefficients come in complex-conjugate pairs corresponding to two basis matrices which are, up to a complex multiplicative factor, hermitian conjugates of each other. Note that we still have the same number of free compression parameters (since for each pair of conjugate basis matrices there are two free parameters in the associated compression coefficients, namely the real and imaginary parts). Furthermore, the compression space (although now complex) still has dimension $D^2 - 1$ (recall that we suppress the dimension corresponding to v_0 as it is identically equal to 1).

Aside 9. *As an aside, we find the exact structure of the compression space. We use the relation $M_{jk}^{\dagger} = \omega^{-jk} M_{-j, -k}$ to see that $M_{jk} = \gamma M_{jk}^{\dagger}$ precisely when $j \equiv -j \pmod{D}$ and $k \equiv -k \pmod{D}$.*

When D is odd the only solution is $j = 0$ and $k = 0$ (i.e. $M_{jk} = \mathbb{I}$). In this case compression space consists of $(D^2 - 1)/2$ complex planes.

When D is even there are four solutions, namely $j, k \in \{0, D/2\}$. In this case, compression space consists of 3 real axes and $(D^2 - 4)/2$ complex planes.

We now prove that, when working in the Heisenberg-Weyl basis, the set of all compression vectors forms a simplex. We then find the extremal channels whose compression vectors lie at the vertices of this simplex.

Theorem 10. *When we work in the Heisenberg-Weyl basis, the set of all compression vectors forms a simplex in compression space.*

Proof. It is evident from equation (6.48) that the eigenvalues of $J(\Phi_{\mathbf{v}})$ are linear in the compression coefficients and so $\lambda_{lm} = 0$ defines a hyperplane in compression space.

Since $\Phi_{\mathbf{v}}$ is completely positive if and only if all eigenvalues λ_{lm} are non-negative, the hyperplanes $\lambda_{lm} = 0$ must enclose precisely the set of vectors which induce completely positive depolarising channels $\Phi_{\mathbf{v}}$. In particular the hyperplanes enclose a finite region of compression space and the shape of this enclosed region must therefore be a simplex. \square

Theorem 11. *The extremal channels are*

$$\Phi^{(jk)}(\rho) := M_{jk}^\dagger \rho M_{jk} \quad j, k \in \{0, \dots, D-1\} \quad (6.49)$$

Proof. We fix j and k and prove that $\Phi^{(jk)}$ is extremal. Using the identity $M_{jk}^\dagger M_{lm} M_{jk} = \omega^{lk-mj} M_{lm}$ we see that

$$\Phi^{(jk)}(\rho) = \sum_{l,m=0}^{D-1} \frac{\text{tr}(M_{lm}\rho)}{D} \omega^{lk-mj} M_{lm} \quad (6.50)$$

and so $\Phi^{(jk)}(\rho) = \Phi_{\mathbf{v}}(\rho)$ is a depolarising channel with compression coefficients $v_{lm} = \omega^{lk-mj}$.

We now show that there exists p and q with $\lambda_{pq} = D$. We see that

$$\lambda_{pq} = \sum_{l,m=0}^{D-1} \frac{\omega^{lk-mj} \omega^{lq-mp}}{D} \quad (6.51)$$

and so picking $p = -j(\text{mod } D)$ and $q = -k(\text{mod } D)$ ensures that $\lambda_{pq} = D$.

As before we make use of the fact that the sum of the eigenvalues of the Choi-Jamiolkowski representation $J(\Phi^{(jk)})$ is equal to the system dimension D : as $\Phi^{(jk)}$ is completely positive the eigenvalues of $J(\Phi^{(jk)})$ are non-negative and sum to D and so we may conclude that

$$\lambda_{pq} = D \quad \text{and} \quad \lambda_{lm} = 0 \quad \text{for all } (l, m) \neq (p, q) \quad (6.52)$$

Therefore the compression vector corresponding to the channel $\Phi^{(jk)}$ lies on all the hyperplanes except one and so it lies at a vertex of the simplex.

To finish, note that there are D^2 vertices and D^2 distinct extremal channels so we have found them all. \square

As any compression vector in the simplex can be written as a convex linear combination of the extremal compression vectors then theorem 11 allows us to conclude that any depolarising channel can be written as a convex linear combination of the extremal channels. Conversely any convex linear combination of the extremal channels is a depolarising channel. Note the similarity here with the Pauli basis.

6.5 Other bases

In sections 6.2, 6.3 and 6.4 we worked in some of the most common bases, but what happens if we wish to work in another basis? What can be said about the set of all compression vectors? Does it form a simplex in compression space? We answer these questions here, giving necessary and sufficient conditions that an arbitrary trace-free and trace-orthogonal basis $\{M_\alpha\}$ must satisfy for the compression vectors to form a simplex.

We know that the set of all compression vectors forms a simplex when the eigenvalues of $J(\Phi_{\mathbf{v}})$ are linear combinations of the compression coefficients v_{α} and we now find all trace-free, trace-orthogonal bases for which this happens.

Theorem 12. *Let $\{M_{\alpha}\}$ be a trace-free and trace-orthogonal basis. Then the set of all compression vectors forms a simplex if and only if*

$$[M_{\alpha} \otimes M_{\alpha}^*, M_{\beta} \otimes M_{\beta}^*] = 0 \quad \text{for all } \alpha, \beta \in \{0, \dots, D^2 - 1\} \quad (6.53)$$

Proof. The set of all compression vectors forms a simplex if and only if all the eigenvalues of $J(\Phi_{\mathbf{v}})$ are linear in the compression coefficients (in which case if λ is an eigenvalue of $J(\Phi_{\mathbf{v}})$ then $\lambda = 0$ defines a hyperplane, which forms one of the facets of the simplex).

Let $|\lambda\rangle$ be an eigenvector of $J(\Phi_{\mathbf{v}})$ with eigenvalue λ . Then by equation (6.7) we see that λ is a linear combination of the compression coefficients v_{α} if and only if $|\lambda\rangle$ is a simultaneous eigenvector of each $M_{\alpha} \otimes M_{\alpha}^*$; this occurs if and only if all the $M_{\alpha} \otimes M_{\alpha}^*$ are simultaneously diagonalisable, which occurs if and only if $[M_{\alpha} \otimes M_{\alpha}^*, M_{\beta} \otimes M_{\beta}^*] = 0$ for all α and β . \square

Note that the condition of theorem 12 is weaker than $[M_{\alpha}, M_{\beta}] = 0$ for all α and β . For example, $M_{\alpha}M_{\beta} = e^{i\theta_{\alpha\beta}}M_{\beta}M_{\alpha}$ satisfies $[M_{\alpha} \otimes M_{\alpha}^*, M_{\beta} \otimes M_{\beta}^*] = 0$ but not $[M_{\alpha}, M_{\beta}] = 0$. Furthermore, it is not hard to show that $M_{\alpha}M_{\beta} = e^{i\theta_{\alpha\beta}}M_{\beta}M_{\alpha}$ is equivalent to $[M_{\alpha} \otimes M_{\alpha}^*, M_{\beta} \otimes M_{\beta}^*] = 0$.

We can now prove the outstanding result from section 6.3, which we present as the following lemma.

Lemma 13. *When $D > 2$ and we work in the Gell-Mann basis then the set of all compression vectors has at least one curved side. Equivalently, at least one of the eigenvalues of $J(\Phi_{\mathbf{v}})$ is non-linear in the compression coefficients v_{α} .*

Proof. It is a simple matter to check that

$$[X_{01} \otimes X_{01}^*, X_{02} \otimes X_{02}^*] \neq 0 \quad (6.54)$$

Then apply theorem 12 above. \square

For the remainder of this section we restrict ourselves to working in a basis which is unitary: $M_\alpha^{-1} = M_\alpha^\dagger$ for all α . When we do this we are able to find the extremal channels whose compression vectors lie at the vertices of the simplex. Before proving this, in a slight change of notation from chapter 5, we first define the maximally entangled states

$$|\Psi\rangle = \sum_{j=0}^{D-1} |j, j\rangle \quad \text{and} \quad |\lambda_\alpha\rangle = (M_\alpha \otimes \mathbb{I}) |\Psi\rangle \quad (6.55)$$

We are now in a position to state and prove a small lemma which states that these maximally entangled states are the eigenstates of the Choi-Jamiolkowski representation of a map Φ_v .

Lemma 14. *If $M_\alpha M_\beta = e^{i\theta_{\alpha\beta}} M_\beta M_\alpha$ for all α and β and if M_α is unitary for all α then the maximally entangled states $|\lambda_\alpha\rangle$ are all the eigenstates of $J(\Phi_v)$.*

Proof. The expansion $M_\alpha = \sum_{j,k=0}^{D-1} \langle j| M_\alpha |k\rangle |j\rangle \langle k|$ allows us to write

$$|\lambda_\alpha\rangle = \sum_{j,k=0}^{D-1} \langle j| M_\alpha |k\rangle |j, k\rangle \quad (6.56)$$

Noting that $\theta_{\alpha\beta} = -\theta_{\beta\alpha}$ allows us to calculate

$$M_\beta \otimes M_\beta^* |\lambda_\alpha\rangle = \sum_{j,k=0}^{D-1} \langle j| M_\beta M_\alpha M_\beta^\dagger |k\rangle |j, k\rangle = e^{-i\theta_{\alpha\beta}} |\lambda_\alpha\rangle \quad (6.57)$$

from which it is easy to see that

$$J(\Phi_v) |\lambda_\alpha\rangle = \sum_{\beta=0}^{D^2-1} \frac{v_\beta e^{-i\theta_{\alpha\beta}}}{\text{tr}(M_\beta^\dagger M_\beta)} |\lambda_\alpha\rangle =: \lambda_\alpha |\lambda_\alpha\rangle \quad (6.58)$$

So $|\lambda_\alpha\rangle$ is indeed an eigenstate of $J(\Phi_v)$. Finally note that there are D^2 eigenstates $|\lambda_\alpha\rangle$ so we have found all the eigenstates of $J(\Phi_v)$. \square

Lemma 14 allows us to find the extremal channels of the simplex.

Theorem 15. *If $M_\alpha M_\beta = e^{i\theta_{\alpha\beta}} M_\beta M_\alpha$ for all α and β and if M_α is unitary for all α then the extremal channels are*

$$\Phi^{(\alpha)}(\rho) \equiv M_\alpha \rho M_\alpha^\dagger \quad (6.59)$$

Proof. First note that $\Phi^{(\alpha)}(\rho)$ is completely positive and trace-preserving, so the eigenvalues of $J(\Phi^{(\alpha)})$ are all positive and sum to D . We can expand

$$\Phi^{(\alpha)}(\rho) = \sum_{\beta=0}^{D^2-1} \frac{\text{tr}(M_{\beta}^{\dagger} \rho)}{\text{tr}(M_{\beta}^{\dagger} M_{\beta})} e^{i\theta_{\alpha\beta}} M_{\beta} \quad (6.60)$$

which shows that $\Phi^{(\alpha)}$ is a depolarising channel with compression coefficients $v_{\beta} = e^{i\theta_{\alpha\beta}}$. Now observe that

$$\lambda_{\alpha} = \sum_{\beta=0}^{D^2-1} \frac{e^{i\theta_{\alpha\beta}} e^{-i\theta_{\alpha\beta}}}{D} = D \quad (6.61)$$

and so $\lambda_{\beta} = 0$ for all $\beta \neq \alpha$. Clearly then $\Phi^{(\alpha)}$ is an extremal channel as its compression vector lies on all the hyperplanes $\lambda_{\beta} = 0$ except $\lambda_{\alpha} = 0$. To finish, note that there are D^2 vertices to the simplex and there are D^2 extremal channels $\Phi^{(\alpha)}$ so we have found them all. \square

6.6 Changing basis

Why does the set of all compression vectors form a simplex in some bases (such as the Heisenberg-Weyl basis) and not in others (such as the Gell-Mann basis)? We answer this question here by investigating the effect of changing basis and we give an explicit example to illustrate our point.

Let $\{M_{\alpha}\}$ and $\{L_{\alpha}\}$ be trace-free, trace-orthogonal bases. Then we may write the relationship between the two bases as

$$\frac{M_{\alpha}}{\sqrt{\text{tr}(M_{\alpha}^{\dagger} M_{\alpha})}} = \sum_{\beta=0}^{D^2-1} u_{\alpha\beta} \frac{L_{\beta}}{\sqrt{\text{tr}(L_{\beta}^{\dagger} L_{\beta})}} \quad \text{where } u_{\alpha\beta} \in \mathbb{C} \quad (6.62)$$

Trace-orthogonality of both bases ensures that the $D^2 \times D^2$ change-of-basis matrix $U = \sum_{\alpha, \beta=0}^{D^2-1} u_{\alpha\beta} |\alpha\rangle \langle \beta|$ is unitary; furthermore, if both bases are hermitian then U is actually a real orthogonal matrix. If we define the vector

$$\widehat{\mathbf{M}} = \left(\frac{M_0}{\sqrt{\text{tr}(M_0^{\dagger} M_0)}}, \dots, \frac{M_{D^2-1}}{\sqrt{\text{tr}(M_{D^2-1}^{\dagger} M_{D^2-1})}} \right) \quad (6.63)$$

and similarly define $\widehat{\mathbf{L}}$, then we may write $\widehat{\mathbf{M}} = U\widehat{\mathbf{L}}$.

We are now in a position to prove a special result for single qubits.

Theorem 16. *The set of single qubit ($D = 2$) compression vectors forms a simplex whenever we work in a hermitian, trace-free and trace-orthogonal basis.*

Furthermore, the extremal channels are conjugations by the basis matrices.

Proof. We know that any single qubit basis which is hermitian, trace-free and trace-orthogonal can be obtained from the Pauli basis by an orthogonal change of basis. Let $\{L_\alpha\}$ be the Pauli basis and $\{M_\alpha\}$ be the new basis. Then the change of basis matrix has the form

$$U = \left(\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & \mathcal{O} & \\ 0 & & & \end{array} \right) \quad (6.64)$$

where \mathcal{O} is a 3×3 orthogonal matrix. Now, any such matrix \mathcal{O} corresponds to a rotation of the Bloch ball and is equivalent to a unitary conjugation $\rho \mapsto V^\dagger \rho V$ where V is a 2×2 unitary matrix. Any such map is completely positive and trace-preserving. Depolarisation with respect to the basis $\{M_\alpha\}$ is the same as applying the reverse of the above rotation, followed by depolarisation in the Pauli basis, followed by the the above rotation. Since the set of all compression vectors forms a simplex when we work in the Pauli basis, the set of all compression vectors also forms a simplex when we work in the new basis $\{M_\alpha\}$.

We know that in the Pauli basis the extremal channels are $\Phi^{(\alpha)}(\rho) = L_\alpha^\dagger \rho L_\alpha$. So when we start in the new basis and rotate to the Pauli basis, the extremal channels will be $\Phi^{(\alpha)}(V \rho V^\dagger) = L_\alpha^\dagger V \rho V^\dagger L_\alpha$; rotating back to the $\{M_\alpha\}$ basis (and noting that $M_\alpha = V^\dagger L_\alpha V$) we see that the extremal channels are $\Phi^{(\alpha)}(\rho) = M_\alpha^\dagger \rho M_\alpha$. \square

In the single qubit case it is true that any unitary change of basis U corresponds to an orthogonal rotation of the polarisation vector \mathbf{a} ; in higher

dimensions this is not so as the following example demonstrates.

Example 17. *Let $D = 4$ and let*

$$\rho = \frac{\mathbb{I}}{4} + \frac{k\sqrt{3}}{4}\sigma^x \otimes \sigma^x \quad k \in \mathbb{R} \quad (6.65)$$

One can explicitly calculate the eigenvalues of ρ : they are $\lambda = (1 \pm k\sqrt{3})/4$ each with multiplicity two. Therefore ρ is a state when $|k| \leq 1/\sqrt{3}$.

Let us now apply any change of basis which maps $\frac{\sigma^x \otimes \sigma^x}{2} \mapsto \frac{Z_3}{\sqrt{2}}$ (that is, we change basis from the hermitian Pauli basis to the hermitian Gell-Mann basis with a unitary change-of-basis matrix U). Then

$$\rho \mapsto \tilde{\rho} = \frac{\mathbb{I}}{4} + \frac{k\sqrt{6}}{4}Z_3 \quad (6.66)$$

which has eigenvalues $\lambda = (1 + k)/4$ (with multiplicity 3) and $\lambda = (1 - 3k)/4$ (with multiplicity 1). But then if $1/\sqrt{3} \geq k > 1/3$ then ρ is a quantum state but $\tilde{\rho}$ is not (as it is not positive).

This example demonstrates that some changes of basis do not map the Bloch “ball” on to itself. It is for this reason that the set of all compression vectors forms a simplex in some bases but not in others.

Theorem 18. *When two trace-free, trace-orthogonal bases $\{M_\alpha\}$ and $\{L_\alpha\}$ are related via*

$$M_\alpha = V^\dagger L_\alpha V \quad (6.67)$$

for all α and some unitary matrix V (that is, the change-of-basis matrix U maps the Bloch “ball” on to itself), then the set of compression vectors either forms a simplex in both bases or does not form a simplex in either basis.

Furthermore, if the extremal channels are conjugations of basis matrices in one basis then they are conjugations of basis matrices in the other basis too.

Proof. One may adapt the proof of theorem 16. (Note that the condition $M_\alpha = V^\dagger L_\alpha V$ for all α guarantees that the Bloch “ball” is rotated onto itself and we avoid the problems exhibited in example 17.) \square

6.7 More general channels

Generalised depolarising channels compress the Bloch “ball” anisotropically and we have now studied them in much detail. However, our method can be extended to deal with a more general class of channels, namely those which both compress the Bloch “ball” and then translate it. Let $\Phi_{\mathbf{v},\mathbf{t}}$ be such a channel defined by its action on the polarisation vector

$$\Phi_{\mathbf{v},\mathbf{t}} : \begin{pmatrix} a_0 \\ \vdots \\ a_{D^2-1} \end{pmatrix} \mapsto \begin{pmatrix} v_0 a_0 \\ \vdots \\ v_{D^2-1} a_{D^2-1} \end{pmatrix} + \begin{pmatrix} t_0 \\ \vdots \\ t_{D^2-1} \end{pmatrix} \quad (6.68)$$

We call the vector $\mathbf{t} = (t_0, \dots, t_{D^2-1})$ the *translation vector* and the t_α the *translation coefficients*. Note that $t_0 = 0$ to ensure that $\Phi_{\mathbf{v},\mathbf{t}}$ is trace-preserving. So far we have only studied channels $\Phi_{\mathbf{v}} := \Phi_{\mathbf{v},\mathbf{0}}$, but we now extend to general \mathbf{t} . We may expand

$$\Phi_{\mathbf{v},\mathbf{t}}(\rho) = \frac{1}{D} \left(\mathbb{I} + \sum_{\alpha=1}^{D^2-1} \sqrt{\frac{D(D-1)}{\text{tr}(M_\alpha^\dagger M_\alpha)}} (v_\alpha a_\alpha + t_\alpha) M_\alpha \right) \quad (6.69)$$

If we now define the compression components of $\Phi_{\mathbf{v},\mathbf{t}}$ to be

$$V_\alpha(\rho) := \frac{\text{tr}(M_\alpha^\dagger \rho)}{\text{tr}(M_\alpha^\dagger M_\alpha)} M_\alpha \quad (6.70)$$

and the translation components to be

$$T_\alpha(\rho) := \frac{\text{tr}(\mathbb{I} \rho) M_\alpha}{D} \sqrt{\frac{D(D-1)}{\text{tr}(M_\alpha^\dagger M_\alpha)}} = \frac{1}{D} \sqrt{\frac{D(D-1)}{\text{tr}(M_\alpha^\dagger M_\alpha)}} M_\alpha \quad (6.71)$$

then we may rewrite $\Phi_{\mathbf{v},\mathbf{t}}$ as a sum of the compression and translation components

$$\Phi_{\mathbf{v},\mathbf{t}}(\rho) = \sum_{\alpha=0}^{D^2-1} v_\alpha V_\alpha(\rho) + t_\alpha T_\alpha(\rho) \quad (6.72)$$

Recall that if a channel χ can be written as $\chi(\rho) = \text{tr}(X^\dagger \rho) Y$ then the Choi-Jamolkowski representation of χ is $J(\chi) = Y \otimes X^*$. This allows us to find the

Choi-Jamiolkowski representations of the compression and translation components

$$J(V_\alpha) = \frac{M_\alpha \otimes M_\alpha^*}{\text{tr}(M_\alpha^\dagger M_\alpha)} \quad J(T_\alpha) = \frac{1}{D} \sqrt{\frac{D(D-1)}{\text{tr}(M_\alpha^\dagger M_\alpha)}} M_\alpha \otimes \mathbb{I} \quad (6.73)$$

which in turn enables us to find the Choi-Jamiolkowski representation of the channel $\Phi_{\mathbf{v},\mathbf{t}}$

$$J(\Phi_{\mathbf{v},\mathbf{t}}) = \sum_{\alpha=0}^{D^2-1} M_\alpha \otimes \left(\frac{v_\alpha M_\alpha^*}{\text{tr}(M_\alpha^\dagger M_\alpha)} + \frac{t_\alpha \mathbb{I}}{D} \sqrt{\frac{D(D-1)}{\text{tr}(M_\alpha^\dagger M_\alpha)}} \right) \quad (6.74)$$

In order to find which parameters (\mathbf{v}, \mathbf{t}) induce completely positive channels one has to find for which parameters all the eigenvalues of $J(\Phi_{\mathbf{v},\mathbf{t}})$ are non-negative. (Recall that these channels are automatically trace-preserving if $v_0 = 1$ and $t_0 = 0$.)

Example 19. *Let us work in the Pauli basis and consider a single-qubit map $\Phi_{\mathbf{v},\mathbf{t}}$. We fix the translation vector $\mathbf{t} = (0, 0, 0, t_z)$ and find the set of all compression vectors \mathbf{v} which induce completely positive channels $\Phi_{\mathbf{v},\mathbf{t}}$. We can write the Choi-Jamiolkowski representation of $\Phi_{\mathbf{v},\mathbf{t}}$ as*

$$J(\Phi_{\mathbf{v},\mathbf{t}}) = \frac{\mathbb{I} \otimes \mathbb{I}}{2} + v_x \frac{\sigma^x \otimes \sigma^x}{2} + v_y \frac{\sigma^y \otimes \sigma^y}{2} + v_z \frac{\sigma^z \otimes \sigma^z}{2} + t_z \frac{\sigma^z \otimes \mathbb{I}}{2} \quad (6.75)$$

It turns out that (for non-zero t_z) the eigenvectors of $J(\Phi_{\mathbf{v},\mathbf{t}})$ are no longer Bell-states, but rather linear combinations of them. Let

$$\begin{aligned} |\lambda_{0cd}\rangle &= c |\Psi_{00}\rangle + d |\Psi_{01}\rangle \\ |\lambda_{1cd}\rangle &= c |\Psi_{10}\rangle + d |\Psi_{11}\rangle \end{aligned} \quad (6.76)$$

where $c, d \in \mathbb{R}$ are real numbers, $c^2 + d^2 = 1$ and $|\Psi_{mn}\rangle$ (with $m, n \in \{0, 1\}$) are the Bell-states defined in section 6.2. One can easily check that

$$\begin{aligned} J(\Phi_{\mathbf{v},\mathbf{t}}) |\lambda_{0cd}\rangle &= \lambda_0 |\lambda_{0cd}\rangle \\ J(\Phi_{\mathbf{v},\mathbf{t}}) |\lambda_{1cd}\rangle &= \lambda_1 |\lambda_{1cd}\rangle \end{aligned} \quad (6.77)$$

where the eigenvalues satisfy the following relations

$$\begin{aligned} \lambda_0 c &= \frac{c}{2}(1 + v_x + v_y + v_z) + \frac{d}{2}t_z \\ \lambda_0 d &= \frac{d}{2}(1 - v_x - v_y + v_z) + \frac{c}{2}t_z \\ \lambda_1 c &= \frac{c}{2}(1 + v_x - v_y - v_z) + \frac{d}{2}t_z \\ \lambda_1 d &= \frac{d}{2}(1 - v_x + v_y - v_z) + \frac{c}{2}t_z \end{aligned} \quad (6.78)$$

We can solve this set of simultaneous equations to find the eigenvalues

$$\begin{aligned}\lambda_0^\pm &= \frac{1}{2}(1 + v_z) \pm \sqrt{(v_x + v_y)^2 + t_z^2} \\ \lambda_1^\pm &= \frac{1}{2}(1 - v_z) \pm \sqrt{(v_x - v_y)^2 + t_z^2}\end{aligned}\tag{6.79}$$

Note that there are two possible values for λ_0 and λ_1 : one for $d = \sqrt{1 - c^2}$ and one for $d = -\sqrt{1 - c^2}$; we have therefore found all four eigenvalues of $J(\Phi_{\mathbf{v}, \mathbf{t}})$. Clearly for $t_z \neq 0$ the surfaces $\lambda_0^\pm = 0$ and $\lambda_1^\pm = 0$ are curved. This reproduces the results of [69] which shows that the set of all compression vectors forms an “asymmetrically rounded tetrahedron”.

We could have worked through the above example with t_x or t_y non-zero but then the generic form of the eigenvectors of $J(\Phi_{\mathbf{v}, \mathbf{t}})$ is more complicated and takes on the form $|\lambda\rangle = c|\Psi_{00}\rangle + d|\Psi_{01}\rangle + e|\Psi_{10}\rangle + f|\Psi_{11}\rangle$ with the coefficients $c, d, e, f \in \mathbb{R}$ satisfying $c^2 + d^2 + e^2 + f^2 = 1$.

Based on the above example we note that even when we work in a basis where the set of all compression vectors forms a simplex for $\mathbf{t} = \mathbf{0}$, a tiny perturbation to $\mathbf{t} \neq \mathbf{0}$ destroys this simplex: the set of compression vectors \mathbf{v} forms a set with a curved boundary. In this situation there are infinitely many extremal channels.

6.8 Mixed unitary channels

In this section we briefly review the definitions of unital, doubly stochastic and mixed unitary quantum channels before quoting a theorem which says that averaging any doubly stochastic quantum channel with the completely depolarising channel results in a mixed unitary channel. Finally we conjecture a generalised version of this theorem involving arbitrary generalised depolarising channels.

A *unital* quantum channel acting on a D -dimensional quantum system maps the completely mixed state to itself $\Phi\left(\frac{\mathbb{I}}{D}\right) = \frac{\mathbb{I}}{D}$. A *doubly stochastic* quantum channel is a completely-positive, trace-preserving, unital channel. Note that all generalised depolarising channels are doubly stochastic.

A quantum channel Φ is said to be *mixed unitary* if there exist unitary matrices U_1, \dots, U_k and a probability distribution p_1, \dots, p_k (that is, $0 \leq p_j \leq 1$ and $\sum_j p_j = 1$) such that Φ is a convex sum of the unitary conjugation channels

$$\Phi(\rho) = \sum_{j=1}^k p_j U_j \rho U_j^\dagger \quad (6.80)$$

It is known that all doubly stochastic single qubit channels are mixed unitary, but that in higher dimensions ($D > 2$) there are doubly stochastic channels which are not mixed unitary. (These channels can however be expressed as an affine sum of unitary channels $\sum_{j=1}^k \lambda_j U_j \rho U_j^\dagger$ where $\lambda_1, \dots, \lambda_k$ are affine parameters: $\lambda_j \in \mathbb{R}$ and $\sum_j \lambda_j = 1$.)

However, the following theorem by Watrous [110] tells us that all doubly stochastic channels (and in particular those which are not mixed unitary) become mixed unitary when appropriately averaged with the isotropic completely depolarising channel $\Omega := \Phi_0$.

Theorem 20. *Let χ be any doubly-stochastic quantum channel acting on D -dimensional quantum systems. Then for any $0 \leq p \leq 1/(D^2 - 1)$ the channel*

$$p\chi + (1 - p)\Omega \quad (6.81)$$

is mixed unitary.

Proof. See [110]. □

It is tempting to think that Ω is not the only channel with which one can average to obtain a mixed unitary channel and we make this suggestion precise in the following conjecture.

Conjecture 21. *Let χ be any doubly-stochastic quantum channel acting on D -dimensional quantum systems and let Φ be any depolarising channel whose compression vector lies within a simplex whose vertices are the compression*

vectors of unitary conjugation depolarising channels (so that Φ is mixed unitary). Then there exists a constant $\delta(\Phi) > 0$ such that for $0 \leq p \leq \delta$ the following channel is mixed unitary

$$p\chi + (1 - p)\Phi \tag{6.82}$$

6.9 Summary

To summarise this chapter, we have made a detailed study of generalised depolarising channels defined with respect to trace-free, trace-orthogonal bases. Each depolarising channel has an associated compression vector lying in compression space, which is a vector space of dimension $D^2 - 1$ (we suppress the other dimension as the trace-preserving condition ensures that the first component of the compression vector is always equal to unity).

We have shown that the set of all compression vectors forms a simplex when we work in either the Pauli basis or the Heisenberg-Weyl basis, but that it does not form a simplex when we work in the Gell-Mann basis. Furthermore, for the Pauli and Heisenberg-Weyl bases we found the extremal channels whose compression vectors lie at the vertices of the corresponding simplex. Working in an arbitrary trace-free and trace-orthogonal basis, we found that the set of all compression vectors forms a simplex if and only if the basis matrices satisfy a certain condition. Furthermore, when the basis matrices were also unitary we were able to find the extremal channels.

We discussed the effects of changing basis and indicated why the set of compression vectors forms a simplex in some bases but not in others. Finally we generalised our methods to deal with a class of more general quantum channels and conjectured that one can average any doubly stochastic quantum channel with any mixed unitary generalised depolarising channel to obtain a mixed unitary channel.

Chapter 7

Conclusions

This chapter briefly summarises the main results of this thesis. We begin by summarising the effects of on-site disorder on the propagation of information through noisy quantum spin chains (§ 7.1) and finish by summarising the results about generalised depolarising channels (§ 7.2).

7.1 Noisy quantum spin chains

7.1.1 Summary of results

We have studied three fundamentally different models of noisy spin chains:

1. Nearest-neighbour XX -model spin chains of finite length experiencing static on-site disorder. The disordered field strengths are drawn from a probability distribution which has bounded probability density function.
2. Nearest-neighbour XX -model spin chains of infinite length (or spin rings of finite size) experiencing on-site disorder which is fixed in direction but fluctuates in strength independently on each site.
3. Finite length spin chains with arbitrary nearest-neighbour interactions experiencing on-site disorder which fluctuates in both strength and direction independently on each site.

These three noise models are all fundamentally different and result in profoundly different effects on the propagation of information:

1. Static disorder leads to Anderson localisation and consequently to exponential localisation of correlation functions, which in turn allows us to derive a new Lieb-Robinson bound which is much tighter than the original Lieb-Robinson bound. This new bound gives rise to a light cone whose radius grows logarithmically with time (in contrast with the original Lieb-Robinson bound which gives rise to a light cone whose radius grows linearly with time).
2. Fluctuating disorder which is fixed in direction results in a finite-strength quantum Zeno effect. This leads to extremely localised averaged dynamics where information can, on average, propagate by at most a constant number of sites before it is exponentially suppressed by the noise. Despite these averaged dynamics we find that individual realisations of the disorder permit less localised dynamics: information may propagate within a light cone whose radius grows in proportion to the square root of the time elapsed. (Note the similarity with a classical random walk.)
3. Disorder which fluctuates in strength and direction gives rise to a potential noise-threshold, below which information could potentially propagate ballistically (in accordance with the original Lieb-Robinson bound) and above which information is on average extremely localised, travelling by at most a constant number of sites before it is lost in the noise.

These results explain the phenomena observed in various numerical simulations of these noise models [81, 111]. The different light cones which result from the different noise regimes are illustrated schematically in figure 7.1.

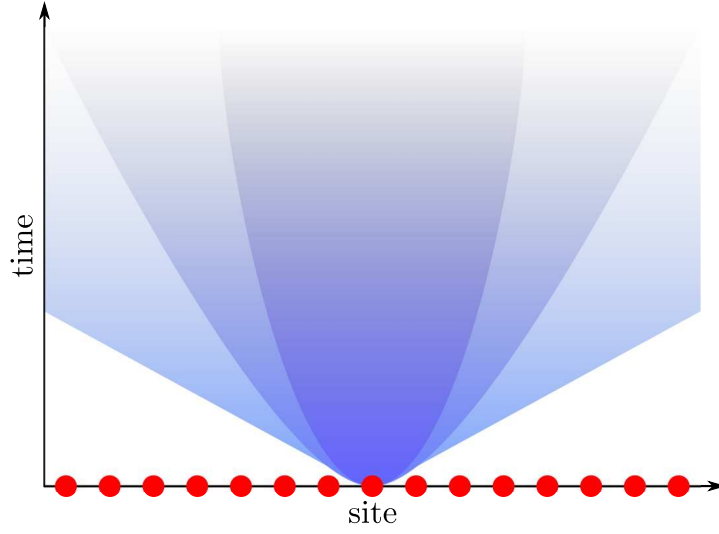


Figure 7.1: Schematic illustration of the different light cones which result from the different noise regimes: the *linear* (outer) light cone corresponding to the noise-free scenario; the *square-root* (middle) light cone resulting from fluctuating disorder whose direction is fixed; and the *logarithmic* (inner) light cone resulting from static disorder.

7.1.2 Comparison of the models

We have seen that the different models of on-site disorder have dramatically different effects upon the propagation of information through the spin chains. This is because there are fundamental differences between the models which we elaborate on below.

Whilst the first two models both preserve the number of excitations in the system they nevertheless produce dramatically different dynamics. In the case of static disorder the noise causes the correlation functions to be exponentially localised, a phenomenon known as Anderson localisation [74]. In contrast, in the model where the on-site disorder is fixed in direction but fluctuates in time, the noise essentially performs continuous finite-strength non-selective measurements on each site and this results in a finite-strength quantum Zeno effect. The root cause of the localisation in each of these two models is different, which explains the differing localisation properties.

The final noise model whose on-site disorder fluctuates in both strength and direction does not preserve the number of excitations in the spin chain: whenever the on-site disorder is oriented in any direction which is not the z -direction it induces bit-flip errors (that is, it creates or destroys excitations). It is unsurprising that this results in quite different localisation properties to those observed in the excitation-preserving models.

7.1.3 Open problems

In chapter 4 where we studied the model with on-site disorder which fluctuates in both strength and direction, we saw that the averaged dynamics are very localised: information can on average propagate by at most a constant number of sites. It would be interesting to determine the amount by which actual dynamics deviate from the averaged dynamics calculated here. It is not clear how one would go about this, but further work is nevertheless warranted.

It would also be interesting to discover if the noise threshold we identified for this model is a real physical phenomenon or merely a function of the way in which we calculated the bound. Numerical simulations should reveal the answer to this question, but either way there is further work to be done. If the threshold is real, why is it present and what causes it? If the threshold is not real, can our bound be extended to cover the low noise regimes too?

7.2 Generalised depolarising channels

We have seen that any quantum channel (in particular a quantum spin chain — whether noisy or noise-free) can be used as a generalised depolarising channel.

For each depolarising channel there is a unique compression vector lying in compression space and we have shown that the set of all compression vectors forms a simplex when we work in either the Pauli basis or the Heisenberg-Weyl basis, but not when we work in the Gell-Mann basis. For arbitrary

trace-free, trace-orthogonal bases we found a necessary and sufficient condition prescribing when the set of all compression vectors forms a simplex. In some bases we were able to find the extremal channels whose compression vectors lie at the vertices of the simplex. We also discussed the effects of changing basis (explaining why the set of compression vectors forms a simplex in some bases but not in others) and generalised our methods to deal with a more general class of quantum channels.

Further work is required to either prove or disprove our conjecture that one can average any doubly stochastic quantum channel with any mixed unitary generalised depolarising channel to obtain a mixed unitary channel.

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Addendum

This addendum briefly describes how equation 2.13 of this thesis is obtained from equation 6.27 of theorem 6.5 in [75].

Equation 6.27 of theorem 6.5 in [75] tells us that if H_ω is a standard ergodic random operator with IAD (independence at a distance) and both the SGEE (strong generalised eigenfunction expansion) and EDI (eigenfunction decay inequality) properties in an open interval \mathcal{I} , then for any $0 < \zeta < 1$, there is a finite constant C_ζ such that

$$\mathbb{E} \left\{ \sup_{\|f\| \leq 1} \|\chi_x f(H_\omega) E_\omega(I) \chi_0\|_2^2 \right\} \leq C_\zeta e^{-|x|^\zeta}$$

for all $x \in \mathbb{Z}^d$, the supremum being taken over all Borel functions f of a real variable, with $\|f\| = \sup_{t \in \mathbb{R}} |f(t)|$.

Note that in [75] the proof of theorem 6.5 is given for systems of infinite size; however the proof technique used can be adapted in a straight-forward manner to systems of finite size [112]: for large enough systems the proof of theorem 3.8 in [113] can be adapted to non-periodic boundary conditions by making small corrections for the boundaries. Note that the introduction section of [113] (below equation 1.5) also justifies the assumptions made here (and in theorem 6.5 of [75]).

Having justified that the above equation applies to the model studied in chapter 2 of this thesis, we now show how to use this to derive equation 2.13 of this thesis.

First note that, for convenience, equation 6.27 in [75] uses lattice sites x and 0 ; but as H_ω is ergodic and probabilities are therefore translation invariant, one could just as easily take any two sites j and k as we do here. \mathbb{E} simply represents an expectation over all possible realisations of the disorder and $E_\omega(I)$ is the spectral projection of an interval; in our case this reduces to the identity. χ_x denotes the characteristic function of the cube of side 1 centered at $x \in \mathbb{Z}^d$; in the one-dimensional case studied here, this reduces to a simple projector $|x\rangle\langle x|$.

Noting that e^{-itH} is a Borel function and employing Markov's inequality therefore allows us to use equation 6.27 of theorem 6.5 in [75] to conclude that with finite probability

$$\| |j\rangle\langle j| e^{itH} |k\rangle\langle k| \|_2^2 \leq \text{const} \times e^{-\text{const} \times |j-k|^\zeta} \quad \text{for all time } t$$

Note that by carefully examining the proof of theorem 6.5 in [75] we can precisely quantify this probability — see equation 2.13 and the associated text in this thesis.

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